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Convergence in variation of the joint laws of multiple Wiener–Itô integrals

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Abstract

The convergence in variation of the laws of multiple Wiener–Itô integrals with respect to their kernel has been studied by Davydov and Martynova in [1987. Limit behavior of multiple stochastic integral. Statistics and Control of Random Process (Preila, 1987), Nauka, Moscow, pp. 55–57 (in Russian)]. Here, we generalize this convergence for the joint laws of multiple Wiener–Itô integrals. In this case, the argument relies on superstructure method which consists in studying related functionals along admissible directions for a Gaussian process.

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Keywords: Convergence in variation; Superstructure method; Wiener–Itô integrals

1. Introduction

We are interested in this paper in the behavior for convergence in variation of the joint laws of multiple Wiener–Itô integrals. The original motivation for such a study comes from the study of the regularity of the law of d-multiple Wiener–Itô integrals $I_d(f)$ given by Davydov (1991): if $f \not\equiv 0$, then the law of $I_d(f)$ is absolutely continuous with respect to the Lebesgue measure λ. Note that this result of absolute continuity was first proved by Shigekawa (1980) (see also Kusuoka (1983) for another proof). The proof of Davydov is based on the stratification method. This method has also been used by Davydov and Martynova (1987) to derive the continuity (for the topology of total variation on the space of signed measures) of the laws of integrals $I_d(f)$ with respect to the kernel $f \in L^2(T^d)$. That is:

Theorem A. *Let $f \in L^2(T^d)$ be symmetric and not zero, then there is $C := C(d,f)$ such that for any sequence of symmetric functions $f_n \in L^2(T^d)$ converging to f:*

$$\|\mathscr{L}(I_d(f)) - \mathscr{L}(I_d(f_n))\| \leqslant C\|f - f_n\|_{L^2(T^d)}^{1/d}. \tag{1}$$

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Here, and in the sequel, $\mathscr{L}(X)$ is the law of a random variable or of a random vector X, $\|\mu\|$ is the total variation of a signed measure μ, and $\xrightarrow{\text{var}}$ stands for the convergence in total variation.

Note that when the limit law has a density, the convergence in variation (1) is equivalent to the convergence in $L^1(\mathbb{R})$ of the densities of these laws.

Theorem A is the starting point for the study in this note. Here, we deal with the multi-dimensional counterpart of Theorem A, that is the continuity for total variation norm of joint laws of multiple Wiener–Itô integrals with respect to their kernels. The difficulties lie in the fact that the stratification method used in Davydov and Martynova (1987) for Theorem A relies on one-dimensional estimates that are no more available in the multi-dimensional setting. Instead, we shall use another argument, based on the superstructure method, which is less sensitive to dimension. It consists in studying the restrictions of related functionals along admissible directions for an underlying Gaussian process. For a complete account on this method, we refer to Davydov et al. (1998). However, the drawback of this approach is the loss of the control (1) of the convergence in variation in this joint law case. It would be interesting to generalize to the multi-dimensional setting the tools used in Davydov and Martynova (1987) in order to recover the control (1). But the main point in Davydov and Martynova (1987) for the control of the variation (1) is the Lemma p. 56 about the distance in variation of the images of a normal law by one-dimensional polynomials. The generalization of this point in our context is a problem we do not succeed to overcome at this time.

Another way to extend Theorem A would be to consider other types of stochastic integrals. Since Gaussian law is a particular case of stable law, the first natural generalization to think about is multiple stable integrals. This has been done for one-dimensional law in Breton (2004). Therein, the argument relies on Breton (2002) instead of Davydov (1991), Breton (2002) states the absolute continuity of the law of multiple stable integrals.

Another easy extension of Theorem A deals with Wiener functionals. Indeed using the chaotic expansion of square integrable Wiener functionals in multiple Wiener–Itô integrals, we can transfer in some cases the convergence in variation from the integrals to the functionals. This is briefly discussed at the end of this introduction.

Since multiple Wiener–Itô integral is the basic object in this paper, we start with a short account of the facts we will need about it. Next, we describe in Section 2 the setting and define the notations we shall need to state the generalization of Theorem A. Section 3 is devoted to the proof of Theorem 4.

Let us consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a measured space $(T, \mathscr{T}, \tau)$ satisfying the following continuity property: for all $A \in \mathscr{T}$ such that $\tau(A) < +\infty$ and for all $\varepsilon > 0$ there exists $B_1, \ldots, B_N$ such that

$$A = \bigcup_{j=1}^{N} B_j, \quad \tau(B_j) < \varepsilon, \ \forall j = 1, \ldots, N.$$

Let $\mathscr{A} = \{A \in \mathscr{T} \,|\, \tau(A) < +\infty\}$ and $\{W(A), A \in \mathscr{A}\}$ be the Gaussian orthogonal measure with control measure τ, that is, the Gaussian process on $(\Omega, \mathscr{F}, \mathbb{P})$ for which:

$$\forall A, B \in \mathscr{A}, \quad E[W(A)] = 0 \quad \text{and} \quad E[W(A)W(B)] = \tau(A \cap B).$$

In the sequel, we consider $\mathscr{M} = \sigma(W(A), A \in \mathscr{A})$ and we shall assume that the space $L^2(\Omega, \mathscr{M}, \mathbb{P})$ is separable. For $d \in \mathbb{N}\backslash\{0\}$, we introduce H_d the Hilbert space of functions $f : T^d \to \mathbb{R}$ for which

$$\|f\|_{H_d}^2 = \int_{T^d} |f(t)|^2 \, \mathrm{d}\tau(t) < +\infty.$$

Let $\langle \cdot, \cdot \rangle_{H_d}$ be the relative inner product and H_0 the space of constants. Let K_d be the subspace of H_d consisting of functions f which are invariant under permutations of the coordinates, that is, functions f such that

$$f(t) = f(S_d(t)) = \frac{1}{d!} \sum_{\sigma \in \Pi_d} f(\sigma(t)),$$

where $\sigma(t) = (t_{\sigma(1)}, \ldots, t_{\sigma(d)})$ and Π_d is the permutation group of d elements. Finally, $\widehat{H}_d$ denotes the set of the so-called simple functions, that is those $f \in H_d$ for which there exists a finite system of sets $\Delta_j \in \mathscr{A}$, $j = 1, \ldots, N$, which are pairwise disjoint and such that f is constant on each $\Delta_{j_1} \times \cdots \times \Delta_{j_d}$.

For such $f \in \widehat{H}_d$, we define the multiple integral $I_d(f)$ by the formula

$$I_d(f) = \sum_{(j_1,\dots,j_d)} f(t_{j_1},\dots,t_{j_d})W(\Delta_{j_1})\cdots W(\Delta_{j_d}),$$

where $t_m \in \Delta_m$ for each m. It is easy to see that the following properties are fulfilled for $f, g \in \widehat{H}_d$:

1. $I_d(f) = I_d(S_d(f))$.
2. $E[I_d(f)I_{d'}(g)] = \mathbf{1}_{\{d=d'\}}\langle S_d(f), S_d(g)\rangle_{H_d}$.

In particular, we have $E[I_d(f)] = 0$ and $E[I_d(f)^2] \leqslant d!\|f\|^2_{H_d}$. Since $\widehat{H}_d$ is dense in H_d, I_d extends to H_d and we shall write

$$I_d(f) = \int_{T^d} f(t_1,\dots,t_d)\,\mathrm{d}W(t_1)\dots\mathrm{d}W(t_d)$$

for all $f \in H_d$, keeping the previous properties 1 and 2 for I_d on H_d.

As an easy generalization of Theorem A, we derive a result of convergence in variation for the laws of square integrable Wiener functionals F exhibiting in their chaotic expansion

$$F = E[F] + \sum_{d=1}^{+\infty} I_d(f_d), \quad f_d \in K_d \tag{2}$$

some summand independent of the remainder. More precisely, we have:

Proposition 1. *Let $k \in \mathbb{N}\backslash\{0\}$ and $(F_n)_n$ be a sequence of Wiener functionals whose kth summand $I_k(f^n_k)$ in its chaotic expansion* (2) *is independent of the other summands $I_d(f^n_d)$, $d \neq k$. If $F_n \to F$ in $L^2(\Omega, \mathscr{M}, \mathbb{P})$ with $f_k \neq 0$ in the expansion* (2) *of F, then*

$$\mathscr{L}(F_n) \overset{\text{var}}{\longrightarrow} \mathscr{L}(F), \quad n \to +\infty. \tag{3}$$

The law of F_n is the convolution of the law of $I_k(f^n_k)$ and of the law of the remainder $\sum_{d\neq k} I_d(f^n_d)$. The same holds true for the law of F. Since $\mathscr{L}(I_k(f^n_k)) \overset{\text{var}}{\longrightarrow} \mathscr{L}(I_k(f_k))$ when $n \to +\infty$ and $\mathscr{L}(I_k(f_k)) \ll \lambda_k$, Proposition 1 is an easy consequence of Theorem A of Davydov (1991, Theorem 1) and of the strong convergence of convolutions given in the following lemma, which is a corollary of a proposition due to Parthasarathy and Steerneman (1985, Theorem 2.1):

Lemma 2. *Let $(X_n, Y_n), n \geqslant 1$, be a sequence of random vectors in $\mathbb{R}^2$ such that for every n, Y_n is independent of X_n, $X_n \overset{\text{var}}{\longrightarrow} X$ and Y_n converges in law to Y. Then $X_n + Y_n \overset{\text{var}}{\longrightarrow} X + Y$, provided the law of X is absolutely continuous with respect to the Lebesgue measure λ.*

Moreover, a rate of strong convergence for convolutions has been given by Davydov (1997, Theorem 1). Note $\pi(v_1, v_2) = \inf\{\varepsilon > 0 | v_1(A) \leqslant v_2(A^\varepsilon) + \varepsilon, \forall A \in \mathscr{B}(\mathbb{R})\}$ the Prokhorov distance between probability measures v_1 and v_2 (here A^ε stands for the ε-neighbourhood of A) and $w_{\mu_1}(t) = \sup_{|x|\leqslant t}\|\mu_1 - \mu_1 T_x^{-1}\|_{\text{var}}$ the modulus of smoothness of μ_1 (where $T_x(y) = x + y$). We have:

Lemma 3. *Let μ_1, μ_2, v_1, v_2 be probability measures. Then*

$$\|\mu_1 * v_1 - \mu_2 * v_2\|_{\text{var}} \leqslant \|\mu_1 - \mu_2\|_{\text{var}} + \pi(v_1, v_2) + w_{\mu_1}(\pi(v_1, v_2)). \tag{4}$$

Using this result, we estimate the rate of convergence in (3): under the hypothesis of Proposition 1, note

$$\mu_n = \mathscr{L}(I_k(f^n_k)), \quad \mu = \mathscr{L}(I_k(f_k)), \quad v_n = \mathscr{L}\left(\sum_{d\neq k} I_d(f^n_d)\right), \quad v = \mathscr{L}\left(\sum_{d\neq k} I_d(f_d)\right).$$

First, from (1): $\|\mu_n - \mu\|_{\mathrm{var}} \leqslant C\|f_k^n - f_k\|_{L^2(T^k)}^{1/k} \leqslant C\|F_n - F\|_{L^2(\Omega)}^{1/k}$. Next using Markov inequality and property 2 of multiple Wiener–Itô integrals:

$$\pi(v_n, v) \leqslant \left\| \sum_{d \neq k} I_d(f_d^n) - \sum_{d \neq k} I_d(f_d) \right\|_{L^2(\Omega)}^{2/3} \leqslant \left\| \sum_{d \neq k} I_d(f_d^n - f_d) \right\|_{L^2(\Omega)}^{2/3}$$
$$\leqslant \left(\sum_{d \neq k} \|f_d^n - f_d\|_{L^2(T^d)} \right)^{2/3} \leqslant \|F_n - F\|_{L^2(\Omega)}^{2/3}.$$

Finally, from (4) we deduce the following rate of convergence in (3):

$$\|\mathscr{L}(F_n) - \mathscr{L}(F)\|_{\mathrm{var}} \leqslant C\|F_n - F\|_{L^2(\Omega)}^{1/k} + \|F_n - F\|_{L^2(\Omega)}^{2/3} + w_{\mathscr{L}(I_k(f_k))}(\|F_n - F\|_{L^2(\Omega)}^{2/3})$$
$$\leqslant (1 + C)\|F_n - F\|_{L^2(\Omega)}^{1/k} + w_{\mathscr{L}(I_k(f_k))}(\|F_n - F\|_{L^2(\Omega)}^{2/3}),$$

when n is large enough. In order to improve this rate of convergence, we need to estimate $w_{\mathscr{L}(I_k(f_k))}$ like in Davydov (1997, Theorem 2), but this requires further information on the density of $I_k(f_k)$.

2. Setting and main result

In this section, let be given $d_1, \ldots, d_p \in \mathbb{N}\backslash\{0\}$ the dimensions of p multiple Wiener–Itô integrals. We are interested in the multi-dimensional laws of vectors $(I_{d_1}(f_1), \ldots, I_{d_p}(f_p))$ where $f_i \in H_{d_i}$, $1 \leqslant i \leqslant p$. Our goal is to derive the continuity for variation norm of these multi-dimensional laws with respect to the integrands $f_1 \in H_{d_1}, \ldots, f_p \in H_{d_p}$. We need first to introduce some notations in order to state the condition for this continuity. We point out that it corresponds in fact to the sufficient condition given in Davydov (1991) for the regularity of the multi-dimensional limit law $(I_{d_1}(f_1), \ldots, I_{d_p}(f_p))$. In the sequel, bold letters are used for multi-dimensional quantities. Define:

- for $i = 1, \ldots, p$, $N_i = d_1 + \cdots + d_i$, $N = N_p$;
- $\mathbf{a}^i = (a_0^i, \ldots, a_p^i) \in \mathbb{N}^{p+1}$ a $(p+1)$-partition of $d_i : d_i = |\mathbf{a}^i| = a_0^i + \cdots + a_p^i$;
- $\mathbf{a} = (\mathbf{a}^1, \ldots, \mathbf{a}^p) \in (\mathbb{N}^{p+1})^p$;
- $M_{\mathbf{a}} = (a_j^i)_{1 \leqslant i,j \leqslant p}$ a p-square matrix such that $d_i = \sum_{k=0}^p a_k^i$ for $1 \leqslant i \leqslant p$ and $b_k = \sum_{i=1}^p a_k^i$ for $0 \leqslant k \leqslant p$;
- for $\mathbf{b} = (b_1, \ldots, b_p) \in \mathbb{N}^p$ (and $b_0 = 0$):

$$E(\mathbf{b}) = \left\{ \mathbf{a} = (\mathbf{a}^1, \ldots, \mathbf{a}^p) | a_1^i + \cdots + a_p^i = d_i, \sum_{i=1}^p a_k^i = b_k, k = 1, \ldots, p \right\},$$

  in particular, for $\mathbf{a} \in E(\mathbf{b})$, note that we have $a_0^i = 0$ for all $1 \leqslant i \leqslant p$;
- for $\mathbf{a} \in E(\mathbf{b})$, let $\sigma_{\mathbf{a}} \in \Pi_N$ the permutation of $\{1, \ldots, N\}$ that sends

$$j = \sum_{u=1}^{k-1} b_u + \sum_{s=1}^{i-1} a_k^s + l, \quad l = 1, \ldots, a_k^i,$$

  to

$$\sigma_{\mathbf{a}}(j) = \sum_{v=1}^{i-1} d_v + \sum_{s=1}^{k-1} a_s^i + l; \tag{5}$$

- for $\mathbf{a} \in E(\mathbf{b})$, let $U_{\mathbf{a}} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be associated with $\sigma_{\mathbf{a}}$ by $U_{\mathbf{a}}(t_1, \ldots, t_N) = (t_{\sigma_{\mathbf{a}}(1)}, \ldots, t_{\sigma_{\mathbf{a}}(N)})$;
- $\phi(\mathbf{t}) = f_1(t_1, \ldots, t_{N_1}) \cdots f_p(t_{N_{p-1}+1}, \ldots, t_{N_p})$, where $\mathbf{t} = (t_1, \ldots, t_N)$;
- $\phi_{\mathbf{b}}(\mathbf{t}) = \sum_{\mathbf{a} \in E(\mathbf{b})} \prod_{i=1}^p \frac{d_i!}{a_0^i! \cdots a_p^i!} \det M_{\mathbf{a}} \phi(U_{\mathbf{a}}(\mathbf{t}))$, where $\mathbf{t} = (t_1, \ldots, t_N)$;

- for $\mathbf{b} = (b_1, \ldots, b_p) \in \mathbb{N}^p$ denoting $\Pi_{\mathbf{b}}$ the sub-group of Π_N of permutations preserving the following so-called "**b**-blocks": $(1, \ldots, b_1)$, $(b_1 + 1, \ldots, b_1 + b_2), \ldots, (b_1 + b_2 + \cdots b_{p-1} + 1, \ldots, b_1 + b_2 + \cdots + b_p = N)$:

$$S_{\mathbf{b}}\phi(\mathbf{t}) = \frac{b_1! \cdots b_p!}{N!} \sum_{\sigma \in \Pi_{b_1,\ldots,b_p}} \phi(t_{\sigma(1)}, \ldots, t_{\sigma(N)}).$$

We can then state our generalization of Theorem A:

Theorem 4. *Let $(f_1^n, \ldots, f_p^n)_{n\in\mathbb{N}}$ be a sequence of elements of $H_{d_1} \times \cdots \times H_{d_p}$ converging to $(f_1, \ldots, f_p)$. If at least for one multi-index $\mathbf{b} = (b_1, \ldots, b_p)$ such that $\sum_{i=1}^p b_i = \sum_{i=1}^p d_i$, we have $S_{\mathbf{b}}\phi_{\mathbf{b}} \neq 0$, then the following convergence holds true when $n \to +\infty$*

$$\mathscr{L}(I_{d_1}(f_1^n), \ldots, I_{d_p}(f_p^n)) \xrightarrow{\text{var}} \mathscr{L}(I_{d_1}(f_1), \ldots, I_{d_p}(f_p)). \tag{6}$$

Remark 5.

- The algebraic condition "$S_{\mathbf{b}}\phi_{\mathbf{b}} \neq 0$ for some $\mathbf{b}$" is mainly required to ensure the non-degeneracy of the limit law $\mathscr{L}(I_{d_1}(f_1), \ldots, I_{d_p}(f_p))$. This is due to Theorem 5 in Davydov (1991). It is difficult to interpret this condition; but, roughly speaking, it deals in a sense with how overlapped are the f_i's.
- However, this algebraic condition is also required in order to apply the forthcoming Proposition 6 in the proof we propose. We can thus not replace this condition by the weaker one $\mathscr{L}(I_{d_1}(f_1), \ldots, I_{d_p}(f_p)) \ll \lambda^p$.
- In fact, the convergence in (6) can fail to hold if the limit law is degenerated. Indeed it is thus possible to choose $(f_1^n, \ldots, f_p^n)$ converging to $(f_1, \ldots, f_p)$ with the function $S_{\mathbf{b}}\phi_{\mathbf{b}}^n$ relative to $(f_1^n, \ldots, f_p^n)$ being not 0 while $S_{\mathbf{b}}\phi_{\mathbf{b}} = 0$.
  In this case, if (6) holds, the sequence of the laws $(\mathscr{L}(I_{d_1}(f_1^n), \ldots, I_{d_p}(f_p^n)))_n$ is in particular a Cauchy sequence in the Banach space of signed measures equipped with the norm of variation. But since $S_{\mathbf{b}}\phi_{\mathbf{b}}^n \neq 0$, the laws $\mathscr{L}(I_{d_1}(f_1^n), \ldots, I_{d_p}(f_p^n))$ have densities and the sequence of the densities is also a Cauchy sequence in $L^1(\mathbb{R}^p)$ and thus must converge to a limit density. But this is in contradiction with the convergence in variation of $\mathscr{L}(I_{d_1}(f_1^n), \ldots, I_{d_p}(f_p^n))$ to a degenerated law, without density.
- Note that since the limit law is not degenerated, the convergence in (6) rewrites also as the convergence of the densities of the joint laws in $L^1(\mathbb{R}^p)$. This is thus also a local limit result.

In order to apply Theorem 4 to the multiple Wiener–Itô integrals of functions $f_1, \ldots, f_p$, we have to check its intricate hypothesis "$S_{\mathbf{b}}\phi_{\mathbf{b}} \neq 0$ for some $\mathbf{b}$". This condition is of the same nature as that in Breton (2005) (in a stable context) and we refer to Breton (2005) for a long illustration in several cases. Here, we only briefly describe some particular setting where it is satisfied.

(1) *Case* $p = 1$, $d_1 = 1$ *with* $\mathbf{b} = 1$: We have $E(\mathbf{b}) = \{1\}$ and $\sigma_1 = id$. The condition is satisfied if $f \neq 0$ a.e. recovering a well-known result for Gaussian law.

(2) *Case* $p = 1$, $d_1 = d > 1$ *with* $\mathbf{b} = d$: We have $E(\mathbf{b}) = \{d\}$ and $\sigma_d = id$. The condition of Theorem 4 is satisfied if $f \neq 0$ a.e. recovering here the convergence part of the one-dimensional result of Davydov and Martynova (Theorem A).

(3) *Case* $p > 1$, $d_1 = \cdots = d_p = 1$ *with* $\mathbf{b} = (1, \ldots, 1)$: The condition is satisfied if

$$\det\{f_i(t_j)\} \neq 0 \quad \text{a.e.}$$

Note that if this sufficient condition does not hold, the limit law $(I_{d_1}(f_1), \ldots, I_{d_p}(f_p))$ may be degenerated (take for instance $p = 3$, $d_1 = d_2 = d_3 = 1$ and $f_3 = f_1 + f_2$).

(4) *Case* $p = 2$, $d_1 = d_2 = 2$ *with* $\mathbf{b} = (2, 2)$ (taken from Davydov, 1991): The condition of Theorem 4 is satisfied if there are no reals c_1, c_2 such that

$$c_1 f_1 = c_2 f_2 \quad \text{a.e.}$$

If this is not the case, the limit law is once more degenerated.

(5) *Case* $p=2$, $d_1=1$, $d_2=d$ *with* $\mathbf{b}=(1,d)$: The condition of Theorem 4 holds in this case if

$$f_1(t_1)f_2(t_2,t_3,\ldots,t_{d+1}) \not\equiv \frac{1}{d}\sum_{i=2}^{d+1} f_1(t_i)\underbrace{f_2(t_2,\ldots,t_{d+1})}_{\substack{\text{with } t_1 \text{ in} \\ i\text{th position}}} \quad \text{a.e.}$$

Moreover, note that at least for examples (1)–(4), when the conditions do not hold, the limit law is degenerated and the convergence in variation in (6) fails to hold in general. In fact these simple examples suggest the sufficient condition "$S_{\mathbf{b}}\phi_{\mathbf{b}} \neq 0$ for some $\mathbf{b}$" is close to the necessary condition "the limit law is not degenerated" for the convergence (6).

Other examples of explicit sufficient conditions for Theorem 4 can be derived, the technical nature of the computations is related to the dimension of the joint laws. This is left to the brave-hearted reader.

3. Proof of Theorem 4

3.1. Setting

We will make an extensive use of the framework exposed in Davydov (1991) with the setting yet described in the Introduction. Like in Proposition 1 in Davydov (1991), we see multiple integrals as functionals on sample path. To this way, let $\mathscr{X}=\mathbb{R}^{\mathscr{A}}$ and $\mathscr{B}(\mathscr{X})=\mathscr{B}^A$ (where $\mathscr{B}$ is the real Borel σ-algebra). We denote $\mathscr{W}$ the mapping from Ω into $\mathscr{X}$ which maps $\omega\in\Omega$ to the corresponding path of the process W. Let $P:=\mathbb{P}\mathscr{W}^{-1}$ be the distribution of $\mathscr{W}$ on $(\mathscr{X},\mathscr{B}(\mathscr{X}))$. Proposition 1 in Davydov (1991) states that for any $f\in H_d$ there exists a measurable mapping $F:\mathscr{X}\to\mathbb{R}$ such that for $\mathbb{P}$-almost all ω:

$$I_d(f(\omega)) = F(\mathscr{W}(\omega)) \tag{7}$$

(first, it is clear for simple functions f and (7) is then easily generalized for any $f\in H_d$). In the sequel, we denote the value of F at $x\in\mathscr{X}$ by $F(x)=\int f\,\mathrm{d}_d x$. Note that, the index d in "$\mathrm{d}_d x$" is related to the space H_d in which f lives.

It is well known also that the admissible shifts of the Gaussian measure P are given by functions $v_h:\mathscr{A}\to\mathbb{R}$ for $h\in H_1$ defined by

$$v_h(A)=\int_A h\,\mathrm{d}\tau,\quad A\in\mathscr{A},$$

see Proposition 2 in Davydov (1991). Moreover,

$$\frac{\mathrm{d}\widehat{P}_h}{\mathrm{d}P}(x)=\exp\left\{\int h\,\mathrm{d}_1 x-\frac{1}{2}\int h^2\,\mathrm{d}\tau\right\},$$

where $\widehat{P}_h$ stands for the distribution of the translated process $W+v_h$.

3.2. Superstructure

We shall prove the convergence in variation in (6) applying the superstructure method in a multi-dimensional setting to functionals related to the kernels $f_1^n,\ldots,f_p^n,f_1,\ldots,f_p$. We refer to Davydov et al. (1998, Section 5) for the description of this method in a one-dimensional setting.

First, let $F_1,\ldots,F_p$ be defined from $f_1,\ldots,f_p$ like in (7) and $F_1^n,\ldots,F_p^n$ similarly defined from $f_1^n,\ldots,f_p^n$. We consider in the sequel $F=(F_1,\ldots,F_p)$ and $F_n=(F_1^n,\ldots,F_p^n)$. The joint laws in (6) can be seen from (7) as the following image-measures:

$$\mathscr{L}(I_{d_1}(f_1),\ldots,I_{d_p}(f_p))=PF^{-1},$$

$$\mathscr{L}(I_{d_1}(f_1^n),\ldots,I_{d_p}(f_p^n))=PF_n^{-1}.$$

The superstructure method is suitable to study such image-measures PF^{-1} for some functionals F defined on $\mathcal{X}$. It consists to introduce a product space $[0,\varepsilon]^p \times \mathcal{X}$ and to twist the product space by a family of transformations $\{G_{\mathbf{c}}\}_{\mathbf{c}}$ adapted to P. The keystone is then to see PF^{-1} as a mixture of (finite dimensional) conditional measures along the orbit of $\{G_{\mathbf{c}}\}_{\mathbf{c}}$. It remains then to study the image-measures with finite-dimensional tools.

More precisely, let for $i = 1, \dots, p$, $h_i \in H_1$ and let $v_i = v_{h_i}$ be p fixed admissible shifts. Consider the following family of transformations of the space $\mathcal{X}$:

$$G_{\mathbf{c}}(x) = x + c_1 v_1 + \cdots + c_p v_p, \quad \mathbf{c} = (c_1, \dots, c_p).$$

This family of transformations is adapted to P in the following sense:

$$PG_{\mathbf{c}}^{-1} \xrightarrow{\text{var}} P, \quad \mathbf{c} \to 0. \tag{8}$$

Indeed, observe that $PG_{\mathbf{c}}^{-1} = \widehat{P}_{\mathbf{c}}$, where $\widehat{P}_{\mathbf{c}}$ stands for the law of the shift process $W + c_1 v_1 + \cdots + c_p v_p = W + v_{c_1 h_1 + \cdots + c_p h_p}$. Moreover, the density of $\widehat{P}_{\mathbf{c}}$ with respect to P is given by

$$\begin{aligned}\frac{\mathrm{d}\widehat{P}_{\mathbf{c}}}{\mathrm{d}P}(x) &= \exp\left\{\int \sum_{i=1}^{p} c_i h_i \,\mathrm{d}_1 x - \frac{1}{2}\int \left(\sum_{i=1}^{p} c_i h_i\right)^2 \mathrm{d}\tau\right\} \\ &= \exp\left\{\sum_{i=1}^{p} c_i \int h_i \,\mathrm{d}_1 x - \sum_{i,j=1}^{p} \frac{c_i c_j}{2} \int h_i h_j \,\mathrm{d}\tau\right\}.\end{aligned}$$

Then, when $\mathbf{c} \to 0$, we have $(\mathrm{d}\widehat{P}_{\mathbf{c}}/\mathrm{d}P)(x) \to 1$ for any $x \in \mathcal{X}$. Next, Scheffé's lemma yields (8).

We define the following auxiliary measures and functionals on the product space $\mathcal{X}_\varepsilon = [0,\varepsilon]^p \times \mathcal{X}$, $\varepsilon > 0$:

$$Q_\varepsilon = \frac{1}{\varepsilon^p} \lambda_{[0,\varepsilon]^p} \times P,$$

$$F_\varepsilon(\mathbf{c}, x) = F(G_{\mathbf{c}}(x)),$$

$$F_{n,\varepsilon}(\mathbf{c}, x) = F_n(G_{\mathbf{c}}(x)).$$

Note that F_ε and $F_{n,\varepsilon}$ depend on ε only through their domain $\mathcal{X}_\varepsilon$. Since $Q_\varepsilon F_{n,\varepsilon}^{-1} = \int_{[0,\varepsilon]^p} PG_c^{-1} F_n^{-1} \,\mathrm{d}\mathbf{c}$, we have the following estimate for the total variation:

$$\|Q_\varepsilon F_{n,\varepsilon}^{-1} - PF_n^{-1}\| \leqslant \frac{1}{\varepsilon^p} \int_{[0,\varepsilon]^p} \|P - PG_{\mathbf{c}}^{-1}\| \,\mathrm{d}\mathbf{c}.$$

Since $\|P - PG_{\mathbf{c}}^{-1}\| \leqslant 2$, from dominated convergence and from (8), we derive

$$\lim_{\varepsilon \to 0} \|Q_\varepsilon F_{n,\varepsilon}^{-1} - PF_n^{-1}\| = 0, \tag{9}$$

where the limit holds uniformly with respect to $n \in \mathbb{N}$. The same limit holds true for the functional F:

$$\|Q_\varepsilon F_\varepsilon^{-1} - PF^{-1}\| \leqslant \frac{1}{\varepsilon^p} \int_{[0,\varepsilon]^p} \|P - PG_{\mathbf{c}}^{-1}\| \,\mathrm{d}\mathbf{c} \longrightarrow 0, \quad \varepsilon \to 0. \tag{10}$$

Next, we express $Q_{n,\varepsilon} F_\varepsilon^{-1}$ as a mixture of finite-dimensional measures as follows: note, for $\mathbf{c} = (c_1, \dots, c_p) \in [0,\varepsilon]^p$, $\varphi_{n,x}(\mathbf{c}) = F_n(G_{\mathbf{c}}(x))$ the restriction of F_n over strata $\{x + c_1 v_1 + \cdots + c_p v_p\}_{\mathbf{c}}$, we have

$$Q_\varepsilon F_{n,\varepsilon}^{-1} = \frac{1}{\varepsilon^p} \int_{\mathcal{X}} \lambda_{[0,\varepsilon]^p} \varphi_{n,x}^{-1} \,\mathrm{d}P. \tag{11}$$

Introducing similarly $\varphi_x(c) = F(x + c_1 v_1 + \cdots + c_p v_p)$, we express also $Q_\varepsilon F_\varepsilon^{-1}$ as

$$Q_\varepsilon F_\varepsilon^{-1} = \frac{1}{\varepsilon^p} \int_{\mathcal{X}} \lambda_{[0,\varepsilon]^p} \varphi_x^{-1} \,\mathrm{d}P. \tag{12}$$

Now, we have

$$\|PF^{-1} - PF_n^{-1}\| \leqslant \|PF^{-1} - Q_\varepsilon F_\varepsilon^{-1}\| + \|Q_\varepsilon F_\varepsilon^{-1} - Q_\varepsilon F_{n,\varepsilon}^{-1}\| + \|Q_\varepsilon F_{n,\varepsilon}^{-1} - PF_n^{-1}\|. \tag{13}$$

From (9) and (10), we deduce that the first and third terms in (13) can be chosen arbitrarily small. Consequently, it remains to deal with the term in the middle of the right-hand side of the previous inequality (13) when $n \to +\infty$ for some $\varepsilon > 0$ arbitrary fixed. But from (11) and (12), we have

$$\|Q_\varepsilon F_\varepsilon^{-1} - Q_\varepsilon F_{n,\varepsilon}^{-1}\| = \frac{1}{\varepsilon^p} \int_{\mathscr{X}} \|\lambda_{[0,\varepsilon]^p} \varphi_x^{-1} - \lambda_{[0,\varepsilon]^p} \varphi_{n,x}^{-1}\| \, \mathrm{d}P.$$

Since $\|\lambda_{[0,\varepsilon]^p} \varphi_x^{-1} - \lambda_{[0,\varepsilon]^p} \varphi_{n,x}^{-1}\| \leqslant 2\varepsilon^p$, it is sufficient, using dominated convergence, to prove for P-almost all x that

$$\lim_{n \to +\infty} \|\lambda_{[0,\varepsilon]^p} \varphi_x^{-1} - \lambda_{[0,\varepsilon]^p} \varphi_{n,x}^{-1}\| = 0. \tag{14}$$

3.3. *Convergence in variation of induced measures*

In order to prove (14), we use the following result from Alexandrova et al. (1999). It gives conditions on finite-dimensional functionals for the convergence in variation of their image-measures.

Proposition 6 (*Alexandrova et al., 1999, Corollary 4*). *Let $G_n, G \in W_{\mathrm{loc}}^{q,1}(\mathbb{R}^p, \mathbb{R}^p)$ where $q \geqslant p$ and let the mappings G_n converge to G with respect to the Sobolev norm $\|\cdot\|_{q,1}$ on every ball. Assume that $E \subset \{\det DG \neq 0\}$ is a set of finite Lebesgue measure. Then when $n \to +\infty$,*

$$\lambda_{|E} G_n^{-1} \xrightarrow{\text{var}} \lambda_{|E} G^{-1}.$$

We apply Proposition 6 to the mappings $G_n = \varphi_{n,x}$ and $G = \varphi_x$ on $[0,\varepsilon]^p$. From Davydov (1991, Proposition 4), the ith component $\varphi_{n,x}^i$ of $\varphi_{n,x}$ has the following polynomial expression:

$$\varphi_{n,x}^i(\mathbf{c}) = F_i^n(x + c_1 v_1 + \cdots + c_p v_p) = \sum_{|\mathbf{a}| = d_i} \mathbf{c}^{\mathbf{a}} B_{\mathbf{a}} \int f_i^n \mathrm{d}_{\mathbf{a}} v, \tag{15}$$

where $\mathbf{a} = (a_0, a_1, \ldots, a_p,) \in \mathbb{N}^{p+1}, \mathbf{c} = (1, c_1, \ldots, c_p) \in \mathbb{R}^{p+1}$ and with

$$\mathbf{c}^{\mathbf{a}} = \prod_{m=0}^{p} c_m^{a_m}, \quad |\mathbf{a}| = \sum_{m=0}^{p} a_m, \quad B_{\mathbf{a}} = n! \Bigg/ \prod_{m=0}^{p} (a_m!), \quad v = (x, v_1, \ldots, v_p),$$

$$\int f \mathrm{d}_{\mathbf{a}} v = \int \left(\int f \mathrm{d}_{a_1} v_1 \ldots \mathrm{d}_{a_p} v_p \right) \mathrm{d}_{a_0} x.$$

The same as for (15) holds also for each component φ_x^i of φ_x. Thus, in (14) we deal in fact with image-measures by multi-dimensional polynomial mappings $\varphi_x = (\varphi_x^1, \ldots, \varphi_x^p)$ and $\varphi_{n,x} = (\varphi_{n,x}^1, \ldots, \varphi_{n,x}^p)$. In order to prove the local Sobolev convergence

$$\varphi_{n,x} \longrightarrow \varphi_x \quad \text{in } W_{\mathrm{loc}}^{q,1}(\mathbb{R}^p, \mathbb{R}^p)$$

for some $q \geqslant p$, it is enough to see the convergences of the coefficients of all monomials $\mathbf{c}^{\mathbf{a}} = c_1^{a_1} \ldots c_p^{a_p}$ in (15). By linearity, we study actually the convergence:

$$\int f_i^n \mathrm{d}_{\mathbf{a}} v = \int \left(\int f_i^n \mathrm{d}_{a_1} v_1 \ldots \mathrm{d}_{a_p} v_p \right) \mathrm{d}_{a_0} x \longrightarrow \int \left(\int f_i \mathrm{d}_{a_1} v_1 \ldots \mathrm{d}_{a_p} v_p \right) \mathrm{d}_{a_0} x = \int f_i \mathrm{d}_{\mathbf{a}} v, \quad n \to +\infty. \tag{16}$$

But $\int (\int f_i^n \mathrm{d}_{a_1} v_1 \ldots \mathrm{d}_{a_p} v_p) \, \mathrm{d}_{a_0} x = \int (\int f_i^n h_1^{\otimes a_1} \ldots h_p^{\otimes a_p} \, \mathrm{d}_{d_i - a_0} \tau) \, \mathrm{d}_{a_0} x$ and since $f_i^n \to f_i$ in H_{d_i}, we derive easily the convergence of the inner integrals in $H_{d_i - a_0}$, using Cauchy–Schwarz's inequality and

Fubini's Theorem:

$$\begin{aligned}
&\left\| \int f_i^n h_1^{\otimes a_1} \dots h_p^{\otimes a_p} \mathrm{d}_{d_i-a_0}\tau - \int f_i h_1^{\otimes a_1} \dots h_p^{\otimes a_p} \mathrm{d}_{d_i-a_0}\tau \right\|_{H_{a_0}}^2 \\
&= \int \left(\int (f_i^n - f_i) h_1^{\otimes a_1} \dots h_p^{\otimes a_p} \mathrm{d}_{d_i-a_0}\tau \right)^2 \mathrm{d}_{a_0}\tau \\
&\leqslant \int \left(\left(\int (f_i^n - f_i)^2 \mathrm{d}_{d_i-a_0}\tau \right) \times \left(\int (h_1^{\otimes a_1} \dots h_p^{\otimes a_p})^2 \mathrm{d}_{d_i-a_0}\tau \right) \mathrm{d}_{a_0}\tau \right. \\
&\leqslant \|h_1\|_{H_1}^{2a_1} \dots \|h_p\|_{H_1}^{2a_p} \int \int (f_i^n - f_i)^2 \mathrm{d}_{d_i-a_0}\tau \mathrm{d}_{a_0}\tau \\
&\leqslant \|h_1\|_{H_1}^{2a_1} \dots \|h_p\|_{H_1}^{2a_p} \times \|f_i^n - f_i\|_{H_{d_i}}^2.
\end{aligned}$$

We get in H_{a_0}:

$$\lim_{n\to+\infty} \int f_i^n h_1^{\otimes a_1} \dots h_p^{\otimes a_p} \mathrm{d}_{d_i-a_0}\tau = \int f_i h_1^{\otimes a_1} \dots h_p^{\otimes a_p} \mathrm{d}_{d_i-a_0}\tau. \tag{17}$$

In the sequel, we denote F_g the measurable mapping from $\mathscr{X}$ to $\mathbb{R}$ associated to some $g \in H_d$ like in (7) from Proposition 1 of Davydov (1991) in order to stress on the dependence of F_g on g. We remind also that for any $x \in \mathscr{X}$, $F_g(x)$ stands for $\int g \mathrm{d}_d x$. From the very definition of F_g in the proof of Proposition 1 in Davydov (1991), we have for any $x \in \mathscr{X}$, $F_{g_n}(x) \to F_g(x)$ whenever $g_n \to g$ in H_d. Thus from (17), we derive when $n \to +\infty$:

$$\int \left(\int f_i^n \mathrm{d}_{a_1} v_1 \dots \mathrm{d}_{a_p} v_p \right) \mathrm{d}_{a_0} x = F_{\int f_i^n \mathrm{d}_{a_1} v_1 \dots \mathrm{d}_{a_p} v_p}(x) \longrightarrow F_{\int f_i \mathrm{d}_{a_1} v_1 \dots \mathrm{d}_{a_p} v_p}(x) = \int \left(\int f_i \mathrm{d}_{a_1} v_1 \dots \mathrm{d}_{a_p} v_p \right) \mathrm{d}_{a_0} x.$$

As a consequence, (16) holds and the coefficients of $\varphi_{n,x}^i$ in (15) converge to their analogues for φ_x^i. It is easy to derive henceforth the convergence of $\varphi_{n,x}^i$ to φ_x^i in the Sobolev sense on every ball.

It remains now to prove $\det D\varphi_x(\mathbf{c}) \neq 0$ a.e on $[0, \varepsilon]^p$ in order to apply Proposition 6; that is, to compute the Jacobian of polynomial mapping φ_x. But Davydov (1991, Section 4) proves this Jacobian is not degenerated (and so, as a polynomial, it is also not zero a.e.) if at least for one multi-index $\mathbf{b} = (b_1, \dots, b_p)$ such that $\sum_{i=1}^p b_i = \sum_{i=1}^p d_i$, we have $S_{\mathbf{b}}\phi_{\mathbf{b}} \neq 0$. This condition is the content of the hypothesis of Theorem 4 (and we repeat: the reason for this condition).

Finally, we can apply Proposition 6 and derive

$$\lim_{n\to+\infty} \|\lambda_{[0,\varepsilon]^p} \varphi_{n,x}^{-1} - \lambda_{[0,\varepsilon]^p} \varphi_x^{-1}\| = 0.$$

This allows to conclude, via (13), that

$$\mathscr{L}(I_{d_1}(f_1^n), \dots, I_{d_p}(f_p^n)) \xrightarrow{\text{var}} \mathscr{L}(I_{d_1}(f_1), \dots, I_{d_p}(f_p)), \quad n \to +\infty.$$

This ends the proof of Theorem 4.

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