Introduction to poroelasticity

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These notes are largely based on chapters 1-4 from *Theory of Linear Poroelasticity* by H. Wang, Princeton University Press, 2000. Roeloffs (1996) provides a more brief and applied presentation.

Contents

1 Constitutive relations for isotropic stress: Biot (1941)

Consider a saturated and isothermal rock. Stresses σ and pore pressure p are the independent variables and we would like to know the strain ϵ and fluid content f of the rock (fluid mass per unit volume divided by fluid density)

$$\epsilon = \epsilon(\sigma, p) \tag{1}$$

$$f = f(\sigma, p) \tag{2}$$

We consider first the case of an isotropic stress, for which $\epsilon = dV/V$ where V is volume. Changes in ϵ and f due to changes in σ and p are thus given by

$$d\epsilon = \left(\frac{\partial \epsilon}{\partial \sigma}\right)_p d\sigma + \left(\frac{\partial \epsilon}{\partial p}\right)_\sigma dp \tag{3}$$

$$df = \left(\frac{\partial f}{\partial \sigma}\right)_p d\sigma + \left(\frac{\partial f}{\partial p}\right)_\sigma dp \tag{4}$$

$1/K$ is the compressibility and K is the bulk modulus. Biot (1941)[1] defined four parameters K, R, H and H_1 such that

$$d\epsilon = \frac{1}{K} d\sigma + \frac{1}{H} dp \tag{5}$$

$$df = \frac{1}{H_1} d\sigma + \frac{1}{R} dp \tag{6}$$

and thus[2]

$$\frac{1}{K} = \left(\frac{\partial \epsilon}{\partial \sigma}\right)_p, \quad \frac{1}{H} = \left(\frac{\partial \epsilon}{\partial p}\right)_\sigma \quad \frac{1}{H_1} = \left(\frac{\partial f}{\partial \sigma}\right)_p, \quad \frac{1}{R} = \left(\frac{\partial f}{\partial p}\right)_\sigma \tag{7}$$

Biot (1941) further assumed there is a potential density function

$$U = \frac{1}{2}\left(\sigma\epsilon + pf\right). \tag{8}$$

Because σ and p are independent variables (hence $\partial U/\partial \sigma = \epsilon/2$ and $\partial U/\partial p = f/2$), then $\partial^2 U/\partial p \partial \sigma = \partial^2 U/\partial \sigma \partial p$ and

$$\left(\frac{\partial \epsilon}{\partial p}\right)_\sigma = \left(\frac{\partial f}{\partial \sigma}\right)_p \tag{9}$$

and hence

$$H = H_1 \tag{10}$$

1.1 Related poroelastic constants

Other (more commonly used) poroelastic constants can be expressed in terms of K, H and R. The **Skempton's coefficient** B is defined as

$$B = -\left(\frac{\partial p}{\partial \sigma}\right)_f = \frac{R}{H} \tag{11}$$

using equation (6).

[1] This was an influential paper. As of May 25, 2018 it had been cited 8600 times.

[2] From (5) we have $d\epsilon = \frac{1}{K}(d\sigma + \frac{K}{H}dp) = \frac{1}{K}(d\sigma + \alpha dp) = \frac{1}{K}d\sigma'$ where σ' is defined as the **effective stress** $d\sigma' = d\sigma + \alpha dp$.

The **Biot-Willis coefficient** α is defined as

$$\alpha = \frac{K}{H} \tag{12}$$

The groundwater flow equations to be discussed later combine conservation of mass and Darcy's law, and typically have a quantity called the specific storage. The specific storage at constant stress is

$$S_\sigma = \left(\frac{\partial f}{\partial p}\right)_\sigma = \frac{1}{R} \tag{13}$$

The specific storage at constant strain $S_\epsilon = \left(\frac{\partial f}{\partial p}\right)_\epsilon$ can be derived by eliminating $d\sigma$ from equations (5) and (6)

$$df = \frac{K}{H}d\epsilon + \left(\frac{1}{R} - \frac{K}{H^2}\right)dp \tag{14}$$

and hence

$$S_\epsilon = \left(\frac{\partial f}{\partial p}\right)_\epsilon = S_\sigma - \frac{K}{H^2} \tag{15}$$

and hence that

$$df = \alpha d\epsilon + S_\epsilon p. \tag{16}$$

With the definitions of B and α, equations (5) and (6) can be written instead as

$$d\epsilon = \frac{1}{K}d\sigma + \frac{\alpha}{K}dp \tag{17}$$

$$df = \frac{\alpha}{K}d\sigma + \frac{\alpha}{KB}dp \tag{18}$$

where

$$S_\sigma = \frac{\alpha}{KB} \tag{19}$$

Equations (17) and (18) can be rearranged for stress and pressure

$$d\sigma = \left(\frac{K}{1-\alpha B}\right)d\epsilon - \left(\frac{K}{1-\alpha B}B\right)df \tag{20}$$

$$dp = -\left(\frac{K}{1-\alpha B}B\right)d\epsilon + \left(\frac{K}{1-\alpha B}\frac{B}{\alpha}\right)df. \tag{21}$$

We can define $K_u = d\sigma/d\epsilon$ for $f = 0$, and hence identify K_u as an undrained bulk modulus (the bulk modulus when there is no fluid gain or loss)

$$K_u = \frac{K}{1-\alpha B} \tag{22}$$

We can rewrite equations (20) and (21) in terms of K_u

$$d\sigma = K_u d\epsilon - K_u B df \tag{23}$$

$$dp = -K_u B d\epsilon + \frac{K_u B}{\alpha} df \tag{24}$$

This last relation (24) is the basis for using water wells as strain meters. If we assume the pore pressure through changes in the water height $dh = \rho_w g dp$ in a well is an undrained response to a volumetric strain $d\epsilon$

$$dh = \frac{1}{\rho_w g} dp|_{f=0} = -\frac{K_u B}{\rho_w g} d\epsilon \tag{25}$$

The validity of this undrained assumption depends on the rate of loading compared to the permeability of the formation.

From equation (24) we obtain

$$df = \alpha d\epsilon + \frac{\alpha}{K_u B} dp \tag{26}$$

which leads to a physical interpretation of α

$$\alpha = \frac{df}{d\epsilon}|_{dp=0}, \tag{27}$$

that is, α is the ratio of the increment in fluid content to volumetric strain at constant pressure. From (26) we can also obtain an expression for the constrained specific storage coefficient

$$S_\epsilon = \frac{df}{dp}|_{\epsilon=0} = \frac{\alpha}{K_u B} \tag{28}$$

and

$$\frac{S_\sigma}{S_\epsilon} = \frac{K_u}{K} = \frac{1}{1 - \alpha B} \tag{29}$$

From equation (23) we obtain

$$d\epsilon = \frac{d\sigma}{K_u} + B df \tag{30}$$

The volumetric strain ϵ is composed of two components: an elastic component under undrained conditions, and a second component from the transfer of fluid. Equation (30) provides a physical interpretation of Skempton's coefficient

$$B = \frac{d\epsilon}{df}|_{d\sigma=0} \tag{31}$$

2 Constitutive relationship for anisotropic stress

Our variables are now pore pressure p, stress σ_{ij}, strain ϵ_{ij} and increment in fluid content df. The mean stress $\sigma_{\text{mean}} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ and the volumetric strain is $\epsilon = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$.

In principal coordinates the constitutive relations are

$$d\epsilon_1 = \frac{1}{E}d\sigma_1 - \frac{\nu}{E}d\sigma_2 - \frac{\nu}{E}d\sigma_3 + \frac{dp}{3H} \tag{32}$$

$$d\epsilon_2 = -\frac{\nu}{E}d\sigma_1 + \frac{1}{E}d\sigma_2 - \frac{\nu}{E}d\sigma_3 + \frac{dp}{3H} \tag{33}$$

$$d\epsilon_3 = -\frac{\nu}{E}d\sigma_1 - \frac{\nu}{E}d\sigma_2 + \frac{1}{E}d\sigma_3 + \frac{dp}{3H} \tag{34}$$

$$df = \frac{1}{H}d\sigma + \frac{1}{R}dp \text{ with } \sigma = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{35}$$

The elastic constants E (Young's modulus) and ν (Poisson ratio) are defined for drained conditions ($dp = 0$).

In general coordinates, there are shear strains and stress and seven equations since the stress and strain tensors are symmetric

$$d\epsilon_{xx} = \frac{1}{E}d\sigma_{xx} - \frac{\nu}{E}d\sigma_{yy} - \frac{\nu}{E}d\sigma_3 + \frac{dp}{3H} \tag{36}$$

$$d\epsilon_{yy} = -\frac{\nu}{E}d\sigma_{xx} + \frac{1}{E}d\sigma_{yy} - \frac{\nu}{E}d\sigma_{zz} + \frac{dp}{3H} \tag{37}$$

$$d\epsilon_{zz} = -\frac{\nu}{E}d\sigma_{xx} - \frac{\nu}{E}d\sigma_{yy} + \frac{1}{E}d\sigma_{zz} + \frac{dp}{3H} \tag{38}$$

$$d\epsilon_{xy} = \frac{1}{2G}d\sigma_{xy} \tag{39}$$

$$d\epsilon_{yz} = \frac{1}{2G}d\sigma_{yz} \tag{40}$$

$$d\epsilon_{xz} = \frac{1}{2G}d\sigma_{xz} \tag{41}$$

$$df = \frac{1}{H}d\sigma + \frac{1}{R}dp \tag{42}$$

where G is the shear modulus which can be related to E and ν via $G = E/2(1+\nu)$.

The first 6 of these equations can be written in standard index notation

$$\epsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right) + \frac{p}{3H}\delta_{ij} \tag{43}$$

and the d has been dropped so that the variable now indicates a change in that variable. Using $\alpha = K/H$ and then $K = \frac{2(1+\nu)}{3(1-2\nu)}G$

$$\epsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij} + \frac{1-2\nu}{1+\nu}\alpha p\delta_{ij}\right) \tag{44}$$

As before, defining the effective stress $\sigma'_{ij} = \sigma_{ij} + \alpha p\delta_{ij}$ then

$$\epsilon_{ij} = \frac{1}{2G}\left(\sigma'_{ij} - \frac{\nu}{1+\nu}\sigma'_{kk}\delta_{ij}\right) \tag{45}$$

Equation (43) can we rearranged into equivalent but useful forms

$$\sigma_{ij} = 2G\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right) - \alpha p\delta_{ij} \tag{46}$$

$$\epsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu_u}{1+\nu_u}\sigma_{kk}\delta_{ij} + \frac{2GB}{3}f\delta_{ij}\right) \tag{47}$$

where ν_u is the undrained Poisson ratio which can be derived from $\nu_u = \epsilon_{jj}/\epsilon_{ii}|_{f=0,\sigma_{jj}=0 \text{ for } j\neq i}$ using equation (42) and $B = -p/\sigma = -3p/\sigma_{kk}$

$$K_u = \frac{2G(1+\nu_u)}{3(1-2\nu_u)}. \tag{48}$$

3 Poroelastic constants

There are a number of poroelastic constants that depend on different constraints on the Representative Elementary Volume (REV).

3.1 Compressibility

Consider an experiment in which a rock is subjected to a confining pressure p_c and an independently controlled pore pressure p. We define a differential pressure

$$p_d = p_c - p \tag{49}$$

and we will use p_d and p and the independent variables. Then

$$\frac{dV}{V} = -\frac{1}{K}dp_d - \frac{1}{K_s'}dp \tag{50}$$

$$\frac{dV_p}{V} = -\frac{1}{K_p}dp_d - \frac{1}{K_\phi}dp \tag{51}$$

where V is the sample volume, V_p is the pore volume, K is the drained bulk modulus. The other constants require more explanation and are called the unjacketed bulk modulus (K_s'), drained pore modulus (K_p) and unjacketed pore modulus (K_ϕ).

In an unjacketed experiment, $p_c = p$ and hence $p_d = 0$. Thus

$$\frac{1}{K_s'} = -\frac{1}{V}\left(\frac{\delta V}{\delta p}\right)_{p_d=0} \quad \text{and} \quad \frac{1}{K_\phi} = -\frac{1}{V_p}\left(\frac{\delta V_p}{\delta p}\right)_{p_d=0} \tag{52}$$

In a drained experiment, $p = 0$ and thus $p_c = p_d$ and

$$\frac{1}{K} = -\frac{1}{V}\left(\frac{\delta V}{\delta p_c}\right)_{p=0} \quad \text{and} \quad \frac{1}{K_p} = -\frac{1}{V_p}\left(\frac{\delta V_p}{\delta p_c}\right)_{p=0} \tag{53}$$

Equation (5) can be rewritten as

$$d\epsilon = \frac{dV}{V} = -\frac{1}{K}dp_c + \frac{1}{H}dp = -\frac{1}{K}dp_d - \left(\frac{1}{K} - \frac{1}{H}\right)dp \tag{54}$$

Using (48), the unjacketed bulk modulus can also be written

$$\frac{1}{K_s'} = \frac{1}{K}(1-\alpha) \tag{55}$$

The drained pore modulus K_p can be related to other poroelastic constants. From the definition of f and the definition of porosity $\phi = V_p/V$, and $\phi = V_f/V$ for the fully saturated limit

$$df = \frac{dV_p - dV_f}{V} = \phi\left(\frac{dV_p}{V_p} - \frac{dV_f}{V_f}\right) = \phi\left(\frac{dV_p}{V_p} - \frac{dp}{K_f}\right). \tag{56}$$

Since the compressibility of the fluid

$$\frac{1}{K_f} = -\frac{1}{V_f}\frac{dV_f}{dp} \tag{57}$$

From (6) we have

$$df = \frac{1}{H}d\sigma + \frac{1}{R}dp = -\frac{1}{H}dp_c + \frac{1}{R}dp \tag{58}$$

Solving (54) for dV_p/V_p using (56) we have

$$\frac{dV_p}{V_p} = -\frac{1}{\phi H}dp_c + \left(\frac{1}{\phi R} - \frac{1}{K_f}\right)dp = -\frac{\alpha}{\phi K}dp_c + \left(\frac{\alpha}{\phi BK} - \frac{1}{K_f}\right)dp \tag{59}$$

and thus

$$\frac{1}{K_p} = -\frac{1}{V_p}\frac{dV_p}{dp_c}|_{p=0} = \frac{\alpha}{\phi K} \tag{60}$$

If we replace p_c in (59) by $p_d + p$ we have

$$\frac{dV_p}{V_p} = -\frac{\alpha}{\phi K}dp_d + \left(\frac{\alpha}{\phi BK} - \frac{\alpha}{\phi K} - \frac{1}{K_f}\right)dp \tag{61}$$

and thus

$$\frac{1}{K_\phi} = -\frac{1}{V_p}\frac{dV_p}{dp}|_{p_p=0} = \frac{\alpha}{\phi K}\left(1 - \frac{1}{B}\right) + \frac{1}{K_f} \tag{62}$$

The importance of these relationships is that the poroelastic constants K_s', K_p and K_ϕ can all be written in terms of measureable constants K, K_f, α, B and ϕ.

3.2 Storage capacity

There are several measures of storage capacity of a rock depending on the constraints on the REV.

The undrained specific storage S_σ is

$$S_\sigma = \frac{\partial f}{\partial p}|_\sigma = \frac{1}{R} = \frac{\alpha}{KB} \tag{63}$$

We can provide a micromechanical interpretation by using (53) and (60) in (61)

$$S_\sigma = \left(\frac{1}{K} - \frac{1}{K_s'}\right) + \phi\left(\frac{1}{K_f} - \frac{1}{K_\phi}\right) \tag{64}$$

The hydrogeological 3D definition of specific storage S' is defined as the volume of water released from the rock per unit volume per unit decline in hydraulic head while holding the mean stress constant. Hence S' and S_σ can be related

$$S' = \rho_f g S_\sigma \tag{65}$$

The constrained specific storage S_ϵ is defined as

$$S_\epsilon = \frac{\partial f}{\partial p}|_\epsilon = S_\sigma - \frac{K}{H^2} = S_\sigma - \frac{\alpha^2}{K} = \frac{\alpha^2}{K_u - K} \tag{66}$$

S_ϵ can also be expressed in terms of G, ν and ν_u (see Detournay and Cheng, 1993)

$$S_\epsilon = \frac{\alpha^2(1-2\nu_u)(1-2\nu)}{2G(\nu_u - \nu)} \tag{67}$$

The uniaxial specific storage is defined

$$S_s = \rho_f g \left(\frac{\partial f}{\partial p}\right)_{\sigma_{zz}=0, \epsilon_{xx}=\epsilon_{yy}=0} \tag{68}$$

where z is the vertical direction, and x and y are two horizontal directions. This is the usual hydrogeological definition of specific storage, i.e., the volume of water released per unit volume per unit decline in head while maintaining zero lateral strain and constant vertical stress.

The constraints $\epsilon_{xx} = \epsilon_{yy} = 0$ can be used in equations (36) and (37) to obtain σ_{xx} and σ_{yy} which can be summed to obtain

$$\sigma_{kk} = -4\eta p \ \text{ with } \ \eta = \frac{1-2\nu}{2(1-\nu)}\alpha. \tag{69}$$

and the change in the mean stress is a scalar multiple of the change in pore pressure (in this limit).

Substituting (69) into (18) with $\sigma_{kk} = 3\sigma$ with $S = S_S/\rho_f g$ the storage coefficient is

$$S = S_\sigma\left(1 - \frac{4\eta B}{3}\right) \tag{70}$$

It can then be shown that

$$S_\sigma \geq S \geq S_\epsilon \tag{71}$$

The different storage coefficients illustrates the importance of boundary conditions on the REV for poroelastic behavior.

The unjacketed specific storage

$$S_\gamma = \left(\frac{\partial f}{\partial p}\right)_{p_d=0} = \phi\left(\frac{1}{K_f} - \frac{1}{K_\phi}\right) \tag{72}$$

3.3 Poroelastic expansion coefficient

One of the Biot (1941) parameters, the poroelastic expansion coefficient, is

$$\frac{1}{H} = \left(\frac{\partial \epsilon}{\partial p}\right)_\sigma = \frac{\alpha}{K} = \frac{1}{K} - \frac{1}{K_s'} \tag{73}$$

The poroelastic expansion coefficient is thus the difference between the bulk compressibility and the unjacketed compressibility.

If there is no horizontal strain $\epsilon_{xx} = \epsilon_{yy} = 0$ the vertical stress

$$\sigma_{zz}|_{\epsilon_{xx}=\epsilon_{yy}=0} = \frac{2G(1-\nu)}{1-2\nu}\epsilon_{zz} - \alpha p = K_v \epsilon_{zz} - \alpha p \tag{74}$$

where K_v is the vertical compressibility. Rearranging for the strain

$$\epsilon_{zz} = \frac{1}{K_v}\sigma_{zz}|_{\epsilon_{xx}=\epsilon_{yy}=0} + \frac{\alpha}{K_v} p \tag{75}$$

Thus for zero vertical stress, the volumetric strain is proportional to the pore pressure change, and the constant of proportionality

$$c_m = \alpha / K_v \tag{76}$$

is known as the Geertsma uniaxial exapansion coefficient (and can be shown to also equal η/G).

3.4 Coefficients of undrained pore pressure buildup

Skempton's coefficient was defined as

$$B = -\left(\frac{\partial p}{\partial \sigma}\right)_{f=0} \tag{77}$$

and can also be expressed in term of the various compressibilities. Using $K_u = K/(1 - \alpha B)$ and $\alpha = 1 - K/K_s'$, we have

$$B = \frac{1 - K/K_u}{1 - K/K_s'} \tag{78}$$

and using (62)

$$B = \frac{1/K - 1/K_s'}{1/K - 1/K_s' + \phi(1/K_k - 1/K_\phi)} \tag{79}$$

B can also be written in terms of Poisson ratios

$$B = \frac{3(\nu_u - \nu)}{\alpha(1 + \nu_u)(1 - 2\nu)} \tag{80}$$

Returning to the limit with no horizontal strains, we can define a loading efficient as

$$\gamma = -\left(\frac{\partial p}{\partial \sigma_{zz}}\right)_{\epsilon_{xx}=\epsilon_{yy}=0, f=0} \tag{81}$$

From the constitutive law (43) with $\epsilon_{xx} = \epsilon_{yy} = 0$ and $f = 0$, we have

$$\sigma_{xx}|_{\epsilon_{xx}=\epsilon_{yy}=0,f=0} = \sigma_{yy}|_{\epsilon_{xx}=\epsilon_{yy}=0,f=0} = \frac{\nu_u}{1-\nu_u}\sigma_{zz} \tag{82}$$

Since $\sigma = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3}\frac{1+\nu_u}{1-\nu_u}\sigma_{zz}$ then

$$\gamma = \frac{B}{3}\frac{1+\nu_u}{1-\nu_u} \tag{83}$$

The tidal efficiency T.E. can be defined as the water level change in a well divided by the water level change in the ocean. The uniaxial strain condition is often assumed for aquifers close to a shoreline. γ can be expressed in terms of other poroelastic constants, e.g.,

$$\text{T.E.} = \gamma = \frac{\alpha}{K_v S} \tag{84}$$

If we assume $1/K_s' = 1/K_\phi = 0$, then $S = \frac{1}{K_v} + \phi\frac{1}{K_f}$ and

$$\text{T.E.} = \frac{K_f}{K_f + \phi K_v} \tag{85}$$

This equation and measurements of water level and then be used to estimate the specific storage of aquifers near the shoreline if ϕ and K_f are known.

Another surface source of loading is variation in atmospheric pressure. The barometric efficiency is defined as the ratio between the change in water level dh in a well to the change in atmospheric pressure dp_{atm} converted to an equivalent head $dp_{atm}/\rho_f g$

$$\text{B.E.} = -\rho_f g \frac{dh}{dp_{atm}} \tag{86}$$

where the negative sign is included to make B.E. positive. The atmospheric pressure exerts a load both on the surface of the Earth and the water surface in the well. The former causes water level to rise by an amount $\gamma dp_{atm}/\rho_f g$ and the latter for the level to drop by $dp_{atm}/\rho_f g$. Thus

$$\text{B.E.} = 1 - \gamma \tag{87}$$

If we again assume $1/K_s' = 1/K_\phi = 0$,

$$\text{B.E.} = \frac{\phi K_v}{K_f + \phi K_v} \tag{88}$$

Independent measures of both T.E. and B.E. yield S and ϕ.

4 Governing equations for fluid flow

Coupling between elastic deformation and fluid flow occurs between the equation for fluid flow (assumed to be Darcy's law) and conservation of mass. We begin with the continuity equation for fluid

$$\frac{\partial f}{\partial t} = -\nabla \cdot \mathbf{q} + Q \tag{89}$$

where $\mathbf{q}$ is the specific discharge vector and Q is the fluid source per unit volume per unit time.

Substituting Darcy's law into the conservation of mass question yields

$$\frac{\partial f}{\partial t} = \frac{k}{\mu} \nabla^2 p + Q \tag{90}$$

where k is permeability (assumed constant in space) and μ the fluid viscosity. With the assumption of uniaxial strain and constant vertical stress, then $f = Sp$, which when used in equation (90) leads to the standard flow equation in hydrogeology

$$S \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla^2 p + Q \tag{91}$$

The assumption of uniaxial strain and constant vertical stress are not satisfied rigorously in 2D and 3D flows in general because the flow distorts the strain field. A more general flow equation can be obtained by using equation (26)

$$f = \frac{\alpha}{K} \frac{\sigma_{kk}}{3} + \frac{\alpha}{KB} p$$

leading to

$$\frac{\alpha}{KB} \left[\frac{B}{3} \frac{\partial \sigma_{kk}}{\partial t} + \frac{\partial p}{\partial t} \right] = \frac{k}{\mu} \nabla^2 p + Q \tag{92}$$

where the first term on the left, the time derivative of the mean stress, is equivalent mathematically to a fluid source.

Since $S_\sigma = \alpha / KB$ (equation 63), then

$$S_\sigma \left[\frac{B}{3} \frac{\partial \sigma_{kk}}{\partial t} + \frac{\partial p}{\partial t} \right] = \frac{k}{\mu} \nabla^2 p + Q \tag{93}$$

Using the relations $\sigma_{kk}/3 = K\epsilon - \alpha p$ (equation yy) and $S_\epsilon = (1 - \alpha B) S_\sigma$, equation (93) can be transformed to

$$\alpha \frac{\partial \epsilon_{kk}}{\partial t} + S_\epsilon \frac{\partial p}{\partial t} = \frac{k}{\mu} \nabla^2 p + Q \tag{94}$$

in which fluid flow is coupled to the time variation of volumetric strain that acts mathematically as a fluid source.

These PDEs are inhomogeneous even when there are no explicit fluid sources Q. Therefore any further simplification to a homogeneous diffusion equation must be justified on the grounds that a constant strain or stress is maintained approximately throughout the region of interest.

4.1 Uncoupling stress (or strain) from fluid flow

The uncoupling is one-way in that the pore-pressure field does produce stress and strain, but under certain conditions the changes in stress and strain do not affect fluid flow. Thus the transient flow equation can be solved independent of stress and strain fields. The resulting fluid pressure field can then be inserted into the elastostatic equation as a parametric function of time which is then solved separately. Two circumstances occur where this is achieved: a highly compressible fluid, an irrotational displacement field in an unbounded domain. For the latter see Wang (2000).

Highly compressible fluid. Because $\alpha/K = 1/K - 1/K_s^{'}$, the Skempton's coefficient can be written as

$$B = \frac{\alpha}{\alpha + \phi\left(\frac{K}{K_f} - \frac{K}{K_\phi}\right)} \tag{95}$$

For a highly compressible fluid $K \gg K_f$. Since both K/K_ϕ and α are < 1,

$$B \approx \frac{\alpha K_f}{\phi K} \ll 1 \tag{96}$$

hence the coupling term $\frac{B}{3}\partial\sigma_{kk}/\partial t$ approaches zero and

$$S_\sigma = \frac{\alpha}{KB} \approx \frac{\alpha\phi K}{K\alpha K_f} = \frac{\phi}{K_f} \tag{97}$$

and we obtain the uncoupled flow equation

$$\frac{\phi}{K_f}\frac{\partial p}{\partial t} = \frac{k}{\mu}\nabla^2 p + Q \tag{98}$$

5 Response of wells to solid Earth tides

The water level in an open well will respond to strains in the aquifer it taps. Because water must flow into the well, there will be a phase lag that depends on permeability. The amplitude of the response will depend on poroelastic properties of the aquifer and geometric properties of the well.

The special case of harmonic forcing offers opportunities to probe aquifer properties, as developed in Hsieh et al. (1989). This forcing could be seismic waves, or solid Earth tides. The hydraulic head fluctuations are assumed to be

$$h = h_0 \exp(i\omega t) \tag{99}$$

with water level response

$$x = x_0 \exp(i\omega t) \tag{100}$$

where h_0 and x_0 are the complex amplitude of pressure head and water level.

Consider the following geometry; r_w is the well radius open to the aquifer with thickness d, r_c the radius of casing, and s is the change in water level. The groundwater flow equation for a homogeneous, isotropic, confined aquifer with only vertical strain and assuming constant vertical stress (see previous sections)

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{S}{T}\frac{\partial s}{\partial t} = 0. \tag{101}$$

where $T = k\rho g d/\mu$ is the transmissivity and $S = dS_s$ is the storativity. The boundary condition as the well is

$$2\pi r_w T \left(\frac{\partial s}{\partial r}\right)_{r=r_w} = -Q_0 \exp(i\omega t) \ \text{ at } \ r = r_w \tag{102}$$

and as $r \to \infty$, $s \to 0$.

Because the equation is linear and the forcing is harmonic, the solution must have the form

$$s(r,t) = G(r)\exp(i\omega t) \tag{103}$$

Substituting into the governing equation and boundary conditions

$$\frac{d^2 G}{dr^2} + \frac{1}{r}\frac{dG}{dr} - \frac{i\omega S}{T}G = 0 \tag{104}$$

with

$$2\pi r_w T \left(\frac{dG}{dr}\right)_{r=r_w} = -Q_0 \exp(i\omega t) \ \text{ at } \ r = r_w \tag{105}$$

and $G \to \infty$, $s \to 0$.

The solution is given by

$$s_w = -\frac{\omega r_c^2 x_0}{2T}\left[(\Psi \mathrm{Ker}(\alpha_w) + \Phi \mathrm{Kei}(\alpha_w)) - i(\Phi \mathrm{Ker}(\alpha_w) - \Psi \mathrm{Kei}(\alpha_w)\right]\exp(i\omega t) \tag{106}$$

where

$$\Phi = -\frac{\mathrm{Ker}_1(\alpha_w) + \mathrm{Kei}_1(\alpha_w)}{\sqrt{2}\alpha_w[\mathrm{Ker}_1^2(\alpha_w) + \mathrm{Kei}_1^2(\alpha_w)]} \tag{107}$$

$$\Psi = -\frac{\mathrm{Ker}_1(\alpha_w) - \mathrm{Kei}_1(\alpha_w)}{\sqrt{2}\alpha_w[\mathrm{Ker}_1^2(\alpha_w) + \mathrm{Kei}_1^2(\alpha_w)]} \tag{108}$$

$$\alpha_w = \left(\frac{\omega S}{T}\right)^{1/2} r_w \tag{109}$$

and Ker and Kei are Kelvin functions of order zero, and Ker_1 and Kei_1 are Kelvin functions of order 1.

Finally we can compute the amplitude response

$$A = |x_0/h_0| = (E^2 + F^2)^{-1/2} \tag{110}$$

and phase shift

$$\eta = -\tan^{-1}(F/E) \tag{111}$$

where the functions F and E are

$$E = 1 - \frac{\omega r_c^2}{2T}\left[\Psi \mathrm{Ker}(\alpha_w) + \Phi \mathrm{Kei}(\alpha_w)\right] \tag{112}$$

$$F = \frac{\omega r_c^2}{2T}\left[\Phi \mathrm{Ker}(\alpha_w) i \Psi \mathrm{Kei}(\alpha_w)\right] \tag{113}$$

This models allows only for horizontal flow and results in a negative phase shift.

A positive phase shift arises from vertical flow through a layer of fixed thickness. The solution in this case, e.g., from Xue et al. (2016), has

$$A = \left[1 - 2\exp(-z/\delta)\cos(z/\delta) + \exp(-2z/\delta)\right]^{1/2} \tag{114}$$

$$\eta = \tan^{-1}\left[\frac{\exp(-z/\delta)\sin(z/\delta)}{1 - \exp(-z/\delta)\cos(z/\delta)}\right] \tag{115}$$

where $\delta = \sqrt{2T/\omega}$.

Because of the sensitivity of phase and amplitude to permeability and storage properties, monitoring the response of water to tides provides a means to document the evolution of these properties. For example, Elkhoury et al. (2006) showed how regional earthquakes changed permeability and how this permeability recovered over time. The actual use of these equations requires smoothing, filtering and windowing data.

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