Conjunction probability of smooth stationary Gaussian fields

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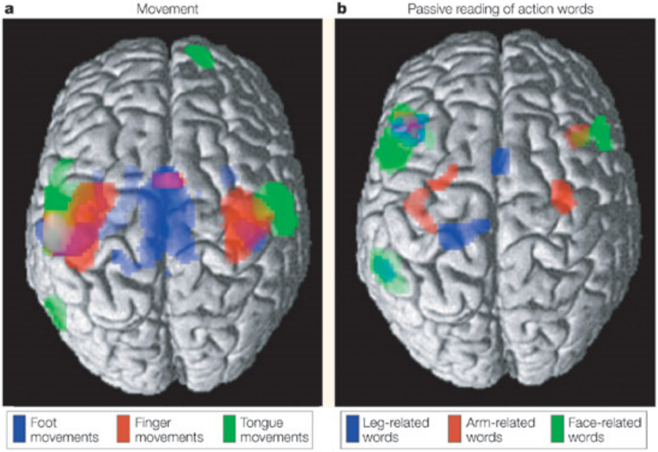
Content

1. Distribution of maximum of Gaussian fields
   - Motivation
   - Euler charactersitic method
   - Non locally convex set

2. Conjunction probability
   - Motivation
   - Heuristic Euler characteristic method
   - Main result

Motivation

- Detect the region on the brain w.r.t the action (Worsley)



Distribution of maximum of Gaussian fields

- Hypothesis testing (at significant level α)

$$\begin{cases} H_0 : \textit{no reaction} \\ H_1 : \textit{reaction} \end{cases}$$

- Problem setting: $\{X(t),\ t \in S \subset \mathbb{R}^d\}$ centered **stationary Gaussian** fields with $\text{Var}(X(t)) = 1,\ \text{Var}(X'(t)) = I_d$.

Distribution of $M_S = \max_{t \in S} X(t)$?,

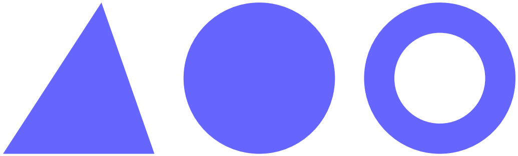
For what threshold u, $\text{P}(M_S \geq u) = \alpha$?

Some methods

- Tube method: Sun, Takemura and Kuriki
- Double-sum method: Piterbarg, Hashorva, Debicki,...
- Euler charactersitic method: Adler and Taylor
- Rice method: Wschebor and Azaïs, Delmas,...

Euler characteritic through Steiner formula

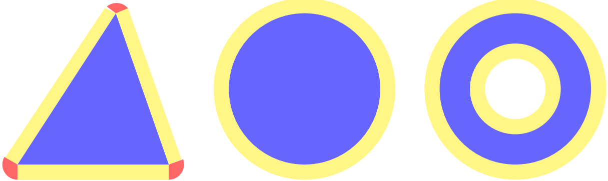
For $S \subset \mathbb{R}^2$,



define ϵ-neighborhood

$$S^\epsilon = \{t \in \mathbb{R}^2 : \text{dist}(t, S) \leq \epsilon\}$$

Steiner formula



$$\lambda_2(S^\epsilon) = \lambda_2(S) + \mathrm{peri}(S)\epsilon + \chi(S)\pi\epsilon^2,$$

where $\chi(S) = \#$ connected components - $\#$ holes,
$\chi(S)$: Euler-Poincaré characteristic of S.

Steiner formula

In general, for $S \subset \mathbb{R}^d$ and $\epsilon < r(S)$ critical radius,

$$\lambda_d(S^\epsilon) = \sum_{j=0}^{d} \mathcal{L}_j(S)\omega_{d-j}\epsilon^{d-j},$$

+ $\mathcal{L}_j(S)$: Minkowski functionals or Lipschitz-Killing curvatures. Example: $\mathcal{L}_d(S) = vol(S)$, $\mathcal{L}_{d-1}(S)$: half of surface area.
+ $\mathcal{L}_0(S) = \chi(S)$, Euler-Poincaré characteristic of S.
+ ω_{d-j}: volume of $d - j$ dimensional unit ball.

Euler-Poincaré characteristic method (Adler-Taylor-Worsley)

- Consider the excursion set

$$C_u = \{t \in S : \ X(t) \geq u\}.$$

- For u large enough,

$$\textit{Heuristic}: \ M_S \geq u \Leftrightarrow C_u \neq \emptyset \Leftrightarrow \chi(C_u) = 1.$$

- Then, by Gaussian kinematic formula,

$$\mathrm{P}(M_S \geq u) \approx \mathrm{E}(\chi(C_u)) = \sum_{j=0}^{d} (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2} \mathcal{L}_j(S),$$

where

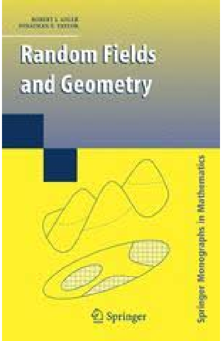
+ $H_{j-1}(.)$: Hermite polynomials.
+ $\mathcal{L}_j(S)$: Minkowski functionals or Lipschitz-Killing curvatures.

Validity of locally convex set

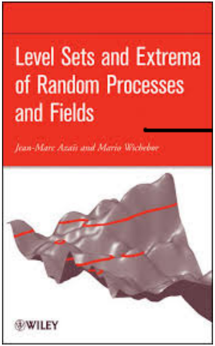
- Validity of EPC method: for S tamed and locally convex,

$$|\mathrm{P}(M_S \geq u) - \mathrm{E}(\chi(C_u))| \leq C \exp\left(\frac{-u^2(1+\delta)}{2}\right).$$

- My bibles



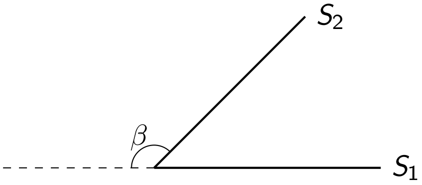
(g) Adler-Taylor



(h) Azaïs-Wschebor

How about non-locally convex set

- Example (Adler and Taylor, Chapter 14)



- By inclusion-exlusion pinciple,

$$P(M_{S_1 \cup S_2} \geq u) = P(M_{S_1} \geq u) + P(M_{S_2} \geq u) - P(\{M_{S_1} \geq u\} \cap \{M_{S_2} \geq u\})$$

Idea

Lemma (Azaïs-Wschebor 2012)

Let B be an open neighborhood of S. Then with high probability,

- *At most one local maximum point t_0 such that $X(t_0) > u$.*
- *The excursion set $C_B(u) = \{t \in B : X(t) \geq u\}$ satisfies*

$$B(t_0, \underline{r}) \subset C_B(u) \subset B(t_0, \overline{r}),$$

where $\underline{r} = \sqrt{2\frac{X(t_0) - u}{X(t_0) + u^\alpha}}$ *and* $\overline{r} = \sqrt{2\frac{X(t_0) - u}{X(t_0) - u^\alpha}}$ *for some* $0 < \alpha < 1$.

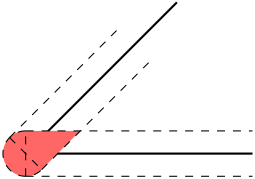
If $M_S > u$ then $S \cap C_B(u) \neq \emptyset$.

+ Upper bound: $S \cap B(t_0, \overline{r}) \neq \emptyset$ or $t_0 \in S^{\overline{r}}$.
+ Lower bound: $S \cap B(t_0, \underline{r}) \neq \emptyset$ or $t_0 \in S^{\underline{r}}$.

Back to example

$$(t_0 \in S_1^{r} \cap S_2^{r}) \subset (\{M_{S_1} \geq u\} \cap \{M_{S_2} \geq u\}) \subset (t_0 \in S_1^{\bar{r}} \cap S_2^{\bar{r}})$$

$\rightarrow$ Idea: Find maximum point t_0 in the intersection $S_1^r \cap S_2^r$



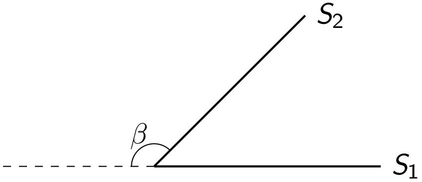
Lemma

If $\lambda_d\left(S_1^{+\epsilon} \cap \ldots \cap S_m^{+\epsilon}\right) = (C + o(1))\,\epsilon^{d-k}$ *as* $\epsilon \to 0$. *Then, as* $u \to +\infty$,

$$\mathrm{P}\left(\forall i = 1 \ldots m: \; M_{S_i} \geq u\right) = u^{k-1}\varphi(u)\left(\frac{C}{2^{k/2}\pi^{d/2}}\Gamma\left(1 + \frac{d-k}{2}\right) + o(1)\right)$$

where Γ *is the Gamma function.*

Back to example



$$P(\{M_{S_1} \geq u\} \cap \{M_{S_2} \geq u\}) = \left(1 + \frac{\tan(\beta/2) - \beta/2}{\pi}\right) \overline{\Phi}(u) + o\left(\overline{\Phi}(u)\right),$$

$$P(M_S \geq u) = \frac{\lambda_1(S_1) + \lambda_1(S_2)}{\sqrt{2\pi}} \varphi(u) + \left(1 - \frac{\tan(\beta/2) - \beta/2}{\pi}\right) \overline{\Phi}(u) + o\left(\overline{\Phi}(u)\right.$$

General case in $\mathbb{R}^2$

Theorem (H. P. and Azaïs 2016)

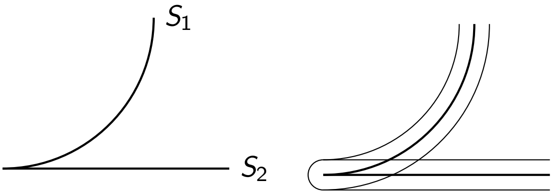
Under some mild conditions, if there exist two constants L_0, L_1 such that as $\epsilon \to 0$,

$$\lambda_2(S^{+\epsilon}) = \lambda_2(S) + \epsilon L_1(S) + \pi\epsilon^2 L_0(S) + o(\epsilon^2),$$

then

$$\mathrm{P}(M_S \geq u) = \lambda_2(S)\frac{u\varphi(u)}{2\pi} + L_1(S)\frac{\varphi(u)}{2\sqrt{2\pi}} + L_0(S)\overline{\Phi}(u) + o\left(u^{-1}\varphi(u)\right).$$

Another example

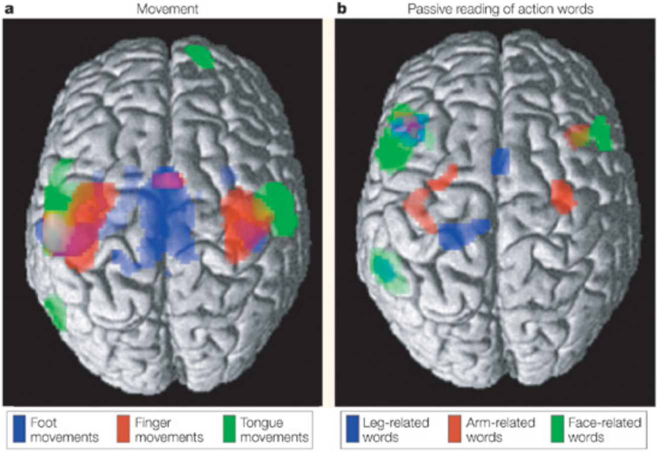


$$P(M_{S_1 \cup S_2} \geq u) = \frac{3\overline{\Phi}(u)}{2} - \frac{8\sqrt{R}}{2^{1/4}3\pi}\Gamma(7/4)u^{-1/2}\varphi(u) + \frac{\lambda_1(S_1) + \lambda_1(S_2)}{\sqrt{2\pi}}\varphi(u) + o\left(u^{-1}\varphi(u)\right).$$

Open: In general, once we know the Steiner formula, whether the corresponding asymptotic formula for the tail of the maximum follows?

Motivation

- Detect the region on the brain w.r.t the action (Worsley)



Conjunction probability

- Question: whether the functional organization of the brain differs according to sex?
- Modelling: Consider two random fields $X_1(t)$ for female and $X_2(t)$ for male.
- General problem: $\{X_i(t), 1 \leq i \leq n\}$: i.i.d. centered stationary Gaussian field and both defined on a compact set $S \subset \mathbb{R}^d$.

$$\mathrm{P}\left(\sup_{t \in S} \min_{1 \leq i \leq n} X_i(t) \geq u\right),$$

$\rightarrow$ Conjunction probability.

Euler characteristic method (Worsley and Friston 2000)

- Consider the excursion set $C_u = \{t \in S : X_i(t) \geq u, \forall 1 \leq i \leq n\}$.
- Then heuristically,

$$\mathrm{P}\left(\sup_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) \approx \mathrm{E}(\chi(C_u)) = (1,0,\ldots,0)R^n\mathcal{L}(S)$$

where

$$R = \begin{pmatrix} \rho_0/b_0 & \rho_1/b_1 & \ldots & \rho_d/b_d \\ 0 & \rho_0/b_0 & \ldots & \rho_{d-1}/b_{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \rho_0/b_0 \end{pmatrix},$$

$\rho_i = (2\pi)^{-(i+1)/2}H_{i-1}(u)e^{-u^2/2}$, $b_i = \Gamma((i+1)/2)/\Gamma(1/2)$, and $\mathcal{L}(S) = (\mathcal{L}_0(S)b_0, \ldots, \mathcal{L}_d(S)b_d)$.
- No validity!

Main result

Theorem (H. P. (2021+))

Let $X_i(t), 1 \leq i \leq n$, be independent copies of a Gaussian field. Then as u tends to infinity,

$$P\left(\max_{t\in S}\min_{1\leq i\leq n} X_i(t) \geq u\right) = (2\pi)^{-d/2}\varphi^n(u)u^{d-n}\lambda_d(S)\times$$
$$\left[\sum_{0\leq h_1\leq\ldots\leq h_{n-1}\leq d} \frac{\Gamma(\frac{1}{2})^{n-1}\Gamma(\frac{d+1}{2})}{\Gamma(\frac{h_1+1}{2})\Gamma(\frac{h_2-h_1+1}{2})\ldots\Gamma(\frac{d-h_{n-1}+1}{2})} + o(1)\right].$$

Sketch proof

$\rightarrow$ Idea: Find a tuple $(t_1, \ldots, t_n)$ of maximum points such that

$$B(t_1, r_1) \cap \ldots \cap B(t_n, r_n) \cap S \neq \emptyset.$$

Lemma

Assume that for a fixed point t_1 and small enough $r_1, r_2, \ldots, r_n$, there exist constants $k > 0$ and C_m such that

$$\lambda_{(n-1)d}\left((t_2, \ldots, t_n) : \bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset\right) = \sum_{\|m\|=k} C_m r^m.$$

Then, as u tends to infinity,

$$\mathrm{P}\left(\max_{t \in S} \min_{1 \leq i \leq n} X_i(t) \geq u\right)$$
$$= u^{nd-n-k} \varphi^n(u) \left(\frac{2^{k/2} \lambda_d(S)}{(2\pi)^{nd/2}} \sum_{\|m\|=k} C_m \prod_{i=1}^{n} \Gamma(1 + m_i/2) + o(1)\right).$$

Example

- $n = 2$, then $B(t_1, r_1) \cap B(t_2, r_2) \neq \emptyset$ implies that $t_2 \in B(t_1, r_1 + r_2)$,

$$\lambda_d \left(B(0, r_1 + r_2) \right) = \omega_d (r_1 + r_2)^d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \sum_{j=0}^{d} \binom{d}{j} r_1^j r_2^{d-j}.$$

- In general,

Lemma

For fixed point t_1 and $r_1, r_2, \ldots, r_n > 0$ small enough,

$$\lambda_{(n-1)d} \left((t_2, \ldots, t_n) \in \mathbb{R}^{d(n-1)} : \bigcap_{1 \le i \le n} B(t_i, r_i) \neq \emptyset \right)$$

$$= \sum_{k_n=0}^{d} \sum_{k_{n-1}=d-k_n}^{d} \cdots \sum_{k_2=(n-2)d-(k_n+k_{n-1}+\ldots+k_3)}^{d} r_1^{(n-1)d-\sum_{i=2}^n k_i} \prod_{i=2}^{n} \left(r_i^{k_i} \omega_{k_i} \right) \times$$

$$\frac{\omega_d \omega_{(n-1)d-\sum_{i=2}^n k_i}}{\omega_{\sum_{i=2}^n k_i-(n-2)d} \prod_{i=2}^n \omega_{d-k_i}} \times \frac{d!}{[\sum_{i=2}^n k_i - (n-2)d]!. \prod_{i=2}^n (d-k_i)!}.$$

Examples

- $d = 1$:

$$\lambda_{n-1}\left(\left(t_2,\ldots,t_n\right)\in\mathbb{R}^{n-1} : \bigcap_{1\le i\le n} B(t_i,r_i)\neq\emptyset\right) = 2^{n-1}\sum_{i=1}^{n}\left(\prod_{j\neq i} r_j\right).$$

$$\rightarrow \mathrm{P}\left(\max_{t\in[0,T]}\min_{1\le i\le n} X_i(t)\ge u\right) = u^{-(n-1)}\varphi^n(u)\left(\frac{nT}{\sqrt{2\pi}}+o(1)\right).$$

- $d = 2$:

$$\lambda_{2(n-1)}\left(\left(t_2,\ldots,t_n\right)\in\mathbb{R}^{2(n-1)} : \bigcap_{1\le i\le n} B(t_i,r_i)\neq\emptyset\right)$$

$$=\pi^{n-1}\sum_{i=1}^{n}\left(\prod_{j\neq i} r_j^2\right) + 2\pi^{n-1}\sum_{1\le i<j\le n}\left(r_i r_j \prod_{k\neq i,j} r_k^2\right).$$

$$\rightarrow \mathrm{P}\left(\max_{t\in S}\min_{1\le i\le n} X_i(t)\ge u\right) = u^{2-n}\varphi^n(u)\left[\frac{\lambda_2(S)}{2\pi}\left(n+\frac{n(n-1)\pi}{4}\right)+o(1)\right]$$

Open questions

Could we prove the full validity of Euler characteristic method for the conjunction probability as for the tail distribution of the maximum of smooth Gaussian field?

Thank you!