Rescaled weighted random balls models and stable self-similar random fields

Jean-Christophe Breton and Clément Dombry

February 17, 2009

Abstract

We consider weighted random balls in $\mathbb{R}^d$ distributed according to a random Poisson measure with heavy-tailed intensity. The asymptotic behaviors of the total weight of some configurations in $\mathbb{R}^d$ are studied. Different scaling, yielding different limit fields, are investigated.

Key words: self-similarity, generalized random fields, stable field, Poisson point process.

AMS Subject classification. Primary: 60G60, Secondary: 60F05, 60G52.

Introduction

In this work, we consider the so-called weighted random balls model and investigate its convergence when suitably rescaled and normalized. We exhibit three different asymptotic regimes driving the macroscopic and microscopic variations of this model, namely a (i) stable, translation and rotation invariant, self-similar random field on $\mathbb{R}^d$, (ii) a Poissonian field and (iii) a stable field with independence. The weighted random balls model is constructed in the following way: the centers of the balls are distributed according to a Poisson point process, with each center $x$ labelled with a random radius $r$ and a random weight $m$. The field under study is, roughly speaking, at each point, the weight density defined as the sum of the weights of the balls containing this point. The overlap of the balls yields non-trivial spatial correlations when the random radii of the balls are heavy-tailed.

This simple geometric construction has found numerous applications and amounts to be pertinent in various modeling situations. Similar stochastic models were considered by Kaj in [Kaj06] when modeling a simplified wireless network that consists of a collection of spatially distributed stations equipped with emitters for transmission over a common communication channel. Here, the location of a station or of a network node is represented by the point $x$, its range by the radius $r$ and its power by the weight $m$. The weight density measures the total power of emission at a given...
point and in this case, \( m \) is supposed to be non-negative. But our model supports more generally real-valued weights.

In [BE], Biermé and Estrade consider similar models in dimension \( d = 2 \) as models in imagery (in this case, the weight intensity stands for the gray level of a pixel in a black and white picture) and in dimension \( d = 3 \) for modeling 3D porous or heterogeneous media (the weight density is here seen as a mass density). They investigate the microscopic properties of the random balls configuration by performing a scaling operation which amounts to zoom in smaller regions of space. In [KLNS], Kaj et al. study similar random grain model by shrinking to zero the volume of the grains. This amounts to analyse the macroscopic properties of the random balls configuration by performing a scaling operation which amounts here to zoom out over larger areas.

Recently, Biermé, Estrade and Kaj introduce in [BEK] a general framework for rescaled random balls model allowing both zoom-in (as in [BE]) and zoom-out (as in [KLNS]). In this zooming procedure, several limit fields arise, which are either of Gaussian or of Poisson type according to the respective asymptotic of the zooming rate and of the Poisson intensity of the balls. Furthermore, they show that essentially all Gaussian, translation and rotation invariant self-similar generalized random fields can be obtained as such a limit.

Note that in the rescaled random balls model of [BE], [KLNS] and [BEK], the weight in the field under study are fixed equal to \( m \equiv 1 \). Models with randomized weights have been less intensively studied. In dimension \( d = 1 \), Kaj and Taqqu study in [KT] limiting schemes for weighted random balls model, deriving Gaussian, Poisson and stable regimes. This model applies in particular to study the random variation in packet networks computer traffic.

Our main contribution in this paper is to introduce a general study of macroscopic and microscopic variations in weighted models in \( \mathbb{R}^d \). This generalizes both [BEK] since the balls are randomly weighted and [KT] since we consider an arbitrary dimension \( d \) and more general configuration on the balls. As in [KLNS] and [KT], three different regimes appear according to the relative behavior of the scaling rate and of the Poisson intensity. In particular, when the random weights are heavy-tailed, the limit generalized random fields are stable, translation and rotation invariant and self-similar. The paper is organized as follows. The model under study is described in Section 1. Our main results under different scaling regimes are stated and discussed in Section 2. Finally, Section 3 is devoted to the proof of technical lemmas and of the main results.

1 Model of weighted random balls

We consider random balls \( B(x, r) = \{ y \in \mathbb{R}^d : \| y - x \| < r \} \) with weight \( m \), the triplet \((x, r, m)\) being distributed according to a Poisson random measure \( N_\lambda(dx, dr, dm) \) on \( \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R} \) with intensity

\[
n(dx, dr, dm) = \lambda dx F(dr) G(dm)
\]

where \( \lambda \) is positive and \( F \) and \( G \) are probability measures on \( \mathbb{R}^+ \) and \( \mathbb{R} \) respectively.
The point process of the centers of the balls in $\mathbb{R}^d$ is the projection of the point process in $\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}$ corresponding to the Poisson random measure $N_\lambda(dx, dr, dm)$. It is easily seen that it is a Poisson point process with intensity $\lambda dx$, and hence the parameter $\lambda$ is interpreted as the intensity of the balls in $\mathbb{R}^d$.

We suppose that the probability $F$ giving the distribution of the radius $r$ has a density $F(dr) = f(r)dr$ satisfying

$$\int_{\mathbb{R}^+} r^d F(dr) < +\infty$$

(1)

and for either $\epsilon = +1$ or $\epsilon = -1$,

$$f(r) \sim_{r \to 0^\epsilon} C r^{-1-\beta}$$

(2)

where by convention $0^+1 = 0$ and $0^-1 = +\infty$. As will be explained later, the case $\epsilon = +1$ will be referred as the zoom-in case, whereas the case $\epsilon = -1$ will be referred as the zoom-out case. Condition (2) assumes a power behaviour of the radius density at the origin (zoom-in case $\epsilon = +1$) or at infinity (zoom-out case $\epsilon = -1$). Condition (1) is equivalent to the finiteness of the volume of the random balls. Note that the conjunction of both assumptions (1) and (2) implies that for $\epsilon = +1$, we must have $\beta < d$, while when $\epsilon = -1$, we must have $\beta > d$.

We suppose that the probability $G$ belongs to the normal domain of attraction of the $\alpha$-stable distribution $S_\alpha(\sigma, b, \tau)$ with $\alpha \in (1, 2]$, i.e. if $X_1, \ldots, X_n$ are i.i.d. according to $G$, $n^{-1/\alpha}(X_1 + \cdots + X_n) \Rightarrow S_\alpha(\sigma, b, \tau)$. We recall the following estimate of the characteristic function of $G$ as $\theta \to 0$ (see [Fel])

$$\phi_G(\theta) = 1 + i\theta \tau - \sigma^\alpha |\theta|^\alpha (1 + i\varepsilon(\theta) \tan(\pi\alpha/2) + o(|\theta|^\alpha)).$$

(3)

Here and in the sequel, where $\varepsilon(a) = +1$ if $a > 0$, $\varepsilon(a) = -1$ if $a < 0$ and $\varepsilon(0) = 0$. In case $\alpha \in (1, 2)$, typical choice for $G$ are heavy-tailed distribution while for $\alpha = 2$, $G$ is any distribution with finite variance. In this latter case, we recover a weighted version of the main results in [BEK] (set $G = \delta_1$ to recover exactly the setting described in [BEK]).

Let $\mathcal{M}$ denote the set of signed measures on $\mathbb{R}^d$ with finite total variation $|\mu|(\mathbb{R}^d)$, where $|\mu|$ is the total variation of measure $\mu$. Equipped with the norm of total variation $||\mu||_\mathcal{M} = |\mu|(\mathbb{R}^d)$, $\mathcal{M}$ is a Banach space. We consider the random field

$$M(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m\mu(B(x, r))N_\lambda(dx, dr, dm)$$

(4)

indexed by signed measures $\mu \in \mathcal{M}$. When $\mu = \delta_y$, $M(\delta_y)$ is the weight density at point $y$ as described in the introduction: it is the sum of the algebraic weights of the balls containing the point $y$.

Note that the stochastic integral in (4) is well defined and has finite expected value since

$$\int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} |m\mu(B(x, r))| n(dx, dr, dm) \leq \int_{\mathbb{R}^d} |m|G(dm) \times \lambda |B(0,1)| |\mu|(\mathbb{R}^d) \int_{\mathbb{R}^+} r^d F(dr) < +\infty.$$
where \(|A|\) stands for the Lebesgue measure of a Borel set \(A\). Furthermore, the expected value is given by

\[
\mathbb{E}[M(\mu)] = \lambda |B(0, 1)| \int_{\mathbb{R}} mG(dm) \int_{\mathbb{R}^+} r^d F(dr) \mu(\mathbb{R}^d).
\]

We are interested in the variations of \(M(\mu)\) at a macroscopic or microscopic level. To do so, we swell, resp. shrink, the volume of the balls replacing the radius \(r\) of a ball by \(\rho r\) and taking the limit \(\rho \to +\infty\), resp. \(\rho \to 0\). In this procedure, the law of the radius is replaced by \(F_\rho(dr) = f(r/\rho)dr/\rho\), the image measure of \(F(dr)\) by the change of scale \(r \mapsto \rho r\). In order to derive non trivial asymptotics, the intensity \(\lambda\) of the balls is changed accordingly and we shall write \(\lambda(\rho)\) to underline that from now on the intensity depends on the scaling parameter \(\rho\). In the sequel, we are thus interested in the following random field:

\[
M_\rho(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m\mu(B(x,r))N_{\lambda(\rho),\rho}(dx,dr,dm)
\]

where \(N_{\lambda(\rho),\rho}(dx,dr,dm)\) is the Poisson measure with intensity \(\lambda(\rho)dx F_\rho(dr)G(dm)\). The limit \(\rho \to 0\) is interpreted as zoom-out in the random configurations of balls and this is pertinent when the behavior of \(f\) is known at \(+\infty\), i.e. \(\epsilon = -1\) in (2). In this case, we investigate the macroscopic variations of \(M\). On the opposite, \(\rho \to +\infty\) is interpreted as zoom-in in space and this is relevant when the behavior of \(f\) is known at 0, i.e. \(\epsilon = +1\) in (2) and this is the microscopic variations that are investigated.

**Remark 1.1** As observed before, the choice \(G = \delta_1\) recovers the setting of [BEK] for non-weighted random balls, see (4) therein. If \(d = 1\), a verbatim replacement of \(B(x,r) = (x-r, x+r)\) by \((x, x+r)\) and the choice \(\mu = |\cdot \cap (0,t)|\) recover the field studied in [KT] in the "continuous flow reward model", see (18) therein.

### 2 Results

We exhibit normalization terms \(n(\rho)\) such that the normalized centered random field \(n(\rho)^{-1}(M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)])\) converges in finite dimensional distribution \(f.d.d.\) to a limit random field. In the sequel, we are interested \(f.d.d.\) convergence on subspaces \(\tilde{\mathcal{M}}\) of \(\mathcal{M}\) and we will denote it by \(\tilde{\mathcal{M}}\rightarrow\). It is natural to investigate first the behavior of the punctual random field \((M_\rho(\delta_y))_{y \in \mathbb{R}^d}\).

The heuristic is the following. The average numbers of balls containing the point \(y\) is given by

\[
\mathbb{E}\left[\int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} 1_{\{y \in B(x,r)\}}N_{\lambda(\rho),\rho}(dx,dr,dm)\right] = V\lambda(\rho)\rho^d,
\]

where \(V = c_d \int r^d F(dr)\) is the expected volume of a random ball and \(c_d\) is the volume of an Euclidean unit ball in \(\mathbb{R}^d\). Since the weights belong to the domain of attraction of an \(\alpha\)-stable distribution, we introduce the scaling \(n_\alpha(\rho) = \lambda(\rho)^{1/\alpha} \rho^{d/\alpha}\). Convergence
of the normalized and centered random variable $M_\rho(\delta_y)$ to an $\alpha$-stable distribution is expected if $n_0(\rho) \to +\infty$. Furthermore, the dependence between $M_\rho(\delta_{y_1})$ and $M_\rho(\delta_{y_2})$ is given by the weights of the balls containing both points. In the zoom-in case ($\rho \to +\infty$), the balls are very large yielding total dependence at the limit. In the zoom-out case ($\rho \to 0$), the balls are very small yielding independence at the limit.

More formally, let $M_{\text{dis}}$ be the subspace $\mathcal{M}$ of measures with finite support. This is the span in $\mathcal{M}$ of the set $\{\delta_y\}_{y \in \mathbb{R}^d}$ of Dirac measures.

**Theorem 2.1** Let $\lambda(\rho)$ be such that $\lambda(\rho)\rho^d \to +\infty$ as $\rho \to 0^\varepsilon$.

**Zoom-out case ($\varepsilon = -1$):** When $\rho \to 0$,

$$n_0(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]) \xrightarrow{\mathcal{M}_{\text{dis}}} W_\alpha(\mu)$$

where $W_\alpha(\mu)$ is a linear random functional on $\mathcal{M}_{\text{dis}}$ such that the random variables $W_\alpha(\delta_y), y \in \mathbb{R}^d$, are independent and identically distributed according to $S_\alpha(\sigma V^{1/\alpha}, b, 0)$.

**Zoom-in case ($\varepsilon = +1$):** When $\rho \to +\infty$,

$$n_0(\rho)^{-1}(X_\rho(\mu) - \mathbb{E}[X_\rho(\mu)]) \xrightarrow{\mathcal{M}_{\text{dis}}} \tilde{W}_\alpha(\mu)$$

where $\tilde{W}_\alpha(\mu)$ is a linear random functional on $\mathcal{M}_{\text{dis}}$ such that the random variables $\tilde{W}_\alpha(\delta_y), y \in \mathbb{R}^d$, are almost surely equal and distributed according to $S_\alpha(\sigma V^{1/\alpha}, b, 0)$.

In the sequel, we are investigating for further asymptotic results for more general measures $\mu$. In contrast to Theorem 2.1, it appears that the normalization of $M_\rho$ and the limit distribution depend really on the measure $\mu$ and on the asymptotic of the parameters $\rho$ and $\lambda(\rho)$.

### 2.1 Preliminaries on measured spaces

We introduce the following subset of configurations $\mu \in \mathcal{M}$ for which we shall derive the variations in the weighted random ball model.

**Definition 2.1** Let $\mathcal{M}_{\alpha,\beta}$ be the subset of measures $\mu \in \mathcal{M}$ satisfying for some finite constant $C$ and some $p < \beta < q$:

$$\int_{\mathbb{R}^d} |\mu(B(x, r))|^\alpha dx \leq C(r^p \wedge r^q) \quad (5)$$

where for reals $a, b$: $a \wedge b = \min(a, b)$.

Here and in the sequel, $C$ is a finite constant that may change at each occurrence. Some elementary properties of the spaces $\mathcal{M}_{\alpha,\beta}$ are given in the following proposition.

**Proposition 2.1**
i) $M_{\alpha,\beta}$ is a linear subspace of $M$ on which
\[ \forall \mu \in M_{\alpha,\beta}, \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x,r))|^\alpha r^{-\beta-1} dx dr < +\infty. \]

ii) $M_{\alpha,\beta}$ is closed under translations, rotations and dilatations, i.e. when $\mu \in M_{\alpha,\beta}$, $\tau_s \mu$, $\Theta \mu$ and $\mu_a$ are also in $M_{\alpha,\beta}$ where for any Borelian set $A$ and for $s \in \mathbb{R}^d$, $\Theta \in O(\mathbb{R}^d)$, $a \in \mathbb{R}^+$
\[ \tau_s \mu(A) = \mu(A-s), \Theta \mu(A) = \mu(\Theta^{-1} A), \mu_a(A) = \mu(a^{-1} A). \]

iii) When $\alpha \leq \alpha'$, we have $M_{\alpha,\beta} \subset M_{\alpha',\beta}$.

iv) When $\beta \geq d$, the space $M_{\alpha,\beta}$ is included in the subspace of diffuse measures.

v) When $\beta \leq d$, the space $M_{\alpha,\beta}$ is included in the subspace of centered measures.

Observe that Dirac measures $\delta_y$, $y \in \mathbb{R}^d$, are not in $M_{\alpha,\beta}$. However, explicit examples of measure in $M_{\alpha,\beta}$ are given in the following proposition.

**Proposition 2.2**

i) If $d < \beta < \alpha d$, any measure $\mu(dx) = f(x) dx$ with bounded integrable density $f$ such that $f(z) = O_{z \to +\infty}(|z|^{-\eta})$ for $\eta > d$ belongs to $M_{\alpha,\beta}$.

ii) If $d - 1 < \beta < d$, any centered measure $\mu(dx) = f(x) dx$ with bounded integrable density $f$ such that $f(z) = O_{z \to +\infty}(|z|^{-\eta})$ for $\eta > d(1 + \alpha)/\alpha$ or any centered measure with finite support belongs to $M_{\alpha,\beta}$.

Note that in particular, when $d < \beta < \alpha d$ (resp. $d - 1 < \beta < d$), $M_{\alpha,\beta}$ contains the space $S$ of measures with density in the Schwartz class (resp. $S_0$ the space of centered measures with density in the Schwartz class). Note also that when $\alpha = 2$, the conditions for the measures $\mu$ in [BEK] (expressed in terms of Riesz energy) implies that $\mu \in M_{2,\beta}$. By analogy with the case $\alpha = 2$, we expect the space $M_{\alpha,\beta}$ to be reduced to $\{0\}$ whenever $\beta \leq d - 1$ or $\beta \geq \alpha d$, but we have unfortunately no rigorous proof of these facts.

2.2 Limit theorems for the rescaled weighted random balls model

We now come to the main result of this paper, viz. limit theorems for the rescaled generalized random fields $M_\rho$ and for configurations $\mu \in M_{\alpha,\beta}$ on the balls. Like in [KLNS] and [KT] (for $\epsilon = -1$), several regimes appear according to the density of large/small balls in the limit. More precisely:

**Zoom-out case:** ($\epsilon = -1$, i.e. $\beta > d$ and $\rho \to 0$). The expected number of balls with radius larger than one that cover the origin is given by
\[ \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{\|x\|<r} 1_{r>1} \lambda(\rho) dx F_\rho(dr) = c_d \lambda(\rho) \int_{1}^{+\infty} r^d F_\rho(dr) \sim_{\rho \to 0} \frac{c_d C_\beta \beta - d \lambda(\rho)}{\beta - d}. \]

Consequently, we distinguish the following three scaling regimes:
• large-balls scaling: \( \lambda(\rho)\rho^\beta \to +\infty \),
• intermediate scaling: \( \lambda(\rho)\rho^\beta \to a \in (0, +\infty) \),
• small-balls scaling: \( \lambda(\rho)\rho^\beta \to 0 \).

**Zoom-in case:** (\( \epsilon = +1 \), i.e. \( \beta < d \) and \( \rho \to +\infty \)). The expected number of balls with radius less than one that cover the origin is given by

\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{|x| < r} 1_{r < 1} \lambda(\rho) dF_\rho(dr) = c_d \lambda(\rho) \int_0^1 r^d F_\rho(dr) \sim \rho \to +\infty \frac{c_d C_{\beta}}{d - \beta} \lambda(\rho)\rho^\beta.
\]

In this case, the three scaling regimes are:

• small-balls scaling: \( \lambda(\rho)\rho^\beta \to +\infty \),
• intermediate scaling: \( \lambda(\rho)\rho^\beta \to a \in (0, +\infty) \),
• large-balls scaling: \( \lambda(\rho)\rho^\beta \to 0 \).

Note that when \( \lambda(\rho)\rho^\beta \to +\infty \), the zooming procedure exhibits balls in accordance with intuition (viz. when zooming out, only the large balls contribute to the limiting shape while when zooming in, only the small ones matter). In this case, we will show that the limiting shape is driven by a stable regime with dependence. On the opposite, when \( \lambda(\rho)\rho^\beta \to 0 \), the zooming procedure exhibits balls contrary to intuition (large balls appear when zooming in and small balls when zooming out). In this case, the limit is driven by a stable regime with independence. In the intermediate case, when \( \lambda(\rho)\rho^\beta \) has a finite non-zero limit, the limiting shape is driven by a Poissonian regime.

In the sequel, we study precisely the limiting shape of the random balls by investigating the fluctuations of \( M(\mu) \) around its mean. As explained just above, three different limit fields are exhibited according to the scaling performed.

### 2.2.1 Stable regime with dependence

In this section, we investigate the behavior of \( M \) under the scaling \( \rho^\beta \lambda(\rho) \to +\infty \). In this case the limiting field is given by a \( \alpha \)-stable integral. We recall that the stable stochastic integral of \( f \) with respect to a \( \alpha \)-stable random measure with control measure \( m \) is well defined whenever \( f \in L^\alpha(dm) \) and in this case, this stochastic integral follows a \( \alpha \)-stable distribution. We refer to [ST] for a complete account on stable measures and integrals. The asymptotic of the rescaled generalized fields \( M_\rho \) is given by the following result:

**Theorem 2.2** Suppose \( \rho^\beta \lambda(\rho) \to +\infty \) when \( \rho \to 0^- \). Let \( n_1(\rho) = \lambda(\rho)^{1/\alpha} \rho^{\beta/\alpha} \). We have

\[
\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n_1(\rho)} \xrightarrow{\mathcal{M}_{\alpha,\beta}} Z_\alpha(\cdot) \quad \rho \to 0^- \tag{6}
\]

where \( Z_\alpha(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r)) M_\alpha(dr,dx) \) is a stable integral with respect to the \( \alpha \)-stable measure \( M_\alpha \) with control measure \( \sigma^\alpha C_{\beta} r^{-1-\beta} dr dx \) and constant skewness function \( b \).
Note that $Z_\alpha(\mu)$ makes sense as soon as $\int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x,r))^{\alpha} r^{\frac{1}{\alpha} - 1 - \beta} dr dx < +\infty$. However, we need the stronger assumption $\mu \in \mathcal{M}_{\alpha,\beta}$ in order to derive (6). Roughly speaking, the control (5) of $\mu \in \mathcal{M}_{\beta,\alpha}$ allows to replace $F$ by its tails behavior given in (1) in asymptotic estimate.

Due to the invariance by translation and rotation of the Lebesgue measure, the self-similarity of stable integral and the (global) invariance by rotation of the balls and because of Proposition 2.1-ii), we derive the following properties for the limit field $Z_\alpha$ in Theorem 2.2:

**Proposition 2.3**

i) The field $Z_\alpha$ is stationary on $\mathcal{M}_{\alpha,\beta}$, that is:

$$\forall \mu \in \mathcal{M}_{\alpha,\beta}, \forall s \in \mathbb{R}^d, Z_\alpha(\tau_s \mu) \overset{fdd}{=} Z_\alpha(\mu).$$

ii) The field $Z_\alpha$ is isotropic on $\mathcal{M}_{\alpha,\beta}$, that is:

$$\forall \mu \in \mathcal{M}_{\alpha,\beta}, \forall \Theta \in \mathcal{O}(\mathbb{R}^d), Z_\alpha(\Theta \mu) \overset{fdd}{=} Z_\alpha(\mu).$$

iii) The field $Z_\alpha$ is self-similar on $\mathcal{M}_{\alpha,\beta}$ with index $(d - \beta)/\alpha$, that is:

$$\forall \mu \in \mathcal{M}_{\alpha,\beta}, \forall a > 0, Z_\alpha(a \mu) \overset{fdd}{=} a^{(d-\beta)/\alpha} Z_\alpha(\mu).$$

**Remark 2.1** The covariation gives an insight into the structure of the spatial dependence of the stable generalized field. It is a generalization of the usual notion of covariance to the stable framework. Here, for $\mu_1, \mu_2 \in \mathcal{M}_{\alpha,\beta}$, the covariation of $Z_\alpha(\mu_1)$ on $Z_\alpha(\mu_2)$ is given by

$$[Z_\alpha(\mu_1), Z_\alpha(\mu_2)]_\alpha = \sigma^\alpha C_\beta \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu_1(B(x,r)) \epsilon(\mu_2(B(x,r))) |\mu_2(B(x,r))^{\alpha - r - \beta - 1} dr dx.$$ 

Note that the integral above is well defined by Hölder’s inequality since $\mu_1$ and $\mu_2$ belong to $\mathcal{M}_{\alpha,\beta}$. We refer to [ST] for a definition and properties of the covariation.

Note that unlike the Gaussian case, the covariation structure is not sufficient to characterize the distribution of the generalized random field. However, since even if $\mu_1$ and $\mu_2$ have disjoint supports, $[Z_\alpha(\mu_1), Z_\alpha(\mu_2)]_\alpha \neq 0$, $Z_\alpha(\mu_1)$ and $Z_\alpha(\mu_2)$ are not independent and the random field $Z_\alpha$ is stable with dependence.

**Remark 2.2** Note that when $d - 1 < \beta < d$, $\mu_z = \delta_z - \delta_0$ for $z \in \mathbb{R}^d$ belongs to $\mathcal{M}_{\alpha,\beta}$. For such a measure, our limiting field rewrites

$$Z_\alpha(\mu_z) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(z,r) \Delta B(0,r)} M_\alpha(dx,dr)$$

when moreover $b = 0$ (i.e. when $G$ in our model is symmetric). In this case, we recover the so-called $(\alpha, H)$-Takenaka field with $H = (d - \beta)/\alpha$. It is self-similar with index $H$, with stationary increments and almost surely with continuous sample paths, see [BEK, p. 25] or [ST, Sect. 8.4].
Remark 2.3 When $d = 1$, $\beta \in (1, \alpha)$ and $\mu_t = | \cdot \cap (0, t)|$, the field $Z_\alpha(\mu_t)$ coincides with the Telecom process obtained in the fast connection rate for the "continuous flow reward model" in [KT, Th. 2], see also Remark 1.1 above. For $\alpha = 2$, $Z_2(\mu_t)$ is a fractional Brownian motion of Hurst index $H = (3 - \beta)/2 \in (1/2, 1)$.

Remark 2.4 When $\alpha = 2$, Theorem 2.2 exhibits a Gaussian limit field and generalizes Theorem 2.1 in [BEK] with random weights. Indeed, in this case, we have (up to some multiplicative constant) $Z_2 = W_\beta$.

2.2.2 Poissonian regime

In this section, we investigate the behavior of $M$ under the scaling $\rho^\beta \lambda(\rho) \to a$. In this case, the limiting field is given by a compensated Poisson integral and we refer to [Kal] for a general description of Poisson integral. We have:

Theorem 2.3 Suppose $\lambda(\rho) \rho^\beta \to a^{d-\beta}$ when $\rho \to 0^{-\epsilon}$ for some $a > 0$. We have

$$M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)] \xrightarrow{\mathcal{M}_{\alpha, \beta}} J(\mu_a), \quad \rho \to 0^{-\epsilon}$$

where $\mu_a$ is the dilatation of $\mu$ and $J$ is the compensated Poisson integral

$$J(\mu) = \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+} m\mu(B(x, r))\tilde{N}_\beta(dx, dr, dm)$$

(7)

with respect to the compensated Poisson random measure $\tilde{N}_\beta$ with intensity given by $C_\beta r^{-\beta-1}dxdrG(dm)$.

Note that the Poisson integral in (7) above is well defined since

$$\int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+} \left( |m\mu(B(x, r))| \wedge (m\mu(B(x, r)))^2 \right) r^{-\beta-1}dxdrG(dm) < +\infty$$

(8)

see Section 3.5. As for the stable field $Z_\alpha$, the Poisson field $J$ enjoys similar properties. However, note that in contrast to $Z_\alpha$, $J$ is not self-similar but (and similarly to [BEK], see also [Kaj05]) $J$ satisfies an aggregate similarity property.

Proposition 2.4 The field $J$ is stationary and isotropic on $\mathcal{M}_{\alpha, \beta}$. Moreover, $J$ is aggregate similar, viz. $\forall \mu \in \mathcal{M}_{\alpha, \beta}, \forall m \geq 1$,

$$J(\mu_{a_m}) \xrightarrow{fdd} \sum_{i=1}^m J^i(\mu)$$

where $J^i$, $1 \leq i \leq m$, are independent copies of $J$ and $a_m = m^{1/(d-\beta)}$.

The proof of this proposition follows from straightforward computation and will be omitted. A comparison of the limiting procedures in Theorem 2.2 where $\lambda(\rho) \rho^\beta \to +\infty$ and in Theorem 2.3 where $\lambda(\rho) \rho^\beta \to a^{d-\beta}$ suggests that when $a^{d-\beta} \to +\infty$, we can recover $Z_\alpha$ from $J$. This is true and precisely stated in the following proposition:
Proposition 2.5 When $a^{d-\beta} \to +\infty$, we have $\frac{1}{a^{d-\beta}} J(\mu_a) \xrightarrow{\mathcal{M}_{a,\beta}} Z_\alpha(\mu)$.

Remark 2.5 Like in Remark 2.3, when $d = 1$ and $\mu_t = |\cdot \cap (0,t)|$, the field $J(\mu_t)$ coincides with the intermediate Telecom process obtained in the intermediate connection rate for the "continuous flow reward model" in [KT, Th. 1], see also Remark 1.1 above.

Remark 2.6 When $\alpha = 2$, Theorem 2.3 generalizes Theorem 2.5 in [BEK] with random weights. The field $J$ recovers $J_\beta$ in [BEK] when the random weight in our model are constant. Otherwise the law of $J$ depends on the law $G$ of the weight.

2.2.3 Stable regime with independence

In this section, we investigate the behavior of $M$ under the scaling $\rho^\beta \lambda(\rho) \to 0$, but we restrict to the case $d < \beta < \alpha d$, i.e. $\epsilon = -1$ and $\rho \to 0$. We show that the asymptotic behavior is given again by a stable field but with index $\gamma = \beta/d$ and defined on $\mathbb{R}^d$. Moreover in contrast to the stable field $Z_\alpha$ of Section 2.2.1, this new field exhibits independence. The asymptotic of the rescaled generalized fields $M_\rho$ is given on the subspace of measure $\mu(dx) = \phi(x)dx$ with $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$. We shall (abusively) note $\mu \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$. We have

Theorem 2.4 Let $d < \beta < \alpha d$ and suppose that $\lambda(\rho) \to +\infty$ and $\lambda(\rho)\rho^\beta \to 0$ as $\rho \to 0$. Then with $n_2(\rho) := \lambda(\rho)^{d/\beta} \rho^d$ and $\gamma = \beta/d \in (1, \alpha)$, we have

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n_2(\rho)} \xrightarrow{L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)} \overline{Z}_\gamma(\cdot)$$

where, for $\mu(dx) = \phi(x)dx$, $\overline{Z}_\gamma(\mu) = \int_{\mathbb{R}^d} \phi(x) \overline{M}_\gamma(dx)$ is a stable integral with respect to the $\gamma$-stable measure $M_\gamma$, with control measure $\sigma_\gamma^2 dx$ for

$$\sigma_\gamma^2 = \frac{c_\gamma^2 C_\beta}{d} \int_{\mathbb{R}^d} \frac{1 - \cos(r)}{r^{1+\gamma}} \int_{\mathbb{R}} |m|^\gamma G(dm)$$

and with constant skewness function equals to

$$b_\gamma = -\frac{\int_{\mathbb{R}} \varepsilon(m) |m|^\gamma G(dm)}{\int_{\mathbb{R}} |m|^\gamma G(dm)}.$$

Note that the integrals above are well defined when $d < \beta < \alpha d$ (see Lemma 3.1 below). discuter les hypotheses.???? The limiting field $\overline{Z}_\gamma$ enjoys similar properties as $Z_\alpha$ and $J$:

Proposition 2.6 The field $\overline{Z}_\gamma$ is stationary, isotropic and invariant by dilatation.

Remark 2.7 Like in Remarks 2.3 and 2.5, when $d = 1$ and $\phi_t = 1_{(0,t)}$, the field $\overline{Z}_\gamma(\phi_t)$ coincides with the process obtained in the slow connection rate for the "continuous flow reward model" in [KT, Th. 3], see also Remark 1.1 above. In this particular case, $\overline{Z}_\gamma(\phi_t)$ is a $\gamma$-stable Lévy process.
3 Proof of the results

In the sequel, note that the linearity of the random functionals $M_\rho$ and of the stochastic integrals in $W_\alpha, \tilde{W}_\alpha, Z_\alpha, J$ and $\tilde{Z}_\gamma$, together with the Cramèr-Wold device imply that the convergence of the finite-dimensional distributions of the centered and renormalized version of $M_\rho$ is equivalent to the convergence of the one-dimensional distributions. To do so, we will explicitely compute the limits of the characteristic functions, denoting $\varphi_X$ for the characteristic function of a random variable $X$.

Observe that the characteristic function of $n(\rho)^{-1}(M_\rho(\mu) - E[M_\rho(\mu)])$ rewrites:

$$\varphi_{n(\rho)^{-1}(M_\rho(\delta_t) - E[M_\rho(\delta_t)])}(\theta) = \exp\left(\int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} \Psi(n(\rho)^{-1}\theta m \mu(B(x,r))) \lambda(\rho) dx F_\rho(dr) G(dm)\right)$$

where $\Psi(u) = e^{iu} - 1 - iu$, see [Kal]. Integrating with respect to the probability $G(dm)$, we have

$$\varphi_{n(\rho)^{-1}(M_\rho(\mu) - E[M_\rho(\mu)])}(\theta) = \exp\left(\int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda(\rho) \Psi_G(n(\rho)^{-1}\theta \mu(B(x,r))) dx F_\rho(dr)\right)$$

(10)

where $\Psi_G(u) = \int_{\mathbb{R}} \Psi(mu) G(dm)$. We recall that also the characteristic function of the stable distribution $S_\alpha(\sigma, b, \tau)$ is given by $\exp(-\sigma^\alpha |x|^\alpha (1 - i\varepsilon(\theta) \tan(\pi\alpha/2)) + i\tau \theta)$.

3.1 Some technical lemmas

We collect here some useful lemmas that will be needed in the proof of our limit theorems 2.2, 2.3 and 2.4. We shall use the following estimate for the characteristic function of such distribution (une ref ???):

**Lemma 3.1** Suppose $X$ is in the domain of attraction of a $\alpha$-stable law $S_\alpha(\sigma, b, 0)$ for some $\alpha > 1$. Then

$$\varphi_X(\theta) - 1 - i\theta E[X] \sim_0 -\sigma^\alpha |\theta|^\alpha (1 - i\varepsilon(\theta) \tan(\pi\alpha/2))b).$$

Furthermore, there is some $C > 0$ such that for any $\theta \in \mathbb{R}$,

$$|\varphi_X(\theta) - 1 - i\theta E[X]| \leq C|\theta|^\alpha.$$

The following lemma is a reformulation from lemma 2.4 in [BEK]. It shows that in the scaling limit $\rho \to 0^{-\epsilon}$, the behaviour of $F_\rho$ is given by the power tail of $F$. This amounts to be crucial in several estimates.

**Lemma 3.2** Let $F$ be as in (1) and $\epsilon = \pm 1$. Assume that $g$ is a continuous function on $\mathbb{R}^+$ such that for some $0 < p < \beta < q$, there exists some $C > 0$ such that

$$|g(r)| \leq C(r^p \wedge r^q).$$

(11)
Assume furthermore that $g_\rho$ is a family of continuous function such that
\[
\lim_{\rho \to 0^-} |g(r) - g_\rho(r)| = 0 \quad \text{and} \quad |g(r) - g_\rho(r)| \leq C(r^p \wedge r^q).
\tag{12}
\]
Then
\[
\int_{\mathbb{R}^+} g_\rho(r) F_\rho(dr) - \int_{\mathbb{R}^+} g(r) F_\rho(dr) \sim C_\beta \rho^\beta \int_{\mathbb{R}^+} g(r)r^{-1-\beta}dr \quad \text{when} \quad \rho \to 0^-.
\]
In the proof of Theorem 2.2 and of Theorem 2.3 below, this lemma will be used in the particular case where $g_\rho = g$ when $g$ satisfying condition (11). Roughly speaking, the proof of Lemma 3.2 consists in taking the limit in the integral. This is authorized by the dominated convergence theorem under (11) and (12). We refer to [BEK] for more details.

3.2 Proof of Theorem 2.1

**Zoom-out case.** Let $\mu = \sum_{i=1}^n a_i \delta_{y_i} \in \mathcal{M}_{\text{dis}}$, where the $y_i$'s are the distinct atoms of the measure $\mu$. From (10), the characteristic function of $n_0(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)])$ rewrites:
\[
\varphi_{n_0(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)])}(\theta) = \exp \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G \left( \theta n_0(\rho)^{-1} \sum_{i=1}^n a_i 1_{B(y_i, \rho r)}(x) \right) \lambda(\rho) dF(dr) \right).
\]
Consider the functions
\[
g_\rho(r) = \int_{\mathbb{R}^d} \Psi_G \left( \theta n_0(\rho)^{-1} \sum_{i=1}^n a_i 1_{B(y_i, \rho r)}(x) \right) \lambda(\rho) dx.
\]
Since the $y_i$'s are distinct, $\Delta = \min\{d(y_i, y_j); i \neq j\} > 0$. Then for fixed $r$ and $\rho$ small enough, the balls $B(y_i, \rho r)$ are disjoint and
\[
g_\rho(r) = \sum_{i=1}^n \Psi_G \left( \theta n_0(\rho)^{-1} a_i \right) \lambda(\rho) c_d r^d.
\]
Using the definition of $n_0(\rho)$ and Lemma 3.1, we observe that
\[
\lim_{\rho \to 0} g_\rho(r) = -c_d r^d \sigma^\alpha \sum_{i=1}^n |a_i| \alpha \left( 1 - \varepsilon(a_i \theta) \tan(\pi \alpha / 2) b \right).
\tag{13}
\]
Furthermore, the following domination condition is satisfied: there is some $0 < C < +\infty$ such that for any $\rho > 0$,
\[
|g_\rho(r)| \leq C \int_{\mathbb{R}^d} \left| \theta n_0(\rho)^{-1} \sum_{i=1}^n a_i 1_{B(y_i, \rho r)}(x) \right|^\alpha \lambda(\rho) dx
\leq C n_0(\rho)^{-\alpha} \lambda(\rho) \sum_{i=1}^n |a_i|^\alpha |\theta|^\alpha c_d r^d
\leq C r^d.
\]

since \( \int_{\mathbb{R}^+} r^d F(dr) < +\infty \). Lebesgue’s convergence theorem applies and gives

\[
\lim_{\rho \to 0} \int_{\mathbb{R}^+} g_\rho(r) F(dr) = -\sigma \alpha \sum_{i=1}^{n} |a_i|^\alpha (1 - \varepsilon (a_i \theta) \tan(\pi \alpha / 2) b).
\]

We finally get

\[
\lim_{\rho \to 0} \varphi_{n_0(\rho)^{-1}, M(\mu) - E[M(\mu)]}(\theta) = \prod_{i=1}^{n} \exp \left( -\sigma \alpha \sum_{i=1}^{n} |a_i|^\alpha (1 - \varepsilon (\theta \sum_{i=1}^{n} a_i) \tan(\pi \alpha / 2) b) \right).
\]

This proves the claim in the zoom-out case.

**Zoom-in case.** The proof follows the same arguments as previously in the zoom-out case. The only difference is that for \( r > 0 \) and as \( \rho \to +\infty \),

\[
\lim_{\rho \to +\infty} g_\rho(r) = -c_d r^d \sigma \alpha \sum_{i=1}^{n} |a_i|^\alpha (1 - \varepsilon (\theta \sum_{i=1}^{n} a_i) \tan(\pi \alpha / 2) b).
\] (14)

Indeed, note that

\[
g_\rho(r) = \sum_{\emptyset \subset \bar{J} \subseteq \{1, \ldots, n\}} \Psi_G \left( \theta n_0(\rho)^{-1} \sum_{j \in \bar{J}} \alpha_j \right) \lambda(\rho) \text{Vol} \left( \bigcap_{j \in \bar{J}} B(y_j, \rho r) \cap \bigcap_{j \notin \bar{J}} B(y_j, \rho r)^c \right).
\]

The term corresponding to \( \bar{J} = \{1, \ldots, n\} \) yields the dominant contribution since

\[
\text{Vol} \left( \bigcap_{1 \leq i \leq n} B(y_i, \rho r) \right) \sim c_d \rho^d r^d, \quad \rho \to +\infty.
\]

The other terms are of smaller order since if \( j_1 \in \bar{J} \) and \( j_2 \notin \bar{J} \),

\[
\text{Vol} \left( \bigcap_{j \in \bar{J}} B(y_j, \rho r) \cap \bigcap_{j \notin \bar{J}} B(y_j, \rho r)^c \right) \leq \text{Vol} \left( B(y_{j_1}, \rho r) \cap B(y_{j_2}, \rho r)^c \right) \leq C(\rho r)^{d-1}.
\]

Using (14) instead of (13), we derive like in the zoom-out case

\[
\lim_{\rho \to +\infty} \varphi_{n_0(\rho)^{-1}, M(\mu) - E[M(\mu)]}(\theta) = \exp \left( -\sigma \alpha \sum_{i=1}^{n} |a_i|^\alpha (1 - \varepsilon (\theta \sum_{i=1}^{n} a_i) \tan(\pi \alpha / 2) b) \right)
\]

and this concludes the proof. \( \square \)

### 3.3 Proof of Propositions 2.1 and 2.1

**Proof of Proposition 2.1. Proof of i).** If (5) holds true for \( \mu_1 \) with \( p_1 < \beta < q_1 \) and for \( \mu_2 \) with \( p_2 < \beta < q_2 \), let \( p = p_1 \lor p_2 < \beta \) and \( q = q_1 \land q_2 > \beta \). Then (5) holds
true for $\mu_1$ and $\mu_2$ with $p < \beta < q$ (possibly with a different constant $C$); so that for all $a_1, a_2 \in \mathbb{R}$:

$$\int_{\mathbb{R}^d} |(a_1\mu_1 + a_2\mu_2)(B(x,r))|^{\alpha} dx = \|(a_1\mu_1 + a_2\mu_2)(B(x,r))\|_\alpha^\alpha$$

$$\leq (|a_1|\|\mu_1(B(x,r))\|_\alpha + |a_2|\|\mu_2(B(x,r))\|_\alpha)^\alpha$$

$$\leq ((|a_1|^{\alpha}C(r^p \wedge r^q))^{1/\alpha} + (|a_2|^{\alpha}C(r^p \wedge r^q))^{1/\alpha})^\alpha$$

$$= C(|a_1| + |a_2|)^{\alpha}r^p \wedge r^q.$$ 

**Proof of ii).** Since $(\tau_s \mu)(B(x,r)) = \mu(B(x-s,r))$, $(\theta \mu)(B(x,r)) = \mu(B(\Theta^{-1}x,r))$, $\mu_\alpha(B(x,r)) = \mu(B(a^{-1}x,a^{-1}r))$, the closeness of $\mathcal{M}_{\alpha,\beta}$ by translations $\tau_s$, by rotations $\Theta$ and by dilatations $x \mapsto ax$ follow straightforwardly from the invariance of the Lebesgue measure by translations, by rotation, and by an immediate change of variable in (5).

**Proof of iii).** Since $|\mu|(\mathbb{R}^d) < +\infty$, for $\mu \in \mathcal{M}_{\alpha,\beta}$, we have

$$\int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha - \alpha'} dx = \int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha} \mu(B(x,r))^{\alpha'} dx$$

$$\leq |\mu|(\mathbb{R}^d)^{\alpha - \alpha'} \int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha'} dx$$

$$\leq C|\mu|(\mathbb{R}^d)(r^p \wedge r^q).$$

**Proof of iv).** Let $\beta > d$, and $\mu \in \mathcal{M}_{\alpha,\beta}$. We prove that $\mu$ is diffuse. Indeed, suppose that $\mu$ has an atom $a$, then for small $r$, $\gamma(r) \geq |\mu(a)|/2|a|^{d\alpha}$. To see this, let $\varepsilon > 0$ be such that $|\mu(B(a,\varepsilon)) - |\mu(a)|| < |\mu(a)|/2$. Then, for every $r < \varepsilon/2$ and $x \in B(a,r)$, $|\mu(B(x,r))| \geq |\mu(a)|/2$. Integrating on $x \in B(a,r)$, we get $\gamma(r) \geq (|\mu(a)|/2)^{d\alpha}$. This is in contradiction with (5) which rewrites $\gamma(r) \leq Cr^d$ for $q > \beta > d$ when $r$ is small.

**Proof of v).** Let $\beta \leq d$ and $\mu \in \mathcal{M}_{\alpha,\beta}$. We prove that $\mu$ is centered. Indeed, when $r$ is large, $\gamma(r) \geq |\mu(\mathbb{R}^d)|/3|a|^{d\alpha}$. To see this, suppose that $\mu(\mathbb{R}^d) \neq 0$, and let $M$ be such that $|\mu|(B(0,M)) \leq |\mu(\mathbb{R}^d)|/3$. Then, for $r \geq M$ and any $x \in B(0,r-M)$, $B(0,M) \subset B(x,r)$ and $|\mu(B(x,r))| \geq |\mu(\mathbb{R}^d)| - |\mu(B(x,r))| \geq 2|\mu(\mathbb{R}^d)|/3$. Integrating on $x \in B(0,r-M)$ yields $\gamma(r) \geq (2|\mu(\mathbb{R}^d)|/3)^{d\alpha}r^d$. This implies the claim and is in contradiction with (5) which rewrites $\gamma(r) \leq Cr^p$ for $p < \beta < d$ when $r$ is large.

**Proof of Proposition 2.1. Proof of i).** We derive asymptotics both when $r$ is large and when $r$ is small for $\gamma(r) = \int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha} dx$. First, when $\mu$ has a density
\[ \int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha} dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} 1_{\{|x-y|<r\}} f(y) dy \right|^{\alpha} dx \]

\[ \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} 1_{\{|x-y|<r\}} |f(y)|^{\alpha} dy \right) \left( \int_{\mathbb{R}^d} 1_{\{|x-y|<r\}} dy \right)^{\alpha-1} dx \]

\[ = (c_d r^d)^{\alpha-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{\{|x-y|<r\}} |f(y)|^{\alpha} dy dx \]

\[ = (c_d r^d)^{\alpha-1} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} 1_{\{|x-y|<r\}} dx \right) |f(y)|^{\alpha} dy \]

\[ = (c_d r^d)^{\alpha} \int_{\mathbb{R}^d} |f(y)|^{\alpha} dy \]  

(15)

where we applied Hölder’s inequality with \( \alpha > 1 \). But under the condition i), \( \int_{\mathbb{R}^d} |f(y)|^{\alpha} dy < +\infty \) and we derive for some \( 0 < C < +\infty : \)

\[ \int_{\mathbb{R}^d} |\mu(B(x,r))|^{\alpha} dx \leq C r^{\alpha d}. \]  

(16)

Next, still under the condition i), there are some finite positive \( A \) and \( M \) such that for any \( z \in B(0, A)^c \), \( |f(z)| \leq M|z|^{-\eta} \). This implies that for any \( x \in B(0, A + r)^c \),

\[ |\mu(B(x,r))| \leq \int_{\mathbb{R}^d} 1_{\{|z|<r\}} f(x + z) |dz| \leq M c_d r^d \bigl(|x| - r\bigr)^{-\eta} \]

Let

\[ \theta \in \left( \frac{\alpha d - \beta}{\alpha \eta - d}, \frac{\beta}{d} \right) \cap (1, +\infty) \]  

(17)

Note that the first interval above is not degenerated since \( \eta > d^2 / \beta \) and the intersection in not empty since \( \beta > d \). Since \( \theta > 1 \), we have first

\[ \int_{|x| \leq r^\theta + r} |\mu(B(x,r))|^{\alpha} dx \leq (|\mu(\mathbb{R}^d)|)^{\alpha} c_d (r^\theta + r)^d = O(r^{\theta d}). \]  

(18)

Next, since for \( r \) large enough, \( r^\theta > A \), we have also:

\[ \int_{|x| \geq r^\theta + r} |\mu(B(x,r))|^{\alpha} dx \]

\[ \leq M_\alpha c_d r^{\alpha d} \int_{|x| \geq r^\theta + r} (|x| - r)^{-\alpha \eta} dx \]

\[ = M_\alpha c_d r^{\alpha d} \int_{\rho > r^\theta} \rho^{-\alpha \eta} c_d^{-1}(\rho + r)^{d-1} d\rho \]

\[ = M_\alpha c_d c_d^{-1} r^{\alpha d} \sum_{k=0}^{d-1} \binom{d-1}{k} (\alpha \eta - k - 1)^{-1} r^{d-1-k+\theta(k+1-\alpha \eta)} \]  

(19)

where the convergence of the integral is ensured by the condition \( \eta > d > d / \alpha \). We derive \( \gamma(r) = O(r^{d \theta (d + (\alpha d + (\theta(d - \alpha \eta))) \eta)} \) as \( r \to +\infty \). Observe that \( \theta d \vee (\alpha d + \theta(d - \alpha \eta)) < \beta \) because of the choice of \( \theta \) in (17).
Finally, we have derived the condition (5) in this case with $p = \theta d \vee (\alpha d + \theta(d - \alpha \eta)) < \beta$ and $q = \alpha d > \beta$.

**Proof of ii).** Note that the bound (15) still applies and also gives (16) when $f$ has a density given by condition v). Next, still under the condition v), there are some finite positive $A$ and $M$ such that for any $z \in B(0, A)^c$, $|f(z)| \leq M|z|^{-\eta}$. This implies that for any $R \geq A$

$$|\mu|(B(0, R)^c) = \int_{\mathbb{R}^d} 1_{|z| > R}|f(z)|dz \leq M c_{d-1}(\eta - d)^{-1} R^{d-\eta}. \quad (20)$$

Now let $r > A$. For any $x \in B(0, A + r)^c$, $B(x, r) \subset B(0, |x| - r)^c$ so that equation (20) with $R = |x| - r > A$ yields

$$|\mu(B(x, r))| \leq |\mu|(B(0, |x| - r)^c) \leq M c_{d-1}(\eta - d)^{-1}(|x| - r)^{d-\eta}$$

and hence,

$$\int_{|x| \geq A + r} |\mu(B(x, r))|^\alpha dx$$

$$\leq C \int_{|x| \geq A + r} (|x| - r)^{\alpha(d-\eta)} dx$$

$$= C \int_{\rho \geq A} \rho^{\alpha(d-\eta)}(\rho + r)^{d-1} d\rho$$

$$= C \sum_{k=0}^{d-1} \binom{d-1}{k} \rho^k(\alpha(d-\eta) + d - k)^{-1} \left[ \rho^{\alpha(d-\eta)+d-k} \right]_{\rho=A}^{\rho=\infty}$$

$$= O(r^{d-1}) \text{ as } r \to +\infty. \quad (21)$$

Note that the condition $\eta > d(1 + \alpha)/\alpha$ ensures that the integrals above are well-defined. In the same way, for any $x \in B(0, r - A)$, $B(x, r)^c \subset B(0, r - |x|)^c$ so that equation (20) with $R = r - |x| > A$ yields

$$|\mu(B(x, r)^c)| \leq |\mu|(B(0, r - |x|)^c) \leq M c_{d-1}(\eta - d)^{-1}(r - |x|)^{d-\eta},$$

and, since $\mu$ is centered

$$|\mu(B(x, r))| = |\mu(B(x, r)^c)| \leq M c_{d-1}(\eta - d)^{-1}(r - |x|)^{d-\eta}.$$

Integrating with respect to $x \in B(0, r - A)$, we get

$$\int_{|x| < r - A} |\mu(B(x, r))|^\alpha dx$$

$$\leq C \int_{|x| < r - A} (r - |x|)^{\alpha(d-\eta)} dx$$

$$= C \int_{\rho=A}^{r} \rho^{\alpha(d-\eta)}(\rho - r)^{d-1} d\rho$$

$$= O(r^{d-1}) \text{ as } r \to +\infty. \quad (22)$$
Finally, when \(|x| \in [r - A, r + A]\), we can use \(|\mu(B(x, r))| \leq |\mu|([R^d])\) and get
\[
\int_{r - A < |x| < r + A} |\mu(B(x, r))|^\alpha dx \leq (|\mu|([R^d]))^\alpha c_d \left((r + A)^d - (r - A)^d\right) = O(r^{d-1})
\]
It follows from these three estimates that \(\gamma(r) = O(r^{d-1})\) as \(r \to +\infty\). This together with equation (16) ensures that \(\mu\) satisfies equation (5) with the choice \(p = d - 1\) and \(q = \alpha d\).

Alternatively, if \(f\) has a finite support \(\{a_1, \ldots, a_p\}\), let \(\delta > 0\) such that for \(1 \leq i \leq p\), \(B(a_i, \delta) \cap \text{Supp}(\mu) = \{a_i\}\). For \(r < \delta/2\),
\[
\gamma(r) = \int_{\mathbb{R}^d} |\mu(B(x, r))|\alpha dx
= \sum_{i=1}^{p} \int_{B(a_i, r)} |\mu(B(x, r))|\alpha dx
= \sum_{i=1}^{p} \int_{B(a_i, r)} |\mu(a_i)|\alpha dx
= c_d \sum_{i=1}^{p} |\mu(a_i)|\alpha r^d = O(r^d).
\]
Next, let \(M\) be such that \(\mu(B(0, M)^c) = 0\) and note that \(\mu(B(x, r)) = 0\) when \(B(x, r) \cap B(0, M) = \emptyset\) or when \(B(0, M) \subset B(x, r)\) since \(\mu([R^d]) = 0\). We derive \(\mu(B(x, r)) = 0\) when \(||x|| \leq r - M\) or when \(||x|| \geq M + r\). Since \(\mu\) is a finite measure, we have
\[
\gamma(r) = \int_{r - M \leq ||x|| \leq r + M} |\mu(B(x, r))|\alpha dx
\leq c_d ((r + M)^d - (r - M)^d) (|\mu([R^d])|^\alpha
= O(r^{d-1}), \quad r \to +\infty.
\]
Together with (23), this yields equation (5) with \(p = d - 1 < \beta\) and \(q = d > \beta\).  

**Remark 3.1 (On the bound for large radii)** Note that in order to derive the bound \(\gamma(r) \leq r^p\) for \(p < \beta\) when \(r\) is large, the existence of a density for \(\mu\) is not required. We can instead suppose that \(\mu\) satisfy some tail condition : for some \(\tilde{\eta} > d/\alpha\)
\[
|\mu|(B(0, R)^c) = O(R^{-\tilde{\eta}}) \text{ as } R \to +\infty.
\]
This appears clearly in the proof of vii) with \(\tilde{\eta} = \eta - d\). An analogous proof would also work for vi) with the tail condition \(\tilde{\eta} > \alpha\) corresponding to \(\eta > d(1 + \alpha)/\alpha\). However, in our proof, the existence of a density allows us to weaken the tail condition and we can consider \(\eta > d\).
3.4 Proof of Theorem 2.2

The characteristic function of the stable integral $Z_\alpha(\mu)$ is given by

$$
\varphi_{Z_\alpha(\mu)}(\theta) = \exp \left( -C_\beta \sigma^\alpha \int_{\mathbb{R}^d \times \mathbb{R}^+} |\theta \mu(B(x, r))|^\alpha (1 - i \varepsilon(\theta \mu(B(x, r)))) \tan(\pi \alpha/2) b r^{-1-\beta} dr dx \right)
$$

(24)

Since the characteristic function of the Poisson integral $n_1(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)])$ is given by (10), comparing (24) and (10), it is sufficient to show that

$$
\lim_{\rho \to 0^-} \int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda(\rho) \Psi_G \left( n_1(\rho)^{-1} \theta \mu(B(x, r)) \right) dx F_\rho(d\rho)
$$

is given by

$$
\int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda(\rho) \Psi_G \left( n_1(\rho)^{-1} \theta \mu(B(x, r)) \right) dx F_\rho(d\rho)
$$

and can be integrated. This yields

$$
\lim_{\rho \to 0^-} \int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda(\rho) \Psi_G \left( n_1(\rho)^{-1} \theta \mu(B(x, r)) \right) dx F_\rho(d\rho)
$$

is given by

$$
\int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda(\rho) \Psi_G \left( n_1(\rho)^{-1} \theta \mu(B(x, r)) \right) dx F_\rho(d\rho)
$$

Finally, Lemma 3.2 applies with $g(r) = \int_{\mathbb{R}^d} |\mu(B(x, r))|^\alpha (1 - i \varepsilon(\theta \mu(B(x, r)))) \tan(\pi \alpha/2) b dx$, condition (11) for $g$ being given by (5) for $\mu \in \mathcal{M}_{\alpha,\beta}$. Consequently,

$$
\int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x, r))|^\alpha (1 - i \varepsilon(\theta \mu(B(x, r)))) \tan(\pi \alpha/2) b dx F_\rho(d\rho)
$$

is given by

$$
C_\beta \rho^\beta \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x, r))|^\alpha (1 - i \varepsilon(\theta \mu(B(x, r)))) \tan(\pi \alpha/2) b r^{-\beta-1} dr.
$$

Finally, (26) and (27) together imply (25), and as explained at the beginning of Section 3, it is sufficient to prove Theorem 2.2. \qed

3.5 Proof of Condition (8)

We prove that Condition (8) for the existence of $J$ is satisfied. Note that this condition splits in two:

$$
\int_{|m_\mu(B(x, r))| \leq 1} (m_\mu(B(x, r)))^2 r^{-\beta-1} dr G(dm) < +\infty
$$

(28)
and
\[
\int_{|m(B(x,r))| \geq 1} |m(\mu(B(x,r)))| r^{-\beta-1} dx dr G(dm) < +\infty.
\] (29)

We shall use the following Lemma for the truncated moments of a distribution in the normal domain attraction of a stable law:

**Lemma 3.3** Let \( G \) be in the normal domain attraction of a \( \alpha \)-stable law for \( \alpha > 1 \). There are \( 0 < C_1, C_2 < +\infty \) such that for all \( x \geq 0 \):

\[
\int_{|m| \geq x} |m| G(dm) \leq C_1 x^{1-\alpha} \quad \text{and} \quad \int_{-x}^{x} m^2 G(dm) \leq C_2 x^{2-\alpha}.
\]

**Proof of the lemma.** From [Fel, XVII.5], we have \( \int_{-x}^{x} m^2 G(dm) \sim x^{2-\alpha} \) when \( x \to +\infty \) (note that since \( G \) is in the normal domain of attraction, there is no slowly varying function in this estimate). But since moreover for \( x \in [0,1] \)

\[
\int_{-x}^{x} m^2 G(dm) = \int_{0}^{x^2} G(m: u \leq m^2 \leq x^2) dx \leq x^2 \leq x^{2-\alpha}
\]

the second part is proved. Next, since \( \lim_{x \to 0} \int_{|m| > x} |m| G(dm) = \int_{\mathbb{R}} |m| G(dm) < +\infty \) while \( x^{1-\alpha} \to +\infty, x \to 0 \), the first part comes from [Fel, Eq. (5.21)]:

\[
\int_{|m| > x} |m| G(dm) \sim \frac{2-\alpha}{\alpha-1} \int_{-x}^{x} m^2 G(dm) \sim \frac{2-\alpha}{\alpha-1} x^{1-\alpha}, \quad x \to +\infty.
\]

\( \square \)

Now, for (28), we have:

\[
\int_{|m(\mu(B(x,r)))| \leq 1} (m(\mu(B(x,r))))^2 r^{-\beta-1} dx dr G(dm)
\]
\[
\leq \int_{\mathbb{R}^d \times \mathbb{R}^+} \left( \int_{|m(B(x,r))|}^{r+1/|m(B(x,r))|} m^2 G(dm) \right) \mu(B(x,r))^2 r^{-\beta-1} dx dr
\]
\[
\leq C_2 \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x,r))|^{2-\alpha} \mu(B(x,r))^2 r^{-\beta-1} dx dr
\]
\[
\leq C_2 \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x,r))|^{\alpha} r^{-\beta-1} dx dr
\]
which is finite when \( \mu \in \mathcal{M}_{\beta,\alpha} \). For (29), we have:

\[
\int_{|m(\mu(B(x,r)))| \geq 1} |m\mu(B(x,r)))| r^{-\beta-1} dx dr G(dm)
\]
\[
\leq \int_{\mathbb{R}^d \times \mathbb{R}^+} \left( \int_{|m| > 1/|\mu(B(x,r))|} |m| G(dm) \right) |\mu(B(x,r))| r^{-\beta-1} dx dr
\]
\[
\leq C_1 \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x,r))|^{1+\alpha} |\mu(B(x,r))| r^{-\beta-1} dx dr
\]
\[
\leq C_1 \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x,r))|^{\alpha} r^{-\beta-1} dx dr
\]
which, again, is finite when \( \mu \in \mathcal{M}_{\beta,\alpha} \). \( \square \)
3.6 Proof of Theorem 2.3

As in the proof of Theorem 2.2, it is enough to consider convergence of one-dimensional marginals. The characteristic function of the Poisson integral \( M_\mu(B(x, r)) = \mathbb{E}[M_\mu(B(x, r))] \) is given by (10) and that of the generalized random field \( J(\mu) \) is given by

\[
\varphi_{J(\mu)}(\theta) = \exp \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi(\theta m\mu(B(a^{-1}x, a^{-1}r)))C_\beta r^{-1-\beta} dr dx G(dm) \right)
\]

\[
= \exp \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G(\theta \mu(B(x, r)))C_\beta a^{d-\beta} r^{-1-\beta} dr dx \right).
\]

From Lemma 3.1 (confirmer quand non symétrique ??), \( |\Psi_G(\theta \mu(B(x, r)))| \leq C|\theta|^{\alpha} |\mu(B(x, r))|^{\alpha} \) for some \( C > 0 \), so that condition (11) for \( g(r) = \int_{\mathbb{R}^d} \Psi_G(\mu(B(x, r))) dx \) is given again by (5) when \( \mu \in M_{\alpha, \beta} \). Thus, Lemma 3.2 applies and together with \( \lim_{\rho \to 0^-} \lambda(\rho)\rho^\beta = a^{d-\beta} \), entail

\[
\lim_{\rho \to 0^-} \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G(\theta \mu(B(x, r))) dx \lambda(\rho) F_\rho(dr) = C_\beta a^{d-\beta} \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G(\theta \mu(B(x, r))) r^{-1-\beta} dr dx.
\]

Since one-dimensional convergence is enough, this achieves the proof of Theorem 2.3.

\[\square\]

3.7 Proof of Proposition 2.5

We consider the subsequence \( a_m = m^{1/(d-\beta)} \). From the aggregate-similarity of the field \( J \) (see (9) in Proposition 2.4), we have:

\[
\frac{1}{a_m^{(d-\beta)/\alpha}} J(\mu_{a_m}) \overset{fdd}{=} \frac{1}{m^{1/\alpha}} \sum_{i=1}^{m} J_i(\mu)
\]

for independent copies \( J_i, 1 \leq i \leq m, \) of \( J \). But

\[
\varphi_{m^{-1/\alpha} \sum_{i=1}^{m} J_i(\mu)}(\theta) = \left( \varphi_{J(\mu)}(m^{-1/\alpha} \theta) \right)^m
\]

\[
= \exp \left( m \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G(m^{-1/\alpha} \theta \mu(B(x, r)))C_\beta r^{-1-\beta} dr dx \right).
\]

But from Lemma 3.1,

\[
\Psi_G(m^{-1/\alpha} \theta \mu(B(x, r))) \sim \sigma^{\alpha} |\theta|^{\alpha} |\mu(B(x, r))|^{\alpha} (1 - i\epsilon \theta \mu(B(x, r)) \tan(\pi\alpha/2)b).
\]

The relation above is uniform in \( x \) and \( r \) and it is thus integrable, yielding

\[
\lim_{m \to +\infty} \varphi_{m^{-1/\alpha} \sum_{i=1}^{m} J_i(\mu)}(\theta) = \exp \left( C_\beta \sigma^{\alpha} |\theta|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu(B(x, r))|^{\alpha} (1 - i\epsilon \theta \mu(B(x, r)) \tan(\pi\alpha/2)b) r^{-1-\beta} dr dx \right)
\]

A standard argument completes the proof of convergence in distribution along an arbitrary sequences. \[\square\]
3.8 Proof of Theorem 2.4

We follow the argument in the proof of Theorem 2 in [KLNS]. Recall that here \( d < \beta < \alpha d \) so that \( \epsilon = -1 \) and the limits are taken with \( \rho \to 0 \). Again, by linearity, using the Cramér-Wold device, it is enough to deal with one-dimensional marginals. From (10) with a change of variable, the characteristic function writes

\[
\varphi_{n_2(\rho)^{-1}(M_\rho(\mu)-\mathbb{E}[M_\rho(\mu)])}(\theta) = \exp \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G \left( \theta n_2(\rho)^{-1} \mu(B(x, n_2(\rho)^{1/d}r)) \right) \lambda(\rho) dx F_{\rho n_2(\rho)^{-1/d}}(dr) \right).
\]

Let \( \mu(dz) = \phi(z)dz \) with \( \phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d) \), then, from Lemma 4 in [KLNS], as \( n_2(\rho) \to 0 \),

\[
n_2(\rho)^{-1} \mu(B(x, n_2(\rho)^{1/d}r)) \to \phi(x)c_d r^d
dx\text{-almost everywhere and}
\]

\[
x \mapsto \phi^*(x) = \sup_{v > 0} \left\{ \frac{c_d^{-1}v^{-d}}{\mu(B(x, v))} \right\} \in L^\alpha(\mathbb{R}^d).
\]

As a consequence,

\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G \left( \theta n_2(\rho)^{-1} \mu(B(x, n_2(\rho)^{1/d}r)) \right) \lambda(\rho) dx F_{\rho n_2(\rho)^{-1/d}}(dr)
\sim C_\beta \lambda(\rho)^{\beta} n_2(\rho)^{-\beta/d} \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G \left( \theta \phi(x)c_d r^d \right) r^{-\beta-1} dr dx. \tag{30}
\]

To see this, apply Lemma 3.2 to

\[
g(r) = \int_{\mathbb{R}^d} \Psi_G \left( \theta \phi(x)c_d r^d \right) dx
\]

and to

\[
g_\rho(r) = \int_{\mathbb{R}^d} \Psi_G \left( \theta n_2(\rho)^{-1} \mu(B(x, n_2(\rho)^{1/d}r)) \right) dx.
\]

Since \( |\Psi_G(u)| \leq C(|u| \wedge |u|^\alpha) \),

\[
|g(r)| \leq C \min \left( c_d \theta \| \phi \|_{L^1} r^d, c_d \theta \| \phi \|_{L^\alpha} r^{\alpha d} \right)
\]

so that condition (11) is satisfied with \( p = d \) and \( q = \alpha d \). Furthermore, since \( \Psi_G \) is a \( K \)-Lipschitzian function for some finite \( K \), we get

\[
|g(r) - g_\rho(r)| \leq K c_d r^d |\theta| r^-d n_2(\rho)^{-1} \mu(B(x, n_2(\rho)^{1/d}r)) - \phi(x)| dx.
\]

The integrand \( |c_d^{-1}r^{-d}n_2(\rho)^{-1} \mu(B(x, n_2(\rho)^{1/d}r)) - \phi(x)| \) converges to zero \( dx\)-almost everywhere. Since its \( L^\alpha \)-norm is bounded by \( \| \phi^* \|_{L^\alpha} + \| \phi \|_{L^\alpha} \), it is uniformly integrable and as a consequence,

\[
\lim_{\rho \to 0} \int_{\mathbb{R}^d} |c_d^{-1}r^{-d}n_2(\rho)^{-1} \mu(B(x, n_2(\rho)^{1/d}r)) - \phi(x)| dx = 0.
\]
On the other hand, for some $C > 0$, $|\Psi_G(v)| \leq C|v|^\alpha$, we obtain
\[ |g(r) - g_\rho(r)| \leq C(\|\phi^*\|_{L^\alpha} + \|\phi\|_{L^\alpha})r^{\alpha d}. \]
Hence, $g_\rho$ satisfy condition (12) with $p = d$ and $q = \alpha d$. This proves (30).

From the definition of $n_2(\rho)$, $\lambda(\rho)\rho^3n_2(\rho)^{-\beta/d} = 1$. Furthermore, by splitting the integration over $\mathbb{R}^d$ into $\{x \in \mathbb{R}^d : \theta \phi(x) \geq 0\}$ and $\{x \in \mathbb{R}^d : \theta \phi(x) < 0\}$ and performing a change of variable, we have
\[ \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G(\theta \phi(x)c_dr^d) r^{-\beta-1}drdx = D \int_{\mathbb{R}^d} (\theta \phi(x))_+^\gamma dx + \bar{D} \int_{\mathbb{R}^d} (\theta \phi(x))_-^\gamma dx, \]
where $\bar{D}$ is the complex conjugate of $D = d^{-1}\epsilon_d^\gamma \int_{\mathbb{R}^+} \Psi_G(r)r^{-\gamma-1}dr$. We deduce
\[ \varphi_{n_2(\rho)^{-1}(M_\rho(\mu) - E[M_\rho(\mu)])}(\theta) = \exp\left(-\sigma^\gamma_\theta |\theta|^\gamma (1 + ib_\phi \varepsilon(\theta) \tan(\frac{\pi\gamma}{2}))\right) \]
where
\[ \sigma^\gamma_\phi = \sigma^\gamma_\phi \int_{\mathbb{R}^d} |\phi(x)|^\gamma dx, \]
and
\[ b_\phi = \frac{\int_{\mathbb{R}^+} r^{1-\gamma}(r - \sin(r))dr}{\tan(\pi\gamma/2)} \frac{\int_{\mathbb{R}^+} \varepsilon(m)|m|^\gamma G(dm) \int_{\mathbb{R}^d} \varepsilon(\phi(x))|\phi(x)|^\gamma dx}{\int_{\mathbb{R}^d} |\phi(x)|^\gamma dx}. \]
But since for $\gamma \in (1, 2)$,
\[ \int_0^{+\infty} e^{ixu} - 1 - ixu \frac{dx}{x^{1+\gamma}} = |u|^\gamma \frac{\Gamma(2 - \gamma)}{(1 - \gamma)(2 - \gamma)} (\cos(\pi\gamma/2) - i\varepsilon(u) \sin(\pi\gamma/2)) \]
see Lemma 2 in [Fel, XVII.4] (with $p = 1, q = 0$ therein), the first ratio in the right-hand side (31) is $-1$ and we have $b_\phi = b_\gamma \frac{\int_{\mathbb{R}^d} \varepsilon(\phi(x))|\phi(x)|^\gamma dx}{\int_{\mathbb{R}^d} |\phi(x)|^\gamma dx}$. This achieves the proof of Theorem 2.4.

\[ \square \]

**Brouillon: le cas $\beta > \alpha d$**

**Theorem 3.1** Let $\beta > \alpha d$ and suppose that equation 2 is satisfied with $\varepsilon = -1$ (or more generally let $F$ be any distribution such that $\int_{\mathbb{R}^+} r^{\alpha d}F(dr) < \infty$). Suppose that $n_2(\rho) := \lambda(\rho)^{1/\alpha} \rho^d \to +\infty$ as $\rho \to 0$. Then,
\[ \frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n_3(\rho)} \overset{L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)}{\to} \tilde{Z}_\alpha(\cdot) \]
where, for $\mu(dx) = \phi(x)dx$, $\tilde{Z}_\alpha(\mu) = \int_{\mathbb{R}^d} \phi(x) \tilde{M}_\alpha(dx)$ is a stable integral with respect to the $\alpha$-stable measure $M_\alpha$ with control measure $\sigma_\alpha dx$ with
\[ \sigma_\alpha = \sigma c_d \left(\int_{\mathbb{R}^+} r^{\alpha d}F(dr)\right)^{1/\alpha} \]
and constant skewness equal to $b$. 

22
Proof:
The proof uses the same tools as the proof of Theorem 2.4 and we only give here the main lines. From (10) and a change of variable, the characteristic function writes

\[
\varphi n_3(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)])(\theta) = \exp \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G (\theta n_3(\rho)^{-1}\mu(B(x, pr))) \lambda(\rho)dx F(dr) \right).
\]

Let \( \mu(dz) = \phi(z)dz \) with \( \phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d) \). Then, as \( n_3(\rho) \to \infty \), and \( \rho \to 0 \)

\[
\lambda(\rho)\Psi_G (\theta n_3(\rho)^{-1}\mu(B(x, pr))) \sim -\lambda(\rho)\sigma^\alpha |\theta| \sigma n_3(\rho)^{-\alpha} |\mu(B(x, pr)|^\alpha (1 - \varepsilon(\theta\mu(B,x,pr)) \tan(\pi\alpha/2))b
\]

\[
= -\sigma^\alpha c_d^\alpha |\theta| |r|^\alpha |c_d^{-1}(pr)^{-d}\mu(B(x, pr)|^\alpha (1 - \varepsilon(\theta\mu(B,x,pr)) \tan(\pi\alpha/2))b
\]

\[
\to -\sigma^\alpha c_d^\alpha |\theta| |r|^\alpha |\phi(x)|^\alpha (1 - \varepsilon(\theta\phi(x)) \tan(\pi\alpha/2))b
\]

\( dx \)-almost everywhere, and this last function is integrable with respect to \( dx F(dr) \)
since \( \phi \in L^\alpha(\mathbb{R}^d) \) and \( \int_{\mathbb{R}^d} |r|^\alpha F(dr) < +\infty \). Furthermore, domination is proved thanks to the maximal function \( \phi^* \) as in the proof of Theorem 2.4: for some \( C > 0 \),

\[
|\lambda(\rho)\Psi_G (\theta n_3(\rho)^{-1}\mu(B(x, pr)))| \leq \lambda(\rho)Cn_3(\rho)^{-\alpha} |\mu(B(x, pr)|^\alpha \leq Cr^\alpha |\phi^*(x)|^\alpha.
\]

This majoration is independent of \( \rho \) and integrable with respect to \( dx F(dr) \) since \( \phi^* \in L^\alpha(\mathbb{R}^d) \). Lebesgue’s Theorem yields the convergence as \( \rho \to 0 \)

\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} \Psi_G (\theta n_3(\rho)^{-1}\mu(B(x, pr))) \lambda(\rho)dx F(dr) \to -\sigma^\alpha c_d^\alpha |\theta| |r|^\alpha \int_{\mathbb{R}^d} |r|^\alpha F(dr) \int_{\mathbb{R}^d} |\phi(x)|^\alpha (1 - \varepsilon(\theta\phi(x)) \tan(\pi\alpha/2))b dx.
\]

This implies Theorem 3.1 \( \square \)

References


