Moments of the number of zeros of one dimensional Gaussian processes.

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1. Literature about the variance and higher moments of the number of zeros of a Gaussian process
2. The Kac density near the diagonal
3. The Kac density far away from the diagonal
4. Extension to higher moments

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- $\sum_{k=1}^{N} a_k P_k(t)$, a_k iid. on $[0, 1]$, and P_k orthogonal polynomials as $N \to +\infty$ (Lubinsky–Pritsker 2021). Kac-Rice method + Explicit computation. Non stationary.

For higher moments, the most recent result is given by (Ancona–Letendre 2021). In dimension one, we have Gaussian moments asymptotics when X is Gaussian stationary and the covariance function is in the Schwartz space. Kac-Rice method.

X Gaussian process of class $\mathcal{C}^1$ on an interval $I = [0, R]$. Kac-Rice formula :

$$\mathbb{E}[N_X^{[k]}(R)] = \int_{[0,R]^k} \rho_k(t_1, \dots, t_k) \mathrm{d}t_1 \dots \mathrm{d}t_k$$

where

$$\rho_k(t_1, \dots, t_k) = p_{t_1,\dots,t_k}(0, \dots, 0) \mathbb{E}\left[\prod_{i=1}^{k} |X'(t_i)| \,|\, X(t_1) = \dots = X(t_k) = 0\right]$$

And $p_{t_1,\dots,t_k}$ is the density of $(X(t_1), \dots, X(t_k))$. In particular,

$$\mathrm{Var}(N_X(R)) = \mathbb{E}[N_X(R)] + \int_{[0,R]^2} (\rho_2(s,t) - \rho_1(s)\rho_1(t)) \,\mathrm{d}s\mathrm{d}t$$

Singularity along the diagonal : Let

$$X[s,t] = \frac{X(t) - X(s)}{t-s}$$
$$X[s,s,t] = \frac{X'(s) - X[s,t]}{t-s}$$
$$X[s,s,t,t] = \frac{X[s,s,t] - X[t,t,s]}{t-s}$$

One has

$$p_{s,t}(0,0) = \frac{1}{\sqrt{\det(\mathrm{Cov}(X(s),X(t))}}$$
$$= \frac{1}{|t-s|}\frac{1}{\det(\mathrm{Cov}(X(s),X[s,t])}$$
$$\simeq \frac{1}{|t-s|}\frac{1}{\det(\mathrm{Cov}(X(s),X'(s))}$$

And also

$$\mathbb{E}[|X'(s)||X'(t)|\,|\,X(s)=X(t)=0]=\mathbb{E}\left[\,|X'(s)||X'(t)|\,\middle|\,X(s)=X[s,t]=0\right]$$

Thr RHS converges to zero.

Geman condition : The quantity

$$\frac{\mathbb{E}\left[\,|X'(s)||X'(t)|\,|\,X(s)=X[s,t]=0\right]}{|t-s|}$$

must be integrable near the diagonal to ensure finite variance.

Stronger condition. Assume X of class $\mathcal{C}^2$. Then

$$
\begin{aligned}
&\mathbb{E}[|X'(s)||X'(t)| \,|\, X(s) = X(t) = 0] \\
&= (t-s)^2 \mathbb{E}\left[|X[s,s,t]||X[t,t,s]| \,|\, X(s) = X[s,t] = 0\right] \\
&\simeq (t-s)^2 \mathbb{E}\left[|X''(s)||-X''(s)| \,\middle|\, X(s) = X'(s) = 0\right] \\
&\simeq (t-s)^2 \mathbb{E}\left[|X''(s)|^2 \,\middle|\, X(s) = X'(s) = 0\right]
\end{aligned}
$$

and thus

$$
\rho_2(s,t) \simeq |t-s| \frac{\mathbb{E}\left[|X''(s)|^2 \,\middle|\, X(s) = X'(s) = 0\right]}{4 \det(\mathrm{Cov}(X(s), X'(s)))}
$$

Even better for upcrossings!

$$
\begin{aligned}
&\mathbb{E}[|X'(s)||X'(t)|\,|\,X(s)=X(t)=0]\\
=&(t-s)^2\mathbb{E}\left[|X[s,s,t]||X[s,s,t]+(t-s)X[s,s,t,t]|\,|\,X(s)=X[s,t]=0\right]\\
\simeq&(t-s)^2\mathbb{E}\left[|X''(s)||X''(s)+(t-s)X'''(s)|\,\middle|\,X(s)=X'(s)=0\right]
\end{aligned}
$$

Smooth function of the covariance of $(X(s), X[s,t], X[s,s,t], X[s,s,t,t])$.

Same thing can be done for higher moments. More complicated, because there is more than one "diagonal", but doable.

Recall that

$$\mathrm{Var}(N_X(R)) = \mathbb{E}[N_X(R)] + \int_{[0,R]^2} (\rho_2(s,t) - \rho_1(s)\rho_1(t))\,\mathrm{d}s\mathrm{d}t$$

Assume that $(X(s), X'(s)) \perp (X(t), X'(t))$.

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Thus

$$\rho_2(s,t) = \rho_1(s)\rho_1(t)$$

Let us prove that

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Let

$$\Sigma(s,t)=\mathrm{Cov}(X(s),X(t),X'(s),X'(t))=\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$$

We have

$$\rho_2(s,t)=\widetilde{\rho_2}(\Sigma(s,t))$$

Where

$$\widetilde{\rho_2}(\Sigma)=\frac{1}{(2\pi)^2\sqrt{\det(\Sigma)}}\int_{\mathbb{R}^2}|x_1||x_2|e^{-{}^T\vec{x}(\Sigma^{-1})_{22}\vec{x}}\mathrm{d}x_1\mathrm{d}x_2$$

For a matrix Σ of $\mathbb{R}^4$, let

$$\Sigma_0 = \begin{pmatrix} \operatorname{diag} \Sigma_{11} & \operatorname{diag} \Sigma_{12} \\ \operatorname{diag} \Sigma_{21} & \operatorname{diag} \Sigma_{22} \end{pmatrix}$$

Then

$$\rho_2(s,t) - \rho_1(s)\rho_1(t) = \widetilde{\rho}_2(\Sigma(s,t)) - \widetilde{\rho}_2(\Sigma_0(s,t))$$

Assume that $\Sigma = \Sigma_0 + \varepsilon e_{ij}$, where e_{ij} is a matrix with only one non-zero coefficient, equal to one.

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Assume that $\Sigma = \Sigma_0 + \varepsilon e_{ij}$, where e_{ij} is a matrix with only one non-zero coefficient, equal to one. Then

$$\det(\Sigma) = \det(\Sigma_0) \quad \text{and} \quad (\Sigma^{-1})_{22} = (\Sigma_0^{-1})_{22} + \widetilde{\varepsilon}\,\widetilde{e_{ij}}$$

where $\widetilde{e_{ij}}$ is a 2×2 matrix with only one non zero coefficient which is off diagonal and equal to one.

It implies that

$$\widetilde{\rho_2}(\Sigma) = \frac{1}{\det(\Sigma_0)} \int_{\mathbb{R}^2} |x_1||x_2| e^{-{}^T\vec{x}(\Sigma_0^{-1})_{22}\vec{x} + x_1 x_2 \widetilde{\varepsilon}} \mathrm{d}x_1 \mathrm{d}x_2$$

$$= \widetilde{\rho_2}(\Sigma_0) + \underbrace{\frac{\widetilde{\varepsilon}}{\det(\Sigma_0)} \int_{\mathbb{R}^2} |x_1||x_2| x_1 x_2 e^{-{}^T\vec{x}(\Sigma_0^{-1})_{22}\vec{x}} \mathrm{d}x_1 \mathrm{d}x_2}_{=0 \text{ by parity}} + O(\varepsilon^2)$$

It implies that

$$\begin{aligned}\widetilde{\rho}_2(\Sigma) &= \frac{1}{\det(\Sigma_0)}\int_{\mathbb{R}^2}|x_1||x_2|e^{-{}^T\vec{x}(\Sigma_0^{-1})_{22}\vec{x}+x_1x_2\widetilde{\varepsilon}}\mathrm{d}x_1\mathrm{d}x_2\\ &= \widetilde{\rho}_2(\Sigma_0)+\underbrace{\frac{\widetilde{\varepsilon}}{\det(\Sigma_0)}\int_{\mathbb{R}^2}|x_1||x_2|x_1x_2e^{-{}^T\vec{x}(\Sigma_0^{-1})_{22}\vec{x}}\mathrm{d}x_1\mathrm{d}x_2}_{=0\text{ by parity}}+O(\varepsilon^2)\end{aligned}$$

We deduce that

$$|\widetilde{\rho}_2(\Sigma)-\widetilde{\rho}_2(\Sigma_0)|\leq C(\Sigma)\|\Sigma-\Sigma_0\|^2$$

And thus

$$|\rho_2(s,t)-\rho_1(s)\rho_1(t)|\leq C\left(\sup_{u,v\leq 1}r^{(u,v)}(s,t)\right)^2$$

Theorem [G.2021]

Let X_n a sequence of random processes on $[0, R]$ such that

(A1) X_n is of class $\mathcal{C}^2$ on $[0, R]$, and $|r_n^{(a,b)}| \leq C$ for $a, b \leq 2$.

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$(A2)$ Uniformly on $x \in [0, R]$, u in a compact set, and $a, b \leq 1$,

$$\lim_{n \to +\infty} r_n^{(a,b)}(x, x+u) = (-1)^b g^{(a+b)}(u).$$

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Then there is an explicit constant $C_\infty(g)$ such that uniformly in n

$$\lim_{(n,R) \to +\infty} \frac{\mathrm{Var}(Z_{X_n}([0, R]))}{R} = C_\infty(g).$$

This theorem covers many models of random polynomials, for instance

• $\sum_{k=1}^{N} a_k \cos(kt) + b_k \sin(kt)$, with $\mathbb{E}[a_k a_l] = \mathbb{E}[b_k b_l] = \rho(k-l)$ and ρ with continuous positive spectral density on $\mathbb{T}$.

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• $\sum_{k=1}^N a_k \cos(kt)$, a_k with $\mathbb{E}[a_k a_l] = \rho(k-l)$.

• $\sum_{k=1}^N a_k P_k(t)$, a_k iid and P_k orthogonal polynomials satisfying assumption in (Pritsker-Lubinsky).

What about higher moments ? If we "forget" the combinatoric of factorial moments then

$$\mathbb{E}\left[(Z_R - \mathbb{E}[Z_R])^k\right] = \sum_{B\subset\{1,\ldots,n\}} (-1)^{|A|-|B|}\mathbb{E}\left[Z^{|B|}\right]\mathbb{E}[Z]^{|A|-|B|}$$

$$= \int_{[0,R]^k} \sum_{B\subset\{1,\ldots,n\}} (-1)^{|A|-|B|}\rho_{|B|}((x_b)_{b\in B}) \prod_{i\in A\setminus B} \rho_1(x_i)\mathrm{d}x_1\ldots\mathrm{d}x_k$$

Let

$$F_k(x_1,\ldots,x_k) = \sum_{B\subset\{1,\ldots,n\}} (-1)^{|A|-|B|}\rho_{|B|}((x_b)_{b\in B}) \prod_{i\in A\setminus B} \rho_1(x_i)$$

Let $\mathcal{I}$ be a partition of $\{1, \ldots, k\}$, and assume that if i and j belongs to different cells of the partition, then $r(x_i, x_j) = 0$. Then

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$$F_k(x_1, \ldots, x_k) = \prod_{I \in I} F_{|I|}((x_i)_{i \in I})$$

Moreover, one has

$$F_1(x) = \rho_1(x) - \rho_1(x) = 0$$

It means that if one variable x_i is very far from the others, the function F_k is small

Competition between :

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The dominant part of the integral is when the variable are clustered into pairs. Thus

$$\int_{[0,R]^k} F_k(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k \simeq \sum_{\mathcal{I}\in\mathcal{PP}_k}\prod_{I\in\mathcal{I}}\int_{\mathbb{R}^2} F_2(x,y)\mathrm{d}x\mathrm{d}y$$
$$\simeq \mathrm{Card}(\mathcal{PP}_k)\mathrm{Var}(Z_R)^{k/2}$$

And thus

$$\mathbb{E}\left[\left(\frac{Z_R - \mathbb{E}[Z_R]}{\sqrt{\mathrm{Var}(Z_R)}}\right)^k\right] \simeq \mu_k(\mathcal{N}(0,1))$$

It requires that the covariance function $r \in L^2$, and all its derivatives.