# Geometric Functionals of Multiple Arithmetic Random Waves, Berry's Cancellation & Wiener Chaos

**NOTARNICOLA MASSIMO**, *University of Luxembourg*

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- Asymptotic study of **local geometric quantities** associated with **random Laplace eigenfunctions** on manifolds (e.g. random spherical harmonics, arithmetic random waves, monochromatic random waves; Zelditch (2009); Canzani & Hanin (2016); Marinucci, Peccati, Rossi & Wigman (2016); Cammarota (2019); Dalmao, Nourdin, Peccati & Rossi (2019))
- Study by means of **Wiener chaos techniques**; typically, the high-energy behaviour of chaos expansions of such geometric objects is **dominated by the projection on a single Wiener chaos**, which is tightly connected with the **Berry's cancellation phenomenon**
- We focus on **Arithmetic Random Waves** (ARW) on the three-dimensional torus (Oravecz, Rudnick & Wigman (2008)) and are interested in characterizing probabilistic fluctuations of two local geometric functionals associated with vectors of ARW: (i) **volumes of zero sets** (nodal volumes) and (ii) notions of **total variation**

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# Outline of the talk

1. Arithmetic Random Waves
2. Nodal Volumes of Arithmetic Random Waves
3. Total Variation of Arithmetic Random Waves

- Eigenvalues of $-\Delta$ on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ are numbers of the form $4\pi^2 n$ with

$$n \in S(d) := \left\{k \in \mathbb{N} : \exists(\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d : \lambda_1^2 + \ldots + \lambda_d^2 = k\right\}.$$

  For $n \in S(d)$, we define the **set of frequencies**

$$\Lambda_n^{(d)} := \left\{\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d : \mu_1^2 + \ldots + \mu_d^2 = n\right\}$$

  with $\mathcal{N}_n^{(d)} := |\Lambda_n^{(d)}|$. The corresponding eigenspaces are generated by complex exponentials $\exp(2\pi i\langle\lambda, x\rangle), x \in \mathbb{T}^d, \lambda \in \Lambda_n^{(d)}$.

- **Arithmetic Random Waves** are Gaussian Laplace eigenfunctions of the form

$$T_n(x) = \frac{1}{\sqrt{\mathcal{N}_n^{(d)}}} \sum_{\lambda \in \Lambda_n^{(d)}} a_\lambda \exp(2\pi i\langle\lambda, x\rangle), \quad x \in \mathbb{T}^d,$$

  where $\{a_\lambda : \lambda \in \Lambda_n\}$ is a collection of complex Gaussian random variables that are independent except for the relation $a_\lambda = \overline{a_{-\lambda}}$.

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- Equivalently, $T_n = \left\{T_n(x) : x \in \mathbb{T}^d\right\}$ is a real-valued centred stationary Gaussian field with covariance function

$$r_n^{(d)}(x-y) = \mathbb{E}\left[T_n(x)T_n(y)\right] = \int_{\mathbb{S}^{d-1}} \exp\left(2\pi i \langle \sqrt{n}\theta, x-y\rangle\right) \mu_n^{(d)}(d\theta),$$

where

$$\mu_n^{(d)} = \frac{1}{\mathcal{N}_n^{(d)}} \sum_{\lambda \in \Lambda_n^{(d)}} \delta_{\frac{\lambda}{\sqrt{n}}}.$$

- **Linnik's problem** on the angular distribution of lattice points on spheres (Linnik (1968)): **equidistribution of lattice points** in dimension three (Golubeva & Fomenko (1987); Duke (1990)), vs. **non-trivial attainable measures** along generic sequences in dimension two (Kurlberg & Wigman (2017))

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- from now on, we denote $\mathcal{N}_n^{(3)} = \mathcal{N}_n, \Lambda_n^{(3)} = \Lambda_n, r_n^{(3)} = r_n$, etc.
- asymptotic analysis (as $\mathcal{N}_n \to \infty$) of geometric functionals of **multiple ARW**

$$\mathbf{T}_n^{(\ell)} := \left\{ \mathbf{T}_n^{(\ell)}(x) := (T_n^{(1)}(x), \dots, T_n^{(\ell)}(x)) : x \in \mathbb{T}^3 \right\}, \quad \ell = 1, 2, 3$$

where $\left\{ T_n^{(i)} \right\}_{i=1,\dots,\ell}$ are independent Arithmetic Random Waves

- For $\ell = 1, 2, 3$, we define **nodal volume**

$$\mathscr{L}_n^{(\ell)} := \mathcal{H}_{3-\ell}\left( \left\{ x \in \mathbb{T}^3 : \mathbf{T}_n^{(\ell)}(x) = \mathbf{0} \right\} \right);$$

in view of the Area/Co-Area formula, one has the representation

$$\mathscr{L}_n^{(\ell)} = \left( \frac{4\pi^2 n}{3} \right)^{\ell/2} \int_{\mathbb{T}^3} \delta_0(\mathbf{T}_n^{(\ell)}(x)) \Phi\left( \widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(x) \right) dx, \quad \Phi(M) := \sqrt{\det(MM^T)}$$

(surface area, length of dislocation lines and number of phase singularities)

- For $\ell = 1, 2, 3$, we define the **total variation**

$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3) := \left( \frac{4\pi^2 n}{3} \right)^{\ell/2} \int_{\mathbb{T}^3} \Phi\left( \widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(x) \right) dx, \quad \Phi(M) := \sqrt{\det(MM^T)}$$

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- For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, the **$k$-th Hermite polynomial** is

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2};$$

$\left\{(k!)^{-1/2} H_k\right\}_{k \geq 0}$ forms a complete orthonormal system of $L^2(\gamma), \gamma(dz) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$

- For any generic separable Gaussian field $\mathbb{G}$,

$$L^2(\Omega, \sigma(\mathbb{G}), \mathbb{P}) = \bigoplus_{q \geq 0} \mathbf{C}_q, \quad \text{i.e.} \quad F = \sum_{q \geq 0} \text{proj}(F | \mathbf{C}_q),$$

where $\mathbf{C}_q$ is the **$q$-th Wiener chaos associated with $\mathbb{G}$** (Wiener chaos expansion)

- **(Fourth Moment Theorem, Nualart & Peccati, 2005)** Let $q \geq 2$ and $\{F_n\}_{n \geq 1} \subset \mathbf{C}_q$.

$$F_n \xrightarrow{\text{LAW}} Z \sim \mathcal{N}(0,1) \iff \mathbb{E}\left[F_n^4\right] \to \mathbb{E}\left[Z^4\right] = 3 \quad (n \to \infty)$$

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- We study the local geometric functional

$$\mathscr{L}_n^{(\ell)} := \mathcal{H}_{3-\ell}\left(\left\{x \in \mathbb{T}^3 : \mathbf{T}_n^{(\ell)}(x) = \mathbf{0}\right\}\right), \quad \ell = 1, 2, 3$$

- The **nodal surface area** (corresponding to the case $\ell = 1$) has been investigated

$$\mathscr{L}_n^{(1)} = \mathcal{H}_2\left(\left\{x \in \mathbb{T}^3 : T_n(x) = 0\right\}\right), \quad \mathcal{N}_n \to \infty$$

- **(Rudnick & Wigman, 2008)** For all $n \in S(3)$,

$$\mathbb{E}\left[\mathscr{L}_n^{(1)}\right] = \sqrt{\frac{4\pi^2 n}{3}}\,\frac{2}{\sqrt{\pi}}$$

- **(Benatar & Maffucci, 2019)** As $\mathcal{N}_n \to \infty$,

$$\mathbf{Var}\left[\mathscr{L}_n^{(1)}\right] \sim \frac{4\pi^2 n}{\mathcal{N}_n^2}\,\frac{8}{375}$$

- **(Cammarota, 2019)** As $\mathcal{N}_n \to \infty$,

$$\widetilde{\mathscr{L}_n^{(1)}} := \frac{\mathscr{L}_n^{(1)} - \mathbb{E}\left[\mathscr{L}_n^{(1)}\right]}{\sqrt{\mathbf{Var}\left[\mathscr{L}_n^{(1)}\right]}} \xrightarrow{\text{LAW}} \frac{1}{\sqrt{10}}\left(5 - \chi^2(5)\right)$$

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## *Theorem I* (Notarnicola, ALEA (2021))

For integers $1 \leq \ell \leq k$, we set $\alpha(\ell,k) := \frac{(k)_\ell \kappa_k}{(2\pi)^{\ell/2}\kappa_{k-\ell}}$.

**(Expected Volume)** For all $n \in S(3)$, $\mathbb{E}\left[\mathscr{L}_n^{(\ell)}\right] = \left(\frac{4\pi^2 n}{3}\right)^{\ell/2} \frac{\alpha(\ell,3)}{(2\pi)^{\ell/2}}$

**(Asymptotic Variance Law)** As $\mathcal{N}_n \to \infty$,

$$\mathbf{Var}\left[\mathscr{L}_n^{(\ell)}\right] \sim \frac{(4\pi^2 n)^\ell}{\mathcal{N}_n^2} \cdot \Xi(\ell), \quad \Xi(\ell) = \frac{4\alpha(\ell,3)^2}{3^\ell (2\pi)^\ell}\left(\ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2} \cdot \frac{76}{375}\right)$$

**(Limiting distribution)** As $\mathcal{N}_n \to \infty$,

$$\widetilde{\mathscr{L}_n^{(\ell)}} \xrightarrow{\text{LAW}} \left(\ell \cdot \frac{1}{250} + \frac{\ell(\ell-1)}{2} \cdot \frac{76}{375}\right)^{-1/2} Y^{(\ell)} M^{(\ell)} (Y^{(\ell)})^T =: \mathscr{L}_\infty^{(\ell)},$$

where $Y^{(\ell)} \sim \mathcal{N}_{\ell(9\ell-4)}(0, \mathbf{I}_{\ell(9\ell-4)})$ and $M^{(\ell)} \in \mathcal{M}_{\ell(9\ell-4)\times\ell(9\ell-4)}(\mathbb{R})$ is given by

$$M^{(\ell)} = \frac{-1}{50}\mathbf{I}_{5\ell} \oplus \frac{-1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{25}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{1}{50}\mathbf{I}_{\frac{5\ell(\ell-1)}{2}} \oplus \frac{-1}{6}\mathbf{I}_{\frac{3\ell(\ell-1)}{2}} \ .$$

- for $\ell = 1$, we recover the results by Benatar & Maffucci (2019) and Cammarota (2019): $\Xi(1) = 8/375$ and $\mathscr{L}_\infty^{(1)} \stackrel{d}{=} \frac{1}{\sqrt{10}}(5 - \chi^2(5))$

# Outline of the proof of Theorem I

- follows a common strategy relying on **Wiener chaos expansions** exploited in a series of works [among others: (Kratz & León (2001); Marinucci, Peccati, Rossi & Wigman (2016); Cammarota (2019); Marinucci, Rossi & Wigman (2020)]
- **(Area/Co-Area formula)** Both $\mathbb{P}$-a.s and in $L^2(\mathbb{P})$,

$$\mathscr{L}_n^{(\ell)} = \left(\frac{4\pi^2 n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} \delta_0(\mathbf{T}_n^{(\ell)}(x)) \Phi\left(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(x)\right) dx, \quad \Phi(M) = \sqrt{\det(MM^T)}$$

- **(Wiener chaos expansion)** In $L^2(\mathbb{P})$,

$$\mathscr{L}_n^{(\ell)} = \sum_{q=0}^{\infty} \mathscr{L}_n^{(\ell)}[2q], \quad \mathscr{L}_n^{(\ell)}[2q] := \mathrm{proj}(\mathscr{L}_n^{(\ell)} | \mathbf{C}_{2q})$$

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# Detour: Berry's Random Plane Waves (M. Berry, 1977, 2002)

- Berry's random plane waves (BRW) $\{b_k(x) : x \in \mathbb{R}^2\}$ are **isotropic** Gaussian Laplace eigenfunctions on $\mathbb{R}^2$ (with eigenvalue $-k^2$) with covariance function

$$\rho_k(x-y) = J_0(k\|x-y\|),$$

  where $J_0$ indicates the **zero-order Bessel function** of the first kind

- BRW are obtained as the **local scaling limit** of ARW on the two-torus, that is, for every fixed $x = (x_1, x_2) \in \mathbb{T}^2$,

$$r_n^{(2)}\left(\sqrt{\frac{k}{4\pi^2 n}} \cdot 2\pi x\right) \to_{n\to\infty} J_0(k \cdot \|x\|) = \rho_k(x),$$

  $r_n^{(2)}$ being the covariance function of ARW on $\mathbb{T}^2$

- **(M. Berry, 2002)** Writing $\mathbf{L}_k(D) := \mathcal{H}_1\{x \in D : b_k(x) = 0\}$,

$$\mathbb{E}[\mathbf{L}_k(D)] = \frac{\text{area}(D)}{2\sqrt{2}} \cdot k, \quad \mathbf{Var}[\mathbf{L}_k(D)] \sim_{k\to\infty} \frac{\text{area}(D)}{256\pi} \log(k \cdot \text{area}(D))$$

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- Consider $\ell$ i.i.d Gaussian fields $\left\{(X_0^{(i)}(z), X_1^{(i)}(z), \dots, X_k^{(i)}(z)) : z \in Z\right\}_{i=1,\dots,\ell}$ and

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***Theorem I'*** (Notarnicola, ALEA (2021))

The projection of $\mathbf{J}(u_1, \dots, u_\ell)$ on the second Wiener chaos is given by

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In particular, if $u_i = \mathbf{D}^{(i)} = 0$ for every $i = 1, \dots, \ell$, then $\mathbf{J}(0, \dots, 0)[2] = 0$.

- yields a general description of **Berry's cancellation phenomenon** observed in various models of random eigenfunctions (Marinucci, Peccati, Rossi & Wigman (2016), Dalmao, Nourdin, Peccati & Rossi (2019), Cammarota (2019), etc.)

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$$\mathbf{V}(\mathbf{T}_n^{(\ell)}; \mathbb{T}^3) := \left(\frac{4\pi^2 n}{3}\right)^{\ell/2} \int_{\mathbb{T}^3} \Phi\left(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(x)\right) dx, \quad \Phi(M) := \sqrt{\det(MM^T)}, \ell = 1, 2, 3$$

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**(Expected total variation)** For all $n \in S(3)$,

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Let $S \in \mathcal{M}_{\ell \times \ell}(\mathbb{R})$ symmetric. **Zonal polynomials** $C_\kappa(S)$ are symmetric polynomials in the eigenvalues of $S$, verifying the relation

$$\{\mathrm{tr}(S)\}^k = \sum_{\kappa \vdash k} C_\kappa(S), \quad k \geq 0$$

E.g.: $C_{(1)}(S) = \mathrm{tr}(S), C_{(2)}(S) = \frac{1}{3}\left(\mathrm{tr}(S)^2 + 2\,\mathrm{tr}(S^2)\right), C_{(1,1)}(S) = \frac{2}{3}\left(\mathrm{tr}(S)^2 - \mathrm{tr}(S^2)\right)$, etc.

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## *Theorem II'* (Notarnicola)

For integers $1 \leq \ell \leq n$, let $X \sim \mathcal{N}_{\ell \times n}(0, \mathbf{I}_\ell \otimes \mathbf{I}_n)$.

(i) $\forall k \geq 0, \kappa \vdash k$, we have that $H_\kappa^{(\ell,n)}(X) \in \mathbf{C}_{2k}$.

(ii) **(Matrix-Chaos expansion)** Let $\mu_X = \sum_{i=1}^{\ell} \delta_{\lambda_i(XX^T)}$. For every $F \in L^2(\Omega, \sigma(\mu_X), \mathbb{P})$,

$$F(X) = \sum_{k \geq 0} \sum_{\kappa \vdash k} \widehat{F}(\kappa) H_\kappa^{(\ell,n)}(X), \quad \widehat{F}(\kappa) = \left(4^{-k} \left(\frac{n}{2}\right)_\kappa^{-1} k! C_\kappa(\mathbf{I}_\ell)\right)^{-1} \mathbb{E}\left[F(X) H_\kappa^{(\ell,n)}(X)\right]$$

In particular, we have that $\mathrm{proj}(F|\mathbf{C}_{2k+1}) = 0$.

(iii) **(Projections of radial functions)** If $F(X) = f_0(XX^T)$, then

$$\widehat{F}(\kappa) = \frac{1}{2^{n\ell/2}\Gamma_\ell(\frac{n}{2})} \frac{(-2)^k}{k! C_\kappa(\mathbf{I}_\ell)} \int_{R>0} f_0(R) L_\kappa^{(\frac{n-\ell-1}{2})}(R/2) \mathrm{etr}(-R/2) \det(R)^{\frac{n-\ell-1}{2}} \nu(dR),$$

where $L_\kappa^{(\gamma)}$ denotes the generalized Laguerre polynomial of order $\gamma > -1$.

(iv) In particular, for $F(X) = \sqrt{\det(XX^T)} \in L^2(\Omega, \sigma(\mu_X), \mathbb{P})$, one has that

$$\widehat{F}(\kappa) = 2^{\ell/2} \frac{\Gamma_\ell(\frac{n+1}{2})}{\Gamma_\ell(\frac{n}{2})} \frac{(-2)^k}{k!} \left(\frac{n}{2}\right)_\kappa \sum_{s=0}^{k} \sum_{\sigma \vdash s} \binom{\kappa}{\sigma} (-1)^s \frac{(\frac{n+1}{2})_\sigma}{(\frac{n}{2})_\sigma}.$$

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$$H_\kappa^{(\ell,n)}(\sqrt{2}X) = (-2)^{-k}\left(\frac{n}{2}\right)_\kappa^{-1} \cdot L_\kappa^{(\frac{n-\ell-1}{2})}(XX^T), \quad \forall \kappa \vdash k, k \geq 0$$

- **(Matrix-polar factorization)**

$$X = R^{1/2}U, \quad R = XX^T > 0, \quad U = (XX^T)^{-1/2}X \in O(n,\ell)$$

Here, $O(n,\ell) = \{Y \in \mathcal{M}_{\ell\times n}(\mathbb{R}) : YY^T = \mathbf{I}_\ell\}$ is the **Stiefel manifold**.

- In a random setting, if $X \sim \mathcal{N}_{\ell\times n}(0, \mathbf{I}_\ell \otimes \Sigma)$, then the density of the pair $(R,U)$ is given by

$$f_{(R,U)}(R,U) = \frac{1}{\Gamma_\ell(\frac{n}{2})}\frac{1}{2^{n\ell/2}}\det(\Sigma)^{-\ell/2}\det(R)^{\frac{n-\ell-1}{2}}\exp\left(\mathrm{tr}\left(-\frac{1}{2}U\Sigma^{-1}U^TR\right)\right)$$

If $\Sigma = \mathbf{I}_n$, then $R \sim \mathbf{Wishart}(\ell,\Sigma)$ and $U \sim \mathbf{Uniform}(O(n,\ell))$ are independent

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- Expansion of total variation in M.H.P: in $L^2(\mathbb{P})$

$$\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3) - \mathbb{E}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)\right] = \left(\frac{4\pi^2 n}{3}\right)^{\ell/2} \sum_{k\geq 1}\sum_{\kappa\vdash k} \widehat{\Phi}(\kappa) \int_{\mathbb{T}^3} H_\kappa^{(\ell,3)}\left(\widetilde{\mathrm{jac}}_{\mathbf{T}_n^{(\ell)}}(x)\right) dx,$$

where $\widehat{\Phi}(\kappa)$ are the projection coefficients associated with $\sqrt{\det(XX^T)}$.

- **(Dominance of 2nd Wiener chaos)** As $\mathcal{N}_n \to \infty$,

$$\widetilde{\mathbf{V}}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3) = \widetilde{\mathbf{V}}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2] + o_{\mathbb{P}}(1)$$

- The **2nd Wiener chaos component** admits the exact representation

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- Establish **asymptotic variance law** for the 2nd Wiener chaos component

$$\mathbf{Var}\left[\mathbf{V}(\mathbf{T}_n^{(\ell)};\mathbb{T}^3)[2]\right] \sim_{\mathcal{N}_n\to\infty} \left(\frac{4\pi^2 n}{3}\right)^{\ell} 2^\ell \frac{\Gamma_\ell(2)^2}{\Gamma_\ell(\frac{3}{2})^2} \frac{\ell}{2\mathcal{N}_n}$$

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**THANK YOU FOR YOUR ATTENTION!**