On the number of roots of random homogeneous polynomials

Based on joint works with D.Armentano, JM. Azaïs, JR. León & K. Kozhasov

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Plan of the talk:

1. Introduction.
2. Random Homogeneous polynomials.
   2.1 Kostlan's classification.
   2.2 Some concrete examples.
3. The expectation of the number of roots.
4. The variance of the number of roots.
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   4.4 On the positivity of the limit variance.

Introduction

Introduction

We are interested in the distribution of the number of roots of random polynomial systems of equations.

This is a natural continuation of the study of the number of roots of random polynomials.

- We shall concentrate on **square systems**, but one can consider instead the geometric measure (depending on the dimension: the length, the area, etc.) of the zero set of non-square systems with the same methods.
- Assuming **homogeneity** is not an important restriction.
- **Invariance under isometries** is natural in some frameworks and greatly simplifies the treatment.

Introduction

In the early nineties, Kostlan and Shub & Smale proposed the study of an invariant Gaussian polynomial system which appears naturally in the analysis of algorithms for solving polynomial systems of equations.

Via geometric/analytic arguments they computed the exact expected number of roots of such systems.

In 2005 Azaïs & Wschebor, using probabilistic tools, gave a new proof for the expectation and computed the asymptotic variance of these systems when the size of the system (number of equations = number of variables) tends to infinity while the degree remains controlled.

Introduction

In the last five years the asymptotic variance of the number of roots of these systems, as the degree tends to infinity, has been obtained by Letendre, Letendre & Puchol and Armentano, Azaïs, D & León.

A Central Limit Theorem was establisehd by the latter authors.

Random Homogeneous Polynomials

Random Homogeneous Polynomials

We are interested in random homogeneous polynomials which distributions are invariant under the action of the orthogonal group $\mathcal{O}(n+1)$:

$$f(Ux) \overset{(d)}{=} f(x), \quad \text{for all } U \in \mathcal{O}(n+1).$$

Here $\overset{(d)}{=}$ means *equal distributed r.v.s.*

Remark

In the complex case there is a unique invariant Gaussian distribution in the space of (complex) homogeneous polynomials. This is not the case for real polynomials.

Random Homogeneous Polynomials

A Gaussian distribution in the space of homogeneous polynomials can be associated with an inner product on the underlying parameter space:

$$p(f) = \mathrm{Const} \cdot \exp\Big(- \frac{1}{2} \langle f, f \rangle \Big).$$

Taking profit of this correspondence Kostlan in 2002 classified all orthogonally invariant centered Gaussian distributions on the space of real homogeneous polynomials.

We start recalling this classification.

Random Homogeneous Polynomials

Let $\mathcal{P}_{d,n}$ be the space of real homogeneous polynomials of degree d in $n+1$ variables and let $\mathcal{H}_{d,n}$ denote the subspace of *harmonic homogeneous polynomials*,

$$\mathcal{H}_{d,n} = \{f \in \mathcal{P}_{d,n} \ : \ \partial^2_{x_0} f + \cdots + \partial^2_{x_n} f \equiv 0\}.$$

Restrictions of harmonic homogeneous polynomials in $\mathcal{H}_{d,n}$ to the unit sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ are called *spherical harmonics*; these are eigenfunctions of the spherical Laplace operator corresponding to the eigenvalue $-d(d+n-1)$.

Random Homogeneous Polynomials

It is well-known that

$$\mathcal{P}_{d,n} = \mathcal{H}_{d,n} \oplus |x|^2\mathcal{H}_{d-2,n} \oplus \cdots \oplus |x|^{2[d/2]}\mathcal{H}_{d-2[d/2],n},$$

is a direct sum decomposition and

$$|x|^{2k}\mathcal{H}_{d-2k,n} = \{|x|^{2k}h \ : \ h \in \mathcal{H}_{d-2k,n}\}, \ k = 0, \ldots, [d/2],$$

are irreducible subspaces of $\mathcal{P}_{d,n}$ under the standard action of the orthogonal group.

Random Homogeneous Polynomials

Kostlan proved that (up to a multplicative constant) there is a unique inner product in $\mathcal{H}_{d-2k,n} : k = 0, \dots, \lfloor d/2 \rfloor$ which is invariant under the action of the orthogonal group.

Therefore, each invariant inner product in $\mathcal{P}_{d,n}$ is a (positive) linear combination of the inner products in $\mathcal{H}_{d-2k,n} : k = 0, \dots, \lfloor d/2 \rfloor$.

From the point of view of the Gaussian distribution this means that:

There is (up to a scale constant) a unique invariant Gaussian distribution on $\mathcal{H}_{d-2k,n} : k = 0, \dots, \lfloor d/2 \rfloor$ and any invariant Gaussian distribution on $\mathcal{P}_{d-2k,n}$ is a convolution of the distributions on $\mathcal{H}_{d-2k,n} : k = 0, \dots, \lfloor d/2 \rfloor$.

Random Homogeneous Polynomials

Hence, a random homogeneous polynomial can be realised as

$$\boxed{f(x) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} w_{d-2k} \sum_{i=1}^{D_{d-2k}} \xi_{d-2k}^{i} Y_{d-2k}^{i}(x), \quad x \in \mathbb{S}^n.}$$

Here, $\{Y^i_{d-2k,n}\}_{i=1}^{D_{d-2k,n}}$ is any $\langle \cdot, \cdot \rangle^{\mathrm{RFS}}_{d-2k,n}$-orthonormal basis of $\mathcal{H}_{d-2k,n}$ and $\xi^i_{d-2k,n}$, $i = 1, \ldots, D_{d-2k,n}, k = 0, \ldots, [d/2]$, are independent standard Gaussian random variables.

The $\{w_{d-2k}\}_k$ are positive weights that parameterise the different Gaussian models.

Some particular examples.

Kostlan Polynomial systems

This distribution is associated with the Bombieri-Weyl inner product. For $m \in \mathbb{N}$, consider homogeneous polynomials

$$f_\ell(x) = \sum_{|j|=d} a_j^{(\ell)} x^j; \quad \ell = 1, \dots, m,$$

where we use the notations

- $j = (j_0, \dots, j_m) \in \mathbb{N}^{m+1}$;
- $x = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}$;
- $|j| = \sum_{k=0}^m j_k$ and $x^j = \prod_{k=0}^m x_k^{j_k}$.
- $a_j^{(\ell)} \in \mathbb{R}$.

We assume that the random variables $a_j^{(\ell)}$ are independent centered Gaussian with variances

$$\operatorname{var}\left(a_j^{(\ell)}\right) = \binom{d}{j} = \frac{d!}{\prod_{k=0}^m j_k!}.$$

Kostlan Polynomials

Remark If we ask the polynomials to be invariant in distribution and the monomials to be independent, then, the polynomials must be Gaussian. Kostlan.

The covariance function for f_ℓ is:

$$r_d(s,t) := \mathbf{E}(f_\ell(s)f_\ell(t)) = \langle s,t\rangle^d ;$$

where $\langle \cdot,\cdot \rangle$ is the standard inner product in $\mathbb{R}^{n+1}$.

This is clearly invariant under orthogonal transformations.

Random Homogeneous Polynomials

The *real Fubini-Study scalar product on* $\mathcal{P}_{d,n}$ (or $L^2(\mathbb{S}^n)$-product) is defined by

$$\langle f, g\rangle_{d,n}^{\mathrm{RFS}} = \int_{\mathbb{S}^n} f(x)g(x)\, d\mathbb{S}^n(x), \quad f, g \in \mathcal{P}_{d,n},$$

where $d\mathbb{S}^n$ is the standard volume measure on the sphere $\mathbb{S}^n$. One can easily see that this scalar product is invariant under orthogonal changes of coordinates on $\mathcal{P}_{d,n}$.

This implies that the restriction of the real Fubini-Study scalar product to $\mathcal{H}_{d,n}$ is the unique (up to a multiple) orthogonally invariant scalar product on $\mathcal{H}_{d,n}$. Equivalently, there is a unique centered Gaussian probability distribution on $\mathcal{H}_{d,n}$ that is invariant under orthogonal changes of variables.

Random Spherical Harmonics

A *random spherical harmonic* $h \in \mathcal{H}_{d,n}$ is modelled as

$$h(x) = \sum_{i=1}^{D_{d,n}} \xi^i_{d,n} Y^i_{d,n}(x), \quad x \in \mathbb{S}^n,$$

where $D_{d,n} = \dim \mathcal{H}_{d,n}$, $\{Y^i_{d,n}\}_{i=1}^{D_{d,n}}$ is any $\langle \cdot, \cdot \rangle^{\mathrm{RFS}}_{d,n}$-orthonormal basis of $\mathcal{H}_{d,n}$ and $\xi^i_{d,n}$, $i = 1, \ldots, D_{d,n}$, are independent standard Gaussian random variables.

Normalization: $h \mapsto \nu h$ with $\nu = \sqrt{\frac{|\mathbb{S}^n|}{dim(\mathcal{H}_{d,n})}}$.

Remark In the case of $\mathbb{S}^2$, if the terms in the above sum are independent then the form is Gaussian Baldi & Marinucci.

Real Fubini-Study

A *random form* $f \in \mathcal{P}_{d,n}$, is modelled as

$$f(x) = \sum_{k=0}^{[d/2]} \sum_{i=1}^{D_{d-2k,n}} \xi^i_{d-2k,n} Y^i_{d-2k,n}(x), \quad x \in \mathbb{S}^n.$$

Here, $\{Y^i_{d-2k,n}\}_{i=1}^{D_{d-2k,n}}$ is any $\langle \cdot, \cdot \rangle^{\mathrm{RFS}}_{d-2k,n}$-orthonormal basis of $\mathcal{H}_{d-2k,n}$ and $\xi^i_{d-2k,n}$, $i = 1, \ldots, D_{d-2k,n}, k = 0, \ldots, [d/2]$, are independent standard Gaussian random variables.

Normalization: $f \mapsto \omega f$ with $\omega = \sqrt{\frac{|\mathbb{S}^n|}{dim(\mathcal{P}_{d,n})}}$.

RSH and RFS as Gaussian fields

Centered Gaussian fields are characterized by its covariance functions.

Addition Theorem for spherical harmonics and the properties of Gegenbauer (a.k.a. ultraspherical) polynomials yield:

$$\mathbf{E}(h(x)h(y)) = \frac{P_d^{(\lambda)}(\langle x, y\rangle)}{P_d^{(\lambda)}(1)};$$

$$\mathbf{E}(f(x)f(y)) = \frac{P_d^{(\lambda+1)}(\langle x, y\rangle)}{P_d^{(\lambda+1)}(1)};$$

where $P_d^{(\lambda)}$ is the *standard Gegenbauer polynomial of parameter* $\lambda = \frac{n-1}{2}$ *of degree* d.

Real Fubini-Study

Using the addition theorem for spherical harmonics: for $x, y \in S^n$ and $t = x \cdot y$,

$$\mathbf{E}(f(x)f(y)) = \omega^2 \sum_{k=0}^{[d/2]} \sum_{i=1}^{D_{d-2k,n}} Y^i_{d-2k,n}\left(\frac{x}{|x|}\right) Y^i_{d-2k,n}\left(\frac{y}{|y|}\right)$$
$$= \sum_{k=0}^{[d/2]} \frac{D_{d-2k,n}}{dim(\mathcal{P}_{d,n})} \frac{P^{(\lambda)}_{d-2k}(t)}{P^{(\lambda)}_{d-2k}(1)},$$

where $P_k^{(\lambda)}$ is the Gegenbauer polynomials of degree k and parameter $\lambda = \frac{n-1}{2}$.

Real Fubini-Study

We have,

$$\frac{D_{d-2k,n}}{P_{d-2k}^{(\lambda)}(1)} = \frac{\binom{n+d-2k}{d-2k} - \binom{n+d-2k-2}{d-2k-2}}{\binom{n+d-2k-2}{d-2k}} = \frac{2(d-2k)+n-1}{n-1}.$$

Now, $dim(\mathcal{P}_{d,n}) = \binom{n+d}{d} = P_d^{(\lambda+1)}(1)$ and $n-1 = 2\lambda$. Hence, we get

$$K(t) = \frac{2}{2\lambda\ dim(\mathcal{P}_{d,n})} \sum_{k=0}^{[d/2]} [d-2k+\lambda] P_{d-2k}^{(\lambda)}(t)$$
$$= \frac{1}{2\lambda P_d^{(\lambda+1)}(1)} \frac{d}{dx} P_{d+1}^{(\lambda)}(t) = \frac{P_d^{(\lambda+1)}(t)}{P_d^{(\lambda+1)}(1)}.$$

Bonus

We consider an extra example based on the spectral function in Nicolaescu defined by

$$f(x) = \sum_{k=0}^{d} \sum_{i=1}^{D_{d-k,n}} \xi_{d-k,n}^{i} Y_{d-k,n}^{i}(x), \quad x \in \mathbb{S}^n.$$

This model does not suit our framework since it includes all $k = 0, \ldots, d$.

Coherent families

When studying the asymptotic behaviour as $d \to \infty$ one can think in changing the model for each d while preserving some consistency in the sequence.

As invariant polynomial distributions are parameterised by the weigths w_{d-2k}, it is natural to put conditions on them.

Fyodorov, Lerario & Lundberg proposed the following coherence condition: there exists $0 < \lambda \leq 1$ such that

$$w_d(d^\lambda x)d^\lambda \to_d \psi(x), \quad x \in [0, 1).$$

Besides, a subgaussian domination takes place.

Sodin asks a similar condition but directly on the covariance of the polynomials.

Main results.

Main results.

Let $F = (f_1, \ldots, f_n)$ with f_i i.i.d. copies of one of the models above.

Then, the expectation of the number of roots on the sphere N_0 can be computed explicitly and

Theorem

There exist constants V_K, V_{RSH}, V_{RFS} such that:

Kostlan $$\lim_{d\to\infty} \frac{\mathrm{Var}(N_0)}{d^{n/2}} = V_K;$$

RFS $$\lim_{d\to\infty} \frac{\mathrm{Var}(N_0)}{d^{n}} = V_{RFS};$$

RSH $$\lim_{d\to\infty} \frac{\mathrm{Var}(N_0)}{d^{n}} = V_{RSH};$$

We know $0 < V_{RFS}, V_K < \infty$ and $V_{RSH} < \infty$.

Remark - Uni- and bi-variate cases

Let us look now at the case $n = 1$.

We have

$$\mathbf{E}(f(x)f(y)) = \frac{1}{\pi}\Big[\frac{1+(-1)^d}{4} + \sum_{k=0}^{[(d-1)/2]} \cos\Big((d-2k)\theta\Big)\Big]$$
$$= \frac{\sin\Big((d+1)(\varphi-\psi)\Big)}{2\pi\,\mathrm{sen}(\varphi-\psi)}.$$

This is similar to the covariance of cosine polynomials which was previously studied.

The bivariate for RSH case was studied by Wigman in 2011.

On the expectation.

On the expectation.

By Rice formula, we have

$$\mathbf{E}(N_0) = \int_{\mathbb{S}^n} \mathbf{E}\big(|F'(s)| \mid F(s) = 0\big) p_{F(s)}(0) ds.$$

Here F' is the derivative (matrix) of F on the tangent space; ds is the surface measure and $|\cdot|$ is a shorthand for $|\det(\cdot)|$.

On the expectation.

Direct computations show that:

For Kostlan polynomials

$$\mathbf{E}(N_0) = 2d^{n/2}.$$

For RFS polynomials

$$\mathbf{E}(N_0) = 2\left[\frac{d(d+n+1)}{n+2}\right]^{n/2} \underset{d\to\infty}{\sim} c_n\, d^n.$$

For RSH (it suffices to replace n by $n-2$ in the variance).

$$\mathbf{E}(N_0) = 2\left[\frac{d(d+n-1)}{n}\right]^{n/2}.$$

This is coherent with Kostlan's results.

On the variance.

Variance - general aspects

To compute the variance we use Rice formula for the second factorial moment.

$$\mathbf{E}(N_0(N_0-1)) \\ = v_d^{2n} \int_{(\mathbb{S}^n)^2} \mathbf{E}[|\det \overline{F'}(s) \det \overline{F'}(t)| \,|F(s) = F(t) = 0] \\ \cdot p_{F(s),F(t)}(0,0) ds dt.$$

Here ds and dt are the (non-normalized) surface measure on $\mathbb{S}^n$, $\overline{F'}$ is the normalized derivative, $p_{F(s),F(t)}(0,0)$ is the joint density of $F(s)$ and $F(t)$ evaluated at $(0,0)$.

This can be strongly simplified thanks to the invariance of the distribution of F as done in [AADL].

Variance - general aspects

Let $\{e_0, e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^{n+1}$ and

$$s = e_0, \quad t = \cos(\theta)e_0 + \sin(\theta)e_1. \tag{1}$$

We fix the bases $\{e_1, \ldots, e_n\}$ and $\{\sin(\theta)e_0 - \cos(\theta)e_1, e_2, \ldots, e_n\}$ for $s^{\perp}$ and $t^{\perp}$.

Then,

$$\mathbf{E}(N_0(N_0 - 1)) = v_d^{2n} \, \frac{|\mathbb{S}^n||\mathbb{S}^{n-1}|}{(2\pi)^n} \int_0^{\pi} \frac{\sin^{n-1}(\theta)\, \mathcal{E}\left(\cos(\theta)\right)}{(1 - \mathcal{C}^2(\cos(\theta)))^{n/2}} d\theta,$$

where $\mathcal{E}(\cos(\theta))$ is the conditional expectation written for s, t as in (1) and $\mathcal{C}(\theta)$ is the covariance function.

Variance - general aspects

In order to compute the conditional expectation $\mathcal{E}(\cos(\theta))$ we need to know the joint distribution of $V(\theta) := \left(f_\ell(s), f_\ell(t), \frac{f'_\ell(s)}{v_d}, \frac{f'_\ell(t)}{v_d} \right)$ at the given basis (1),

Its variance-covariance matrix at the given basis (1), can be written in the following form

$$\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{12}^\top & I_n & A_{23} \\ \hline A_{13}^\top & A_{23}^\top & I_n \end{array}\right], \qquad (2)$$

where I_n is the $n \times n$ identity matrix...

Variance - general aspects

$$A_{11} = \begin{bmatrix} 1 & \mathcal{C} \\ \mathcal{C} & 1 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\mathcal{A} & 0 & \cdots & 0 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} \mathcal{A} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and A_{23} is the $n \times n$ diagonal matrix $diag(\mathcal{B}, \mathcal{D}, \ldots, \mathcal{D})$.

Here, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are the covariances between the different coordinates of V.

Variance - general aspects

Gaussian regression formulas imply that the conditional distribution of the vector $\left(\frac{f_\ell'(s)}{v_d}, \frac{f_\ell'(t)}{v_d}\right)$ (conditioned on $F(s) = F(t) = 0$) is centered normal with variance-covariance matrix given by

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{12}^\top & B_{22} \end{array}\right], \tag{3}$$

with $B_{11} = B_{22} = diag(\sigma^2, 1, \dots, 1)$ and $B_{12} = diag(\sigma^2\rho, \mathcal{D}, \dots, \mathcal{D})$.

Here,

$$\sigma^2 = 1 - \frac{\mathcal{A}^2}{1 - \mathcal{C}^2}, \quad \rho = \frac{\mathcal{B}(1 - \mathcal{C}^2) - \mathcal{A}^2\mathcal{C}}{1 - \mathcal{C}^2 - \mathcal{A}^2}.$$

Variance - particular aspects

In the case of RFS polynomials:

$$
\begin{aligned}
\mathcal{A}(\theta) &= \frac{2(\lambda+1)}{v_d} \frac{P_{d-1}^{(\lambda+2)}(\cos(\theta))}{P_d^{(\lambda+1)}(1)} \sin(\theta). \\
\mathcal{B}(\theta) &= \frac{4(\lambda+1)(\lambda+2)}{v_d^2 \cdot P_d^{(\lambda+1)}(1)} \cdot P_{d-2}^{(\lambda+3)}(\cos(\theta)) \sin^2(\theta) \\
&\quad - \frac{2(\lambda+1)}{v_d^2 \cdot P_d^{(\lambda+1)}(1)} \cdot P_{d-1}^{(\lambda+2)}(\cos(\theta)) \cos(\theta). \\
\mathcal{C}(\theta) &= \frac{P_d^{(\lambda+1)}(\cos(\theta))}{P_d^{(\lambda+1)}(1)}. \\
\mathcal{D}(\theta) &= \frac{2(\lambda+1)}{v_d^2} \frac{P_{d-1}^{(\lambda+2)}(\cos(\theta))}{P_d^{(\lambda+1)}(1)}.
\end{aligned}
$$

On the variance - the limit.

Variance - particular aspects

After the scaling $\theta \mapsto \frac{\theta}{\sqrt{d}}$ or $\theta \mapsto \frac{\theta}{d}$ we get pointwise limits and dominations in order to pass to the limit.

In Kostlan's case:

$$\mathcal{C}\Big(\frac{\theta}{\sqrt{d}}\Big) = \cos^d\Big(\frac{\theta}{\sqrt{d}}\Big) \underset{d\to\infty}{\to} e^{-\frac{1}{2}\theta^2}.$$

In the RFS case:

$$\mathcal{C}\Big(\frac{\theta}{d}\Big) = \frac{P_d^{(\lambda+1)}(\cos(\theta/d))}{P_d^{(\lambda+1)}(1)} \underset{d\to\infty}{\to} \Gamma\Big(\lambda+\frac{3}{2}\Big)\Big(\frac{2}{\theta}\Big)^{\lambda+\frac{1}{2}} J_{\lambda+\frac{1}{2}}(\theta),$$

where $J_{\lambda+\frac{1}{2}}$ is Bessel function of the first kind of order $\lambda+\frac{1}{2}$.

Variance - particular aspects

The domination in Kostlan's case is obtained by direct computations.

In the RFS case we use a bound by Elderly, Magnus & Nevai which in our setting can be written as: for $d \geq 0$

$$\max_{x\in[-1,1]} (1-x^2)^\lambda \left[\frac{P_d^{(\lambda)}(x)}{d^{\lambda-1}}\right]^2 \leq \kappa_\lambda, \tag{4}$$

for some positive constant κ_λ.

we get

$$\mathcal{C}\left(\frac{\theta}{d}\right) \leq \frac{\mathbf{C}}{\theta^{\lambda+1}}.$$

We used that for $u \in [0, \pi/2]$, we have $\sin(u) \geq \frac{2u}{\pi}$.

Some words about the positivity of the limit variance.

Some words about the positivity of the limit variance.

We borrow the chaotic expansion from [AADL],

$$N_d = v_d^n \sum_{q=0}^{\infty} \sum_{|\alpha|+|\beta|=q} b_\alpha f_\beta \int_{\mathbb{S}^n} H_\alpha(\widetilde{F}(t)) H_\beta(\frac{\widetilde{F}'(t)}{v_d}) dt$$

Here, the v_d^n comes from the homogeneity of the determinant.

Now, for $q = 2$ we have:

$$I_{2,d} = v_d^n \sum_{\ell=1}^{n} b_0^{n-1} \int_{\mathbb{S}^n} \Big[b_2 f_0 H_2(f_\ell(t)) + b_0 f_2 \sum_{r=1}^{n} H_2(\frac{\partial_r f_\ell(t)}{v_d}) \Big] dt,$$

where $v_d^2 = \mathbf{Var}(\partial_r f_\ell(t)) = \frac{d(d+n+1)}{n+2}$. Besides,

$$f_0 = n f_2 \qquad \text{and} \qquad \mathrm{b}_2 = -\mathrm{b}_0.$$

Some words about the positivity of the limit variance.

Now, using Green theorem we get

$$\sum_{r=1}^{n}\int_{\mathbb{S}^n} H_2\Big(\frac{\partial_r f_\ell(t)}{v_d}\Big)dt = \sum_{r=1}^{n}\int_{\mathbb{S}^n}\Big(\frac{\partial_r f_\ell(t)^2}{v_d^2}-1\Big)dt$$
$$= \sum_{r=1}^{n}\int_{\mathbb{S}^n}\Big(-\frac{f_\ell(t)\partial_{rr}^2 f_\ell(t)}{v_d^2}-1\Big)dt$$
$$= \int_{\mathbb{S}^n}\Big(\frac{f_\ell(t)(-\Delta)f_\ell(t)}{v_d^2}-n\Big)dt.$$

But,

$$-\Delta f(t) = \omega \sum_{k=0}^{[d/2]}\sum_{i=1}^{D_{d-2k,n}} \xi_{d-2k,n}^i(-\Delta)Y_{d-2k,n}^i(t)$$
$$= \omega \sum_{k=0}^{[d/2]}\sum_{i=1}^{D_{d-2k,n}} \xi_{d-2k,n}^i\lambda_{d-2k}Y_{d-2k,n}^i(t),$$

where $\lambda_{d-2k}=(d-2k)(d-2k+n-1)$ are the eigenvalues of $-\Delta$.

Some words about the positivity of the limit variance.

Hence,

$$\sum_{r=1}^{n}\int_{\mathcal{S}^n} H_2\Big(\frac{\partial_r f_\ell(t)}{v_d}\Big)dt = \int_{\mathcal{S}^n}\Big(\frac{f_\ell(t)(-\Delta)f_\ell(t)}{v_d^2} \pm n f_\ell(t)^2 - n\Big)dt$$

$$= \omega^2 \sum_{k=0}^{[d/2]}\Big(\frac{\lambda_{d-2k}}{v_d^2} - n\Big)\sum_{i=1}^{D_{d-2k,n}} (\xi^i_{d-2k,n})^2$$

$$+ \int_{\mathcal{S}^n} n H_2(f_\ell(t))dt.$$

Here we used the orthogonality of the Y's.

Some words about the positivity of the limit variance.

Putting all toghether we have

$$I_{2,d} = v_d^n \sum_{\ell=1}^{n} b_0^{n-1} (\underbrace{b_2 f_0 + n b_0 f_2}_{=0}) \int_{\mathcal{S}^n} H_2(f_\ell(t)) dt$$

$$+ b_0^n f_2 \underbrace{v_d^n \omega^2 \sum_{k=0}^{[d/2]} \left(\frac{\lambda_{d-2k}}{v_d^2} - n \right) \sum_{i=1}^{D_{d-2k,n}} (\xi_{d-2k,n}^i)^2}_{:=\zeta_d} .$$

Now,

$$\frac{\lambda_{d-2k}}{v_d^2} - n = (n+2) \frac{(d-2k)(d-2k+n-1)}{d(d+n+1)} - n \to_d 2.$$

So, since $\sum_{k=0}^{[d/2]} \sum_{i=1}^{D_{d-2k,n}} (\xi_{d-2k,n}^i)^2 \sim \chi^2_{\dim(\mathcal{P}_{n,d})}$, we have

$$\mathbf{Var}(\zeta_d) \approx \underbrace{v_d^{2n}}_{\sim d^{2n}} \underbrace{\omega^4}_{\sim d^{-2n}} \; 2 \underbrace{\binom{d+n}{d}}_{\sim d^n} \sim_{d \to \infty} d^n.$$

¡THANKS FOR YOUR ATTENTION!