Fluctuations for Brownian bridge expansions and convergence rates of Lévy area approximations

Karen Habermann

University of Warwick

Stochastic Differential Geometry and Mathematical Physics
8 June 2021

Brownian motion conditioned on vanishing time integrals

Condition Brownian motion $(W_t)_{t\in[0,1]}$ in $\mathbb{R}$ on $\boldsymbol{W}_1^N = 0$ for

$$\boldsymbol{W}_1^N = \left(W_1, \int_0^1 W_s\,\mathrm{d}s, \ldots, \int_0^1 \int_0^{s_{N-1}} \cdots \int_0^{s_2} W_{s_1}\,\mathrm{d}s_1 \ldots \mathrm{d}s_{N-1}\right)$$

Strategy for determining conditioned process.

Find explicit expression by writing

$$W_t = L_t^N + \sum_{l=1}^N \beta_l(t) \boldsymbol{W}_1^{N,l}$$

for functions $\beta_1, \ldots, \beta_N$ on $[0,1]$ such that, for all $l \in \{1, \ldots, N\}$,

$$\mathbb{E}\left[L_t^N \boldsymbol{W}_1^{N,l}\right] = 0\ .$$

Two Gaussian random variables are independent if and only if they have zero covariance. $\square$

Brownian motion conditioned on vanishing time integrals

Proposition (H., JLMS, 2021)

Let Q_n be the shifted Legendre polynomial of degree n on $[0,1]$. For $N \in \mathbb{N}$, the stochastic process $\left(L_t^N\right)_{t\in[0,1]}$ in $\mathbb{R}$ defined by

$$L_t^N = W_t - \sum_{n=0}^{N-1} (2n+1) \int_0^t Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r)\,\mathrm{d}W_r$$

has the same law as $(W_t)_{t\in[0,1]}$ conditioned on $\boldsymbol{W}_1^N = 0$.

Proof of the Proposition.

By integration by parts, we have $\boldsymbol{W}_1^{N,l} = \frac{1}{(l-1)!} \int_0^1 (1-r)^{l-1}\,\mathrm{d}W_r$.
For all $m \in \{0, \ldots, N-1\}$, we obtain, for all $t \in [0,1]$,

$$\mathbb{E}\left[L_t^N \int_0^1 Q_m(r)\,\mathrm{d}W_r\right] = 0\,. \qquad \square$$

Brownian motion conditioned on vanishing time integrals

Proposition (H., JLMS, 2021)

Let Q_n be the shifted Legendre polynomial of degree n on $[0,1]$. For $N \in \mathbb{N}$, the stochastic process $\left(L_t^N\right)_{t\in[0,1]}$ in $\mathbb{R}$ defined by

$$L_t^N = W_t - \sum_{n=0}^{N-1} (2n+1) \int_0^t Q_n(r)\,\mathrm{d}r \int_0^1 Q_n(r)\,\mathrm{d}W_r$$

has the same law as $(W_t)_{t\in[0,1]}$ conditioned on $\boldsymbol{W}_1^N = 0$.

The stochastic process $\left(L_t^N\right)_{t\in[0,1]}$ is a zero-mean Gaussian process in $\mathbb{R}$ with covariance C_N given, for $s,t \in [0,1]$, by

$$C_N(s,t) = \min(s,t) - \sum_{n=0}^{N-1} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r\,.$$

Brownian motion conditioned on vanishing time integrals

Theorem (H., JLMS, 2021)

Let $\Omega^{0,0} = \{w \in C([0,1],\mathbb{R}) \colon w_0 = w_1 = 0\}$ be the set of continuous loops in $\mathbb{R}$ at zero. The laws of $(W_t)_{t\in[0,1]}$ conditioned on $\boldsymbol{W}_1^N = 0$ converge weakly on $\Omega^{0,0}$ as $N \to \infty$ to the unit mass δ_0 at the zero path.

Proof.

Adapting the usual proof of Mercer's theorem shows that as $N \to \infty$ the sequence of covariances

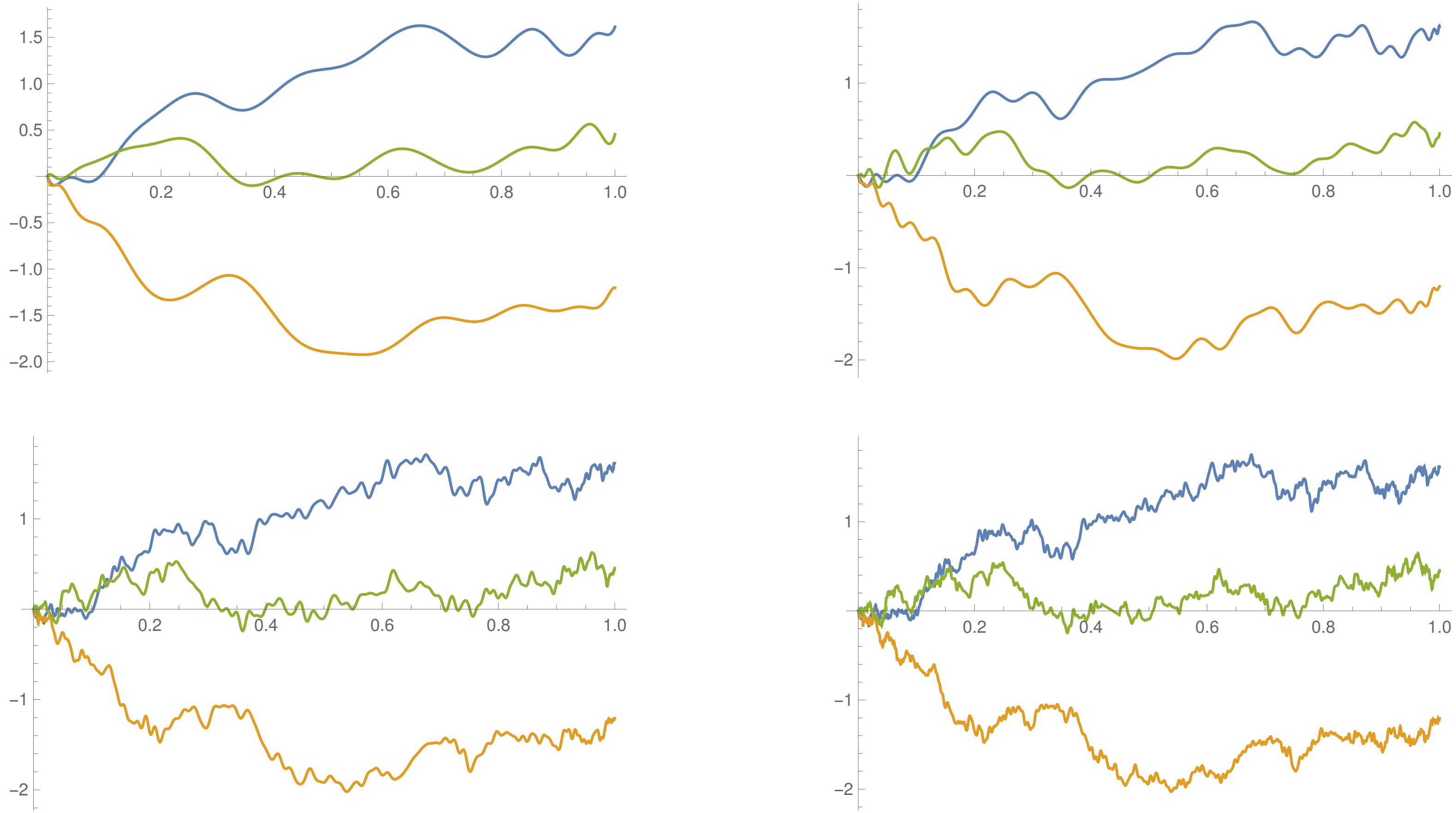
$$C_N(s,t) = \min(s,t) - \sum_{n=0}^{N-1} (2n+1) \int_0^s Q_n(r)\,\mathrm{d}r \int_0^t Q_n(r)\,\mathrm{d}r$$

converges uniformly on $[0,1] \times [0,1]$ to the zero function. $\square$

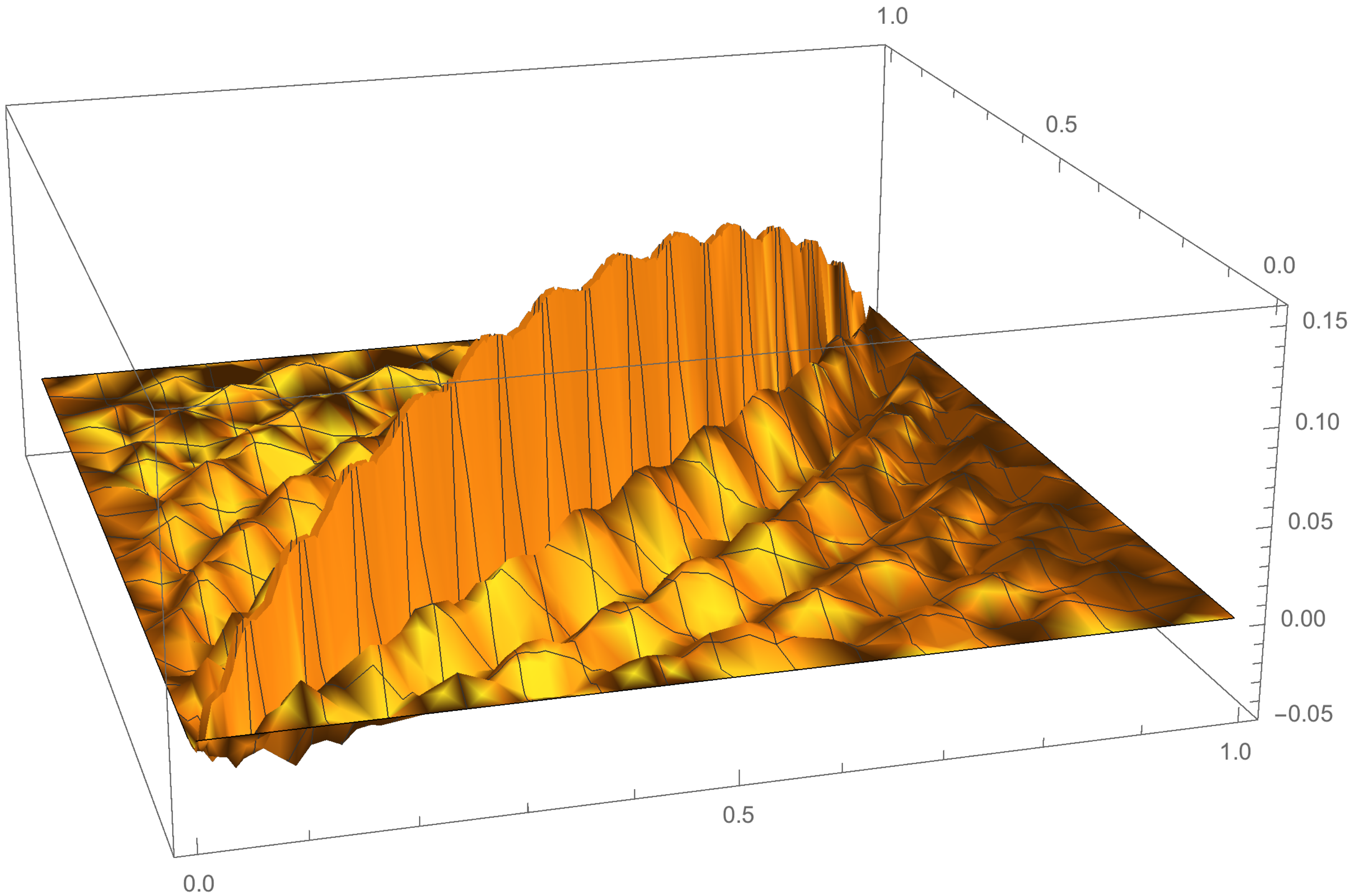
Brownian motion $(W_t)_{t\in[0,1]}$ admits the polynomial approximation

$$\left(\sum_{n=0}^{N-1}(2n+1)\int_0^t Q_n(r)\,\mathrm{d}r\int_0^1 Q_n(r)\,\mathrm{d}W_r\right)_{t\in[0,1]}.$$

Approximations plotted for $N=20$, $N=50$, $N=200$, $N=500$.



Fluctuations for the polynomial approximation



Plot of $(s,t) \mapsto NC_N(s,t)$ for $N = 16$

Fluctuations for the polynomial approximation

We notice that pointwise, for $s, t \in [0,1]$,

$$\lim_{N\to\infty} NC_N(s,t) = \begin{cases} \frac{1}{\pi}\sqrt{t(1-t)} & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases} .$$

Theorem (H., JLMS, 2021)

The fluctuation processes $\left(F_t^N\right)_{t\in[0,1]}$ *where* $F_t^N = \sqrt{N} L_t^N$ *converge in finite dimensional distributions as* $N \to \infty$ *to the collection* $(F_t)_{t\in[0,1]}$ *of independent zero-mean Gaussian random variables whose variances are given, for* $t \in [0,1]$*, by*

$$\mathbb{E}\left[(F_t)^2\right] = \frac{1}{\pi}\sqrt{t(1-t)} .$$

The collection $(F_t)_{t\in[0,1]}$ neither has a realisation as a process in $C([0,1],\mathbb{R})$ nor is it equivalent to a measurable process.

Fluctuations for Legendre polynomials

Theorem (H., JLMS, 2021)

Let P_n be the Legendre polynomial of degree n on $[-1,1]$.
Fix $x, y \in [-1,1]$ and, for $N \in \mathbb{N}$, set

$$R_N(x,y) =$$
$$N\left(\min(1+x,1+y) - \sum_{n=0}^{N-1} \frac{2n+1}{2} \int_{-1}^{x} P_n(z)\,\mathrm{d}z \int_{-1}^{y} P_n(z)\,\mathrm{d}z\right).$$

Then, we have pointwise

$$\lim_{N\to\infty} R_N(x,y) = \begin{cases} \frac{1}{\pi}\sqrt{1-x^2} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}.$$

Quantifies an integrated version of the completeness and orthogonality property $\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x) P_n(y) = \delta(x-y)$.

Fluctuations for Legendre polynomials

For the on-diagonal moment argument, we introduce complex-valued polynomials on $[-1,1]$ by setting

$$P_{-n-1}(x) = \mathrm{i}\, P_n(x) \quad \text{for } n \in \mathbb{N}_0 \ .$$

Properties of Legendre polynomials extend well

(a) $\int_{-1}^{1} (P_n(x))^2 \,\mathrm{d}x = \frac{2}{2n+1}$ for all $n \in \mathbb{Z}$

(b) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ for all $n \in \mathbb{Z}$

We can improve the off-diagonal argument to give, for $y \neq 0$,

$$\lim_{N\to\infty} R_N\left(x, x + N^{-\beta} y\right) = 0 \quad \text{for} \quad 0 \le \beta < 1 \ ,$$

which provides a bound on the decorrelation scale.

Fluctuations for Brownian bridge expansions

For the standard Brownian bridge $(B_t)_{t\in[0,1]}$, define fluctuation processes $(F_t^{N,1})_{t\in(0,1)}$ and $(F_t^{N,2})_{t\in(0,1)}$ by

$$F_t^{N,1} = \sqrt{N}\left(B_t - \sum_{k=1}^{N} \frac{2\sin(k\pi t)}{k\pi}\int_0^1 \cos(k\pi r)\,\mathrm{d}B_r\right) ,$$

$$F_t^{N,2} = \sqrt{2N}\left(B_t - \frac{1}{2}a_0 - \sum_{k=1}^{N} (a_k\cos(2k\pi t) + b_k\sin(2k\pi t))\right) .$$

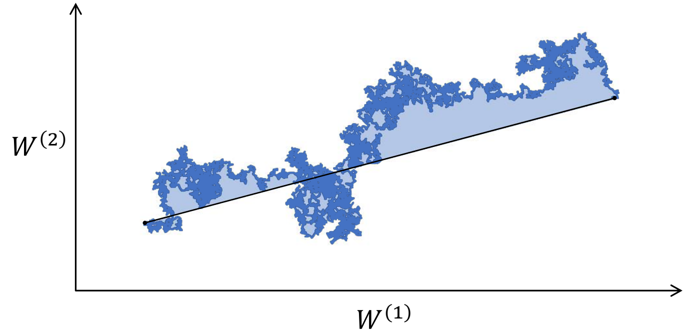
Theorem (Foster–H., 2021)

The fluctuation processes converge in finite dimensional distributions as $N \to \infty$ to the collection $(F_t)_{t\in(0,1)}$ of independent Gaussian random variables with mean zero and variance

$$\mathbb{E}\left[(F_t)^2\right] = \frac{1}{\pi^2} .$$

Asymptotic convergence rates of Lévy area approximations

The Lévy area of a Brownian motion $(W_t)_{t\in[0,1]}$ in $\mathbb{R}^d$ is the antisymmetric $d \times d$ matrix $A_{0,1}$ with the entries



$$A_{0,1}^{(i,j)} = \frac{1}{2}\left(\int_0^1 W_r^{(i)}\,\mathrm{d}W_r^{(j)} - \int_0^1 W_r^{(j)}\,\mathrm{d}W_r^{(i)}\right) .$$

Let $(B_t)_{t\in[0,1]}$ given by $B_t = W_t - tW_1$ be the Brownian bridge in $\mathbb{R}^d$ associated with $(W_t)_{t\in[0,1]}$, and set, for $k \in \mathbb{N}_0$,

$$a_k^{(i)} = 2\int_0^1 \cos(2k\pi r) B_r^{(i)}\,\mathrm{d}r , \quad b_k^{(i)} = 2\int_0^1 \sin(2k\pi r) B_r^{(i)}\,\mathrm{d}r$$

and, for $k \in \mathbb{N}$,

$$c_k^{(i)} = \int_0^1 Q_k(r)\,\mathrm{d}B_r^{(i)} .$$

Asymptotic convergence rates of Lévy area approximations

Theorem (Foster–H., 2021)

Define approximations $\widehat{A}_n$, $\widetilde{A}_n$ *and* $\bar{A}_{2n}$ *of the Lévy area* $A_{0,1}$ *by*

$$\widehat{A}_n^{(i,j)} = \frac{1}{2}\left(a_0^{(i)}W_1^{(j)} - W_1^{(i)}a_0^{(j)}\right) + \pi\sum_{k=1}^{n-1} k\left(a_k^{(i)}b_k^{(j)} - b_k^{(i)}a_k^{(j)}\right),$$

$$\widetilde{A}_n^{(i,j)} = \pi\sum_{k=1}^{n-1} k\left(a_k^{(i)}\left(b_k^{(j)} - \frac{1}{k\pi}W_1^{(j)}\right) - \left(b_k^{(i)} - \frac{1}{k\pi}W_1^{(i)}\right)a_k^{(j)}\right),$$

$$\bar{A}_{2n}^{(i,j)} = \frac{1}{2}\left(W_1^{(i)}c_1^{(j)} - c_1^{(i)}W_1^{(j)}\right) + \frac{1}{2}\sum_{k=1}^{2n-1}\left(c_k^{(i)}c_{k+1}^{(j)} - c_{k+1}^{(i)}c_k^{(j)}\right).$$

Then, for $i \neq j$ *and as* $n \to \infty$, *we have the mean squared errors*

$$m_{KP,M} \sim \frac{1}{\pi^2}\left(\frac{1}{2n}\right), \quad m_{KPW} \sim \frac{3}{\pi^2}\left(\frac{1}{2n}\right), \quad m_P \sim \frac{1}{8}\left(\frac{1}{2n}\right).$$

Link fluctuations and asymptotic convergence rates

The error in approximating Brownian Lévy area using N random vectors is essentially given by

$$\int_0^1 W_t^{(i)}\,\mathrm{d}W_t^{(j)} - \int_0^1 S_t^{N,(i)}\,\mathrm{d}S_t^{N,(j)}\;.$$

If one can argue that

$$\int_0^1 S_t^{N,(i)}\,\mathrm{d}\left(W_t^{(j)} - S_t^{N,(j)}\right) = O\left(\frac{1}{N}\right)\;,$$

then in terms of $(F_t^N)_{t\in[0,1]}$ defined by $F_t^N = \sqrt{N}\left(W_t - S_t^N\right)$, the error of the Lévy area approximation can be expressed as

$$\frac{1}{\sqrt{N}}\int_0^1 F_t^{N,(i)}\,\mathrm{d}W_t^{(j)} + O\left(\frac{1}{N}\right)\;.$$

It remains to apply Itô's isometry and Fubini's theorem.