Zeros of random orthogonal polynomials

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Random Nodal Domains
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Real zeros of random trigonometric polynomials

Dunnage, 1966: $\mathbb{E}\left[N_n([0,2\pi])\right] = \dfrac{2n}{\sqrt{3}} + o(n)$ *as* $n \to \infty$

for $T_n(x) = \sum_{k=0}^{n} A_k \cos(kx)$ with $A_k \sim \mathcal{N}(0,1)$.

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for $T_n(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (C_j \sin jx + D_j \cos jx)$ with $C_j, D_j \sim \mathcal{N}(0,1)$.

Granville&Wigman, 2010: $\mathrm{Var}[N_n([0,2\pi])] \sim cn$, $c \approx 0.55826$, as predicted by Bogomolny-Bohigas-Leboeuf. Moreover,

$$\frac{N_n([0,2\pi]) - \mathbb{E}[N_n([0,2\pi])]}{\sqrt{cn}} \xrightarrow{d} \mathcal{N}(0,1).$$

Universality results: Angst&Poly, 2015; Flasche, 2017; Iksanov&Kabluchko&Marynych, 2016; O.Nguyen&Vu, 2018; etc.

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Let $\{A_k\}_{k=0}^{\infty}$ be i.i.d. real Gaussian random variables and $\{p_k\}_{k=0}^{\infty}$ be orthonormal Legendre polynomials. Consider $P_n(x) = \sum_{k=0}^{n} A_k p_k(x)$.

Das, 1971, and Wilkins, 1997: $\mathbb{E}[N_n([-1,1])] = n/\sqrt{3} + o(n^{\varepsilon}),\ \varepsilon > 0$

$$\mathbb{E}N_n([a,b]) = \frac{1}{\pi}\int_a^b \sqrt{\frac{S_n(x)+R_n(x)}{D_n(x)} - \frac{1}{4}\frac{Q_n^2(x)}{D_n^2(x)}}\,dx,$$

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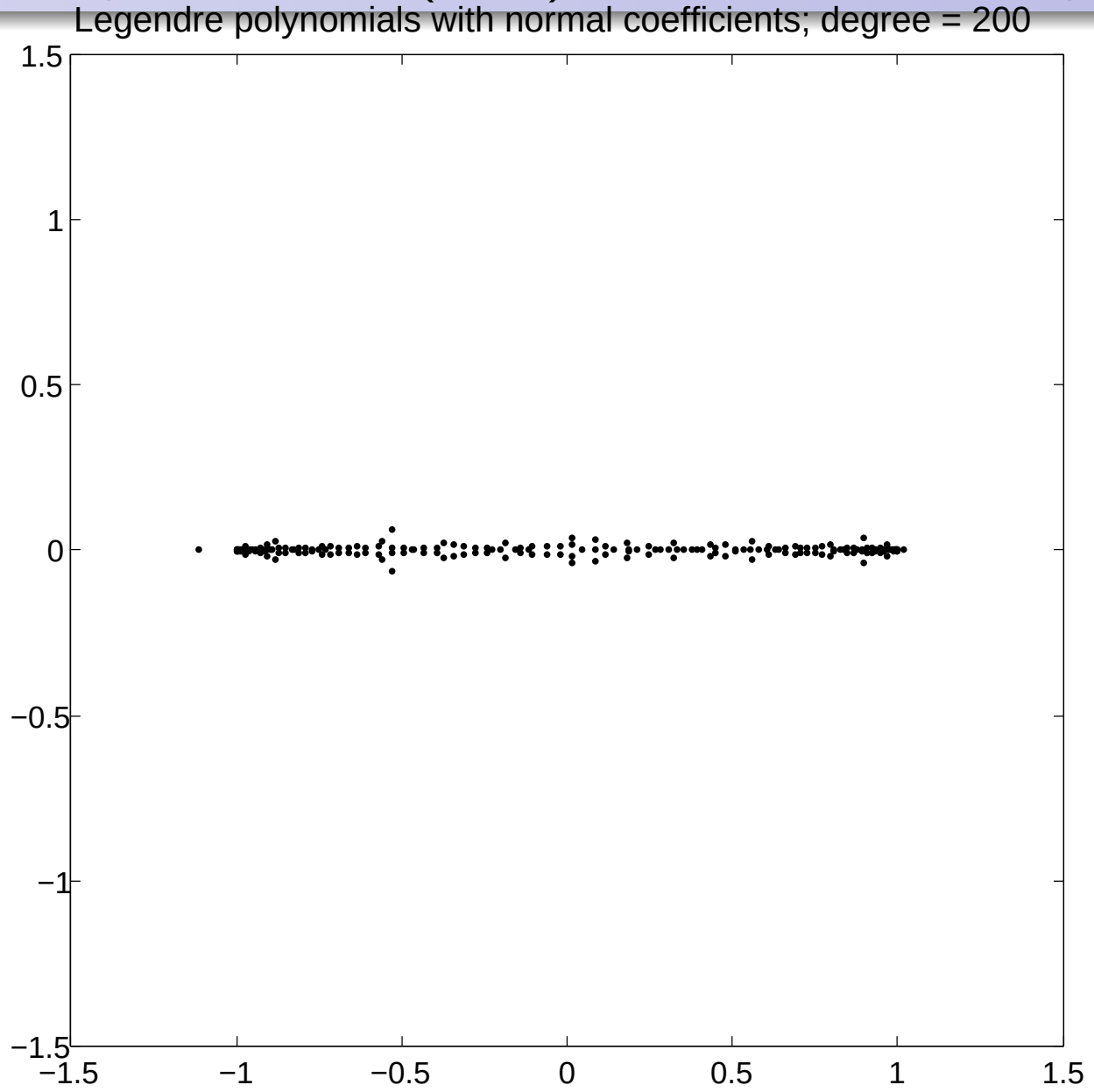
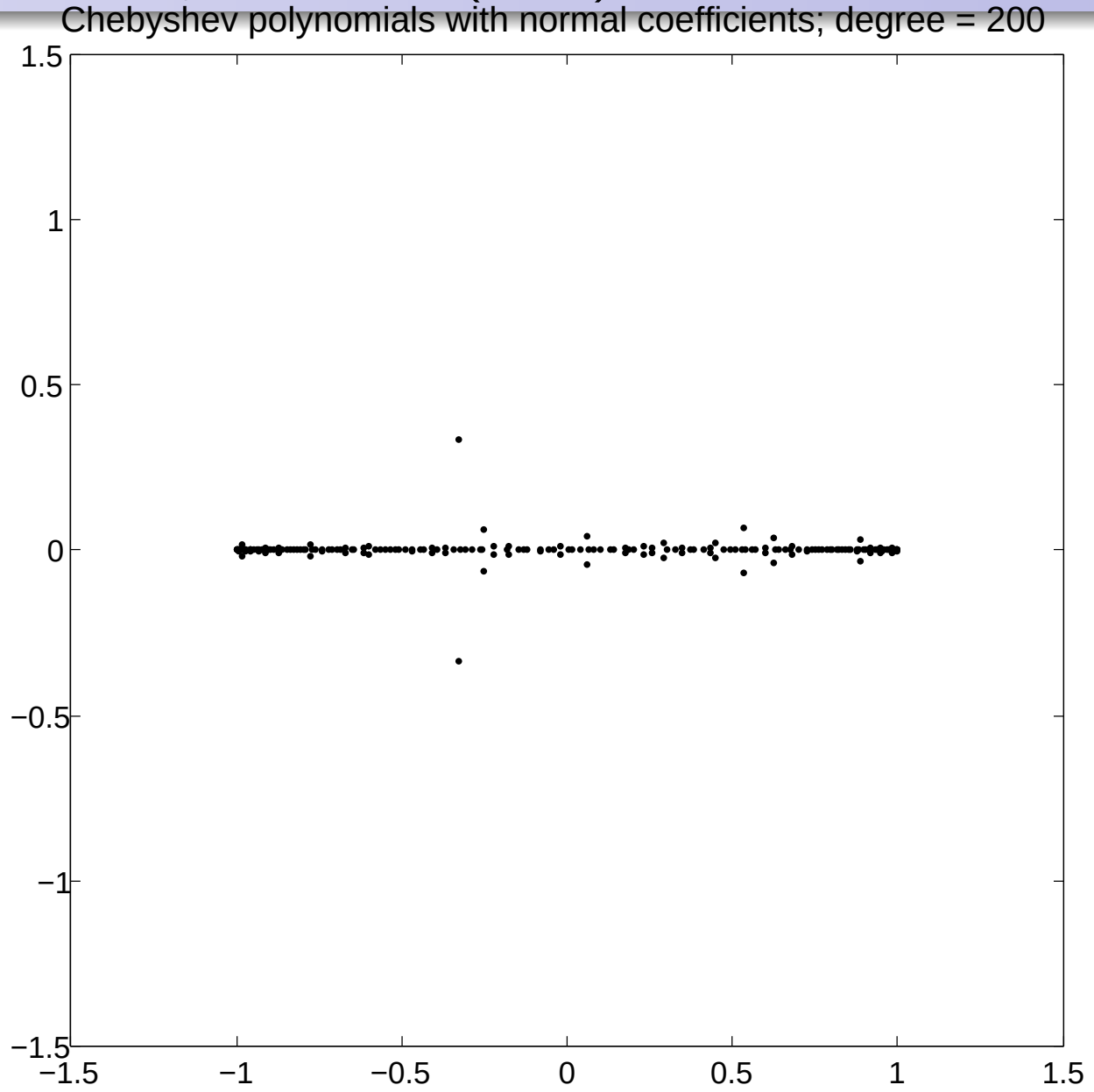
Random Legendre, $\mathcal{N}(0,1)$ coefficients, deg=200



Random Chebyshev, $\mathcal{N}(0,1)$ coefficients, deg=200



General random orthogonal polynomials on intervals

Let $K \subset \mathbb{R}$ be a finite union of compact intervals, and let μ be a Borel measure such that $d\mu(x) = w(x)dx$ and $w > 0$ a.e. on K. Let $\{p_k(x)\}_{k=0}^{\infty}$ be the orthonormal polynomials associated with μ.

Lubinsky&Pritsker&Xie, 2016: If $\forall\, \varepsilon > 0\ \exists\, S \subset K$ of Lebesgue measure $|S| < \varepsilon$ and $C > 1$ such that $C^{-1} < w < C$ a.e. on $K \setminus S$, then the random orthogonal polynomials $P_n(x) = \sum_{k=0}^{n} A_k p_k(x)$ with i.i.d. real Gaussian coefficients satisfy

$$\lim_{n\to\infty} \mathbb{E}[N_n(\mathbb{R})]/n = 1/\sqrt{3}.$$

Reproducing kernels: $K_{n+1}^{(r,s)}(x,y) = \sum_{k=0}^{n} p_k^{(r)}(x) p_k^{(s)}(y),\ r,s = 0,1.$

$$\mathbb{E}[N_n([a,b])] = \frac{1}{\pi}\int_a^b \sqrt{\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)} - \left[\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)}\right]^2}\, dx,\ [a,b] \subset \mathbb{R}.$$

Use asymptotic results for reproducing kernels on $K \setminus S$, and use potential theory to show that $\mathbb{E}[N_n(\mathbb{R} \setminus K)] = o(n)$.

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A local result for expected real zeros

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$$\lim_{n\to\infty} \mathbb{E}\left[N_n([a,b])\right]/n = \nu_K([a,b])/\sqrt{3}.$$

Proposition: Let $\{A_k\}_{k=0}^{\infty}$ be i.i.d. with $\mathbb{E}[\log^+ |A_0|] < \infty$, and $\tau_n := \frac{1}{n}\sum_{P_n(Z_k)=0} \delta_{Z_k}$. If $E \subset \mathbb{C}$ is a compact set s.t. $\nu_K(\partial E) = 0$, then

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Example: For $K = [c, d] \subset \mathbb{R}$, we have $d\nu_{[c,d]}(x) = \frac{dx}{\pi\sqrt{(x-c)(d-x)}}$

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Variance for real zeros in the bulk

Let $S(u) = \frac{\sin \pi u}{\pi u}$, and define

$$F(u) = \det \begin{bmatrix} 1 & S(u) & 0 & S'(u) \\ S(u) & 1 & -S'(u) & 0 \\ 0 & -S'(u) & -S''(0) & -S''(u) \\ S'(u) & 0 & -S''(u) & -S''(0) \end{bmatrix},$$

$$G(u) = \det \begin{bmatrix} 1 & S(u) & -S'(u) \\ S(u) & 1 & 0 \\ -S'(u) & 0 & -S''(0) \end{bmatrix},$$

$$H(u) = \det \begin{bmatrix} 1 & S(u) & 0 \\ S(u) & 1 & -S'(u) \\ S'(u) & 0 & -S''(u) \end{bmatrix},$$

$$\Phi(u) = \frac{1}{\pi^2} \left(\frac{\sqrt{F(u)}}{1 - S(u)^2} + \frac{1}{\left(1 - S(u)^2\right)^{3/2}} H(u) \arcsin\left(\frac{H(u)}{G(u)}\right) \right) - \frac{1}{3},$$

$$c_v = \int_{-\infty}^{\infty} \Phi(u)\, du + \frac{1}{\sqrt{3}}.$$

Variance for real zeros in the bulk

Let μ be the orthogonality measure, and let $\nu_K(x)$ be the equilibrium measure of $K = \operatorname{supp}\mu$. We consider random orthogonal polynomials $P_n(x) = \sum_{k=0}^{n} A_k p_k(x)$ with i.i.d. real Gaussian coefficients.

Lubinsky&Pritsker, 2021: Suppose that μ is absolutely continuous, and μ' is positive and continuous on $[a', b'] \subset K$. Assume that

$$\sup_{n\geq 1} \|p_n\|_{L_\infty[a',b']} < \infty.$$

If $[a, b] \subset (a', b')$ then

$$\lim_{n\to\infty} \operatorname{Var}\left[N_n([a,b])\right]/n = c_v\, \nu_K([a,b]).$$

Corollary. Let μ be absolutely continuous on $K = [-1, 1]$, satisfy

$$\int_{-1}^{1} \log \mu'(x)\, \frac{dx}{\pi\sqrt{1-x^2}} > -\infty.$$

If μ' is positive and Dini-continuous on $[a', b'] \subset K$, then for any $[a, b] \subset [a', b']$ we have

$$\lim_{n\to\infty} \operatorname{Var}\left[N_n([a,b])\right]/n = c_v \frac{\arcsin b - \arcsin a}{\pi}.$$

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Let $w(x) = \exp(-c|x|^\lambda)$ on $\mathbb{R}$, where $c > 0,\ \lambda > 1$.

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Scaled polynomials $P_n^*(s) := P_n(a_n s)$, where $a_n = \gamma_\lambda^{\frac{1}{\lambda}} c^{-\frac{1}{\lambda}} n^{\frac{1}{\lambda}}$ is the MRS number for w and $\gamma_\lambda = \Gamma(\frac{1}{2})\Gamma(\frac{\lambda}{2})/\Gamma(\frac{\lambda+1}{2})$.

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If $[a,b] \subset (-1,1)$ is any closed interval, then

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Example: If $\lambda = c = 2$ then $a_n = \sqrt{n}$ and $d\mu_2(s) = \frac{2}{\pi}\sqrt{1-s^2}\,ds$.

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