Spectral asymptotics for large quantum graphs

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1) Introduction: spectral asymptotic
2) Quantum graphs
    2.1) General definitions
    2.2) Benjamini-Schramm convergence of quantum graphs

1) Introduction: spectral asymptotic

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with $\partial\Omega$ regular enough.

There exists a non-decreasing sequence (λ_k) of positive numbers going to infinity, and a family of smooth functions (φ_k) forming an orthonormal basis of $L^2(\Omega)$ such that

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega \\ \varphi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

The sequence (λ_k) is called the *Dirichlet spectrum* of Ω.
What does it asymptotically look like?

Weyl's law

For any $\lambda > 0$, let us write

$$N_\Omega(\lambda) := \sharp\{k \in \mathbb{N} \,;\; \lambda_k \leq \lambda\}.$$

(Weyl, 1911) : We have

$$N_\Omega(\lambda) \sim_{\lambda\to\infty} (2\pi)^{-d} \lambda^{d/2} \omega_d \mathrm{Vol}(\Omega).$$

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Variants of Weyl's law also hold for the Laplace-Beltrami operator on compact manifolds, for operators $\hbar^2\Delta + V$ in the limit $\hbar \to 0$...

An alternative point of view on Weyl's law

If $\Omega \subset \mathbb{R}^d$ and $r > 1$, let us write $r\Omega$ for a rescaled version of Ω where all the distances are multiplied by r.
If (λ_k) is the Dirichlet spectum of Ω, then the Dirichlet spectrum of $r\Omega$ is $\left(\frac{\lambda_k}{r^2}\right)$.

$$N_{r\Omega}(\lambda) \sim_{r\to\infty} r^d \lambda^{d/2} \omega_d \mathrm{Vol}(\Omega).$$

Here, we work in a fixed spectral interval, but let the **size of the domain go to infinity**.

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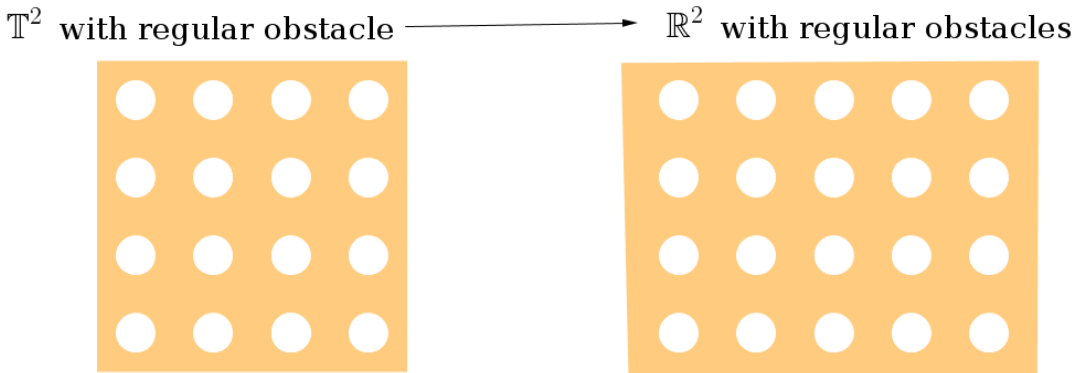


Why is the large domain limit interesting ?

$\mathbb{T}^2$ with regular obstacle $\longrightarrow$ $\mathbb{R}^2$ with regular obstacles

$\mathbb{T}^2$ with a complicated or random obstacles $\longrightarrow$ $\mathbb{R}^2$ with random obstacles

The case of manifolds of constant negative curvature

(Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, Samet, 2017) Suppose that (X_k) is a sequence of compact hyperbolic manifolds whose injectivity radius goes to infinity

$$N_{X_k}(\lambda) \sim_{k\to\infty} \mathrm{Vol}(X_k) f(\lambda).$$

Here, f is a function depending only on the spectral data of the hyperbolic space.

2) Quantum graphs

2.1) General definitions

What is a quantum graph?

A **quantum graph** $\mathcal{Q} = (V, B, L)$ is the data of:

- A connected graph (V, B), where V is the set of vertices, and B is the set of *oriented* edges or *bonds*. If $b \in B$, we write $\widehat{b}$ for the reverse bond, and $o_b, t_b \in V$ for the origin and terminus of b.
- A map $L : B \longrightarrow (0, \infty)$ assigning a length $L_b = L(b)$ to each bond, and such that for all $b \in B$, $L(b) = L(\widehat{b})$.

A quantum graph will be called *finite* if the graph (V, B) is finite. If the graph is infinite, we will always suppose that all the lengths and degrees are bounded from above and from below.

What is a quantum graph? (2)

We may also add to the definition:

- For each $b \in B$, a potential $W_b \in C^2([0, L_b], \mathbb{R})$, such that $W_{\widehat{b}}(L_b - x) = W_b(x)$.
- For each $v \in V$, a real number α_v.

In the sequel, we will often consider the case $W_b \equiv 0$, $\alpha \equiv 0$.

The Schrödinger operator on a quantum graph

In the sequel, we will work in the Hilbert space

$$\mathcal{H}_\mathcal{Q} := \Big\{ f = (f_b)_{b\in B} \in \bigoplus_{b\in B} L^2[0, L_b] \text{ such that } f_b(L_b - \cdot) = f_{\widehat{b}}(\cdot) \text{ and } \sum_{b\in B} \|f_b\|^2_{L^2([0,L_b])} < \infty \Big\},$$

whose scalar product we denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}_\mathcal{Q}}$, and with the subspace

$$\mathcal{H}^2_\mathcal{Q} := \Big\{ f = (f_b)_{b\in B} \in \bigoplus_{b\in B} H^2[0, L_b] \text{ such that } f_b(L_b - \cdot) = f_{\widehat{b}}(\cdot) \text{ and } \sum_{b\in B} \|f_b\|^2_{H^2([0,L_b])} < \infty \Big\} \subset \mathcal{H}.$$

We define an operator $H_\mathcal{Q} : \mathcal{H}^2_\mathcal{Q} \to \mathcal{H}_\mathcal{Q}$ acting on $\psi = (\psi_b)_{b\in B}$ by

$$(H_\mathcal{Q}\psi_b)(x) = -\psi_b''(x) + W_b(x)\psi_b(x). \tag{1}$$

Thus defined, $H_{\mathcal{Q}}$ is not self-adjoint: we need to impose boundary conditions at each vertex.

- **Continuity**: For all $b, b' \in B$, we have $f_b(0) = f_{b'}(0) =: f(v)$ if $o_b = o_{b'} = v$.
- **Current conservation:** For all $v \in V$,

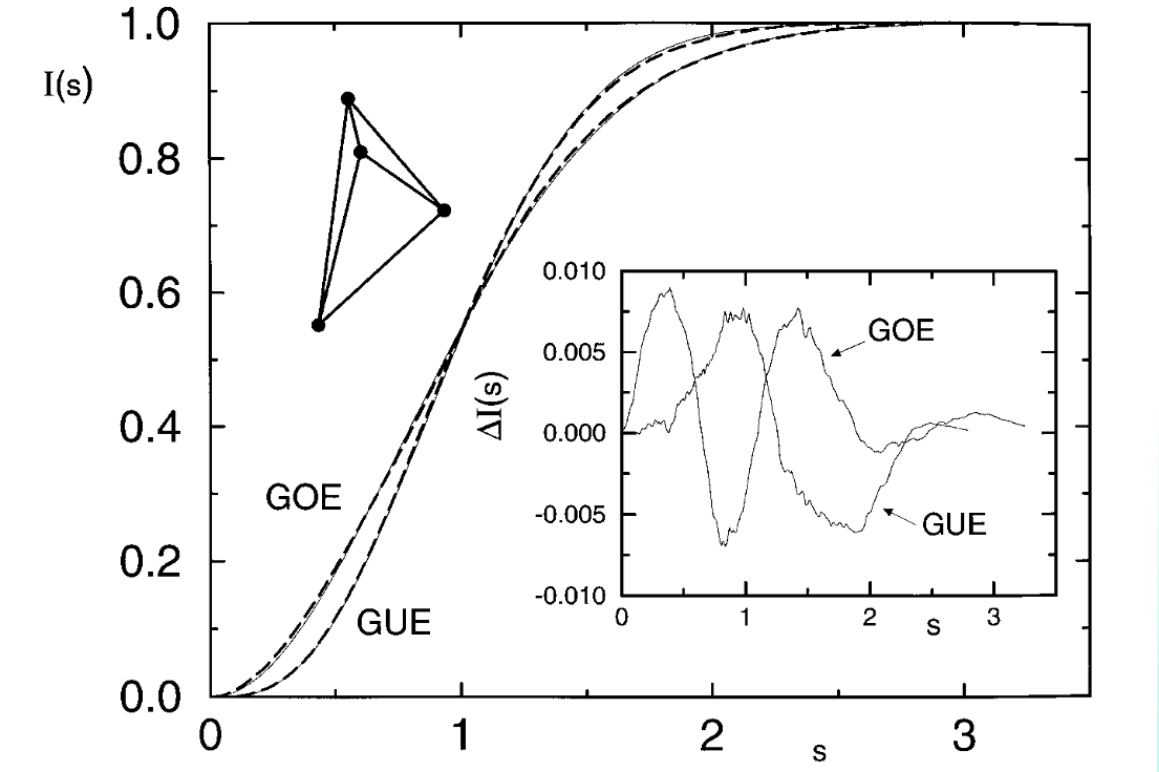
$$\sum_{b:o_b=v} f_b'(0) = \alpha_v f(v).$$

We define the space

$$\mathcal{H}^{b.c.}_{\mathcal{Q}} := \Big\{ f \in \mathcal{H}^2 \text{ satisfying } \textbf{Continuity} \text{ and } \textbf{Current conservation} \Big\}.$$

Fact: The operator $H_{\mathcal{Q}}$ with domain $\mathcal{H}^{b.c.}_{\mathcal{Q}}$ is self-adjoint.
If $\mathcal{Q}$ is finite, then $H_{\mathcal{Q}}$ has compact resolvent.

Quantum chaos on a fixed quantum graph



Kottos and Smilansky (1997) showed numerically that the eigenvalue spacing on some simple quantum graphs follow the same law as in some random matrix models. This is supposed to hold for all chaotic systems (Bohigas, Giannoni and Schmit conjecture, 1984).

Some recent developments in the theory of quantum graphs

- Counting nodal domains on quantum graphs (Gnutzman-Smilansky-Weber '03, Alon-Band-Berkolaiko '18...)
- Anderson delocalization on infinite quantum trees (Aizenmann-Warzel '05, Anantharaman-I.-Sabri-Winn '20).
- Colin de Verdière '13 : Classification of semiclassical measures on quantum graphs $\rightsquigarrow$ No Quantum Ergodicity on a given quantum graph.
- Quantum Ergodicity on families of (larger and larger) quantum graphs : Berkolaiko-Keating-Smilansky '04 & '07, Bramall-Winn '15, Anantharaman-I.-Sabri-Winn '19 & '20.
- Kurasov-Sarnak (Work in progress) : No (non-trivial) arithmetic progressions in the spectrum of quantum graphs.

2) Quantum graphs

2.2) Benjamini-Schramm convergence of quantum graphs

Benjamini-Schramm convergence for quantum graphs (1)

Let $\mathcal{Q}_1 = (V_1, B_1, L_1, v_1)$ and $\mathcal{Q}_2 = (V_2, B_2, L_2, v_2)$ be two **rooted** quantum graphs with marked vertices $v_1 \in V_1$, $v_2 \in V_2$.
If $r \in \mathbb{N}$, we denote by $B_{\mathcal{Q}_i}(v_i, r)$ the set of vertices of V_i at a graph-distance $\leq r$ from v_i.

We define $d(\mathcal{Q}_1, \mathcal{Q}_2)$ to be the infimum of the $\varepsilon > 0$ such that

- There exists a graph-isomorphism $\phi : B_{\mathcal{Q}_1}(v_1, \lfloor \varepsilon^{-1} \rfloor) \longrightarrow B_{\mathcal{Q}_2}(v_2, \lfloor \varepsilon^{-1} \rfloor)$ such that $\phi(v_1) = v_2$.
- If b_1 is a a bond between two vertices of $B_{\mathcal{Q}_1}(v_1, \lfloor \varepsilon^{-1} \rfloor)$, then

$$\left| L_1(b_1) - L_2(\phi(b_1)) \right| < \varepsilon.$$

Benjamini-Schramm convergence for quantum graphs (2)

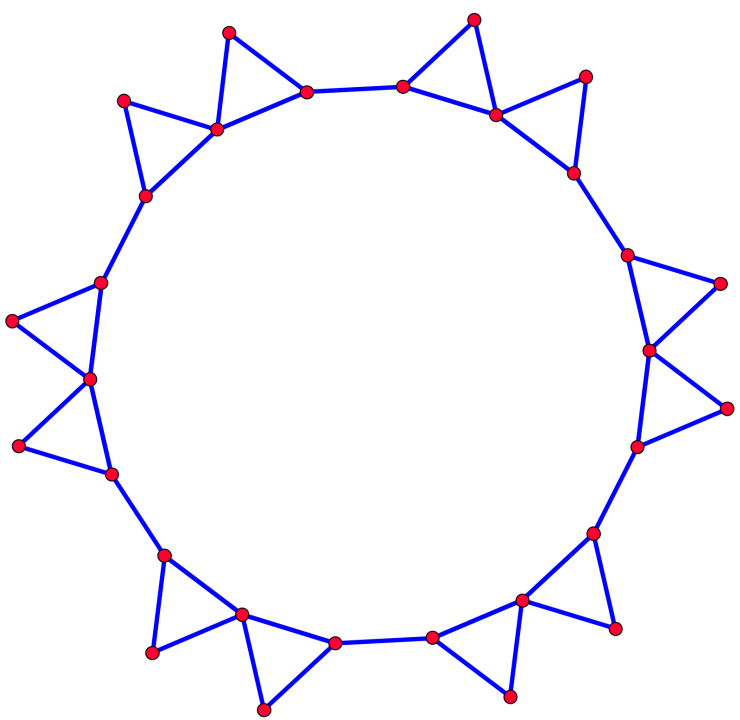
We denote by Q the set of rooted quantum graphs, equipped with the distance d. We write $\mathcal{P}(\mathrm{Q})$ for the set of probability measures on Q, and $\mathcal{C}(\mathrm{Q})$ for the set of bounded continuous maps on Q. If $\mathcal{Q} = (V, B, L)$ is a quantum graph, it induces naturally a probability measure in $\mathcal{P}(\mathrm{Q})$, obtained by choosing the root uniformly at random:

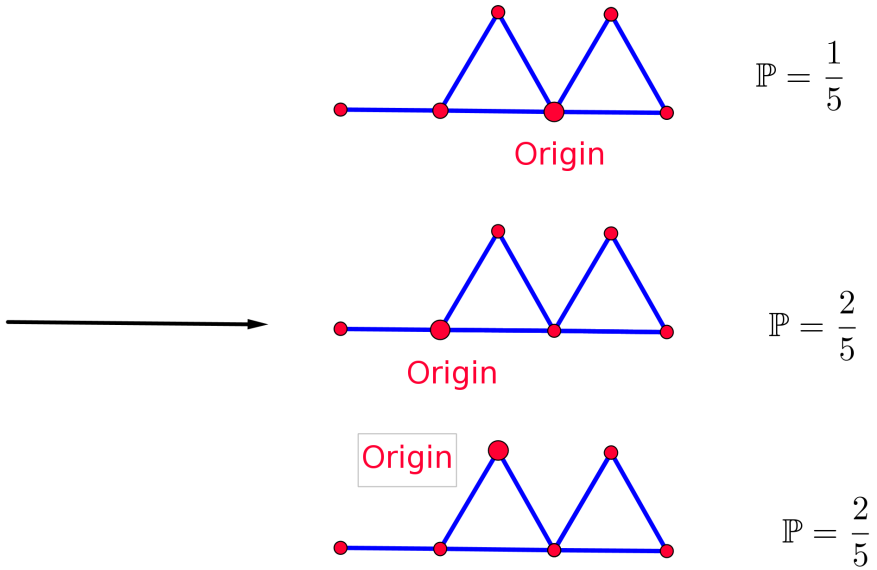
$$\nu_{\mathcal{Q}} := \frac{1}{|V|} \sum_{v \in V} \delta_{(V,B,L,v)}.$$

We say that a sequence of quantum graphs $\mathcal{Q}_n$ converges **in the sense of Benjamini-Schramm** to a measure $\mathbb{P} \in \mathcal{P}(\mathrm{Q})$ if $\nu_{\mathcal{Q}_n}$ converges weak-* to $\mathbb{P}$, i.e., if, for any $F \in \mathcal{C}(\mathrm{Q})$, we have

$$\langle \nu_{\mathcal{Q}_n}, F \rangle \longrightarrow \langle \mathbb{P}, F \rangle.$$

An example of Benjamini-Schramm convergence





Consequences of Benjamini-Schramm convergence

Fact: If $(\mathcal{Q}_N)$ is a sequence of quantum graphs with uniformly bounded data, we can extract a subsequence that converges in the sense of Benjamini-Schramm.

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Examples of functionals F one could take:

- The degree of v.
- The average of the lengths of the edges touching v.
- The number of cycles in a ball $B(v, R)$, with R fixed.

A priori, it is not clear that we can take spectral quantities.

If $\mathcal{Q}$ is a quantum graph, we denote by $\mathcal{L}(\mathcal{Q})$ the total length of the graph and by $\Sigma(\mathcal{Q})$ the spectrum of the Laplacian on $\mathcal{Q}$. We define its **empirical spectral measure** $\mu_{\mathcal{Q}}$ by

$$\mu_{\mathcal{Q}} := \frac{1}{\mathcal{L}(\mathcal{Q})} \sum_{k \in \Sigma(\mathcal{Q})} \delta_k.$$

BS convergence and empirical spectral measures

Theorem 1[Anantharaman, I., Sabri, Winn, 2020]
Let $(\mathcal{Q}_N)$ be a sequence of finite quantum graphs with degree and lengths bounded from above and below, converging in the sense of Benjamini-Schramm to a measure $\mathbb{P} \in \mathcal{P}(\mathrm{Q})$.
Let $\chi \in C_c(\mathbb{R})$. We have

$$\lim_{N\to\infty} \int_{\mathbb{R}} \chi \mathrm{d}\mu_{\mathcal{Q}_N}$$
$$= \frac{1}{2} \int_{\mathrm{Q}} \mathrm{d}\mathbb{P}(\mathcal{Q}, v) \sum_{b; o(b)=v} \int_0^{L_b} \left(\chi(H_{\mathcal{Q}})\right)(x_b, x_b)\mathrm{d}x_b.$$

The theorem is still true if we have potentials and coupling constants that are uniformly bounded.

Proof (1) : The scattering approach to quantum graphs

Suppose that $f = (f_b)_{b \in B}$ is an eigenfunction of $H_{\mathcal{Q}}$ with eigenvalue $z \neq 0$. For each b, we can write $f_b(x) = a(b)e^{i\sqrt{z}x} + a'(b)e^{-i\sqrt{z}x}$, with $a'(b) = e^{iL_b x}a(\widehat{b})$, so that

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Now, for each vertex v of degree d, we write $\sigma^v_{b,b'} := \frac{2}{d} - \delta_{b,b'}$. The boundary conditions may then be rewritten as

$$a(b) = \sum_{b';\, t(b')=o(b)} \sigma^{(o(b))}_{b,\widehat{b'}} e^{i\sqrt{z}L_{b'}} a(b')\,.$$

Proof (2) : The spectrum of a finite quantum graph

Let us define the *evolution operator* $U(z)$, which is a matrix of size $|B| \times |B|$ defined as

$$U(z) = SD(z)\,,$$

where $D(z)$ is the diagonal matrix $D(z)_{b,b} = e^{i\sqrt{z}L_b}$, and S is the unitary matrix

$$S_{b,b'} = \delta_{t(b')=o(b)}\sigma^{(o(b))}_{b,\widehat{b'}}\,.$$

If $\vec{a} = \big(a(b)\big)_{b\in B}$, then we have

$$U(z)\vec{a} = \vec{a}.$$

Therefore, $z \neq 0$ is an eigenvalue if and only if

$$\det\big(U(z) - \mathrm{Id}\big) = 0\,.$$

Sketch of proof (3)

Let $z \in \mathbb{C}$ with $\mathfrak{Im}\, z > 0$, and $v \in V$. We write $g_{z,v} := \left(H_{\mathcal{Q}} - z\right)^{-1}(v, \cdot)$ for the Green's function at v.
It satisfies $-f'' = zf$, so we may find coefficients $\vec{a} = \big(a(b)\big)$ such that for any $b \in B$, we have on L_b

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A simple computation shows that $g_{z,v}$ is continuous at v, and that the coefficients $\vec{a}$ must satisfy

$$\big(\mathrm{Id} - SD(z)\big)\vec{a} = \Lambda_v, \quad \text{where} \quad \Lambda_v(b) := \frac{1}{d(o(b))\mathrm{i}\sqrt{z}}\delta_{o(b)=v}.$$

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We therefore have $a = \sum_{k=0}^{\infty} \big(SD(z)\big)^k \Lambda_v$, so $g_{z,v}$ is continuous in the Benjamini-Schramm topology.

Thank you for your attention !