

Hydrogeology  
Group



etcg



idæ<sup>a</sup>

# Flow through permeable media

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# Introduction and contents

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- Defining fluid flow of any kind of medium in any kind of circumstances involves:
  - Momentum conservation
  - Mass conservation
- For permeable media and slow laminar flow momentum conservation is described by Darcy's Law.
- This Chapter es devoted to:
  - Study Darcy's law and its terms:
  - Head
  - Viscosity
  - Permeability
  - The meaning of Darcy's law
  - Its limits of validity
  - The mass conservation equation
  - Storage coefficient

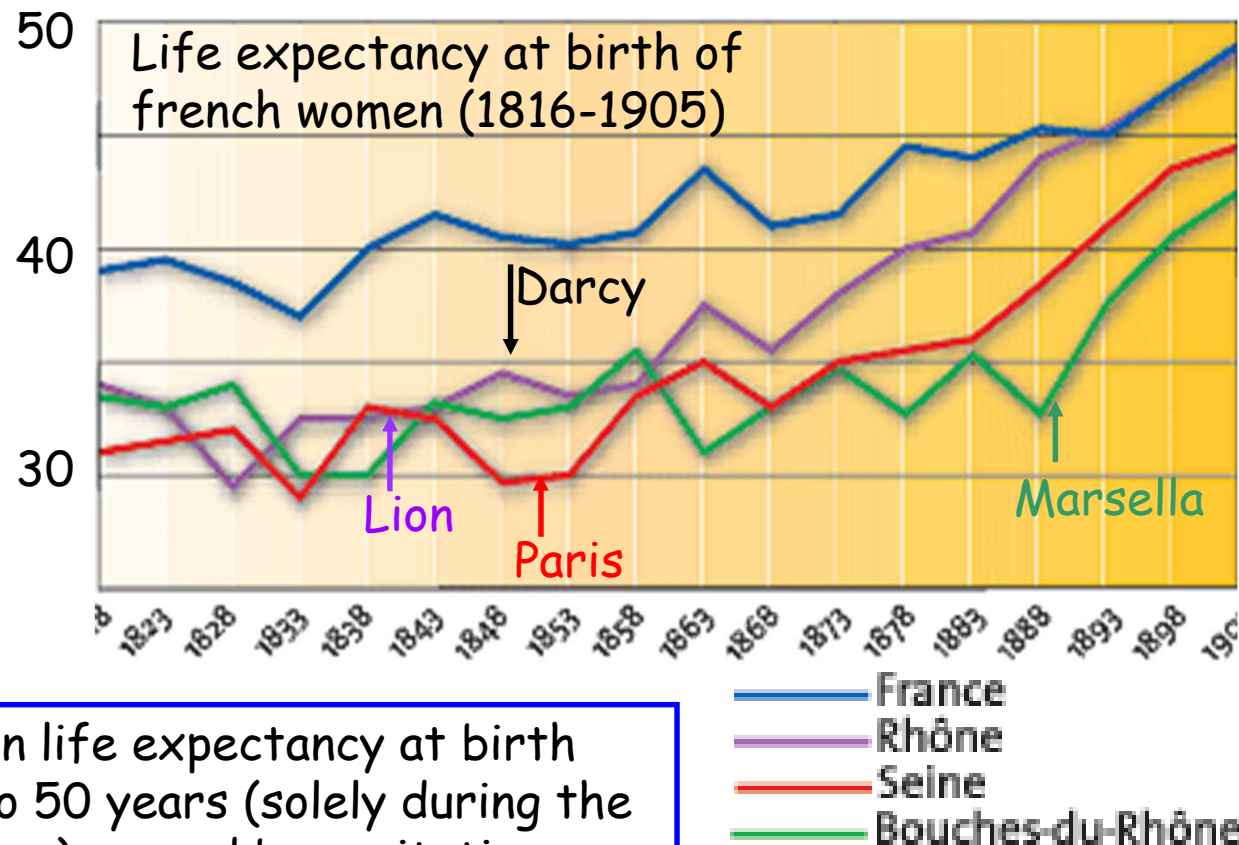
# Our context



During the course we will often forget it, but our business is water: understanding it for supplying and protecting it...

# Darcy's context

XIXth century engineers researched potabilization of water for drinking and treatment of waste water. Sand filtering was one of the key elements: size of grains and filters?



Increase in life expectancy at birth from 32 to 50 years (solely during the XIX century) caused by sanitation (Preston, 1978)

# Henry Philibert Gaspard Darcy (1803-1858)

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He did numerous civil works and was a good "conventional" civil engineer.

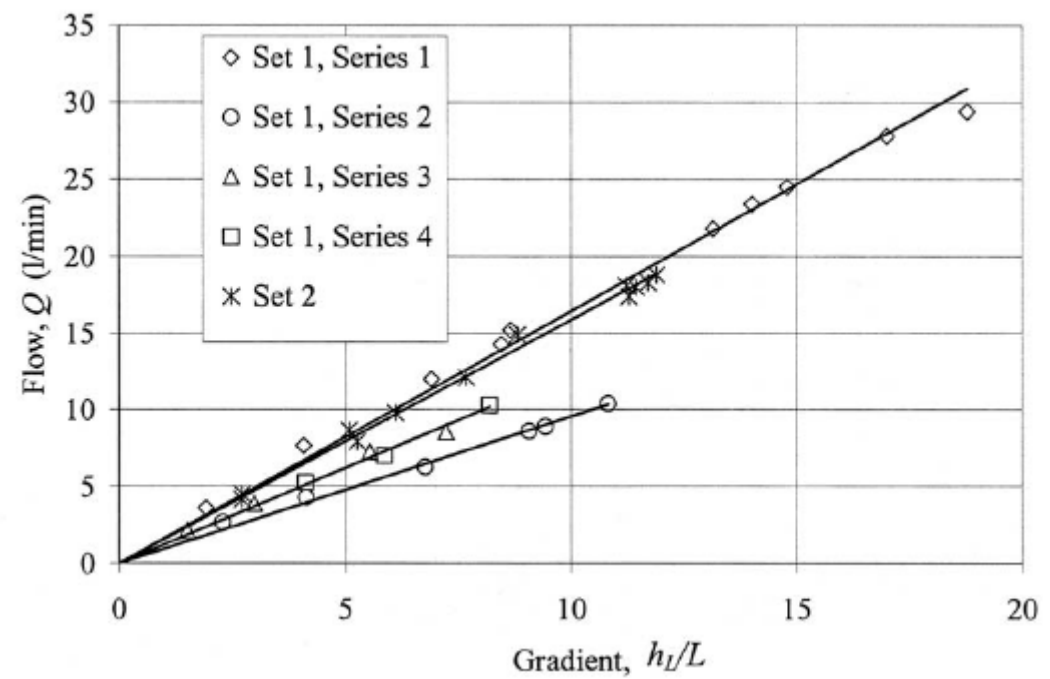
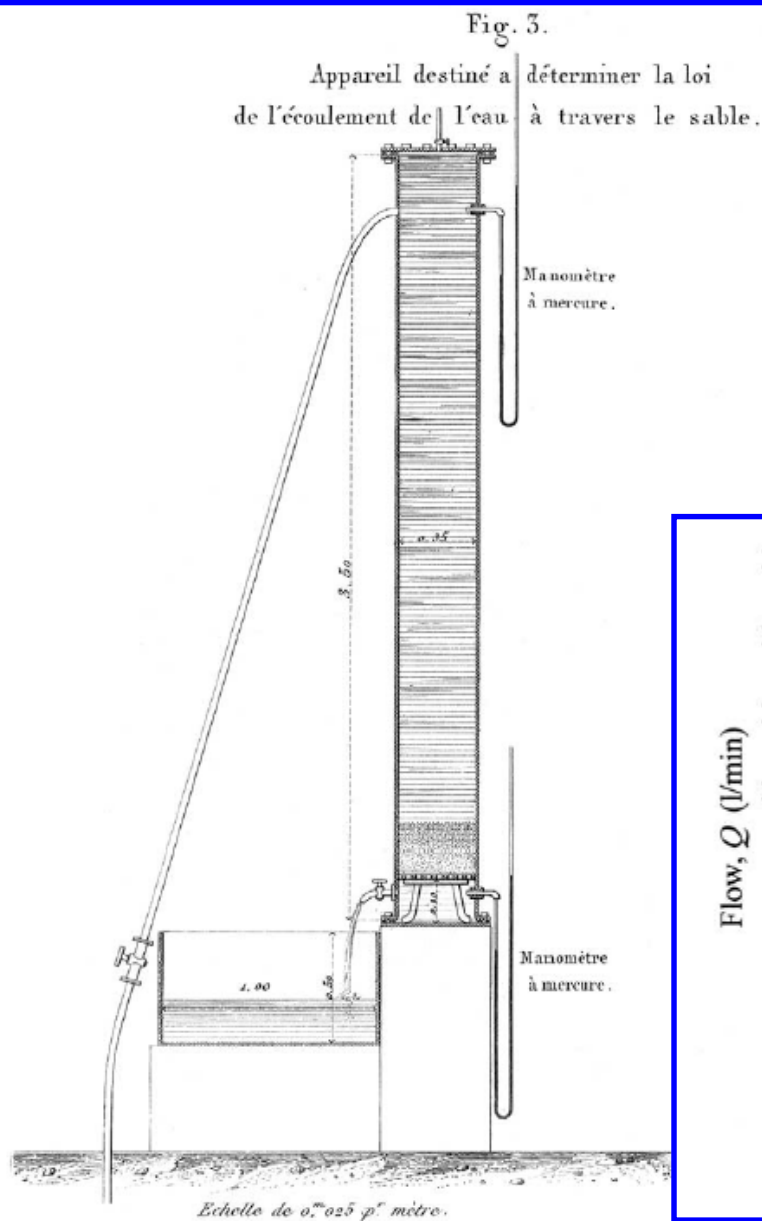
He had no idea of groundwater (his well hydraulics concepts are very primitive)

He designed the Dijon municipal water system. After retiring, he investigated water related issues, performed numerous experiments singularly:

- flow through pipes, which led to the Darcy-Weisbach equation
- flow through porous media for the design of sand filters. The results of these experiments were published as an appendix to the *Les Fontaines Publiques de la Ville de Dijon* [Darcy, 1856].

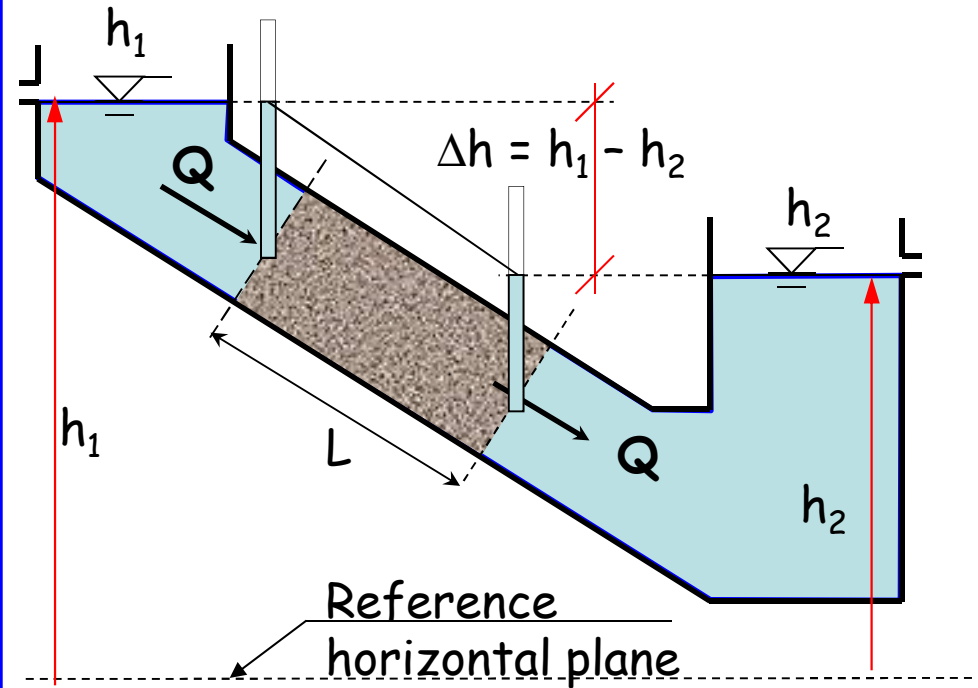


# Darcy (1856) experiment



# DARCY's LAW: an EXPERIMENTAL LAW

- Darcy showed that the flow through a sand column is:
  - Proportional to cross section  $A$
  - Inversely proportional to length  $L$
  - Proportional to head drop
  - Proportional to the square of grain size
- Therefore,
  - $Q = Cd^2 A \Delta h / L$
- Currently written as
  - $q = Q/A = -K \text{ grad } h$



# Generalizing Darcy's law

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- What is exactly  $h$ ? Is it a potential?
- Does Darcy's law apply to different fluids?
- Does it apply in open systems (as opposed to a pipe)?
- Which properties of the fluid control it?
- Does the nature of the solid affect it (or only its geometry)?
- What are the limitations of Darcy's law?
- Is it valid for heterogeneous media?
- Does flow need to be steady?

You should know the answer to these questions, but do you know the whys?

# Is there a potential for flow?

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- First, what does "**potential**" mean?
  - Potential is a field (normally, energy per unit mass), from which fluxes can be derived (typically fluxes are proportional to the gradient of potential). Examples: Electrical potential, temperature, chemical potential (concentration), etc.
- Second, under some conditions, **yes, HEAD** (Bernouilli, 1738)
- It is our state variable. It represents energy of fluid per unit weight.

$$h = \frac{p}{\gamma} + z \left[ + \frac{v^2}{2g} \right]$$

- ... water elevation in wells...

# Bernoulli's equation: energy conservation

Daniel Bernoulli derived his equation from the conservation of energy, although the concept of energy was not well-developed in his time. Using energy concepts, the equation can be extended to compressible fluids and thermodynamic processes.

Energy in = Energy out on the volume of fluid  $Q = A \cdot V \cdot t$ , which disappears at one point and reappears at another imaginary pistons move with the speed of the fluid. Capital letters are used for quantities at one point, small letters for the same quantities at the second point.

Energy made of ( $Q$ : Volume of water =  $VA t$ ):

$$\text{Kinetic: } MV^2/2/t = Q\rho V^2/2$$

$$\text{Potential: } Mgz/t = Q\rho gz$$

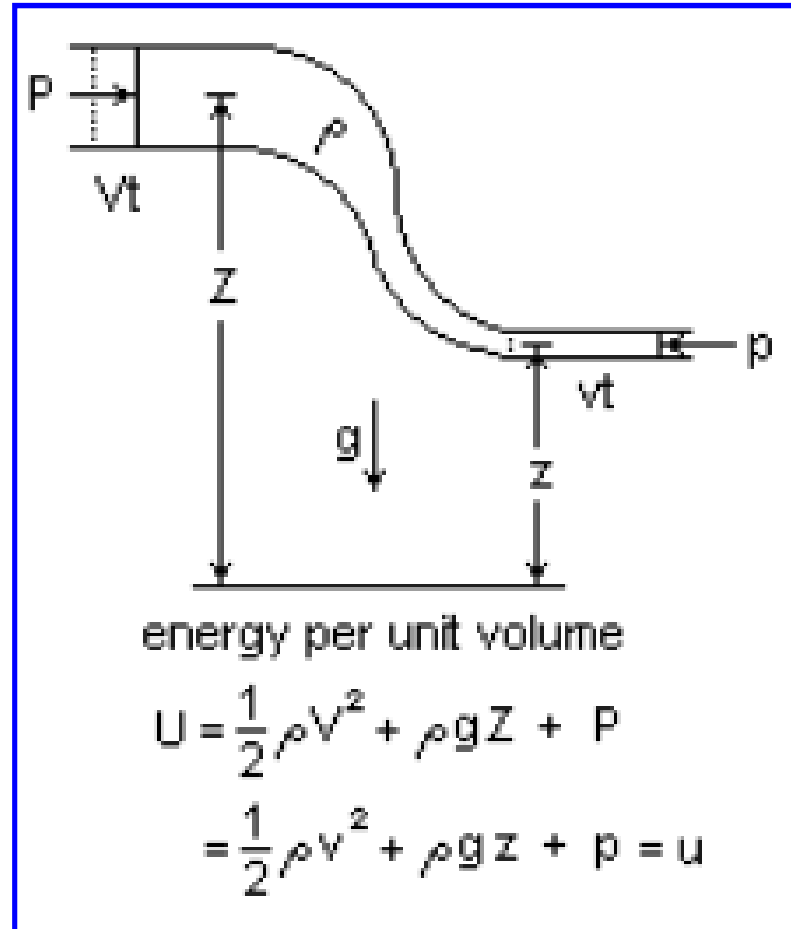
$$\text{Pressure Work}/t = F \cdot X/t = (P \cdot A) \cdot V = Q \cdot P$$

Total energy of the piston:

$$Q \cdot (P + \rho gz + \rho V^2/2)$$

Divide by  $Q$  to get energy per unit volume,

Divide by  $Q\rho g$  to get energy per unit weight



# Bernoulli equation: from momentum conserv.

From momentum conservation:  
(Eulerian equations)

Assuming:

velocity must derive from a  
potential ( $\mathbf{v} = \text{grad}\phi$ )

external forces are conservative  
(they derive from a potential)

density is constant, or a function  
of the pressure alone. That,  
density differences caused by  
temperature or concentration  
variations are neglected)

Bernoulli's Equation follows on  
integration

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{F} - \frac{1}{\rho} \nabla p \quad \text{Euler's equation}$$

$$\mathbf{v} = -\nabla \phi \quad \text{so } \nabla \times \mathbf{v} = 0 \quad \text{irrotational}$$

$$\mathbf{F} = -\nabla \Omega \quad \text{conservative}$$

$$\rho = \text{const. or } f(p) \quad \text{incompressible}$$

$$\frac{\partial}{\partial t} (-\nabla \phi) + \nabla \phi \cdot \nabla \nabla \phi = -\nabla \Omega - \frac{1}{\rho} \nabla p$$

$$\nabla \left[ -\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \Omega + \frac{p}{\rho} \right] = 0$$

$$-\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \Omega + \frac{p}{\rho} = C$$

$$\frac{v^2}{2} + \Omega + \frac{p}{\rho} = C \quad \text{Bernoulli's equation}$$

$\frac{\nabla p}{\rho} = \nabla \int_{p_{ref}}^p \frac{dp}{\rho(p)} \dots$  Therefore, Bernoulli (and all we are going to say) is

still valid for compressible fluids, the piezometric term is  $\int_{p_{ref}}^p \frac{dp}{\rho(p)}$ .

If  $\rho(c)$  or  $\rho(T)$ , there is no potential

# Bernoulli derived simpler momentum conserv.

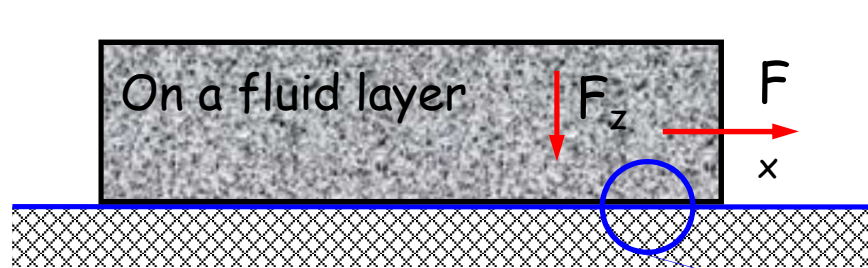
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- The second form of Bernoulli's Equation arises from the fact that in steady flow the particles of fluid move along fixed streamlines, as on rails, and are accelerated and decelerated by the forces acting tangent to the streamlines.
- Under the same assumptions for the external forces and the density, but without demanding irrotational flow, we have for an equation of motion  $dv/dt = v(dv/ds) = -dz/ds - (1/\rho)dp/ds$ , where  $s$  is distance along the streamline.
- This integrates immediately to  $v^2/2 + z + p/\rho = c$ . In this case, the constant  $c$  is for the streamline considered alone; nothing can be said about other streamlines.
- This form of Bernoulli's Equation is more generally applicable, but less powerful than the preceding one. It is the form most often applicable to typical engineering problems.
- The derivation is easy and straightforward, clearly showing the hypotheses, and also that the motion is assumed frictionless.

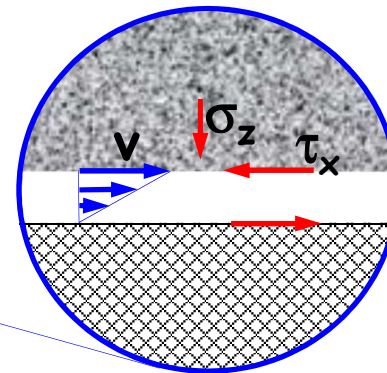
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- The concept of potential only addresses the energy of the fluid. It does not say anything about how the fluid flows. For this, we need to either
    - Consider dissipated energy, or
    - Balance the forces affecting the fluid (momentum conservation)
  - **In either case, we need to evaluate the forces opposing fluid flow**

# On the resistance of a fluid to flow

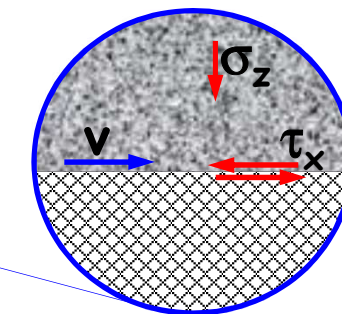
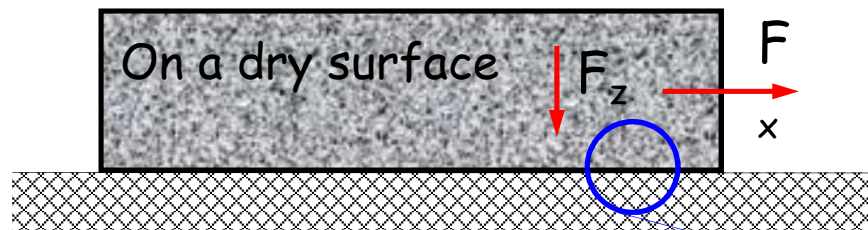
Slide a solid at a constant velocity,  
what is the resistance? Is it proportional to velocity?  
Does it depend on the weight of the object?



$$\tau_x = F_x / A \quad \sigma_z = F_z / A$$



On a fluid layer, shear stress,  $\tau_x$ , is usually proportional to velocity  $v$  (for a given fluid thickness)



On a dry surface, shear stress,  $\tau_x$ , is usually proportional to normal stress  $\sigma_z$

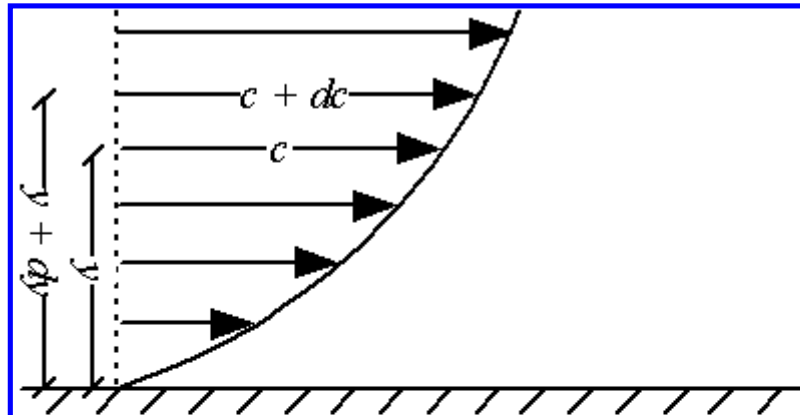
# Viscosity: A sticky subject

- We can say that viscosity is the resistance a material has to change in form. This property can be thought of as an internal friction.
- Viscosity is defined as the degree to which a fluid resists flow under an applied force, measured by the tangential friction force per unit area divided by the velocity gradient under conditions of streamline flow; coefficient of viscosity.

**Dynamic (absolute) Viscosity** is the tangential force per unit area (shear stress) required to move one horizontal plane with respect to the other at unit velocity when maintained a unit distance apart by the fluid.

Units are **N s/m<sup>2</sup>**, **Pa s** or **kg/m s** where  
1 Pa s = 1 N s/m<sup>2</sup> = 1 kg/m s  
Often expressed in the CGS system as  
**g/cm.s**, **dyne.s/cm<sup>2</sup>** or **poise (p)** where  
1 poise = dyne s/cm<sup>2</sup> = g/cm s = 1/10 Pa s  
= 100 centipoise (cP)

**Viscosity of water at 20.2 °C = 1 cP**



**Newtons Law of Friction.**

$$\tau = \mu \frac{dc}{dy} \quad (1)$$

where

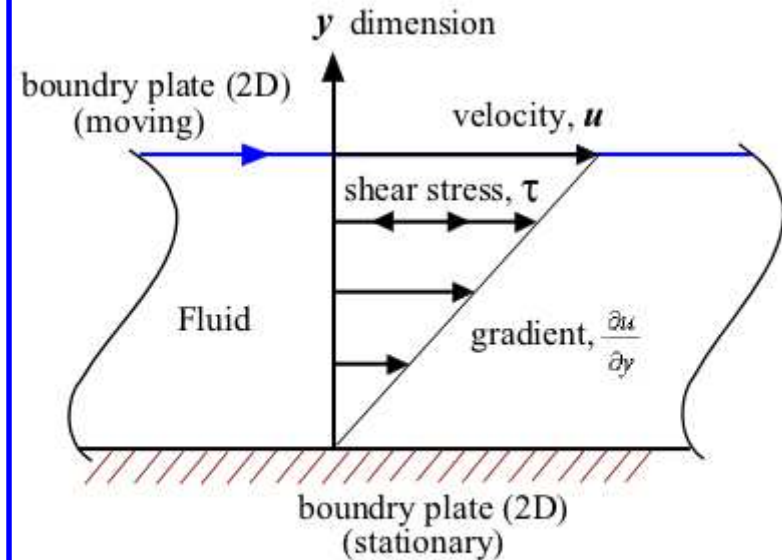
$\tau$  = shearing stress

$\mu$  = dynamic viscosity

# More on viscosity: Newton's law

Isaac Newton postulated that, for straight, parallel and uniform flow, the shear stress,  $\tau$ , between layers is proportional to the velocity gradient,  $\partial u / \partial y$ , in the direction perpendicular to the layers, in other words, the relative motion of the layers.

Here, the constant  $\mu$  is known as the *coefficient of viscosity*, *viscosity*, or *dynamic viscosity*. Many fluids, such as water and most gases, satisfy Newton's criterion and are known as Newtonian fluids. Non-Newtonian fluids exhibit a more complicated relationship between shear stress and velocity gradient than simple linearity.



$$\tau = \mu \frac{\partial u}{\partial y}$$

Viscosity is the principal means by which energy is dissipated in fluid motion, typically as heat.

# Molecular origins

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The viscosity of a system is determined by how molecules constituting the system interact. There are no simple but correct expressions for the viscosity of a fluid. The simplest exact expressions are the Green-Kubo relations for the linear shear viscosity or the Transient Time Correlation Function expressions derived by Evans and Morriss in 1985. Although these expressions are each exact in order to calculate the viscosity of a dense fluid, using these relations requires the use of molecular dynamics computer simulation.

# Viscosity of gases

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Viscosity in gases arises principally from the molecular diffusion that transports momentum between layers of flow. The kinetic theory of gases allows accurate prediction of the behaviour of gaseous viscosity, in particular that, within the regime where the theory is applicable: Viscosity is independent of pressure; and Viscosity increases as temperature increases.

**Gases (at 0 °C):**

**viscosity (Pa · s)**

hydrogen	$8.4 \times 10^{-6}$
air	$17.4 \times 10^{-6}$
xenon	$21.2 \times 10^{-6}$

# Viscosity of Liquids

In liquids, the additional forces between molecules become important. This leads to an additional contribution to the shear stress though the exact mechanics of this are still controversial. Thus, in liquids:

- Viscosity is independent of pressure (except at very high pressure); and
- Viscosity tends to fall as temperature increases (for example, water viscosity goes from 1.79 cP to 0.28 cP in the temperature range from 0°C to 100°C)

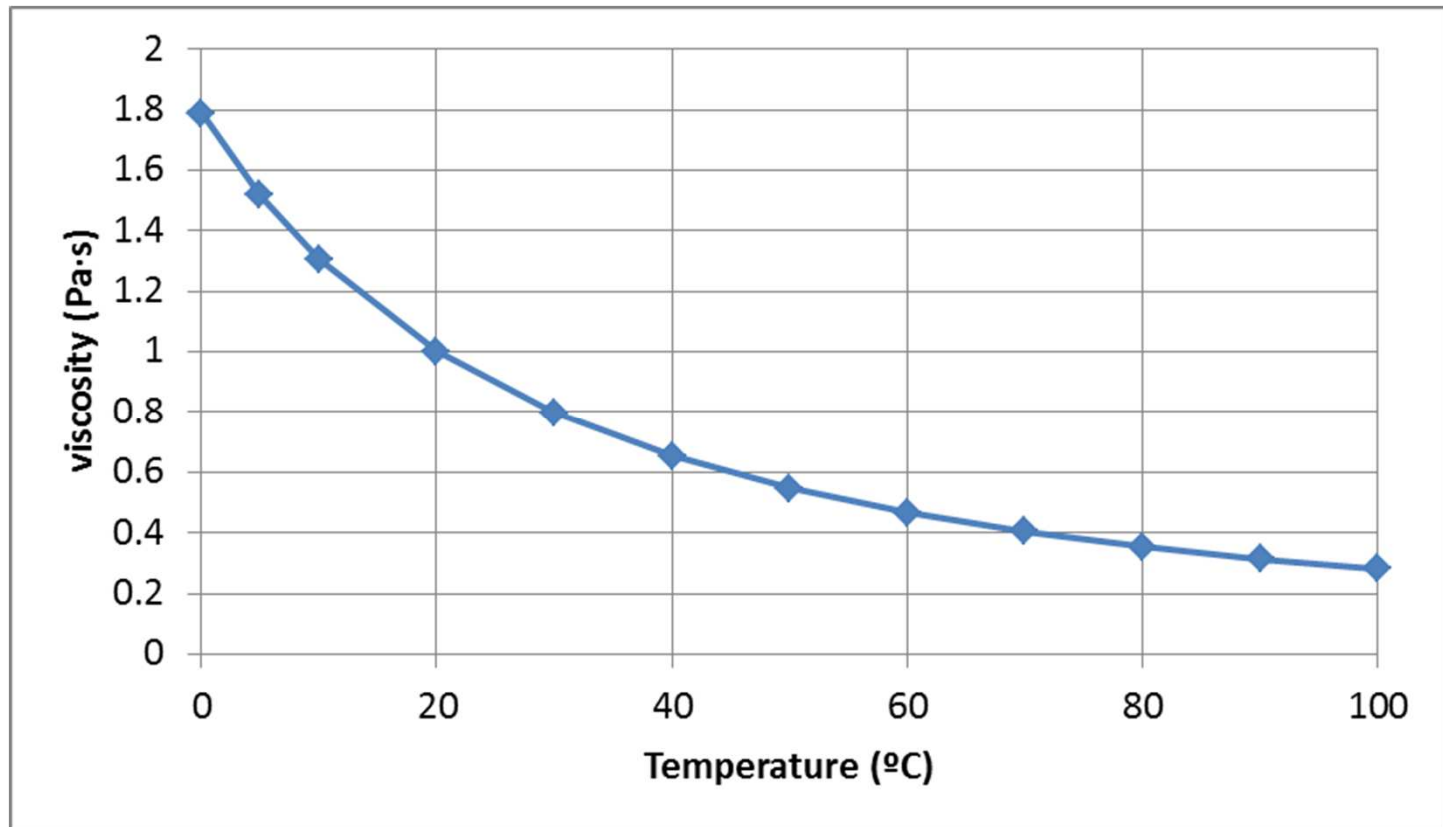
[www.answers.com/topic/viscosity](http://www.answers.com/topic/viscosity)

Liquids (at 20 °)	viscosity (Pa · s)
<a href="#">ethyl alcohol</a>	$0.248 \times 10^{-3}$
<a href="#">acetone</a>	$0.326 \times 10^{-3}$
<a href="#">methanol</a>	$0.597 \times 10^{-3}$
<a href="#">propyl alcohol</a>	$2.256 \times 10^{-3}$
<a href="#">benzene</a>	$0.64 \times 10^{-3}$
<a href="#">water</a>	$1.0030 \times 10^{-3}$
<a href="#">nitrobenzene</a>	$2.0 \times 10^{-3}$
<a href="#">mercury</a>	$17.0 \times 10^{-3}$
<a href="#">sulfuric acid</a>	$30 \times 10^{-3}$
<a href="#">olive oil</a>	$81 \times 10^{-3}$
<a href="#">castor oil</a>	0.985
<a href="#">glycerol</a>	1.485
<a href="#">molten polymers</a>	$10^3$
<a href="#">pitch</a>	$10^7$
<a href="#">glass</a>	10

# Temperature dependence of water viscosity

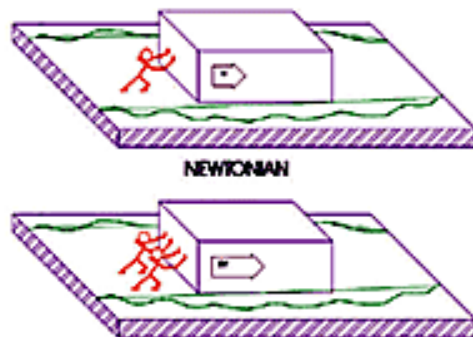
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temp	din visc
0	1.787
5	1.519
10	1.307
20	1.002
30	0.798
40	0.653
50	0.547
60	0.467
70	0.404
80	0.355
90	0.315
100	0.282

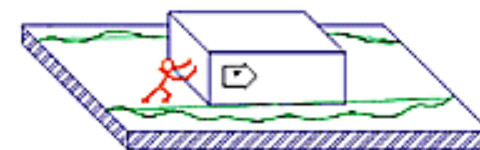


# Viscosity: Newtonian and non-newtonian fluids

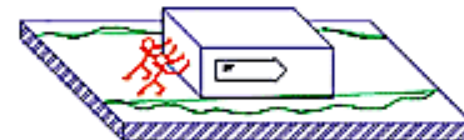
- Imagine two surfaces with a fluid between them. A force is applied to the top surface and thus it moves at a certain velocity. The ratio of the Shear Stress / Shear Rate will be the viscosity.
- Note that as the force is doubled then the velocity doubles. This is indicative of a Newtonian fluid, such as motor oil.



- When measuring a Non-Newtonian fluid, such as an ink or coating, The change in velocity is non-linear. While the force is doubled in each case the ratio of increase in speed is not the same for the two speeds



NON-NEWTONIAN (THIXOTROPIC)



NON-NEWTONIAN (THIXOTROPIC)



# Poiseuille

Poiseuille was interested in the forces that affected the blood flow in small blood vessels. He performed meticulous tests on the resistance of flow of liquids through capillary tubes. Using compressed air, Poiseuille (1838-46) forced water (instead of blood due to the lack of anti-coagulants) through capillary tubes. Poiseuille's measurement of the amount of fluid flowing showed there was a relationship between the applied pressure and the diameter of the tubes. He discovered that the rate of flow through a tube increases linearly with pressure applied and the fourth power of the tube diameter. The constant of proportionality, found by Hagen (?) is  $\pi/8$ . In honor of his early work the equation for *flow of liquids through a tube* is called *Poiseuille's Law* (or *Hagen-Poiseuille's Law*).



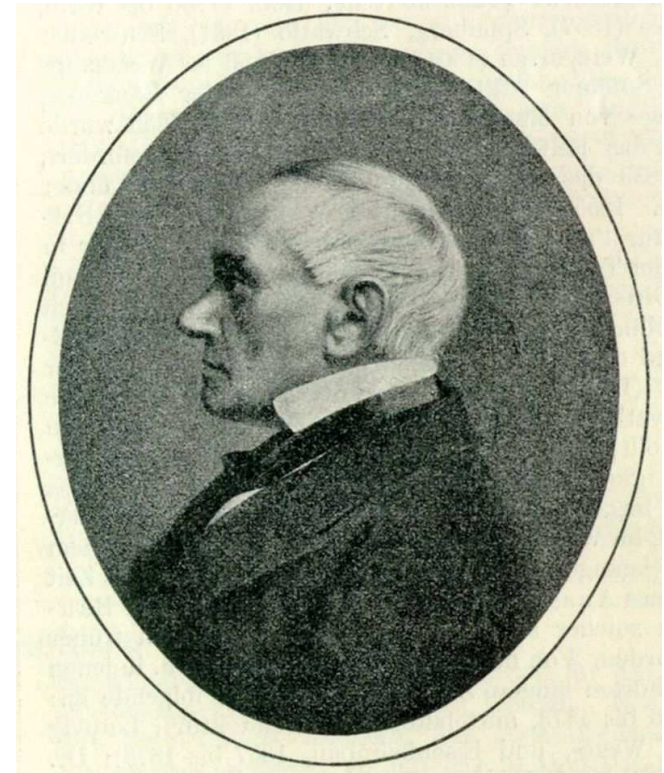
<http://xtronics.com/reference/viscosity.htm>

Ironically, blood is not a newtonian fluid. The viscosity of blood declines in capillaries as the cells line up single file

# Gotthilf Hagen

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Hagen was a civil engineer who was interested in hydraulics. His 1839 measurements of water flow through small diameter tubes showed that the flow rate was proportional to the radius raised to the power of 4.12. But, a true engineer, he suggested in view of possible measurement errors that a value of 4.0 be assumed. It must be acknowledged that he was an expert statistician, having demonstrated why measurement errors should follow a Gaussian distribution. At his time, he was better known for his works in coastal engineering



[https://en.wikipedia.org/wiki/Gotthilf\\_Hagen#/media/File:Gotthilf\\_Hagen.jpg](https://en.wikipedia.org/wiki/Gotthilf_Hagen#/media/File:Gotthilf_Hagen.jpg)

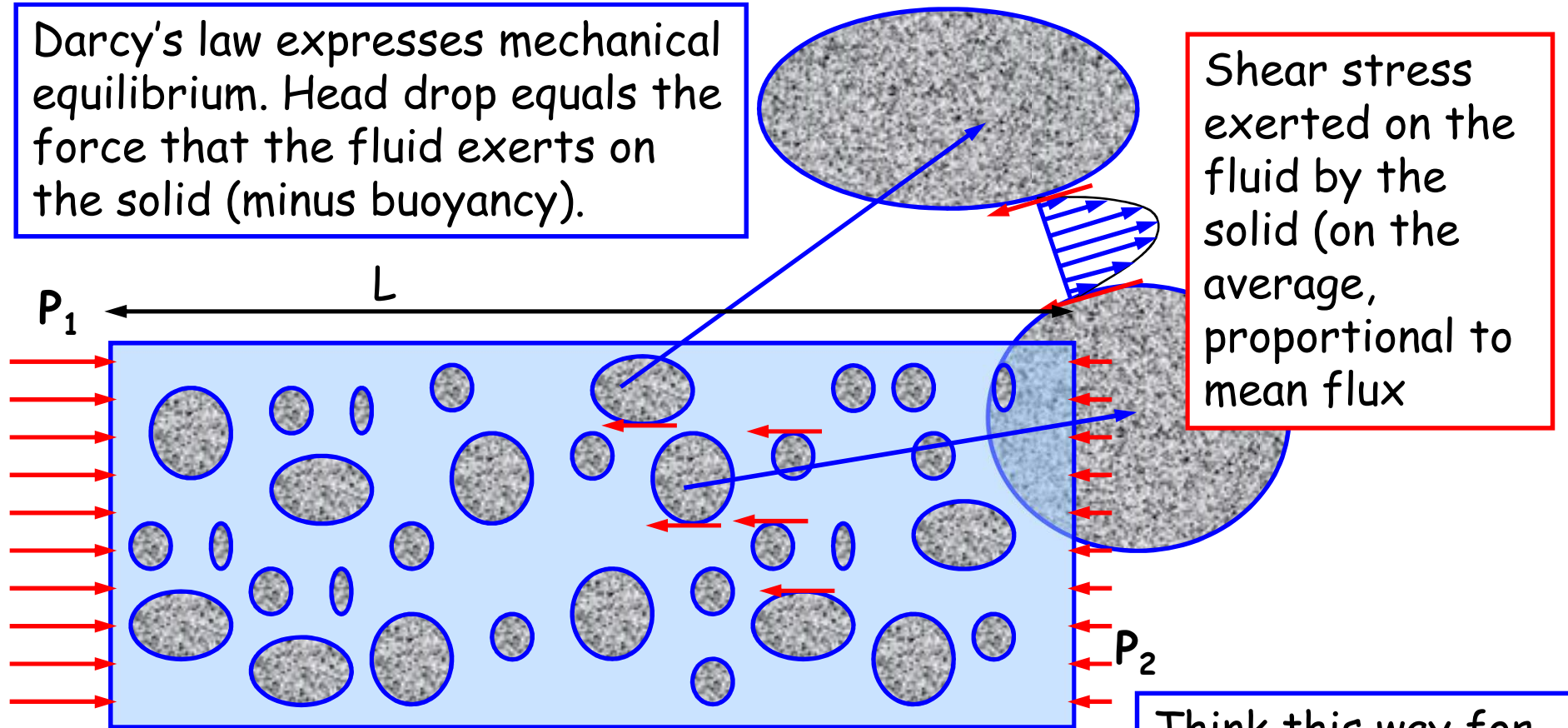
# Flow through capillary tubes

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- Derive Hagen-Poisellieu equation

# Darcy's law expresses momentum conservation

Darcy's law expresses mechanical equilibrium. Head drop equals the force that the fluid exerts on the solid (minus buoyancy).



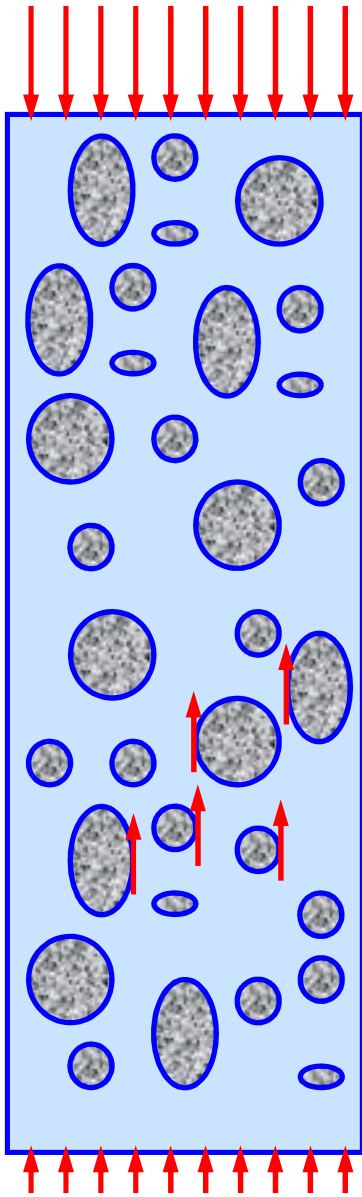
Shear stress exerted on the fluid by the solid (on the average, proportional to mean flux)

Pressure forces  $(P_1 - P_2)A = LACq$  Viscous forces

$$(P_1 - P_2)/LC = q \quad \dots \text{ or } \dots q = (k/\mu) \cdot (P_1 - P_2)/L$$

Think this way for multi-phase flow and especially for hydro-mechanical coupling

# Momentum conservation for variable density



Perform the same analysis for a vertical column.

One must add the weight of water

$$(g\rho LA) + (P_1 - P_2)A = LACq \text{ Viscous forces}$$

$$[(g\rho) + (P_1 - P_2)/L]C = q$$

... or ...

$$q = (k/\mu)(\text{grad } P + \rho g)$$

Or, with proper signs (positive upwards, and gravity downwards)

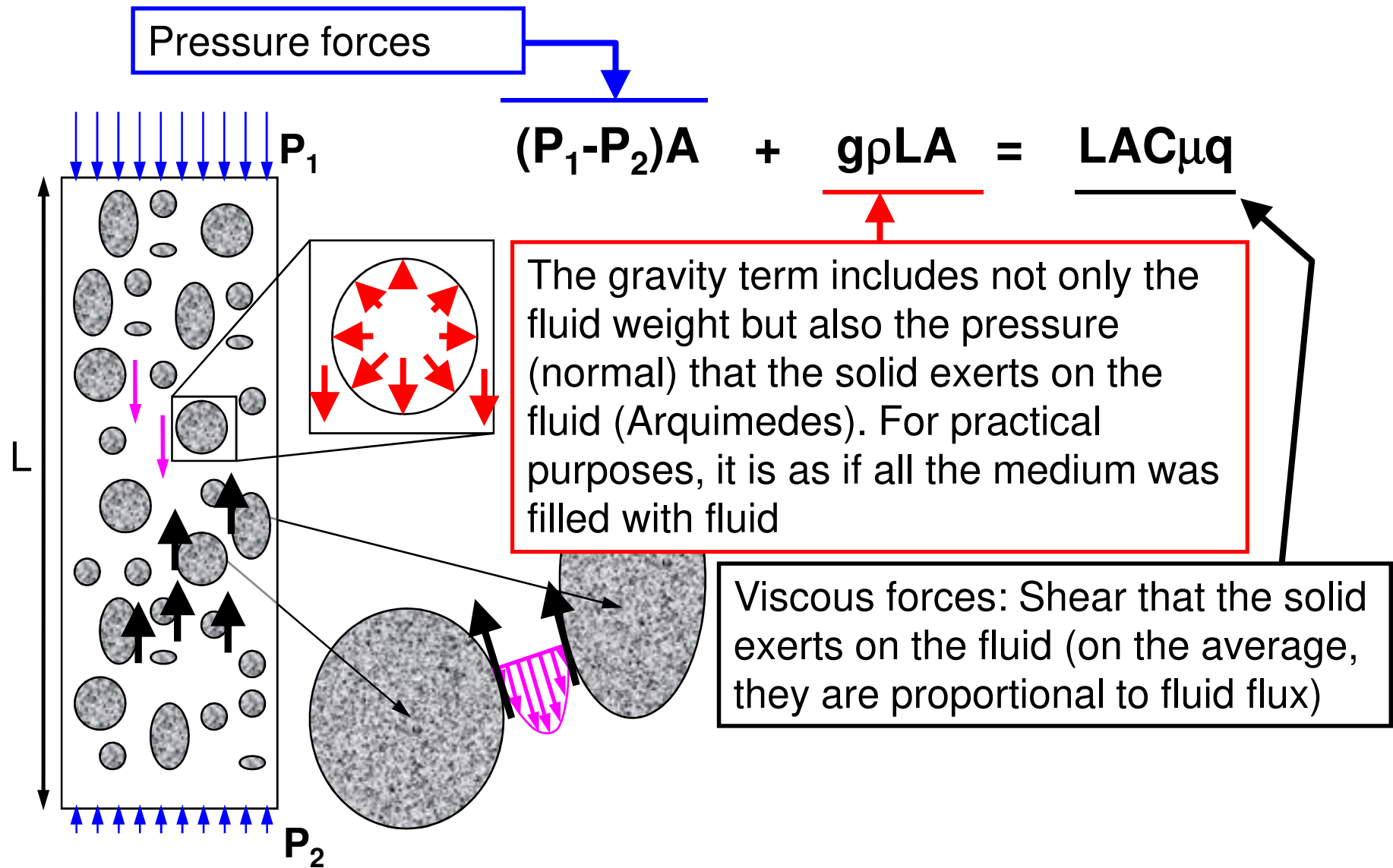
$$q = - (k/\mu)(\text{grad } P - \rho g) \quad \text{Best form of Darcy's Law!!!}$$

If constant density,

$$q = -K \cdot \text{grad } h$$

$$\text{With } h = z + P/\rho g$$

# Further details about the forces involved



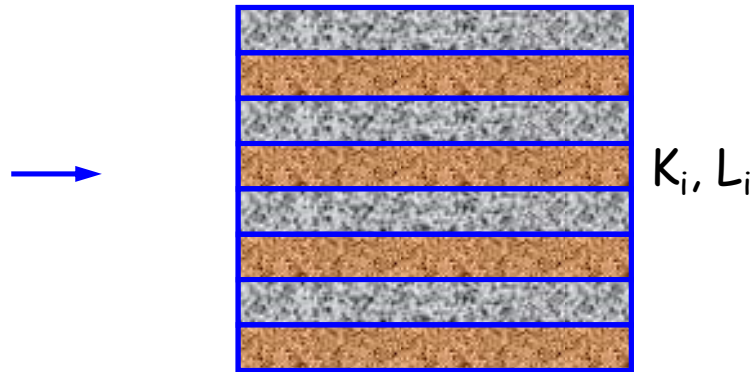
# Energy dissipation

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- Derive expression for energy dissipation

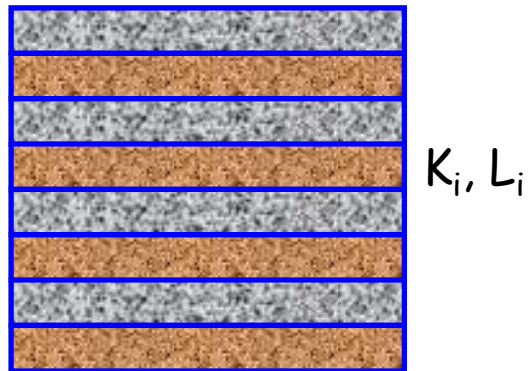
# Tensorial nature of Darcy's law

For complex media,  $K$  depends on flow direction:



$$Q = \sum Q_i = \sum K_i L_i (h_1 - h_2) / L$$

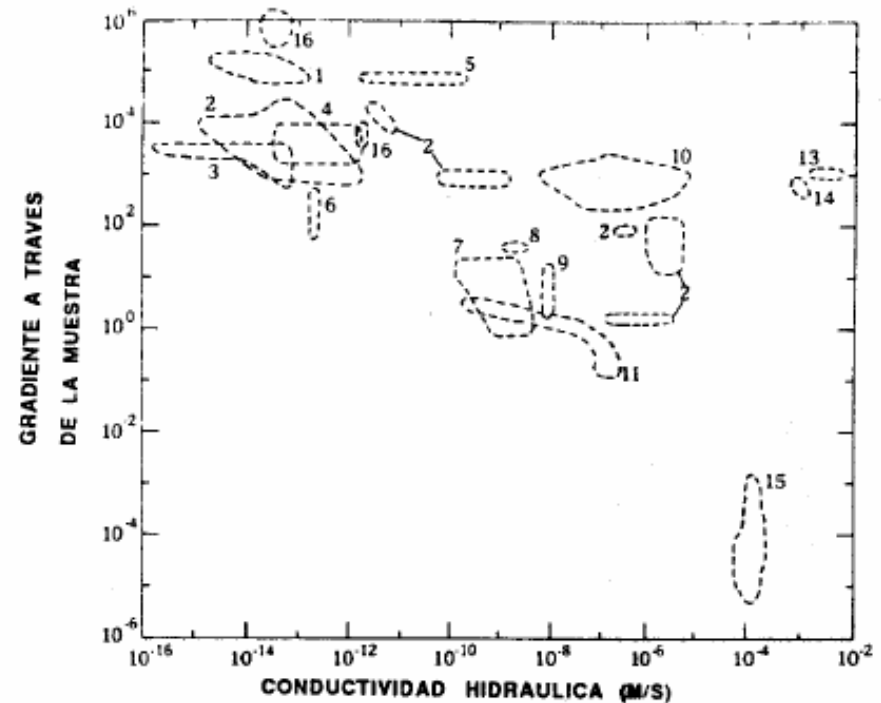
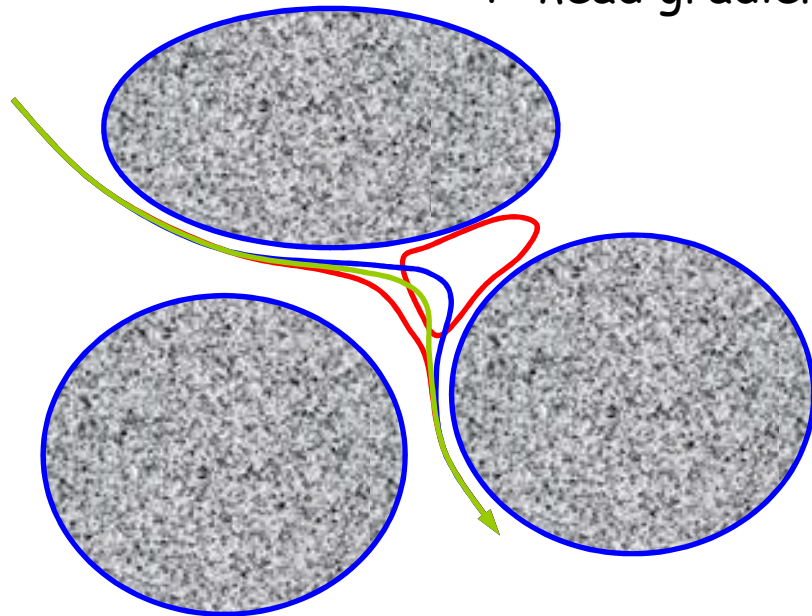
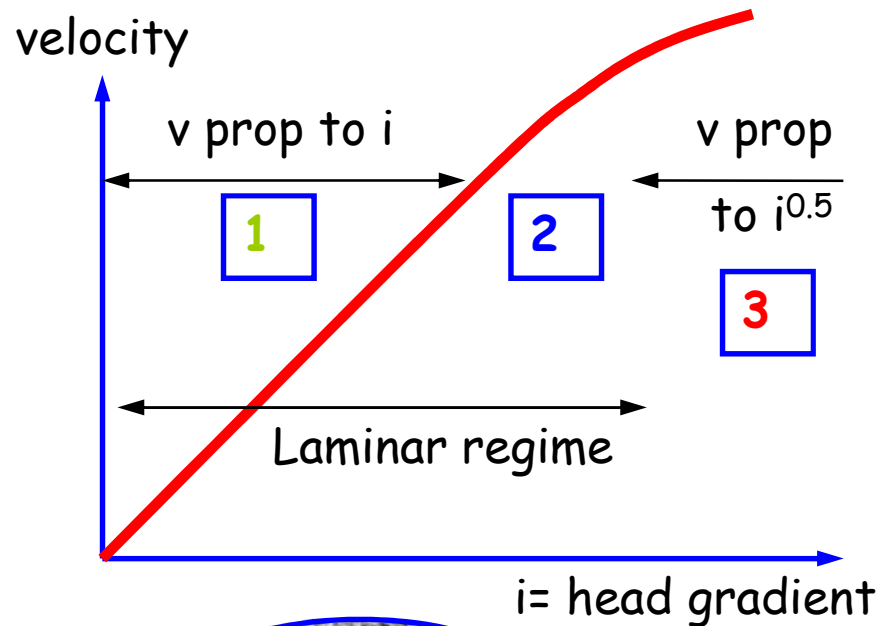
$$K_h = \dots$$



$$K_v = \dots$$

$$\mathbf{q} = \mathbf{K} \nabla h$$

# Is there a lower limit for Darcy's law validity?



There is no experimental validated evidence for a lower limit of Darcy's law, but would not be surprising (I'd expect a threshold gradient for adsorbed water)

# The basic processes

Phenomena / Flux	Heat conduction	Electrical current	Molecular diffusion	Elasticity /stress
State variable/ potential	Temperature $T$	Electrical potential, $V$	Concentration (chemical potential), $c$	displacement $u$ (vector!)
Law	Fourier	Ohm	Fick	Hooke
Constant	Thermal cond.	Electrical conductivity	Molecular diffusion coeff.	Elasticity modulus
Conservation principle	Energy	Electrical charge	Solute mass	Momentum
Capacity term	Thermal capacity	Elect capac. (not really!)	Porosity	Mass Inertia (not really)
Equation	$\frac{\partial T}{\partial t} = \nabla(\lambda \nabla T)$	$\nabla(C \nabla V) = 0$	$\frac{\partial c}{\partial t} = \nabla(D \nabla c)$	$\rho \frac{\partial^2 u}{\partial t^2} = \nabla(E \nabla u)$

# Transmissivity

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- Aquifers are essentially 2D
- Hydrogeologists can hardly measure hydraulic conductivity. We perform pump tests!
- Pump tests measure the resistance to "horizontal" water flow over the whole aquifer thickness.
- $q_b = -T \text{grad} h$
- If the aquifer is vertically homogeneous
  - $T = K(z_t - z_b) = Kb$
- If the aquifer is vertically homogeneous and unconfined
  - $T = K(h - z_b)$

# Is Transmissivity ever Kb?

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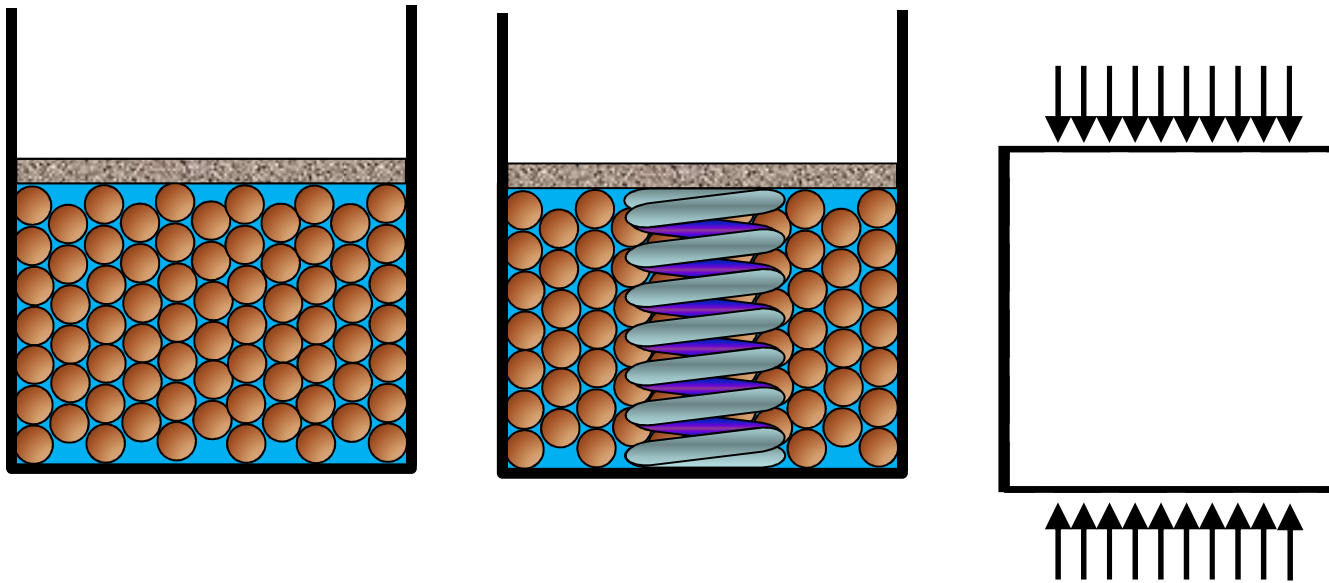
# Storage

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- Where does ground water come from?

# Permeable media behave elastically for small deformations

(They behave like a spring)



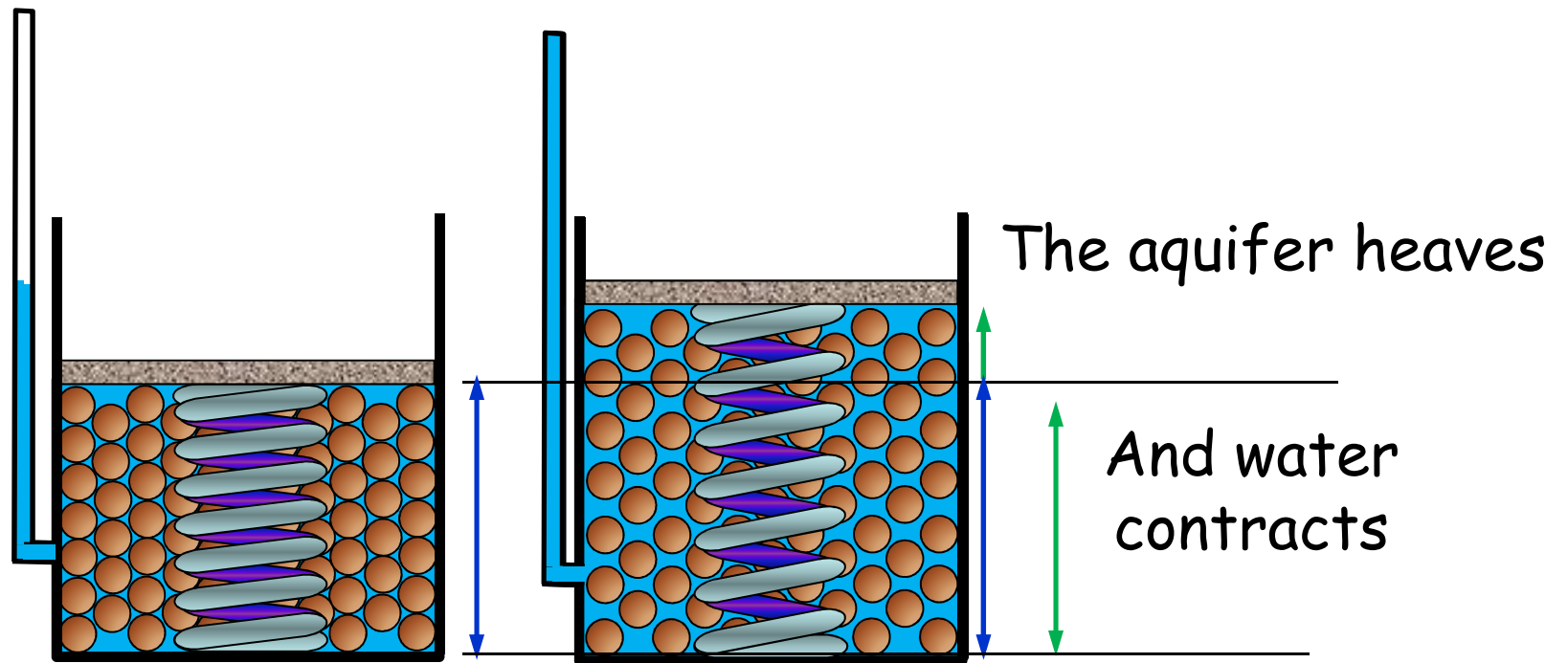
Hooke's law

$$\varepsilon_z = \sigma'_z / E$$

$$\varepsilon_x = \nu \sigma'_z / E$$

Lateral expansion is hindered, but, still, net volume change

When pressure increases, the medium expands, but the volume of water and solids is reduced



# Storage

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- Where does water come from:
- Elastic storage:  $S_s$  = Decrease in Volume of stored water **per unit volume of medium** and unit head drop)

$\beta$ : Compressibility of water (water expands when head drops)

$\alpha$  ( $\beta_s$ ) Compressibility of medium (porosity reduced when head drops)

$$S_s = \rho g (\alpha + \phi \beta)$$

- Drainage at the phreatic level:  $S_y$  = Decrease in Volume of stored water **per unit surface of aquifer** and unit head drop)
  - Specific yield:  $S_y = \phi - \theta_f$
- Total storage coefficient:
  - $S = S_y + S_s b$  with  $b$  = aquifer thickness
  - Usually  $S_s$  negligible

# Vertical, drained compressibilities

$$\beta = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)$$

Material	$\beta$ (m <sup>2</sup> /N)
Plastic clay	$2 \times 10^{-6} - 2.6 \times 10^{-7}$
Stiff clay	$2.6 \times 10^{-7} - 1.3 \times 10^{-7}$
Medium-hard clay	$1.3 \times 10^{-7} - 6.9 \times 10^{-8}$
Loose sand	$1 \times 10^{-7} - 5.2 \times 10^{-8}$
Dense sand	$2 \times 10^{-8} - 1.3 \times 10^{-8}$
Dense, sandy gravel	$1 \times 10^{-8} - 5.2 \times 10^{-9}$
<b>Rock, fissured</b>	<b><math>6.9 \times 10^{-10} - 3.3 \times 10^{-10}</math></b>
<b>Rock, sound</b>	<b><math>&lt; 3.3 \times 10^{-10}</math></b>
<b>Water at 25°C (undrained)</b>	<b><math>4.6 \times 10^{-10}</math></b>

$$\frac{1}{V} \frac{\partial V_p}{\partial p} = (\beta_p + \phi \beta_w + (1 - \phi) \beta_w)$$

$$\Delta V_p = V (\beta_p + \phi \beta_w + (1 - \phi) \beta_w) \Delta p = S_s \Delta h$$

# Vertical, drained compressibilities[2]

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$$\beta = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)$$

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Dense, sandy gravel	$1 \times 10^{-8} - 5.2 \times 10^{-9}$
Rock, fissured	$6.9 \times 10^{-10} - 3.3 \times 10^{-10}$
Rock, sound	$< 3.3 \times 10^{-10}$
Water at 25°C (undrained)[3]	$4.6 \times 10^{-10}$

- △ Domenico, P.A. and Mifflin, M.D. (1965). "Water from low permeability sediments and land subsidence". *Water Resources Research* **1** (4): 563–576. [OSTI:5917760](#).
- △ Fine, R.A. and Millero, F.J. (1973). "Compressibility of water as a function of temperature and pressure". *Journal of Chemical Physics* **59** (10). [doi:10.1063/1.1679903](#).

## Values of specific yield, from Johnson (1967)

				Consolidated deposits		Warning: highly site specific
	min	avg	max			
Unconsolidated deposits				Fine-grained sandstone	21	
Clay	0	2	5	Medium- grained sandstone	27	
Sandy clay (mud)	3	7	12	Limestone	14	
Silt	3	18	19	Schist	26	
Fine sand	10	21	28	Siltstone	12	
Medium sand	15	26	32	Tuff	21	
Coarse sand	20	27	35	Other deposits		
Gravelly sand	20	25	35	Dune sand	38	
Fine gravel	21	25	35	Loess	18	
Medium gravel	13	23	26	Peat	44	
Coarse gravel	12	22	26	Till, predominantly silt	6	
Johnson, A.I. 1967. <i>Specific yield — compilation of specific yields for various materials</i> . U.S. Geological Survey Water Supply Paper 1662-D, 74 p.				Till, predominantly sand	16	
				Till, predominantly gravel	16	

## Flow equation:

---

- Use divergence theorem to write mass balance

# How is the fluid flow equation

- Conservation principle: Fluid mass (Fluid, not water!)
- Capacity term: Storativity
- Flow equation
- Derive from mass conservation

$$S_s \frac{\partial h}{\partial t} = \nabla(K \nabla h)$$

Other forms:

2D  $S \frac{\partial h}{\partial t} = \nabla(T \nabla h)$   $S$ : storage coefficient,  $T$  transmissivity

With source terms  $S \frac{\partial h}{\partial t} = \nabla(T \nabla h) + r$   $r$  recharge

Dimensionless form  $\frac{\partial h_b}{\partial t_D} = \nabla(\nabla h_b)$   $t_D = Tt/(SL^2)$   $h_D$ : B.C.'s

$t_c = SL^2 / T$

Second (only to Darcy's law) most important equation in GW hydraulics

## Flow equation:

---

- Write for radial flow
- Write in dimensionless form

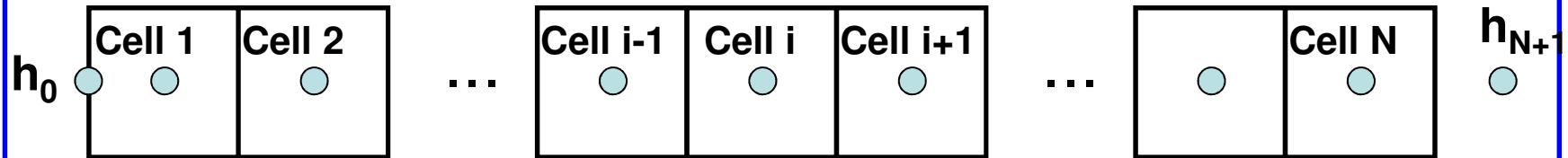
# The flow equation can be solved using...

---

- Analytical solutions: we'll see a few below
- Graphical solutions: flow nets. very handy. Give it a try.
- Numerical solutions:
  1. Discretize
  2. Write discrete equations: Finite differences, Finite volumes, Finite Element,... in general mass balance. Be careful with boundary conditions.
  3. Solve sequentially in time: starting from initial conditions ( $k=0$ ), obtain  $h$  at time  $k=1$ . From  $k=1$ , obtain  $h$  at time  $k=2$ , etc.

# Example: Finite volumes (same for FD)

- Discretize



- Mass balance at a generic cell  $i$

$$\underbrace{\frac{S.A}{\Delta t} (h_i^k - h_i^{k-1})}_{\text{Storage variation}} = T \Delta y \left( \underbrace{\frac{h_{i+1}^{k-1} - h_i^{k-1}}{\Delta x}}_{\text{Flow rate from i+1 to i}} - \underbrace{\frac{(h_i^{k-1} - h_{i-1}^{k-1})}{\Delta x}}_{\text{Flow rate from i to i-1}} \right)$$

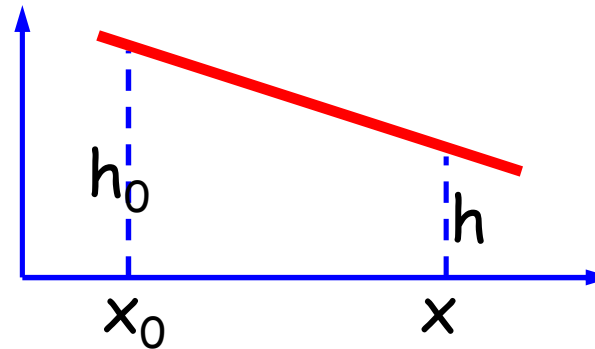
$$\begin{array}{l} \text{BC at } x=0 \\ \text{Dirichlet} \end{array} \quad \frac{S.A}{\Delta t} (h_1^k - h_1^{k-1}) = T \Delta y \left( \frac{h_2^{k-1} - h_1^{k-1}}{\Delta x} - \frac{(h_1^{k-1} - h_0^{k-1})}{\Delta x / 2} \right)$$

$$\begin{array}{l} \text{BC at } x=L \\ \text{Neuman} \end{array} \quad \frac{S.A}{\Delta t} (h_N^k - h_N^{k-1}) = T \Delta y \left( 0 - \frac{(h_N^{k-1} - h_{N-1}^{k-1})}{\Delta x} \right)$$

# Basic solutions to flow equation: Steady-state

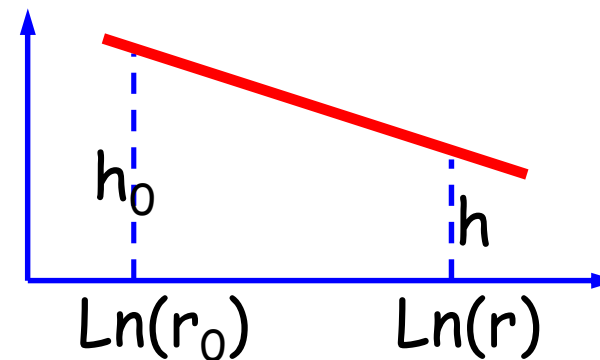
- Parallel flow:

- $q_b = -Tdh/dx$
- $Q = -aTdh/dx$
- $dh = -Qdx/(aT)$
- $h = h_0 - Q(x - x_0)/(aT)$



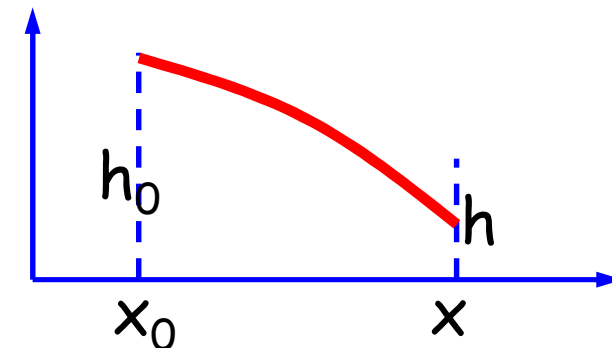
- Radial flow:

- $q_b = -2\pi rTdh/dr$
- $Q = -2\pi rTdh/dr$
- $dh = - (Q/2\pi T)dr/r$
- $h = h_0 - (Q/2\pi T)\ln(r/r_0)$  (**Thiem**)



- Parallel flow with recharge  $w$ :

- $q_b = -Tdh/dx$
- $Q = Q_0 + wa(x - x_0) = -aTdh/dx$
- $dh = - (Q_0 + wa(x - x_0))dx/(aT)$
- $h = h_0 - [Q(x - x_0) + (wa(x - x_0))^2/2] / (aT)$



# Basic solutions to flow equation: Steady-state

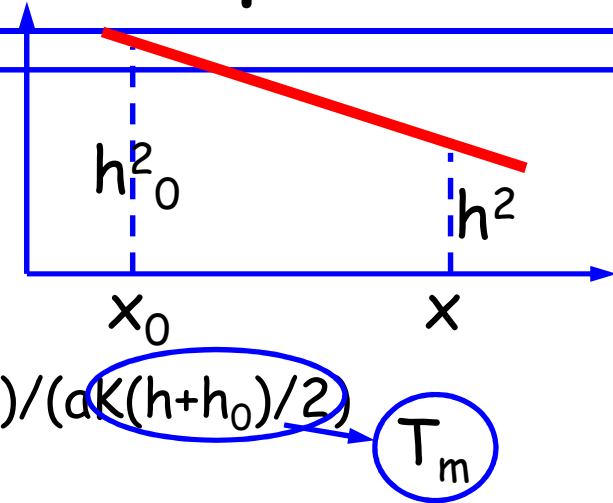
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- Arbitrary geometry: flow nets

# Basic solutions to unconfined flow equation

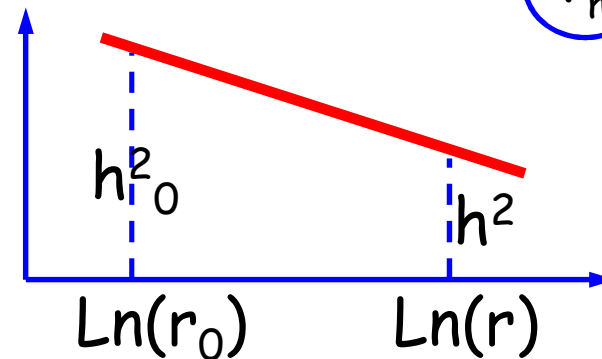
- Parallel flow (assume  $T=Kh$ ):

- $q_b = -Kh \frac{dh}{dx}$
- $Q = -(aK/2) \frac{dh^2}{dx}$
- $dh^2 = -2Q dx / aK$
- $h^2 = h_0^2 - 2Q(x-x_0)/aK \rightarrow h-h_0 = -Q(x-x_0)/(aK(h+h_0)/2)$



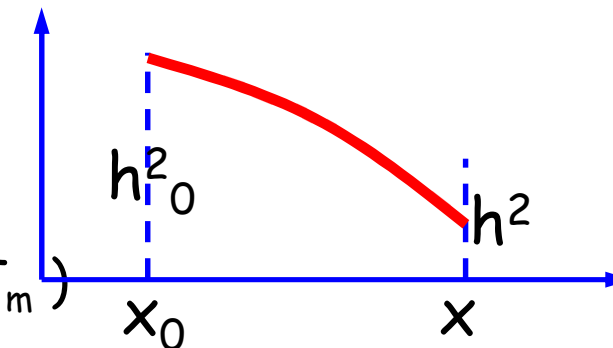
- Radial flow:

- $q_b = -2\pi rKh \frac{dh}{dr}$
- $Q = -\pi rK \frac{dh^2}{dr}$
- $dh^2 = - (Q/\pi K) \frac{dr}{r}$
- $h^2 = h_0^2 - (Q/\pi K) \ln(r/r_0)$



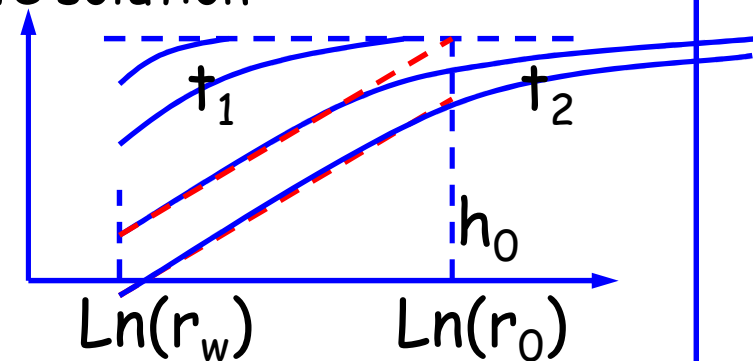
- Parallel flow with recharge  $w$ :

- $q_b = -K \frac{dh^2}{dx}$
- $Q = Q_0 + wa(x-x_0) = -(aK/2) \frac{dh^2}{dx}$
- $dh^2 = - (Q_0 + wa(x-x_0)) dx / (aK/2)$
- $h^2 = h_0^2 - [Q(x-x_0) + (wa(x-x_0))^2/2] / (aT_m)$

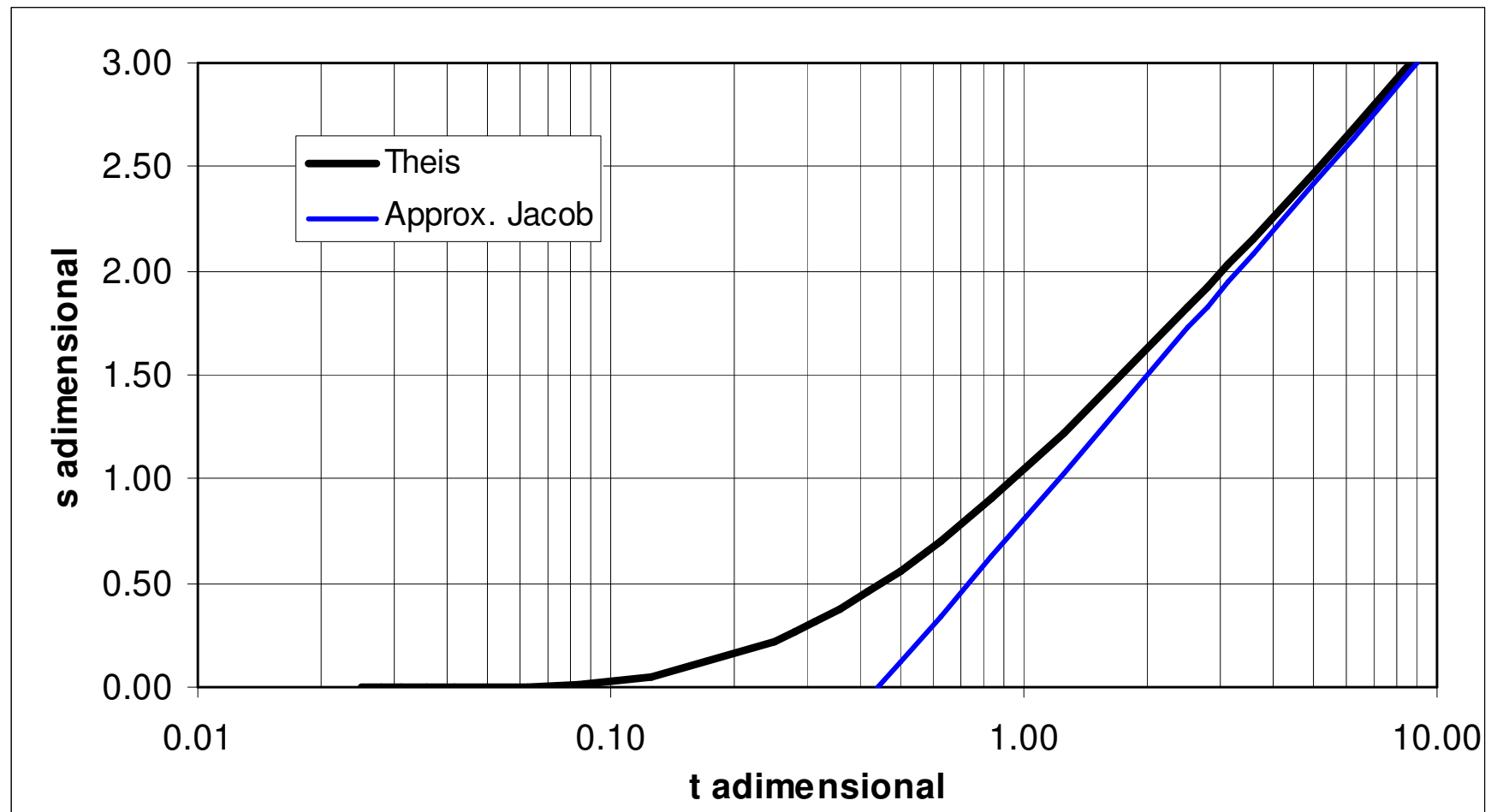


# Basic solutions to flow equation: Transient

- Find final state and use characteristic time to guess how long it will take to reach it
- Beautiful example: Jacob equation
  - Transient response to constant pumping
  - Radial flow, therefore, steady state solution
  - $h=h_0 - (Q/2\pi T)\ln(r/r_0)$



1.  $r_0$  will start noticing  $Q$  at  $t_1 = 0.1t_c$
2.  $r_0$  will reach steady cone at  $t_2 = t_c$   
(en realidad, un poco mas  $t_2 = 364t_c$ )
3.  $r_0$  intersected by radial gradient at well for  $t = t_c/2.25$
4. That is  $t = Sr_0^2/(2.25T) \rightarrow r_0^2 = 2.25Tt/S$
5.  $h = h_0 - (Q/4\pi T)\ln(r^2/r_0^2) = h_0 + (Q/4\pi T)\ln(2.25Tt/Sr^2)$



# Exercise

---

- Alluvial aquifer 100 m long x 10 km wide
- Heads are prescribed to zero at the boundary
- $T=100 \text{ m}^2/\text{day}$
- 200 m thick ( $z_b=-200$ )
- At  $t=0$  starts recharging 100 mm/yr
- What will be the maximum head, how long it will take to reach it?

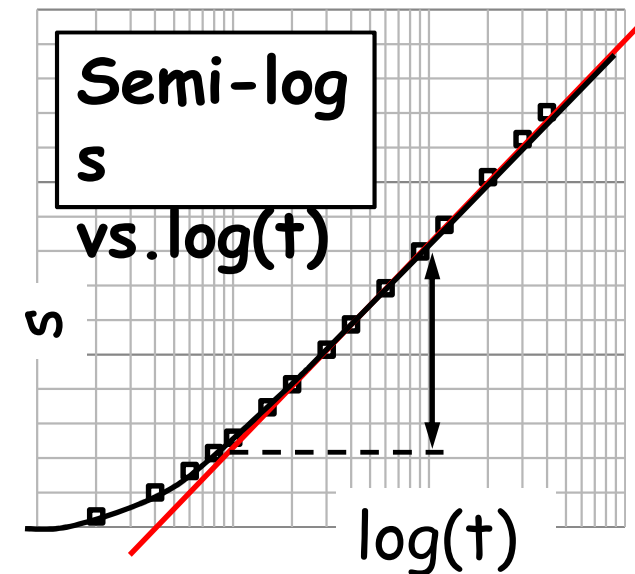
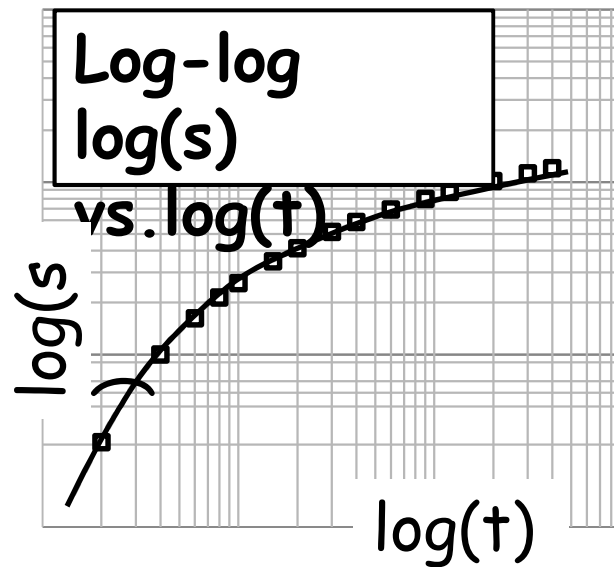
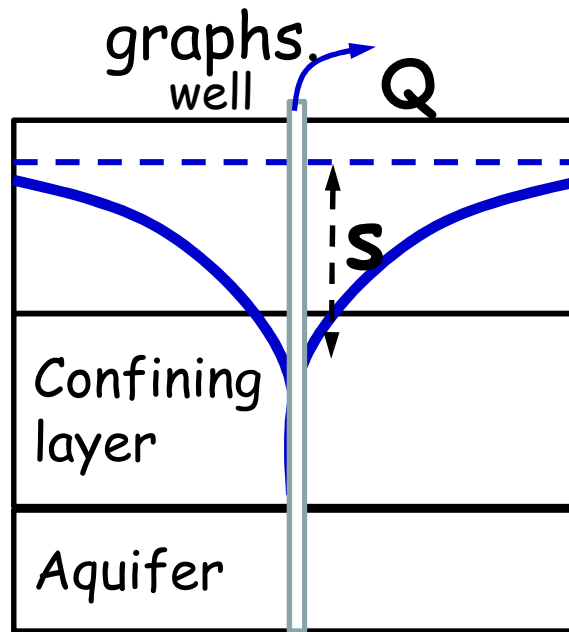
# Well hydraulics: Why all this mess?

---

- **Permeability** is the most **important** parameter controlling just about everything in the underground (water flow, solute transport, multiphase flow, and what not)
- Permeability is the most **variable** parameter.
- **Cannot be measured** directly.
- We can only peer through wells (but wait for geophysics), hence **pumping tests**

Hydraulic tests are performed by

- 1) pumping (or injecting) a flow rate  $Q$ , and measuring drawdown  $s$ .
- 2) Parameters are obtained by fitting a model to measurements,
- 3) This can be done graphically using log-log or semi-log



# Radial flow: Basics

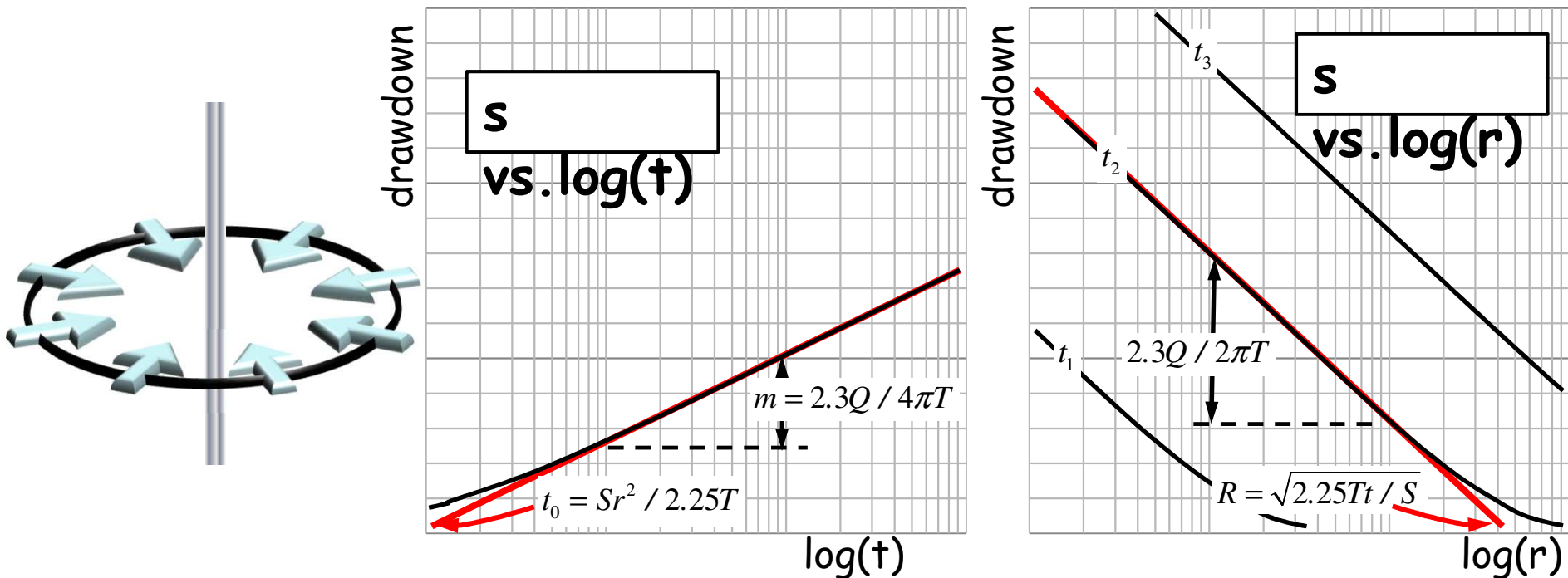
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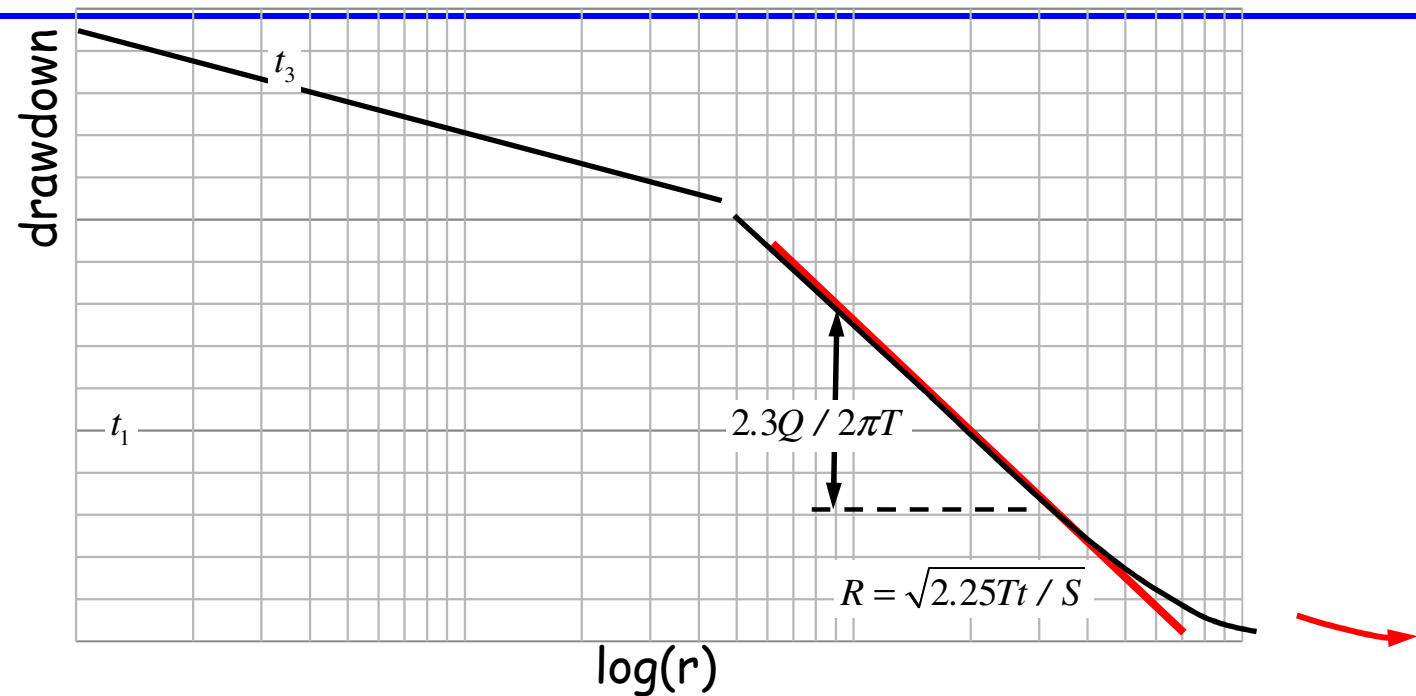
$$2\pi rT \frac{dh}{dr} = Q$$

$$dh = \frac{Q}{2\pi rT} dr$$

$$s = \frac{2.3Q}{4\pi T} \log \left( \frac{2.25Tt}{Sr^2} \right) = \frac{2.3Q}{2\pi T} \log \left( \frac{R}{r} \right) = \frac{2.3Q}{4\pi T} \log \left( \frac{t}{t_0} \right)$$

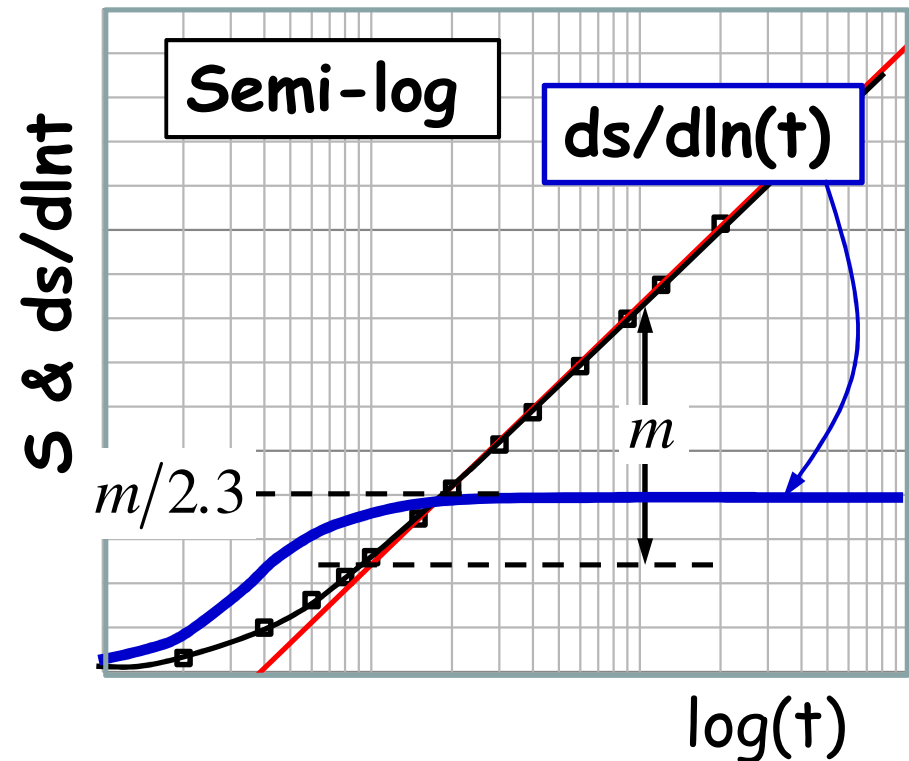
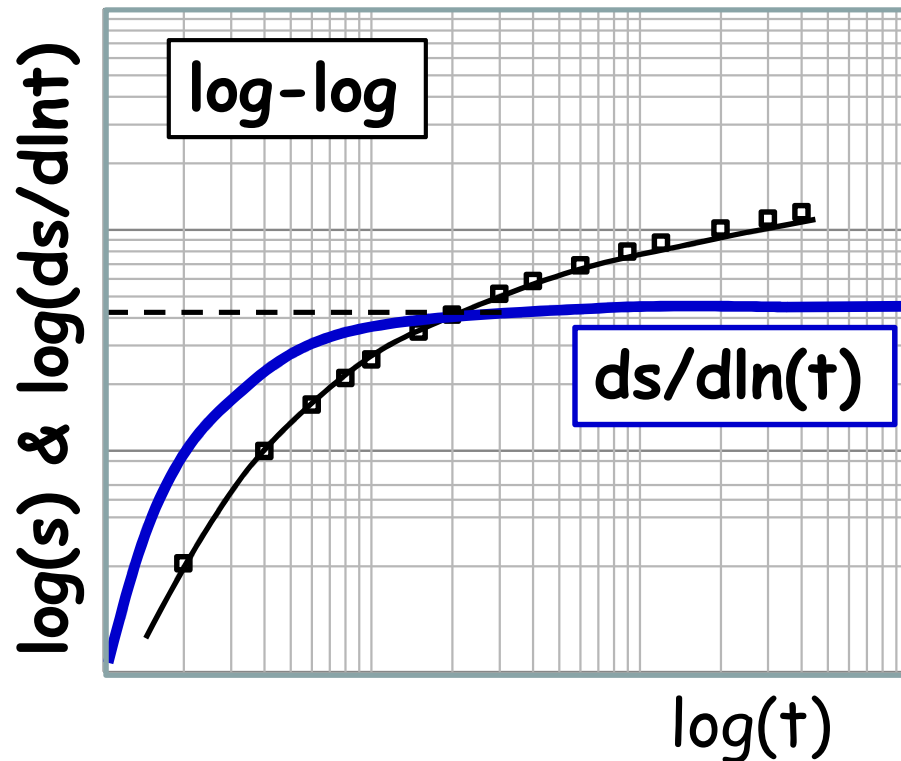
Under radial flow conditions, drawdowns tend to plot as a straight line when plotted either versus  $\log(t)$ , which allows defining the response time  $t_0$ , or versus  $\log(r)$ , which allows defining the radius of influence,  $R$ . As time grows, so does the cone of depression, but its shape does not change.



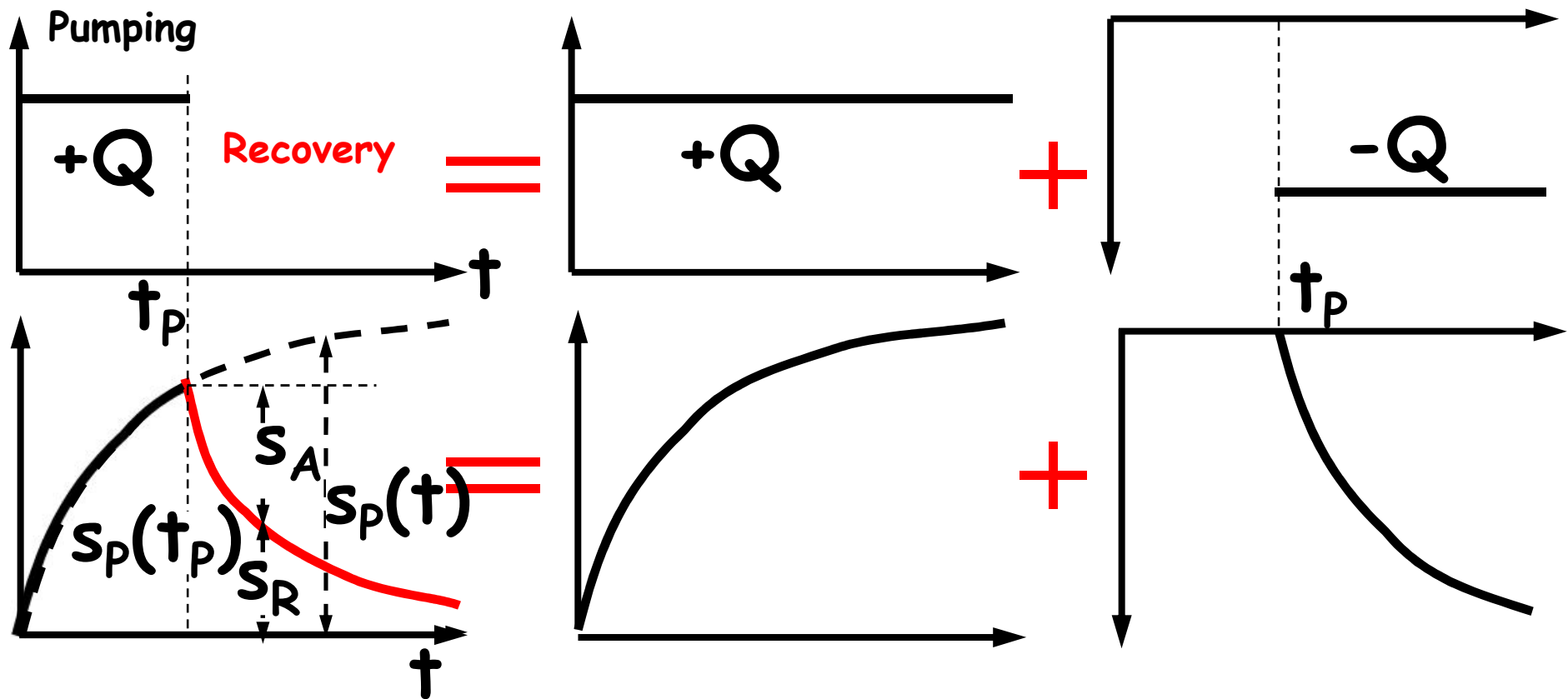


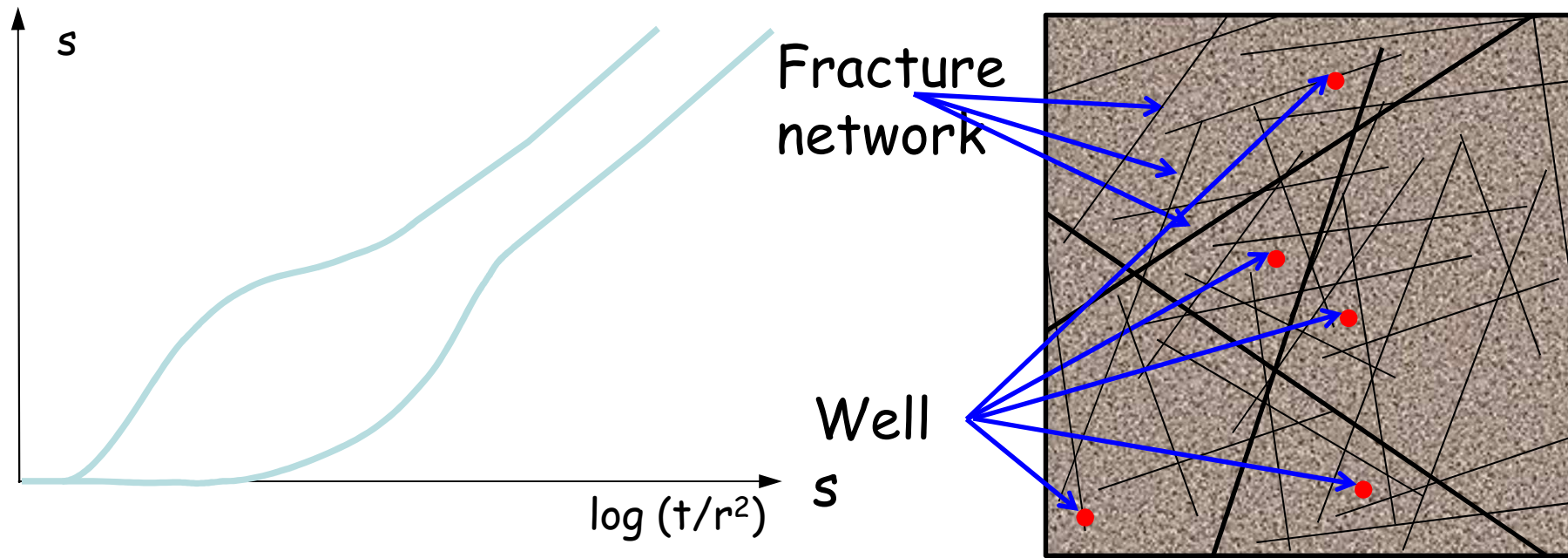
Under radial flow conditions, drawdowns tend to plot as a straight line when plotted either versus  $\log(t)$ , which allows defining the response time  $t_0$ , or versus  $\log(r)$ , which allows defining the radius of influence,  $R$ . As time grows, so does the cone of depression, but its shape does not change.

Diagnostic plots include drawdown and derivative graphs. Both log-log (left) and semi log (right) graphs are useful. In the pure radial flow case, the derivative tends to a constant ( $m/2.3$ ). The 2.3 factor reflects that derivatives are taken with respect to  $\ln(t)$ , while the slope  $m$  is obtained graphically from the semi-log graph, where the logarithm is decimal.

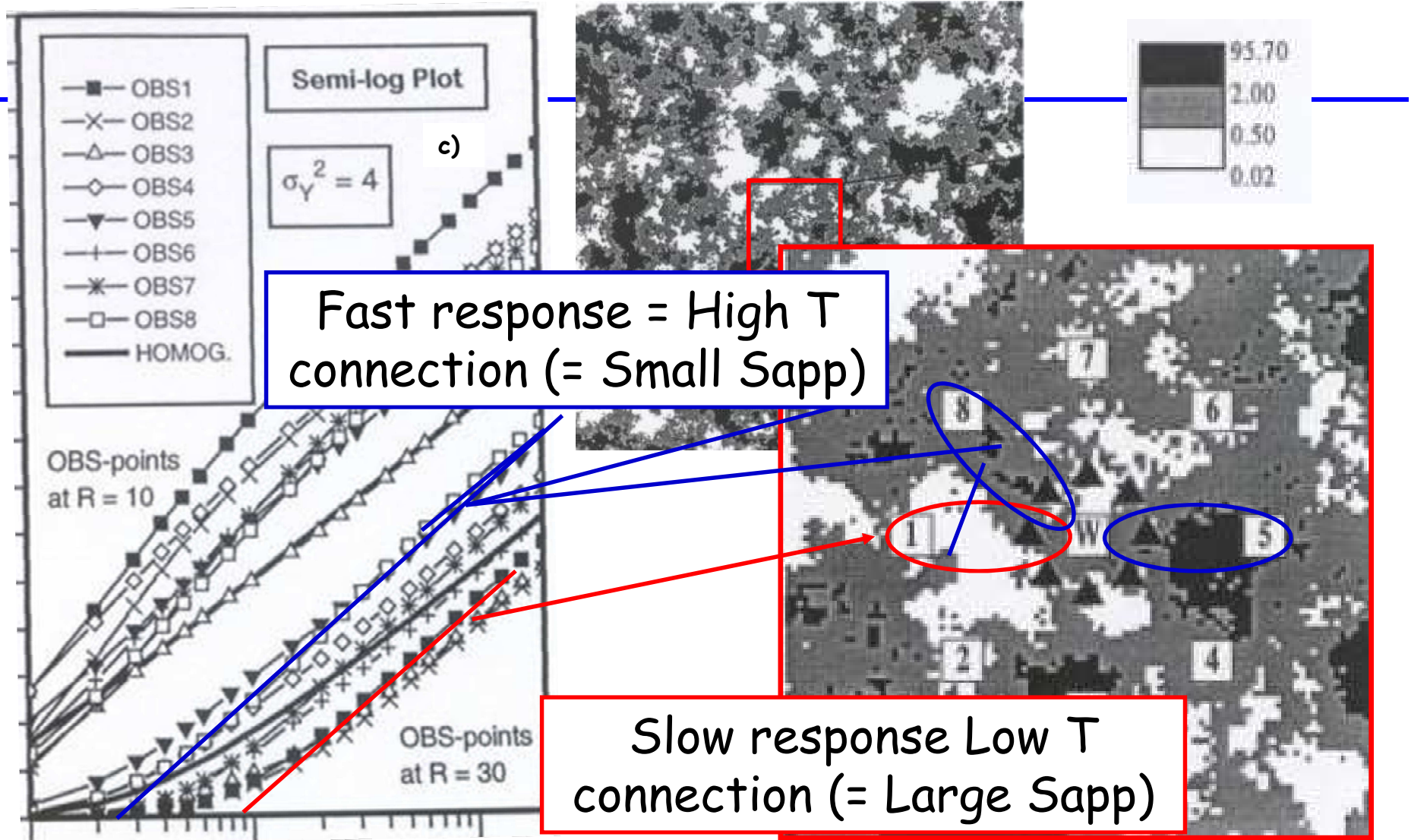


Superposition to obtain recovery plots. This recovery method consists of plotting  $s_R$  versus  $t_p / (t - t_p)$ . Agarwal method consists of plotting  $s_p(t_p) - s_R$  versus  $t_p (t - t_p) / t$ , which should yield a graph very similar to that obtained with a constant pumping rate and makes it a very attractive method.





Large scale transmissivity is controlled by preferential flow paths (long, permeable, well connected fractures). The probability of intersecting these fractures is low., so that transmissivity derived from injectivity or short term tests tends to be much smaller. Large scale transmissivity may be derived from the slope of long term tests , which is the same regardless of whether the pumping or observation wells are located in high or low permeability areas. Local features are reflected in the early portion of the drawdown curve.



Simulations of Meier et al (1998) to show the effect of heterogeneity on pumping tests interpretation. a) transmissivity field; b) local region around the pumping well with a few observation points; c) sem-log drawdown plots. Note that all curves display the same late time slope (i.e., yield the same T), but response times (t<sub>0</sub>) are highly variable. Yet, response times (storativity) are highly variable.

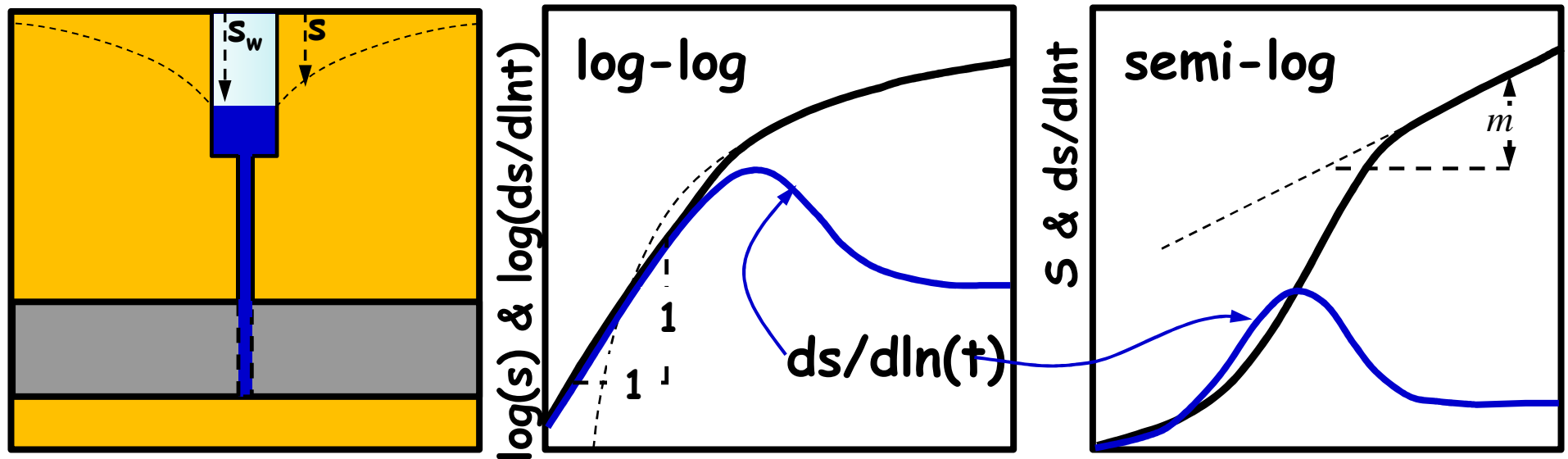


Illustration of well bore storage effects. Well bore storage affects the shape of the response curves while  $Qt$  is comparable to  $A_w s_w$  - During this period, both drawdown and derivative plots display a straight line with slope equal to 1, in log-log scale. In semi-log scale, the drawdown curve displays a concave shape, while the derivative curve displays a maximum.

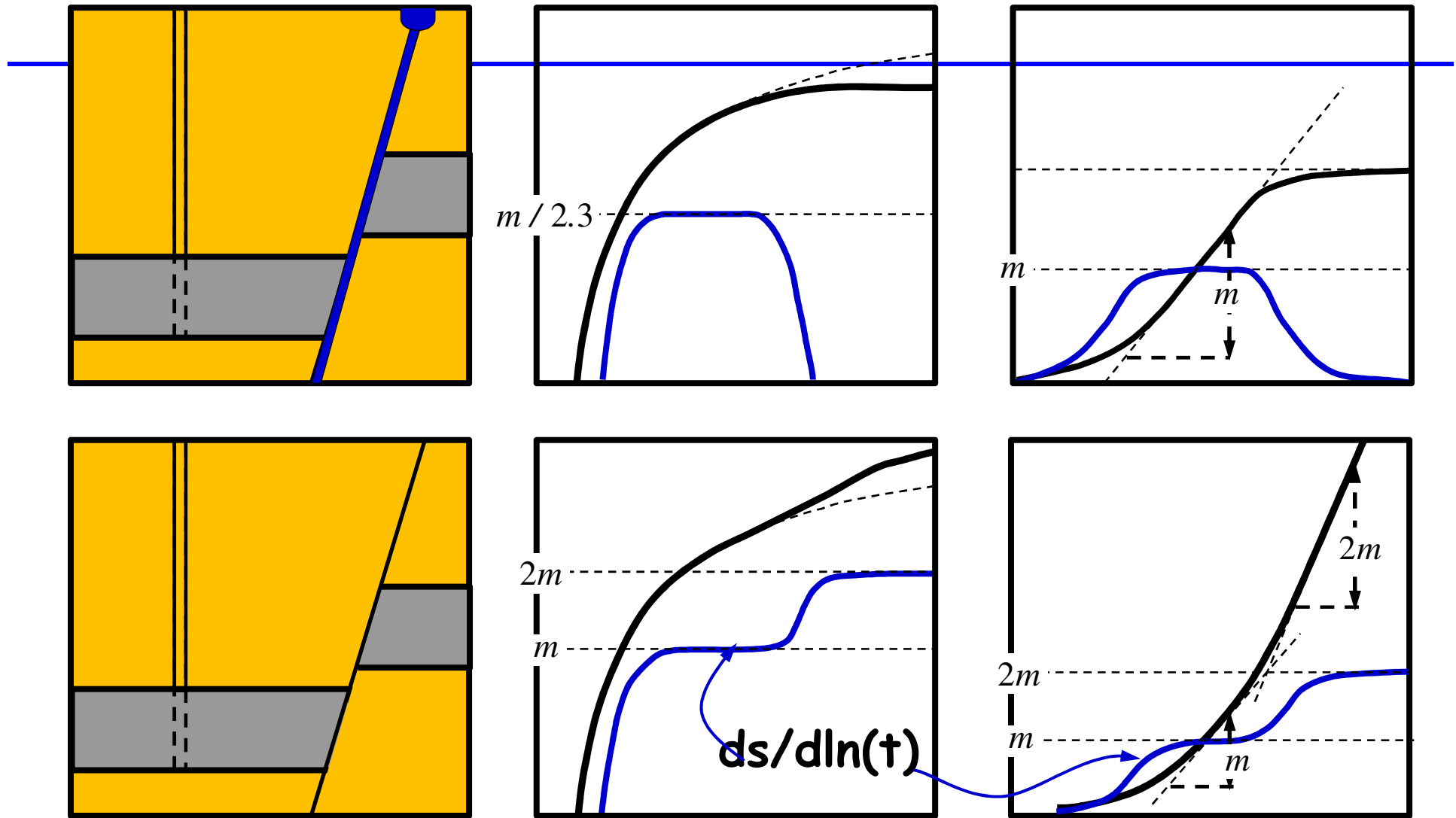


Illustration of boundary effects. A linear prescribed head boundary (e.g., a high transmissivity fault connected to a constant head water body) can be identified by a zero derivative (drawdown becomes constant). If that fault does not allow flow-through, the derivative is multiplied by a factor of 2.

# Analytical study of Jacob's method (Sánchez-Vila et al, WRR, 1999)

---

1) Write flow equation for spatially variable.

$$Y = \log T = \langle Y \rangle + Y' \quad (\text{constant } S)$$

2) perturbation expansion for  $h(\underline{x}, t)$

$$h = h^{(0)} + h^{(1)} + h^{(2)} + \dots$$

3) Write perturbation equations.

4) Solve them! (Green's function method).

5) Interpret solution. Approximate late time behavior as

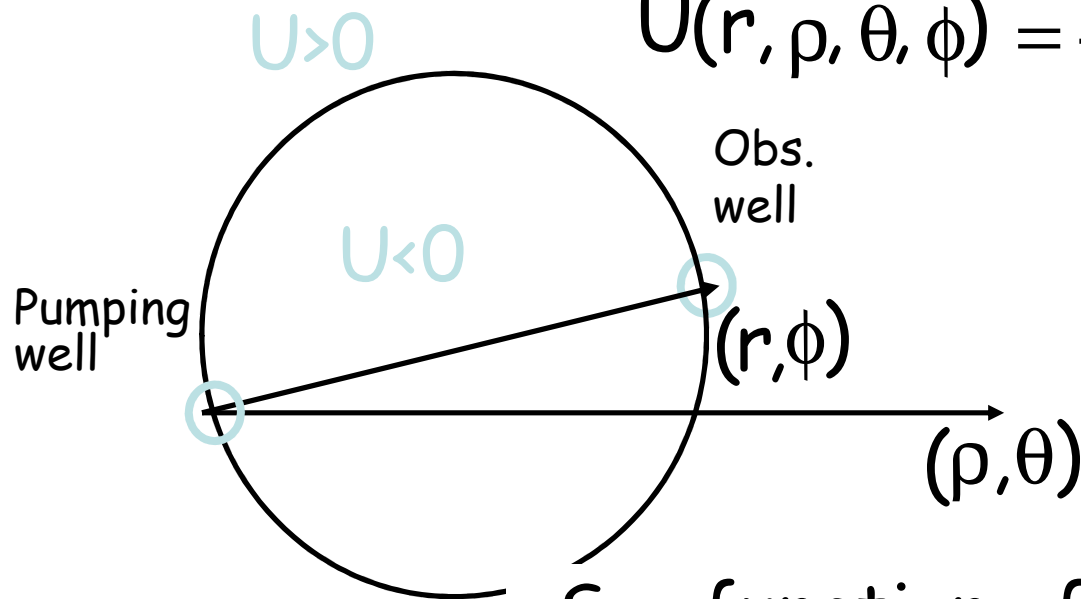
$$h = m \log (t/t_0).$$

6) Use  $m$  and  $t_0$  to obtain  $T_{Jac}$  and  $S_{Jac}$ .

# Jacob's storage coefficient ( $S_{Jac}$ )

$$S_{Jac}(r, \theta) = S \exp\left[\frac{1}{\pi_V} \int \gamma' U(r, \rho, \theta, \phi) \frac{dV}{\rho}\right]$$

$$U(r, \rho, \theta, \phi) = \frac{\rho - r \cos(\theta - \phi)}{r^2 + \rho^2 - 2\rho r \cos(\theta - \phi)}$$



- $S_{jac}$  function of  $\gamma'$
- Depends on T between pumping and obs. Wells (measures connectivity!)

# Analytical solution for T

---

$$T_{Jac}(r, \phi) = T_G \exp \left\{ \lim_{\rho \rightarrow \infty} \left[ \iiint_{\theta, \theta', \rho'} Y'(\rho, \phi) Y'(\rho', \phi') W(\rho, \rho', \theta, \theta') dV' d\theta \right] \right\}$$

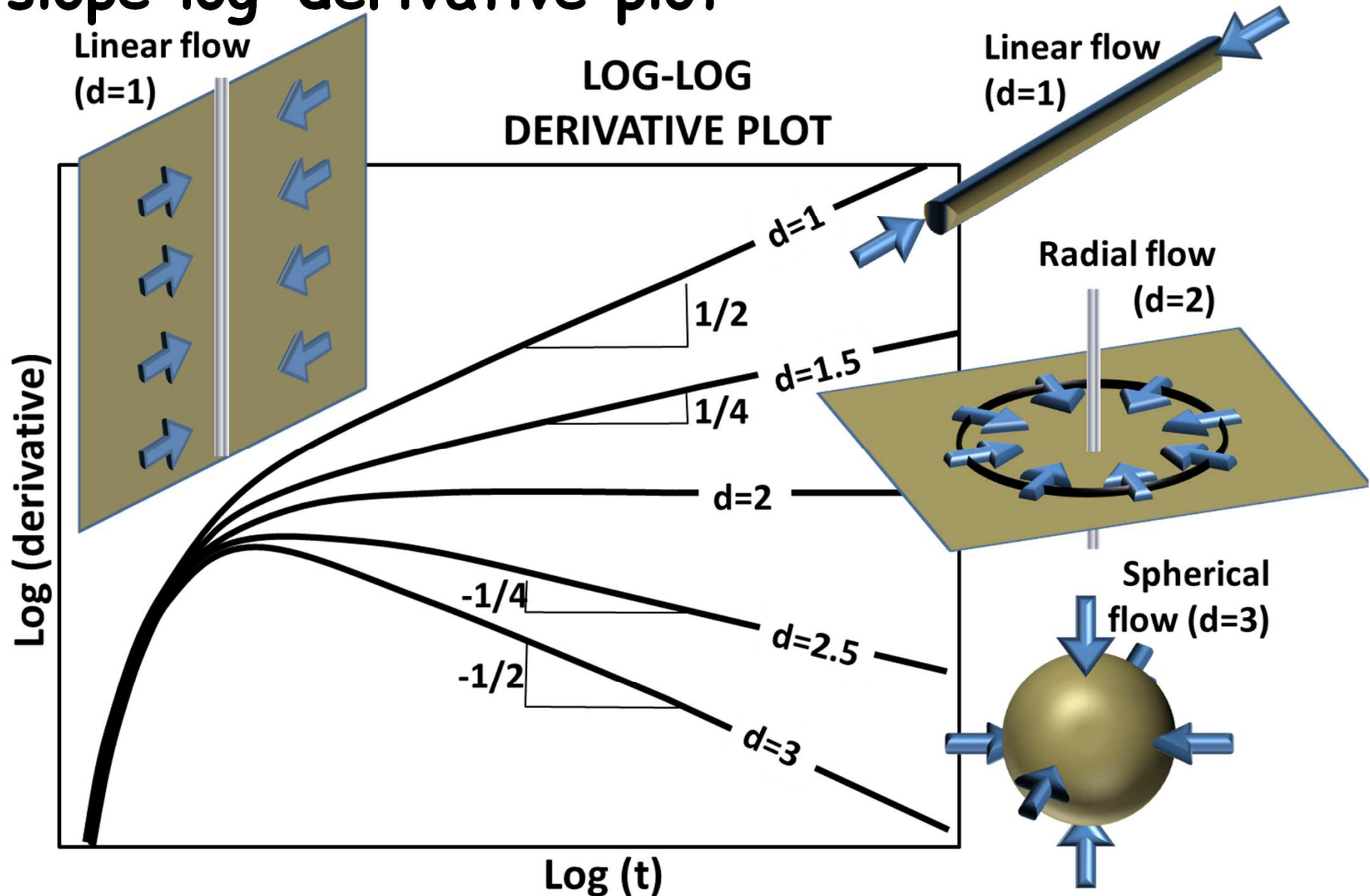
- $T_{Jac}(r, \phi)$  is independent of  $(r, \phi)$   
Jacob's T does not depend on the location of the observation point (regardless of the assumptions on T!!!, just validity of 2nd order approximation)

If multigaussian Y, then  $T_{Jac}$  equals  $T_G$   
If not, it does not!

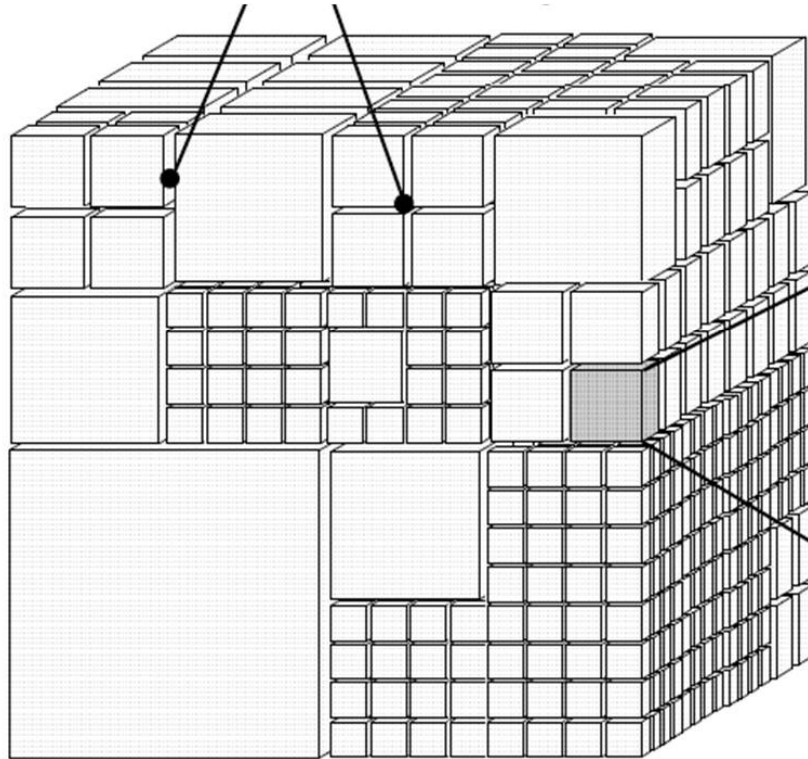
Late time growth reflects resistance to  
flow in the area where the cone of  
depression is growing

---

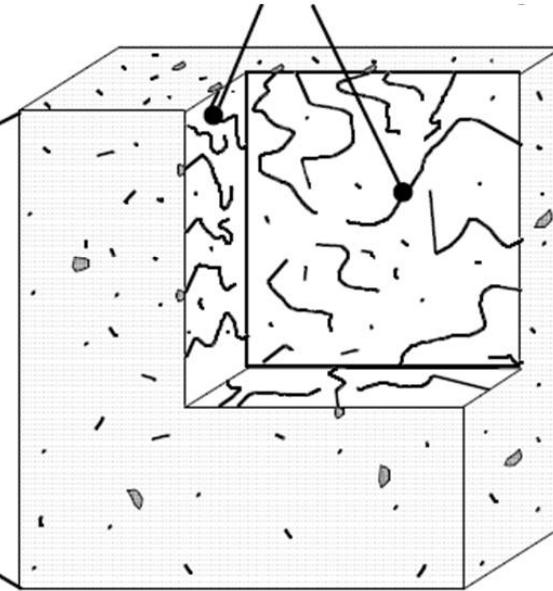
# Flow dimension is reflected in late time slope log-log derivative plot



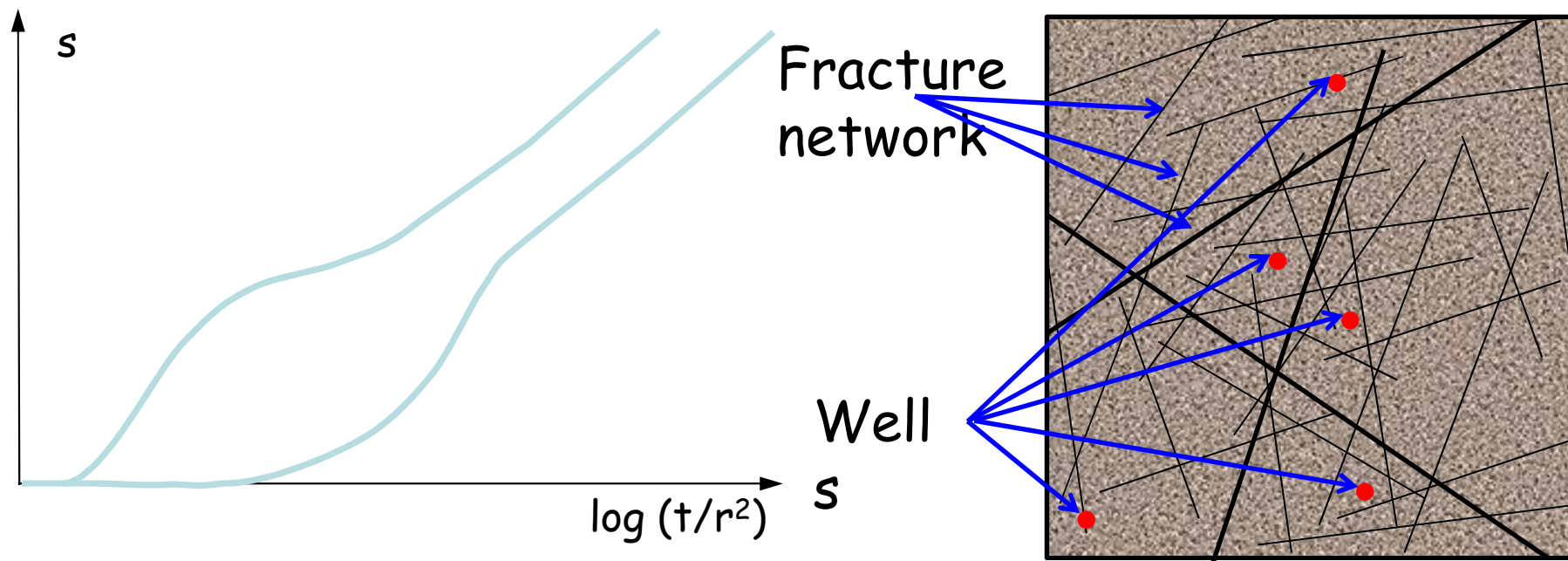
## Porosidad Advectiva



## Porosidad Difusiva

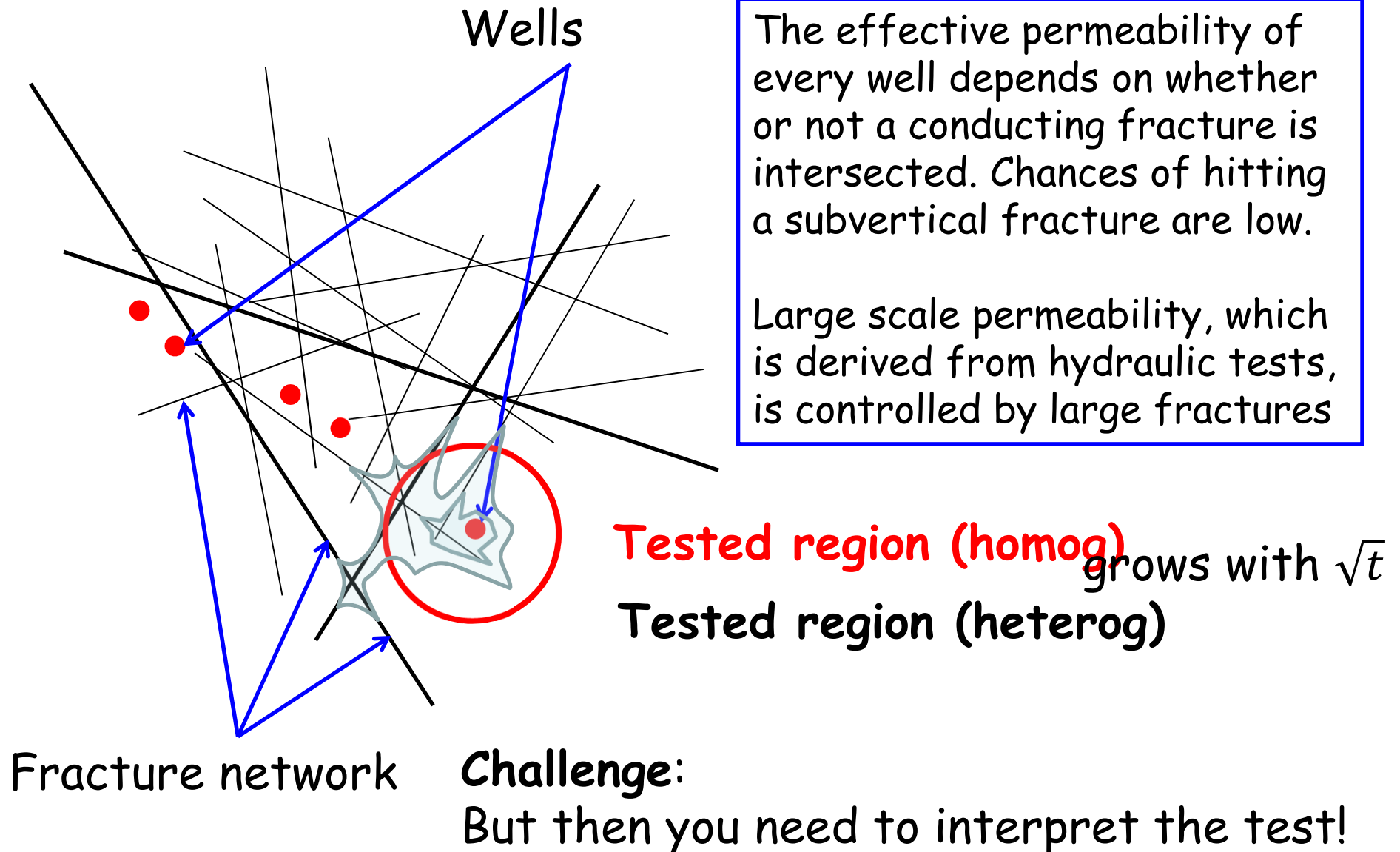


Esquema de un modelo de transferencia de masa de porosidad múltiple para representar el transporte de solutos y el flujo multifase. La porosidad advectiva (móvil) se encuentra en medio de una mezcla de bloques de matriz de varios tamaños, cada uno de los cuales contiene un rango de distintos tipos de porosidad difusiva (tomado de Haggerty y McKenna., 1999).



Large scale transmissivity is controlled by preferential flow paths (long, permeable, well connected fractures). The probability of intersecting these fractures is low, so that transmissivity derived from injectivity or short term tests tends to be much smaller. Large scale transmissivity may be derived from the slope of long term tests, which is the same regardless of whether the pumping or observation wells are located in high or low permeability areas. Local features are reflected in the early portion of the drawdown curve.

# Cause of scale effects



# Flow equation for fractal dimension

---

$$\frac{\partial p_D}{\partial t_D} = \frac{1}{r_D^{D^*-1}} \frac{\partial}{\partial r_D} \left( r_D^{D^*-1} \frac{\partial p_D}{\partial r_D} \right)$$

$$r_D^{\gamma+\beta-1} \frac{\partial p_D}{\partial t_D} = \frac{\partial}{\partial r_D} \left( r_D^{\beta+1} \frac{\partial p_D}{\partial r_D} \right)$$

---

### 3 MODELO TRANSITORIO FRACTAL PARA YACIMIENTOS DE TRIPLE POROSIDAD

**Flujo en Fractura:** 
$$\omega_F \frac{\partial p_{DF}}{\partial t} - r_D^{1-\gamma-\beta} \frac{\partial}{\partial r_D} \left[ r_D^{\beta+1} \frac{\partial p_{DF}}{\partial r_D} \right] - \lambda_{FV} (p_{DV} - p_{DF}) - \lambda_{FM} (p_{DM} - p_{DF}) = 0$$

**Flujo en Cavidad:** 
$$\omega_V \frac{\partial p_{DV}}{\partial t_D} + \lambda_{FV} (p_{DV} - p_{DF}) + \lambda_{VM} (p_{DV} - p_{DM}) = 0$$

**Flujo en Matriz:** 
$$\omega_M \frac{\partial p_{DM}}{\partial t_D} + \lambda_{FM} (p_{DM} - p_{DF}) + \lambda_{VM} (p_{DM} - p_{DV}) = 0$$

---

$$S_m \frac{\partial h_m}{\partial t} + \sum_{j=1}^N S_{im,j} \frac{\partial h_{im,j}}{\partial t} = -\nabla \cdot \mathbf{q} + q$$

$$S_{im,j} \frac{\partial h_{im,j}}{\partial t} = \sigma_{im,j} \frac{K_{im,j}}{L_{im,j}} (h_m - h_{im,j})$$

$$F(x,t) = \int_0^t g(t-\tau) \frac{\partial u_m(x,\tau)}{\partial \tau} d\tau = g * \frac{\partial u_m}{\partial t} + g(t)u_m(x,0)$$

$$g(t) = \sum_{j=1}^{\infty} \alpha_j \beta_j e^{-\alpha_j t}$$

# MRMT can be made equivalent to fractional derivatives by appropriate choice of $\alpha$ and $\beta$

---

The truncated power law memory function only requires specifying the slope of the memory function in log-log scale,  $m_g$ , and the interval of time  $[t_1, t_N]$  on which this function displays a power-law behavior. A practical method to calculate the distribution coefficients  $b_j$  consists of, first, calculate the  $\alpha_j$  values assuming they are evenly distributed on a logarithmic scale while fixing  $\alpha_1 = t_N^{-1}$  and  $\alpha_N = t_1^{-1}$ . Secondly, we obtain a recursive relationship for  $\beta_j$  values by approximating the memory function with expressions of successive increasing orders, i.e.

$$\log\left(\sum_{i=j}^N \beta_i \alpha_i\right) - \log\left(\sum_{i=j+1}^N \beta_i \alpha_i\right) = m_g (\log t_j - \log t_{j+1})$$

where  $t_j = \alpha_{j-1}$ . This leads to

$$\beta_j = \frac{\sum_{i=j+1}^N \beta_i \alpha_i}{\alpha_j} \left[ \left( \frac{\alpha_j}{\alpha_{j+1}} \right)^{m_g} - 1 \right] \quad j = 1 \dots N-1$$

# MRMT can be easily incorporated into conventional simulators

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