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LOCAL LIMIT THEOREM FOR THE SUPREMUM OF AN EMPIRICAL PROCESS FOR I.I.D. RANDOM VARIABLES

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Abstract. In this paper, we establish the convergence in total-variation norm of the law of the supremum of an empirical process constructed from a sequence of i.i.d. random variables to the law of the supremum of a (generalized) Brownian bridge.

Keywords: convergence in variation, empirical process, local limit theorem.

INTRODUCTION

Let $(\xi_i)_{i\in\mathbb{N}^*}$ be a sequence of independent identically distributed random variables with rather smooth distribution function F. We consider the empirical process given by

$$\zeta_n^F(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \big[\mathbf{1}_{(-\infty,t]}(\xi_i) - F(t)\big]. \tag{0.1}$$

It is well known that, for uniformly distributed random variables ξ_i, $1 \leqslant i \leqslant n$, the empirical process ζ_n^U weakly converges, as $n \to +\infty$, to a standard Brownian bridge W_U° on [0, 1]. In fact, the same result holds for a sequence of i.i.d. random variables $(\xi_i)_i$ with smooth distribution function F. In this case, the limit process is a generalized Brownian bridge W_F°, i.e., a continuous Gaussian process with independent increments with the covariance function

$$F(t) \wedge F(s) - F(t)F(s).$$

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More precisely, denoting by $\Longrightarrow$ the weak convergence, we have (see [1], Theorem 14.3):

$$\mathcal{L}(\zeta_n^F) \Longrightarrow \mathcal{L}(W_F^0), \quad n \to +\infty, \tag{0.2}$$

where the convergence in (0.2) holds in $\mathbb{D}(\mathbb{R})$ (the space of *cadlag* functions equipped with a Skorokhod-type topology), and $\mathcal{L}(X)$ denotes the distribution of X.

In this paper, we study the behavior of the suprema of empirical processes and their asymptotic distributions. Clearly, (0.2) immediately implies the weak convergence

$$\mathcal{L}\Big(\sup_{t\in\mathbb{R}} \zeta_n^F(t)\Big) \Longrightarrow \mathcal{L}\Big(\sup_{t\in\mathbb{R}} W_F^0(t)\Big), \quad n \to +\infty.$$

Our aim is to strengthen this convergence to the convergence in variation. In the sequel, we denote the latter by $\xrightarrow{var}$; $\|\mu\|$ denotes the (total) variation of a signed measure μ, while μ_A is the restriction of μ to a measurable set A.

Note that the set $\mathcal{Z}(\mathbb{R})$ of all signed measures on $\mathbb{R}$ is a Banach space, provided that it is equipped with the variation norm $\|\cdot\|$. We also denote by $\xrightarrow{\mu}$ the convergence in measure μ.

Since the distribution of the supremum of a Brownian bridge is absolutely continuous (see [6]), the convergence in variation of $\sup_{t\in\mathbb{R}} \zeta_n^F(t)$ is equivalent to the convergence in $L^1(\mathbb{R})$ of the densities of $\sup_{t\in\mathbb{R}} \zeta_n^F(t)$. Thus, this implies a local limit theorem for this distribution.

In the setting of the classical invariance principle of Donsker–Prokhorov (i.e., in the case of linear piecewise processes X_n constructed on the basis of $(\xi_i)_i$), such a stronger convergence in variation $f(X_n) \xrightarrow{var} f(X_\infty)$ is also proved in [5], Section 20, and in [2], [3] for a large class of functionals f for which the key is the existence and nondegeneracy of some directional derivatives. For example, the smooth functionals and supremum- or integral-type functionals belong to this class. However, in the case of empirical processes, the problem is much more complicated. For instance, in the uniform case, even for the simplest functionals of estimation $f_{t_0}(x) = x(t_0)$ (with $t_0 \neq 0, 1$), the law of estimated empirical process $f_{t_0}(\zeta_n)$ is atomic and cannot, in principle, converge in variation to the Gaussian law of $W_F^0(t_0)$.

For this reason, we deal with specific functionals of interest and begin, in this paper, with supremum-type functionals. The main result is:

THEOREM 1. *Let $(\xi_i)_{i>0}$ be a sequence of i.i.d. random variables with continuous distribution function F, and let ζ_n^F be the related empirical process given by* (0.1). *Then, as $n \to +\infty$, we have*

$$\mathcal{L}\Big(\sup_{t\in\mathbb{R}} \zeta_n^F(t)\Big) \xrightarrow{var} \mathcal{L}\Big(\sup_{t\in\mathbb{R}} W_F^0(t)\Big)$$

and

$$\mathcal{L}\Big(\sup_{t\in\mathbb{R}} \big|\zeta_n^F(t)\big|\Big) \xrightarrow{var} \mathcal{L}\Big(\sup_{t\in\mathbb{R}} \big|W_F^0(t)\big|\Big).$$

As far as we know, these limit theorems are the first results stating a local limit theorem for suprema of empirical processes.

The proof of the second part of Theorem 1 is similar to that of the first part with easy modifications and will be omitted.

Since the laws of $\sup_{t\in\mathbb{R}} \zeta_n^F(t)$ and $\sup_{t\in\mathbb{R}} W_F^0(t)$ do not depend on the distribution F, it suffices to prove Theorem 1 for a Gaussian i.i.d. sequence, for which the calculations are easier. We denote by Φ the distribution function of the standard normal law, and by $p(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)$ its density.

The sequel is thus devoted to the (long) proof of Theorem 1 for an i.i.d. standard Gaussian sequence $(\xi_i)_{i>0}$. In order to derive such a convergence in variation, we apply the so-called superstructure method relying on the behavior of the laws of empirical processes near admissible directions for the limit law (of a Brownian bridge). For a complete description of this method, we refer to [5]. In particular, we fundamentally use the following result:

THEOREM A ([5], Theorem 18.4). *Let $\{P_n, n \in \overline{\mathbb{N}}\}$ be a sequence of probability measures defined on the Borel σ-algebra $\mathcal{B}_{\mathcal{X}}$ of a complete separable metric space $(\mathcal{X}, d)$. Suppose that $P_n \Rightarrow P_\infty$. Moreover, suppose that, for P_∞-almost all x, there exist an open ball V centered at x, a number $\varepsilon > 0$, and a family $(G_{n,c},\ n \in \overline{\mathbb{N}},\ c \in (0,\varepsilon])$ of measurable transformations of $\mathcal{X}$ such that the following five conditions are fulfilled*:

(i) *for all $c \in (0,\varepsilon)$ and $\delta > 0$,*

$$\lim_{n\to+\infty} P_n\big(z \mid d(G_{n,c}\, z, G_{\infty,c}\, z) \geqslant \delta\big) = 0;$$

(ii) *for every $c \in (0,\varepsilon)$, the mapping $G_{\infty,c}$ is P_∞-almost everywhere continuous*; *moreover, suppose that*

$$\rho(S,c) = \sup_{z\in S} d\left(z, G_{\infty,c} z\right) \to 0 \quad \text{as } c \to 0$$

for all open balls S;

(iii) $\lim_{c\to0} \overline{\lim}_{n\to\infty} \| P_n G_{n,c}^{-1} - P_n \| = 0$;

(iv) *for all $\delta \in (0,\varepsilon)$,*

$$\lim_{n\to+\infty} \int_{V(x)} \left\| \lambda_{[0,\delta]} \varphi_{n,z}^{-1} - \lambda_{[0,\delta]} \varphi_{\infty,z}^{-1} \right\| P_n(\mathrm{d}z) = 0,$$

where $\varphi_{n,z}(c) = f(G_{n,c} z)$ with $n \in \overline{\mathbb{N}}$ and $c \in (0,\varepsilon]$, and $\lambda_{[0,\delta]}$ is the restriction of the Lebesgue measure to the interval $[0,\delta]$;

(v) *for all $\delta \in (0,\varepsilon)$, the mapping $z \longmapsto \lambda_{[0,\delta]} \varphi_{\infty,z}^{-1}$ of V into $\mathcal{Z}(\mathbb{R})$, the Banach space of signed measures on $\mathbb{R}$ with the total-variation norm, is P_∞-almost everywhere continuous.*

Then

$$P_n f^{-1} \xrightarrow{var} P_\infty f^{-1}, \quad n \to +\infty.$$

Since the empirical processes ζ_n lie in the Skorokhod space $\mathbb{D}(\mathbb{R})$, we are going to apply Theorem A to $\mathcal{X} = \mathbb{D}_0(\mathbb{R})$, the subspace of $\mathbb{D}(\mathbb{R})$ defined below), to the laws P_n of ζ_n and P_∞ of a Brownian bridge W_Φ^0, and to the functional $f(x) = \sup_{t\in\mathbb{R}} x(t)$.

We first define suitable transformations $(G_{n,c})_c$ and a single $\varepsilon > 0$ for P_∞-almost every x. Then we show that the conditions of Theorem A are satisfied.

The rest of the paper is organized as follows. We begin with a precise description of a setting for application of Theorem A and then check that all the conditions are satisfied. The proof of conditions (i) and (iv) in Sections 1 and 4 are lengthy and complicated.

Notation and setting

We recall some notation for the Skorokhod space. The space of sample paths of the empirical processes (0.1) is the Skorokhod space $\mathbb{D}(\mathbb{R})$. However, since $\zeta_n(t) \to 0$ as $t \to \pm\infty$, we rather consider the subspace

$$\mathbb{D}_0(\mathbb{R}) = \left\{ x \in \mathbb{D}(\mathbb{R}) \mid \lim_{t\to\pm\infty} x(t) = 0 \right\}$$

as the sample space.

We equip this space with a complete separable topology brought from the usual Skorokhod space $\mathbb{D}([0,1])$ by the bijection Φ from $\mathbb{R}$ to $]0,1[$. We refer to [1] for a detailed study of the Skorokhod space $\mathbb{D}([0,1])$. We denote byΨ its inverse and define the following Skorokhod metrics on $\mathbb{D}_0(\mathbb{R})$:

$$\begin{aligned} d_{0,\mathbb{R}}(x,y) &= d_0(x\circ\Psi, y\circ\Psi),\\ d_{\mathbb{R}}(x,y) &= d(x\circ\Psi, y\circ\Psi), \end{aligned}\qquad \text{for } x,y\in\mathbb{D}_0(\mathbb{R}),$$

where d_0 and d are the usual Skorokhod metrics on $\mathbb{D}([0,1])$ given by

$$d_0(x,y) = \inf_{\lambda\in\Lambda}\Bigl(\sup_{t\in[0,1]}\bigl|x(\lambda(t)) - y(t)\bigr| + \sup_{t\in[0,1]}\bigl|\lambda(t)-t\bigr|\Bigr),$$

$$d(x,y) = \inf_{\lambda\in\Lambda}\Bigl(\sup_{t\in[0,1]}\bigl|x(\lambda(t)) - y(t)\bigr| + \sup_{\substack{s<t,\\ s,t\in[0,1]}}\log\Bigl|\frac{\lambda t-\lambda s}{t-s}\Bigr|\Bigr),$$

$x,y\in\mathbb{D}([0,1])$, $\Lambda=\{\lambda\colon\ [0,1]\to[0,1]$ nondecreasing continuous bijection$\}$. Moreover, by convention, we set

$$x\circ\Psi(0) = \lim_{t\to 0} x\circ\Psi(t) = \lim_{s\to-\infty} x(s) = 0,$$

$$x\circ\Psi(1) = \lim_{t\to 1} x\circ\Psi(t) = \lim_{s\to+\infty} x(s) = 0.$$

Since d is a complete metric for $\mathbb{D}([0,1])$, one easily sees that $(\mathbb{D}_0(\mathbb{R}), d_{\mathbb{R}})$ also is a complete separable metric space. But, since d_0 and d define the same convergence on $\mathbb{D}([0,1])$, we will henceforth work with the simpler metric $d_{0,\mathbb{R}}$.

The first step consists of the definition of the transformations $G_{n,c}$ and $G_{\infty,c}$. For $n\in\mathbb{N}^*$, we consider

$$G_{n,c}\zeta_n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bigl[\mathbf{1}_{(-\infty,t]}(\xi_i + c/\sqrt{n}) - \Phi(t)\bigr]. \tag{0.3}$$

More precisely, since the support of P_n is measurable, we define $G_{n,c}$ globally as follows: $G_{n,c}x$ is given by (0.3) if $x=\zeta_n(\omega)\in$ Supp(P_n) and equals 0 if $x\notin$ Supp(P_n). Note that, in fact, $G_{n,c}$ act as translations on the underlying Gausian variables ξ_i, $1\leqslant i\leqslant n$.

The asymptotic transformation is chosen to be the mere translation

$$G_{\infty,c}x(t) = x(t) - c\Phi'(t).$$

Applying the Skorokhod representation theorem (Theorem 6.7 of [1]), we can suppose that we are in a probability space $(\Omega,\mathcal{F},\mathbb{P})$ where the weak convergence (0.2) turns into the almost sure one: $\zeta_n \longrightarrow W_\Phi^0$ a.s. in $\mathbb{D}_0(\mathbb{R})$, $n\to+\infty$. Moreover, since W_Φ^0 is a continuous process, the convergence also holds for the uniform metric (see [1], p. 124).

However, note that when we apply the Skorokhod theorem, the nth empirical process now is constructed on the basis of the nth line of a random array $(\tilde\xi_{1,1},\ldots,\tilde\xi_{n,n})$, instead of the nth first terms of the sequence $(\xi_i)_{i\in\mathbb{N}}$. In our study, this is not a cumbersome point, since we work with the triangular arrays of order statistics.

Anyway, when we deal with probabilities involving the vectors $(\tilde\xi_{1,1},\ldots,\tilde\xi_{n,n})$, the probabilities remain the same if we replace $(\tilde\xi_{1,1},\ldots,\tilde\xi_{n,n})$ by $(\xi_1,\ldots,\xi_n)$, since the laws of these vectors coincide. Henceforth, we forget the tildas in the array $(\tilde\xi_{1,1},\ldots,\tilde\xi_{n,n})$ and denote by $(\xi_1^n,\ldots,\xi_n^n)$ the triangular array of order statistics of $(\xi_1,\ldots,\xi_n)$.

Localization

For a function $f\colon\ X\to\mathbb{R}$, we denote Argmax$(f)=\{t\in X,\ f(t)=\max_{s\in X} f(s)\}$. If Argmax$(f)$ consists of a single point, we denote the latter by argmax(f).

The study of Argmax of $G_{n,c}x$ and $G_{\infty,c}x$ for $x=\zeta_n$ or $x=W_\Phi^0$ has some importance in the proof of Theorem 1. In order to ensure the argmax to occur at finite points, we use the following localization procedure of choosing, in Theorem A, an open ball $V(x)$ and a single $\varepsilon>0$ for P_∞-almost all x.

For P_∞-almost all x, #Argmax$(x) = 1$ and argmax$(x) \in \mathbb{R}$ (i.e., the maximum does not occur at $\pm\infty$; see [6]). For such x, we define

$$\varepsilon := \varepsilon(x) = \frac{\sqrt{2\pi}}{4} \sup_{t\in\mathbb{R}} x(t), \qquad V(x) = \big\{y \in \mathbb{D}_0 \mid d_{0,\mathbb{R}}(x, y) < \varepsilon\big\}.$$

Then, for $\omega \in \Omega(x) := (W_\Phi^0)^{-1}\{V(x)\}$, we have that $W_\Phi^0(\omega) \in V(x)$ and $d_{0,\mathbb{R}}(W_\Phi^0(\omega), x) < \varepsilon$. This ensures that$G_{\infty,c}(W_\Phi^0(\omega))$ takes positive values, so that its argmax does not occur at $\pm\infty$.

Since by the Skorokhod representation theorem $\zeta_n \to W_\Phi^0$ in $\mathbb{D}_0(\mathbb{R})$ almost surely as $n \to +\infty$, for $n = n(\omega)$ large enough, we have that $\zeta_n(\omega) \in V(x)$ for $x \in \Omega(x)$ and that $G_{\infty,c}(\zeta_n(\omega))$ takes positive values, so that its argmax does not occur at $\pm\infty$.

The following sections are devoted to the study of conditions (i)–(v) of Theorem A.

1. FIRST CONDITION

In our setting, condition (i) of Theorem A can be rewritten in terms of the process ζ_n:

$$\forall \mathrm{e} > 0, \qquad \lim_{n\to+\infty} \mathbb{P}\big(d_{0,\mathbb{R}}\big(G_{n,c}(\zeta_n), G_{\infty,c}(\zeta_n)\big) > \mathrm{e}\big) = 0, \tag{1.1}$$

where $d_{0,\mathbb{R}}$ denotes the Skorokhod metric on $\mathbb{D}_0(\mathbb{R})$. Note that, for $t \in [0, 1]$,

$$\zeta_n \circ \Psi(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big(\mathbf{1}_{[0,t]}\big(\Phi(\xi_i)\big) - t\Big),$$

$$G_{n,c}\zeta_n \circ \Psi(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big(\mathbf{1}_{[0,t]}\big(\Phi(\xi_i + c/\sqrt{n})\big) - t\Big).$$

We have

$$\begin{aligned} d_{0,\mathbb{R}}\big(G_{n,c}(\zeta_n), G_{\infty,c}(\zeta_n)\big) &= d_0\big(G_{n,c}(\zeta_n)\circ\Psi, G_{\infty,c}(\zeta_n)\circ\Psi\big) \\ &= \inf_{\lambda\in\Lambda} \left\{ \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big(\mathbf{1}_{[0,\lambda t]}\big(\Phi(\xi_i + c/\sqrt{n})\big) - \lambda t - \mathbf{1}_{[0,t]}\big(\Phi(\xi_i)\big) + t\Big) \right.\right. \\ &\qquad \left.\left. + c\Phi'\big(\Psi(t)\big) \right\|_\infty + \|\lambda t - t\|_\infty \right\}. \end{aligned} \tag{1.2}$$

We estimate $\inf_{\lambda\in\Lambda}$ by the appropriate λ_n that cancels the indicator terms in (1.3). Precisely, let λ_n be the piecewise linear function in Λ given by

$$\lambda_n\Phi(\xi_i^n) = \Phi\bigg(\xi_i^n + \frac{c}{\sqrt{n}}\bigg), \quad i = 1, \dots, n, \quad \text{and} \quad \lambda_n(0) = 0, \quad \lambda_n(1) = 1,$$

so that

$$\mathbf{1}_{[0,\lambda_n t]}\bigg(\Phi\Big(\xi_i^n + \frac{c}{\sqrt{n}}\Big)\bigg) = \mathbf{1}_{[0,t]}\big(\Phi(\xi_i^n)\big) \quad \text{for } 1 \leqslant i \leqslant n.$$

Moreover, $\|\lambda_n t - t\|_\infty$ is obviously reached at some $t = \Phi(\xi_i^n)$ and can be estimated by

$$\|\lambda_n t - t\|_\infty = \sup_{1\leqslant i\leqslant n} \big|\lambda_n\Phi(\xi_i^n) - \Phi(\xi_i^n)\big|$$

$$
\begin{aligned}
&= \sup_{1\leqslant i\leqslant n} \left|\Phi(\xi_i^n + c/\sqrt{n}) - \Phi(\xi_i^n)\right| \\
&\leqslant c/\sqrt{n},
\end{aligned}
$$

since Φ obviously is 1-Lipschitz. Thus, almost surely

$$
\|\lambda_n t - t\|_\infty \longrightarrow 0, \quad n \to +\infty. \tag{1.3}
$$

Using (1.1), (1.3), and (1.3), it now suffices to show that, for all $\mathrm{e} > 0$,

$$
\lim_{n\to+\infty} \mathbb{P}\left(\sqrt{n}\left\|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big)\right\|_\infty > \mathrm{e}\right) = 0.
$$

First, note that

$$
(\Phi' \circ \Psi)(t) = \frac{\mathrm{e}^{-\Psi(t)^2/2}}{\sqrt{2\pi}} \longrightarrow 0 \quad \text{as } t \to 0 \text{ or } t \to 1.
$$

Fix $[\alpha, 1-\alpha] \subset]0,1[$ such that, for $t \in]0,1[\backslash[3\alpha, 1-3\alpha]$,

$$
(\Phi' \circ \Psi)(t) < \mathrm{e}/(2c). \tag{1.4}
$$

Denoting, for simplicity,

$$
A_n(t) = \lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big),
$$

we have

$$
\begin{aligned}
\mathbb{P}\bigg(\sqrt{n}\sup_{t\in[0,1]}\left|A_n(t)\right| > \varepsilon\bigg) &\leqslant \mathbb{P}\bigg(\sqrt{n}\sup_{t\in[0,\alpha]}\left|A_n(t)\right| > \varepsilon\bigg) \\
&\quad + \mathbb{P}\bigg(\sqrt{n}\sup_{t\in[\alpha,1-\alpha]}\left|A_n(t)\right| > \varepsilon\bigg) + \mathbb{P}\bigg(\sqrt{n}\sup_{t\in[1-\alpha,1]}\left|A_n(t)\right| > \varepsilon\bigg)
\end{aligned} \tag{1.5}
$$

and we begin to study three terms on the right-hand side of the inequality. We need the following:

LEMMA 1. *We have*

$$
\mathbb{P}\text{-}\lim_{n\to+\infty}\sup_{1\leqslant i\leqslant n}\left|\Phi(\xi_{i+1}^n) - \Phi(\xi_i^n)\right| = 0.
$$

Proof. The statement follows from the inequality

$$
\sup_{1\leqslant i\leqslant n}\left|\Phi(\xi_{i+1}^n) - \Phi(\xi_i^n)\right| \leqslant \frac{1}{n} + 2\sup_{1\leqslant i\leqslant n}\left|\Phi(\xi_i^n) - \frac{i}{n}\right|
$$

and from the Glivenko–Cantelli theorem applied to the uniform order statistics $\Phi(\xi_i^n)$, $1 \leqslant i \leqslant n$.

First, we deal with the first summand on the right-hand side of (1.6). For $t \in [0,\alpha]$ and n large enough, we have

$$
\begin{aligned}
&\sqrt{n}\sup_{t\in[0,\alpha]}\left|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big)\right| \\
&\quad\leqslant \sup_{t\in[0,\alpha]}\sqrt{n}|\lambda_n t - t| + c\sup_{t\in[0,\alpha]}\left|\Phi'\big(\Psi(t)\big)\right|
\end{aligned}
$$

$$\leqslant \sup_{\{i \mid \Phi(\xi_i^n) \text{ or } \Phi(\xi_{i-1}^n)\in[0,\alpha]\}} \sqrt{n}\left|\lambda_n\Phi(\xi_i^n)-\Phi(\xi_i^n)\right|+\varepsilon/2 \tag{1.6}$$

$$< \sup_{\{i \mid \Phi(\xi_i^n)\in[0,2\alpha]\}} \sqrt{n}\left|\Phi\left(\xi_i^n+\frac{c}{\sqrt{n}}\right)-\Phi(\xi_i^n)\right|+\varepsilon/2 \tag{1.7}$$

where, by Lemma 1, for n large enough, the supremum in (1.6) is bounded by that in (1.7). Now note that, for $n > (c/\alpha)^2$ and for indices i such that $\Phi(\xi_i^n) < 2\alpha$, we have

$$\Phi(t) \leqslant \Phi\left(\xi_i^n+\frac{c}{\sqrt{n}}\right) \leqslant \Phi(\xi_i^n)+\frac{c}{\sqrt{n}} \leqslant 2\alpha+\alpha=3\alpha, \quad t\in\left[\xi_i^n, \xi_i^n+c/\sqrt{n}\right].$$

Therefore, by the choice of α in (1.4), we have that $\Phi'\circ\Psi(\Phi(t)) \leqslant \varepsilon/(2c)$ and we can estimate the first term on the right-hand side of (1.7) by

$$\begin{aligned}\sqrt{n}\left|\Phi\left(\xi_i^n+\frac{c}{\sqrt{n}}\right)-\Phi(\xi_i^n)\right| &= \sqrt{n}\left|\int_{\xi_i^n}^{\xi_i^n+c/\sqrt{n}} \Phi'(t)\,\mathrm{d}t\right| \\ &= \sqrt{n}\left|\int_{\xi_i^n}^{\xi_i^n+c/\sqrt{n}} \Phi'\circ\Psi\big(\Phi(t)\big)\,\mathrm{d}t\right| \\ &\leqslant \sqrt{n}\left|\int_{\xi_i^n}^{\xi_i^n+c/\sqrt{n}} \frac{\varepsilon}{2c}\,\mathrm{d}t\right| \\ &\leqslant \varepsilon/2.\end{aligned} \tag{1.8}$$

For n large enough, (1.7) and (1.9) imply that

$$\sqrt{n}\sup_{t\in[0,\alpha]}\left|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big)\right| < \mathrm{e}.$$

Thus, the probability

$$\mathbb{P}\left\{\sqrt{n}\sup_{t\in[0,\alpha]}\left|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big)\right| > \mathrm{e}\right\}$$

is zero for n large enough as the probability of an empty set.

The same is true for the third summand relative to the probability of the supremum over $\{t\in[1-\alpha,1]\}$. It remains to deal with the second term relative to the supremum over $\{t\in[\alpha,1-\alpha]\}$. Roughly speaking, note that, for $t=\Phi(\xi_i^n)$,

$$\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big) = \Phi(\xi_i^n+c/\sqrt{n})-\Phi(\xi_i^n)-\frac{c}{\sqrt{n}}\Phi'(\xi_i^n) \simeq \left(\frac{c}{\sqrt{n}}\right)^2\Phi''(\xi_i^n).$$

Let us do this more precisely. First, the extremum of $\lambda_n t - t$ over $[\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)]$ is achieved at $\Phi(\xi_i^n)$ or at $\Phi(\xi_{i+1}^n)$. Using the previous notation $A_n(t)$, we rewrite

$$A_n(t) = \lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n) + \frac{c}{\sqrt{n}}\Phi'(\xi_i^n) - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big)$$

and first take the supremum over $[\Phi(\xi_i^n), \Phi(\xi_{i+1}^n)]$:

$$\sqrt{n}\big\|A_n(t)\big\|_{\infty,i} \leqslant \sqrt{n}\left\|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right\|_{\infty,i} + c\big\|\Phi'(\xi_i^n) - \Phi'(\Psi(t))\big\|_{\infty,i} \tag{1.9}$$

where we denote

$$\|\cdot\|_{\infty,i} = \|\cdot\|_{\infty,[\Phi(\xi_i^n),\Phi(\xi_{i+1}^n)]} = \sup_{x\in[\Phi(\xi_i^n),\Phi(\xi_{i+1}^n)]}\{\cdot\}.$$

Then we take the supremum over the indices i such that $\Phi(\xi_i^n) \in [\alpha, 1-\alpha]$. We need the following:

LEMMA 2. *For every finite interval $[a,b]$, we have*

$$\mathbb{P}\text{-}\lim_{n\to+\infty} \sup_{\substack{1\leqslant i\leqslant n\\ \xi_i^n,\xi_{i+1}^n\in[a,b]}} \left|\xi_{i+1}^n - \xi_i^n\right| = 0.$$

Proof. Indeed, we have

$$\begin{aligned}\sup_{\substack{1\leqslant i\leqslant n\\ \xi_i^n,\xi_{i+1}^n\in[a,b]}} \left|\xi_{i+1}^n - \xi_i^n\right| &= \sup_{\substack{1\leqslant i\leqslant n\\ \xi_i^n,\xi_{i+1}^n\in[a,b]}} \left|\Phi^{-1}\big(\Phi(\xi_{i+1}^n)\big) - \Phi^{-1}\big(\Phi(\xi_i^n)\big)\right| \\ &\leqslant w_{\Phi^{-1},[\Phi(a),\Phi(b)]}\left(\sup_{1\leqslant i\leqslant n}\left|\Phi(\xi_{i+1}^n) - \Phi(\xi_i^n)\right|\right),\end{aligned} \tag{1.10}$$

where $r \mapsto w_{f,\Delta}(r)$ denotes the modulus of uniform continuity of a function f on an interval Δ. Since Φ^{-1} is continuous on $[\Phi(a), \Phi(b)]$, (2) follows from (1.11) and Lemma 1.

The second term in (1.9) can now be handled as follows:

$$\begin{aligned}\sup_{\substack{1\leqslant i\leqslant n\\ i\,|\,\Phi(\xi_i^n)\in[\frac{\alpha}{2},1-\frac{\alpha}{2}]}} \big\|\Phi'(\xi_i^n) - \Phi'(\Psi(t))\big\|_{\infty,i} &= \sup_{\substack{1\leqslant i\leqslant n\\ i\,|\,\Phi(\xi_i^n)\in[\frac{\alpha}{2},1-\frac{\alpha}{2}]}} \sup_{s\in[\xi_i^n,\xi_{i+1}^n]} \left|\Phi'(\xi_i^n) - \Phi'(s)\right| \\ &\leqslant w_{\Phi',\mathbb{R}}\left(\sup_{\substack{1\leqslant i\leqslant n\\ i\,|\,\Phi(\xi_i^n)\in[\frac{\alpha}{2},1-\frac{\alpha}{2}]}} \left|\xi_{i+1}^n - \xi_i^n\right|\right),\end{aligned}$$

where the latter tends to 0 by (2), since Φ' is uniformly continuous.

Next, we deal with the first term on the right-hand side of (1.9). The supremum is achieved

– either at $\Phi(\xi_i^n)$ and is equal to

$$\sqrt{n}\left|\lambda_n\Phi(\xi_i^n) - \Phi(\xi_i^n) - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right| = \sqrt{n}\left|\int_{\xi_i^n}^{\xi_i^n+c/\sqrt{n}} \big(\Phi'(t) - \Phi'(\xi_i^n)\big)\,\mathrm{d}t\right|;$$

since Φ' is uniformly continuous, the latter term is arbitrarily small for large values of n uniformly with respect to $1 \leqslant i \leqslant n$;

– either at $\Phi(\xi_{i+1}^n)$ and is equal to

$$\sqrt{n}\left|\lambda_n\Phi(\xi_{i+1}^n) - \Phi(\xi_{i+1}^n) - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right|$$

$$
\begin{aligned}
&= \sqrt{n}\left|\Phi\left(\xi_{i+1}^n + \frac{c}{\sqrt{n}}\right) - \Phi(\xi_{i+1}^n) - \frac{c}{\sqrt{n}}\Phi'(\xi_i^n)\right| \\
&= \sqrt{n}\left|\int_{\xi_{i+1}^n}^{\xi_{i+1}^n + c/\sqrt{n}} \big(\Phi'(t) - \Phi'(\xi_i^n)\big)\,\mathrm{d}t\right| \\
&\leqslant \sqrt{n}\int_{\xi_{i+1}^n}^{\xi_{i+1}^n + c/\sqrt{n}} w_{\Phi',\mathbb{R}}\left(\left|\xi_{i+1}^n - \xi_i^n\right| + \frac{c}{\sqrt{n}}\right)\mathrm{d}t \\
&= c\, w_{\Phi',\mathbb{R}}\left(\left|\xi_{i+1}^n - \xi_i^n\right| + \frac{c}{\sqrt{n}}\right).
\end{aligned}
$$

Since Φ' is uniformly continuous, by Lemma 2 the latter expression tends to 0 as $n \to +\infty$, uniformly with respect to the indices $1 \leqslant i \leqslant n$ for which $\Phi(\xi_i^n) \in [\frac{\alpha}{2}, 1 - \frac{\alpha}{2}]$.

Finally, (1.9) tends to zero uniformly with respect to all i for which $\Phi(\xi_i^n) \in [\alpha/2, 1 - \alpha/2]$. Therefore, we have

$$\lim_{n\to+\infty} \mathbb{P}\left(\sqrt{n}\sup_{t\in[\alpha,1-\alpha]}\left|\lambda_n t - t - \frac{c}{\sqrt{n}}\Phi'\big(\Psi(t)\big)\right| \geqslant \mathrm{e}\right) = 0.$$

Gathering all the intermediate results from (1.6), we get that condition (i) of Theorem A is satisfied.

2. ITEM (ii)

In our setting, this point is straightforward. The function $G_{\infty,c}$ is the translation $z \mapsto z - c\Phi'$ and, thus, is continuous. Moreover, since Φ' is bounded, we have

$$d_{0,\mathbb{R}}(G_{\infty,c}x, x) = d(G_{\infty,c}x \circ \Psi, x \circ \Psi) \leqslant \|G_{\infty,c}x \circ \Psi - x \circ \Psi\|_\infty = c\|\Phi' \circ \Psi\|_\infty \to 0$$

as $c \to 0$, uniformly in $x \in \mathbb{D}_0(\mathbb{R})$.

3. ITEM (iii)

The purpose of this section is to prove that

$$\lim_{c\to 0}\overline{\lim}_{n\to+\infty}\left\|P_n G_{n,c}^{-1} - P_n\right\| = 0.$$

But from expressions (0.1) and (0.3) we have

$$P_n = \mathcal{L}\big(\Theta_n(\xi_1, \dots, \xi_n)\big), \qquad P_n G_{n,c}^{-1} = \mathcal{L}\big(\Theta_n(\xi_1 + c/\sqrt{n}, \dots, \xi_n + c/\sqrt{n})\big),$$

where Θ_n: $\mathbb{R}^n \longrightarrow \mathbb{D}(\mathbb{R})$ is defined by

$$\Theta_n(x_1, \dots, x_n)(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \big(\mathbf{1}_{]-\infty,t]}(x_i) - \Phi(t)\big).$$

We then derive

$$\|P_n G_{n,c}^{-1} - P_n\| \leqslant \left\|\mathcal{L}(\xi_1 + c/\sqrt{n}, \dots, \xi_n + c/\sqrt{n}) - \mathcal{L}(\xi_1, \dots, \xi_n)\right\|$$

$$\leqslant \int_{\mathbb{R}^n} \left| \prod_{i=1}^n p\Big(x_i - \frac{c}{\sqrt{n}}\Big) - \prod_{i=1}^n p(x_i) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_n$$
$$\leqslant c$$

where the last majorization follows from the following lemma (Lemma 20.1 of [5]) applied to the function $l(t) = t$:

LEMMA 3. *For l in the Cameron–Martin space H_1, define*

$$l_n^i = \sqrt{n}\bigg(l\Big(\frac{i}{n}\Big) - l\Big(\frac{i-1}{n}\Big)\bigg), \quad 1 \leqslant i \leqslant n.$$

Then we have the estimate

$$\int_{\mathbb{R}^n} \left| \prod_{i=1}^n p\big(x_i - c l_n^i\big) - \prod_{i=1}^n p(x_i) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_n \leqslant c \|l'\|_{L^2([0,1])}.$$

From this, condition (iii) immediately follows.

4. ITEM (iv)

In this section, we study condition (iv) of Theorem A. We have to prove that, for P_∞-almost all x and all $\delta \in (0, \varepsilon)$,

$$\lim_{n\to+\infty} \int_V \left\| \lambda_{[0,\delta]} \varphi_{n,z}^{-1} - \lambda_{[0,\delta]} \varphi_{\infty,z}^{-1} \right\| P_n(\mathrm{d}z) = 0 \tag{4.1}$$

where

$$\varphi_{n,z}(c) = \sup_{t\in\mathbb{R}} \big(G_{n,c} z(t)\big), \quad n \in \mathbb{N} \cup \{\infty\}, \ \ c \in (0, \delta],$$

and $V = V(x)$ is defined in the localization procedure.

Since $P_n \Rightarrow P_\infty$, we obtain (4.1) if we show that, for P_∞-almost all z, the convergence $z_n \to z$ implies

$$\lim_{n\to+\infty} \left\| \lambda_{[0,\delta]} \varphi_{n,z_n}^{-1} - \lambda_{[0,\delta]} \varphi_{\infty,z_n}^{-1} \right\| = 0. \tag{4.2}$$

LEMMA 4. *Let h_n and h be measurable mappings of X into $\mathbb{R}$. Let P_n, $n \in \mathbb{N} \cup \{+\infty\}$, be probability measures on X. Suppose that $P_n \Rightarrow P_\infty$ and $h_n(x) \to h(x)$ for P_∞-almost all x. Then*

$$\lim_{n\to+\infty} \int_X h_n \ \mathrm{d}P_n = \int_X h \ \mathrm{d}P_\infty.$$

To show the convergence in variation (4.2), we shall use the following result of Davydov [4] for one-dimensional image measures:

PROPOSITION 1. *Let f_n: $[0,1] \to \mathbb{R}$, $n \in \mathbb{N} \cup \{\infty\}$, be a sequence of absolutely continuous functions such that*

– $f_n(0) \to f_\infty(0)$,

– $f_n' \to f_\infty'$ *in* $L^1([0,1])$,

– $f_\infty' \neq 0$ *a.e.*

Then $\lambda_{[0,1]} f_n^{-1} \xrightarrow{var} \lambda_{[0,1]} f_\infty^{-1}$.

Unfortunately, Proposition 1 cannot be directly applied in our setting, since (4.2) is concerned with the asymptotic distance in variation of two sequences of measures, whereas Proposition 1 deals with a single converging sequence. Hence, we introduce an auxiliary image measure $\lambda_{[0,\delta]}\varphi^{-1}_{\infty,W^0_\Phi}$ and split the study of (4.2) into two parts

$$\lim_{n\to+\infty} \left\| \lambda_{[0,\delta]}\varphi^{-1}_{\infty,\zeta_n} - \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W^0_\Phi} \right\| = 0, \tag{4.3}$$

$$\lim_{n\to+\infty} \left\| \lambda_{[0,\delta]}\varphi^{-1}_{n,\zeta_n} - \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W^0_\Phi} \right\| = 0, \tag{4.4}$$

Proposition 1 (or, at least, Proposition 2) is applicable for both. In this section, $\|\cdot\|$ denotes the variation norm on $\Omega(x)$, the sub-probability space related to $V(x)$ in the localization procedure.

4.1. Derivatives

In order to apply Proposition 1 (or Proposition 2), the first step consists in calculating the derivatives of the functions φ_{n,ζ_n}, φ_{∞,ζ_n}, and $\varphi_{\infty,W^0_\Phi}$. This is the purpose of this section.

4.1.1. Derivative of φ_{n,ζ_n}. First, ζ_n obviously reaches its maximum at some a.s. unique point ξ_i. Similarly, for almost every c, $G_{n,c}\zeta_n$ reaches its maximum once at another point $\xi_j + c/\sqrt{n}$ (with *a priori* different index j).

We now calculate the derivative of φ_{n,ζ_n} at almost all c for which the uniqueness at Argmax holds. Let i_0 be the index (almost surely well defined) realizing the argmax, that is,

$$\operatorname{argmax} G_{n,c}\zeta_n = \xi^n_{i_0} + \frac{c}{\sqrt{n}}.$$

For d near enough to c, $G_{n,d}\zeta_n$ reaches its maximum at $\xi^n_{i_0} + d/\sqrt{n}$ with the same index i_0. Then we have

$$\begin{aligned} \varphi'_{n,\zeta_n}(c) &= \lim_{d\to c} \frac{\varphi_{n,\zeta_n}(d) - \varphi_{n,\zeta_n}(c)}{d-c} \\ &= -\lim_{d\to c} \sqrt{n}\frac{\Phi(\xi^n_{i_0} + d/\sqrt{n}) - \Phi(\xi^n_{i_0} + c/\sqrt{n})}{d-c} \\ &= -\Phi'(\xi^n_{i_0} + c/\sqrt{n}) \\ &= -\Phi'\big(\operatorname{argmax}(G_{n,c}\zeta_n)\big). \end{aligned} \tag{4.5}$$

4.1.2. Derivative of φ_{∞,ζ_n}. In order to calculate the derivative of φ_{∞,ζ_n}, we rewrite

$$\varphi_{\infty,\zeta_n}(c) = \sup_{t\in\mathbb{R}} \big(\zeta_n(t) - c\Phi'(t)\big) = \sup_{t\in\mathbb{R}} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \big\{\mathbf{1}_{]-\infty,t]}(\xi^n_i) - \Phi(t)\big\} - c\Phi'(t) \right).$$

The function $G_{\infty,c}\zeta_n = \zeta_n - c\Phi'$ reaches its maximum almost surely once at a finite point. Indeed, since $\omega \in \Omega(x)$, the supremum must occur at a finite point. Since, for $t \geqslant \sqrt{n}/c$, $G_{\infty,c}\zeta_n$ is an increasing function with zero limit at $+\infty$, the maximum must occur at some $t \leqslant \sqrt{n}/c$. However, for all $t \leqslant \sqrt{n}/c$, $G_{\infty,c}\zeta_n$ is obtained by adding positive jumps at the points ξ_i to a decreasing function. Therefore, the local maxima occur at the points ξ_i. Next, it is easy to see that, for almost all c, the local maxima do not coincide and that the global maximum is thus unique. Therefore, $\operatorname{argmax}(G_{\infty,c}\zeta_n)$ is well defined for almost all c.

For such c_0, denote $t_0 = \xi_{i_0}^n = \operatorname{argmax}(G_{\infty,c_0}\zeta_n)$. Then we have

$$\varphi_{\infty,\zeta_n}(c_0) = G_{\infty,c_0}\zeta_n(t_0) = \frac{i_0}{\sqrt{n}} - \sqrt{n}\Phi(t_0) - c_0\Phi'(t_0).$$

LEMMA 5. *For any fixed n, let $c_p \to c_0$ and $t_p \in \operatorname{Argmax}(G_{\infty,c_p}\zeta_n)$. Then $t_p \to t_0$.*

Proof. Since t_p belongs to the finite set $\{\xi_1^n, \dots, \xi_n^n\}$, from any subsequence $(t_{p'})_{p'}$, we can extract another subsequence $(t_{p''})_{p''}$ converging to some t_∞. To simplify the notation, we always write t_p instead of $t_{p''}$. We first show that $t_\infty = t_0$.

Note that $(\zeta_n(t_p))_p$ must converge to $\zeta_n(t_\infty)$: this is obvious if ζ_n is continuous at t_∞ or, by the right-continuity, if $(t_p)_p$ decreases to t_∞. Since we can always supppose that $t_p \nearrow t_\infty$ or $t_p \searrow t_\infty$ (by passing, if necessary, to a subsubsequence), it remains to consider the case where t_p increasingly converges to t_∞. In this case, we have

$$\zeta_n(t_p) \to \zeta_n(t_\infty^-) = \zeta_n(t_\infty) - 1/\sqrt{n} \quad \text{as } p \to +\infty$$

and, for p large enough,

$$\zeta_n(t_p) - c_p\Phi'(t_p) < \zeta_n(t_\infty) - c_p\Phi'(t_\infty).$$

This is a contradiction to $t_p \in \operatorname{argmax}(G_{\infty,c_p}\zeta_n)$. Thus, $(\zeta_n(t_p))_p$ must converge to $\zeta_n(t_\infty)$. Now, since

$$G_{\infty,c_p}\zeta_n(t_p) = \zeta_n(t_p) - c_p\Phi'(t_p) \geqslant \zeta_n(t) - c_p\Phi'(t)$$

for all t, taking the limit as $p \to \infty$ yields, for all t,

$$\zeta_n(t_\infty) - c\Phi'(t_\infty) \geqslant \zeta_n(t) - c\Phi'(t),$$

that is, $t_\infty \in \operatorname{argmax}\{G_{\infty,c}\zeta_n\} = \{t_0\}$. This justifies the convergence of the whole sequence $(t_p)_p$ to t_0.

Now

$$\varphi_{\infty,\zeta_n}(c_p) = G_{\infty,c_p}\zeta_n(t_p) \geqslant G_{\infty,c_p}\zeta_n(t_0)$$

implies

$$\varphi_{\infty,\zeta_n}(c_p) - \varphi_{\infty,\zeta_n}(c) \geqslant G_{\infty,c_p}\zeta_n(t_0) - G_{\infty,c}\zeta_n(t_0) = (c - c_p)\Phi'(t_0).$$

Similarly, $\varphi_{\infty,\zeta_n}(c) \geqslant G_{\infty,c}\zeta_n(t_p)$ implies

$$\varphi_{\infty,\zeta_n}(c_p) - \varphi_{\infty,\zeta_n}(c) \leqslant G_{\infty,c_p}\zeta_n(t_p) - G_{\infty,c}\zeta_n(t_p) = (c - c_p)\Phi'(t_p).$$

From both inequalities we get

$$(c - c_p)\Phi'(t_0) \leqslant \varphi_{\infty,\zeta_n}(c_p) - \varphi_{\infty,\zeta_n}(c) \leqslant (c - c_p)\Phi'(t_p).$$

Finally, for almost all c, we have

$$\varphi'_{\infty,\zeta_n}(c) = -\Phi'(t_0) = -\Phi'\big(\operatorname{argmax}(G_{\infty,c}\zeta_n)\big). \tag{4.6}$$

4.1.3. Derivative of $\varphi_{\infty,W_\Phi^0}$. In this section, the first step consists in proving that $\#\operatorname{Argmax}\{G_{\infty,c}W_\Phi^0\} = 1$ for almost all c. Calculations similar to those for φ_{∞,ζ_n} allow us to to derive the expression of the derivative given in (4.7).

For $c = 0$, $G_{\infty,0}W_\Phi^0 = W_\Phi^0$ has an almost surely unique argmax (see [6]).

Since $\Phi' \in H_0^1$ (the subspace of Cameron–Martin space of functions vanishing at infinity), the function Φ is an admissible direction for W_Φ^0, and we have $\mathcal{L}(W_\Phi^0 - c\Phi') \ll \mathcal{L}(W_\Phi^0)$. Since

$$1 = \mathbb{P}\big(\#\mathrm{Argmax}\,\big(W_\Phi^0\big) = 1\big) = \mathbb{P}_{W_\Phi^0}\big(x \mid \#\mathrm{Argmax}\,(x) = 1\big)$$

and $\{x \mid \#\mathrm{Argmax}\,(x) = 1\}$ is measurable in the space $\mathcal{C}(\mathbb{R})$ of real continuous functions, we also have that

$$\mathbb{P}\big(\#\mathrm{Argmax}\,\big(W_\Phi^0 - c\Phi'\big) = 1\big) = \mathbb{P}_{W_\Phi^0 - c\Phi'}\big(x \mid \#\mathrm{Argmax}\,(x) = 1\big) = 1$$

that is, $W_\Phi^0 - c\Phi'$ has a unique argmax,

Since

$$(\omega, c) \mapsto G_{\infty,c} W_\Phi^0(\omega) = W_\Phi^0(\omega) - c\Phi'$$

is bimeasurable, using the Fubini Theorem, we derive that, almost surely,

$$\#\mathrm{argmax}\,\big\{G_{\infty,c} W_\Phi^0\big\} = 1$$

for almost all $c \in [0, \delta]$.

To adopt the calculations for φ_{∞,ζ_n} to the function $\varphi_{\infty,W_\Phi^0}$, we need to revise Lemma 5. Let $t_0 = \mathrm{argmax}\,(G_{\infty,c_0} W_\Phi^0)$.

LEMMA 6. *Let* $c_p \to c_0$ *and* $t_p \in \mathrm{Argmax}\,(G_{\infty,c_p} W_\Phi^0)$. *Then* $t_p \to t_0$.

Proof. Let $I(t_0) = [t_0 - 1, t_0 + 1]$ be a nighborhood of t_0. One easily sees that, for c sufficiently near to c_0,

$$\sup_{t\in\mathbb{R}} \big(W_\Phi^0(t) - c\Phi'(t)\big) = \sup_{t\in I(t_0)} \big(W_\Phi^0(t) - c\Phi'(t)\big).$$

For p large enough, we thus have $t_p \in I(t_0)$ and from any subsequence $(t_{p'})$ we can extract another subsequence $(t_{p''})$ which converges to t_∞. We finish the proof as in Lemma 5.

Finally, we finish the calculations as those for $\varphi'_{\infty,\zeta_n}$ and obtain that, for almost all c,

$$\varphi'_{\infty,W_\Phi^0}(c) = -\Phi'\big(\mathrm{argmax}\,(G_{\infty,c} W_\Phi^0)\big). \tag{4.7}$$

4.2. Convergence of image measures in (4.3) *and* (4.4)

4.2.1. The case of $\lambda_{[0,\delta]}\varphi_{\infty,\zeta_n}^{-1}$. In this section, we apply Proposition 1 to derive (4.3). Recall that we work on a probability space given by the Skorokhod representation theorem on which the weak convergence (0.2) is strengthened to the $\mathbb{P}$-almost sure

$$\|\zeta_n - W_\Phi^0\|_{\infty,\mathbb{R}} \longrightarrow 0, \quad n \to +\infty. \tag{4.8}$$

Since the functions

$$\varphi_{\infty,\zeta_n}(c) = \sup_{t\in\mathbb{R}} G_{\infty,c}\zeta_n(t) = \sup_{t\in\mathbb{R}} \big(\zeta_n(t) - c\Phi'(t)\big),$$

$$\varphi_{\infty,W_\Phi^0}(c) = \sup_{t\in\mathbb{R}} G_{\infty,c} W_\Phi^0(t) = \sup_{t\in\mathbb{R}} \big(W_\Phi^0(t) - c\Phi'(t)\big),$$

obviously are 1-Lipschitzian, they are absolutely continuous, with the derivatives calculated in (4.6) and (4.7).

We can now apply Proposition 1. Indeed, since, first, $\zeta_n \to W_\Phi^0$ uniformly, we have that $\varphi_{\infty,\zeta_n}(c) \longrightarrow \varphi_{\infty,W_\Phi^0}(c)$ for all c. Second, since

$$\varphi'_{\infty,\zeta_n}(c) = -\Phi'\big(\text{argmax}\,(G_{\infty,c}\zeta_n)\big),$$
$$\varphi'_{\infty,W_\Phi^0}(c) = -\Phi'\big(\text{argmax}\,\big(G_{\infty,c}W_\Phi^0\big)\big),$$

and the function Φ' is continuous and bounded, it suffices to prove that

$$\text{argmax}\,(G_{\infty,c}\zeta_n) \longrightarrow \text{argmax}\,\big(G_{\infty,c}W_\Phi^0\big), \quad n \to +\infty.$$

Now we need the following elementary result:

LEMMA 7. *Let f_n and f be real functions such that $f_n \to f$ uniformly and* $\#\text{argmax}\{f\} = 1$. *Then, for any sequence $(t_n)_n$ with $t_n \in$* argmax $\{f_n\}$, *we have*

$$t_n \to \text{argmax}\, f, \quad n \to +\infty.$$

Moreover, on the set of functions reaching their maxima only once, argmax *is a continuous function.*

Since the following argmax's are unique, applying Lemma 7, we get that

$$\text{argmax}\,\{\zeta_n - c\Phi'\} \longrightarrow \text{argmax}\,\{W_\Phi^0 - c\Phi'\}, \quad n \to +\infty$$

for almost all c. Using the dominated convergence theorem, we easily get the convergence of the derivatives (4.6) to (4.7) in $L^1([0,\delta])$.

Finally,

$$\varphi'_{\infty,W_\Phi^0}(c) = -\Phi'\big(\text{argmax}\, G_{\infty,c}W_\Phi^0\big) \neq 0 \quad \text{a.e.},$$

since argmax $G_{\infty,c}W_\Phi^0$ is finite and Φ' vanishes only at $\pm\infty$ for $\omega \in \Omega(x)$.

We can thus apply Proposition 1 and derive (4.3): almost surely on $\Omega(x)$,

$$\lambda_{[0,\delta]}\varphi^{-1}_{\infty,\zeta_n} \xrightarrow{var} \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W_\Phi^0}, \quad n \to +\infty. \tag{4.9}$$

4.2.2. The case of $\lambda_{[0,\delta]}\varphi^{-1}_{n,\zeta_n}$. Let us now prove (4.4), that is,

$$\lim_{n\to+\infty} \|\lambda_{[0,\delta]}\varphi^{-1}_{n,\zeta_n} - \lambda_{[0,\delta]}\varphi^{-1}_{\infty,W_\Phi^0}\| = 0.$$

We again use the convergence (4.8). The functions φ_{n,ζ_n} and $\varphi_{\infty,W_\Phi^0}$ are absolutely continuous with derivatives given by (4.6) and (4.7), respectively. The case of $\varphi_{\infty,W_\Phi^0}$ has been derived in Section 4.2.1. Next, note that φ_{n,ζ_n} can be specified on the subsets where argmax $G_{n,c}\zeta_n$ is identified. More precisely,

$$\varphi_{n,\zeta_n}(c) = \sum_{i=1}^{n}\left[\frac{i}{\sqrt{n}} - \sqrt{n}\,\Phi(\xi_i^n + c/\sqrt{n})\right]\mathbf{1}_{A_i}(c), \tag{4.10}$$

where

$$A_i := \big\{c \in [0,\delta] \mid \text{argmax}\, G_{n,c}\zeta_n = \xi_i^n + c/\sqrt{n}\big\}.$$

The set A_i is a finite union of open intervals

$$A_i = \bigcup_{j=1}^{p_i} I_{i,j}$$

with the union of their closures equal to $[0, \delta]$:

$$A_i = \bigcup_{j=1}^{p_i} I_{i,j}, \qquad [0,1] = \bigcup_{j=1}^{p_i} \bar{I}_{i,j}.$$

The restrictions of φ_{n,ζ_n} to those intervals are absolutely continuous (see expression (4.10). Since the values of φ_{n,ζ_n} coincide on the common ends of any two adjacent intervals $I_{i,j}$, the absolute continuity of φ_{n,ζ_n} follows from the following elementary lemma:

LEMMA 8. *Let $a_1 < \cdots < a_p$, and let $f\colon [a_1, a_p] \to \mathbb{R}$ be such that $f|_{[a_i,a_{i+1}]} = f_i$, where every f_i is an absolutely continuous function on $[a_i, a_{i+1}]$ with derivative g_i and $f_i(a_{i+1}) = f_{i+1}(a_{i+1})$. Then f is absolutely continuous on $[a_1, a_p]$ and*

$$f(x) = f(a_1) + \int_{a_1}^{x} g(t)\ \mathrm{d}t$$

with derivative

$$g(x) = \sum_{i=1}^{p-1} g_i(x)\mathbf{1}_{[a_i,a_{i+1}]}(x).$$

The last hypothesis of Proposition 1 cannot be verified (at least, easily) for the function φ_{n,ζ_n} of (4.4). Therefore, we use instead the following version of Proposition 1 with the last hypothesis satisfied for φ_{n,ζ_n}. The proof of this proposition can be found in [3], p. 44–45.

PROPOSITION 2. *Let, for $n \in \mathbb{N} \cup \{\infty\}$,*

$$f_n\colon \big(\Omega \times [0,\delta], \mathcal{F} \times \mathcal{B}([0,\delta]), \mathbb{P} \otimes \lambda\big) \longrightarrow \mathbb{R}, \quad \Omega^* \in \mathcal{F},\ \Omega^* \subset \Omega,$$

be such that

1. $\forall \omega \in \Omega^*$, $\exists N_1(\omega)$, $\forall n \geqslant N_1(\omega)$, $f_n(\omega, \cdot)$ *is absolutely continuous*;
2. $f_n(\omega, 0) \xrightarrow{\mathbb{P}} f_\infty(\omega, 0)$ *on* Ω^*;
3. $f_n(\omega, \delta) \xrightarrow{\mathbb{P}} f_\infty(\omega, \delta)$ *on* Ω^*;
4. $\forall \omega \in \Omega^*$, $\exists N_4(\omega)$, $\forall n \geqslant N_4(\omega)$, $\frac{\partial}{\partial c} f_n(\omega, c) > 0$ *λ-a.e. for* $c \in (0, \delta)$;
5. $\frac{\partial}{\partial c} f_n(\omega, c) \xrightarrow{\mathbb{P}\otimes\bar{\lambda}} \frac{\partial}{\partial c} f_\infty(\omega, c)$ *on* Ω^*.

   Then

$$\big\| \lambda_{[0,\delta]} f_n(\omega, \cdot)^{-1} - \lambda_{[0,\delta]} f_\infty(\omega, \cdot)^{-1} \big\| \xrightarrow{\mathbb{P}} 0$$

on Ω^*.

We apply this proposition to the functions $f_n = \varphi_{n,\zeta_n}$ and $f_\infty = \varphi_{\infty, W_\Phi^0}$ with $\Omega^* = \Omega(x)$. First note that their absolute continuity (necessary for condition 1) is already known.

Next, for $c = 0$, we have

$$\varphi_{n,\zeta_n}(0) = \sup_{t\in\mathbb{R}} \zeta_n(t) \quad \text{and} \quad \varphi_{\infty, W_\Phi^0}(0) = \sup_{t\in\mathbb{R}} W_\Phi^0(t).$$

Since $\zeta_n \to W_\Phi^0$ uniformly, we still have $\varphi_{n,\zeta_n}(0) \longrightarrow \varphi_{\infty, W_\Phi^0}(0)$.

For condition 3, we use condition (i) of Theorem A already justified in Section 1:

$$\mathbb{P}\text{-}\lim_{n\to+\infty} d_{0,\mathbb{R}}(G_{n,c}\zeta_n, G_{\infty,c}\zeta_n) = 0 \quad \text{for all } c \in [0, \varepsilon].$$

Together with the uniform convergence of ζ_n to W_Φ^0, we derive

$$\mathbb{P}\text{-}\lim_{n\to+\infty} d_{0,\mathbb{R}}\big(G_{n,c}\zeta_n, G_{\infty,c} W_\Phi^0\big) = 0. \tag{4.11}$$

We obtain the same convergence for the uniform norm (instead of $d_{0,\mathbb{R}}$) as follows. First, recall that by the definition of the Skorokhod metric $d_{0,\mathbb{R}}$ on $\mathbb{R}$ we have

$$d_{0,\mathbb{R}}\big(G_{n,c}\zeta_n, G_{\infty,c} W_\Phi^0\big) = d_0\big(G_{n,c}\zeta_n \circ \Psi, G_{\infty,c} W_\Phi^0 \circ \Psi\big).$$

Note that

$$G_{\infty,c} W_\Phi^0 \circ \Psi = W_U^0 - c\Phi' \circ \Psi,$$

where W_U^0 is a standard Brownian bridge (on $[0, 1]$) and that $\Phi' \circ \Psi$ can be extended to a continuous function on $[0, 1]$ vanishing at 0 and 1. Thus, $G_{\infty,c} W_\Phi^0 \circ \Psi$ is a uniformly continuous function. By the definition of d_0, we can choose $\lambda_n \in \Lambda([0, 1])$ such that

$$\begin{aligned} &\big\| G_{n,c}\zeta_n \circ \Psi - G_{\infty,c} W_\Phi^0 \circ \Psi \circ \lambda_n \big\|_{\infty,[0,1]} + \|\lambda_n - id\|_{\infty,[0,1]} \\ &\quad \leqslant 2 d_0\big(G_{n,c}\zeta_n \circ \Psi, G_{\infty,c} W_\Phi^0 \circ \Psi\big). \end{aligned} \tag{4.12}$$

From this we derive that, for all $t \in [0, 1]$,

$$\begin{aligned} \big|G_{n,c}\zeta_n \circ \Psi(t) - G_{\infty,c} W_\Phi^0 \circ \Psi(t)\big| &\leqslant \big|G_{n,c}\zeta_n \circ \Psi(t) - G_{\infty,c} W_\Phi^0 \circ \Psi(\lambda_n t)\big| \\ &\quad + \big|G_{\infty,c} W_\Phi^0 \circ \Psi(\lambda_n t) - G_{\infty,c} W_\Phi^0 \circ \Psi(t)\big| \\ &\leqslant 2 d_n + w_{G_{\infty,c} W_\Phi^0 \circ \Psi, \mathbb{R}}\big(|\lambda_n t - t|\big) \end{aligned}$$

where, for simplicity, we denoted

$$d_n = d_0\big(G_{n,c}\zeta_n \circ \Psi, G_{\infty,c} W_\Phi^0 \circ \Psi\big).$$

By (4.13) this gives

$$\big\| G_{n,c}\zeta_n \circ \Psi - G_{\infty,c} W_\Phi^0 \circ \Psi \big\|_{\infty,[0,1]} \leqslant 2 d_n + w_{G_{\infty,c} W_\Phi^0 \circ \Psi, \mathbb{R}}(2 d_n). \tag{4.13}$$

Since $G_{\infty,c} W_\Phi^0 \circ \Psi$ is uniformly continuous, for all $\mathrm{e} > 0$, there exists $\alpha > 0$ such that

$$\mathbb{P}\big\{w_{G_{\infty,c} W_\Phi^0 \circ \Psi, \mathbb{R}}(2 d_n) > \mathrm{e}\big\} \leqslant \mathbb{P}\{d_n > \alpha/2\}.$$

The convergence of $w_{G_{\infty,c} W_\Phi^0 \circ \Psi, \mathbb{R}}(2 d_n)$ in probability now follows from (4.11). Finally, from (4.11) and (4.13) we deduce that

$$\big\| G_{n,c}\zeta_n - G_{\infty,c} W_\Phi^0 \big\|_{\infty,\mathbb{R}} = \big\| G_{n,c}\zeta_n \circ \Psi - G_{\infty,c} W_\Phi^0 \circ \Psi \big\|_{\infty,[0,1]} \xrightarrow{\mathbb{P}} 0. \tag{4.14}$$

as $n \to +\infty$, and condition 3 now easily follows for the functions φ_{n,ζ_n}.

The condition 4 follows from the expression (4.6) of φ'_{n,ζ_n}. Indeed, (4.6) ensures the a.e. nondegeneracy of the derivatives for $\omega \in \Omega(x)$.

Next, to see the convergence of the derivatives, we first establish the convergence

$$\operatorname{argmax}(G_{n,c}\zeta_n) \longrightarrow \operatorname{argmax}(G_{\infty,c}W_\Phi^0), \quad n \to +\infty.$$

Since both

$$G_{\infty,c}W_\Phi^0(t) = W_\Phi^0(t) - c\Phi'(t)$$

and

$$G_{n,c}\zeta_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \big[\mathbf{1}_{]-\infty,t]}(\xi_i + c/\sqrt{n}) - \Phi(t)\big]$$

are bimeasurable as functions of (ω, c), the Fubini theorem, together with (4.14), yields

$$(\lambda \otimes \mathbb{P}) \big\{ \big\| G_{n,c}\zeta_n - G_{\infty,c}W_\Phi^0 \big\|_\infty \geqslant \mathrm{e} \big\} \longrightarrow 0, \quad n \to +\infty.$$

LEMMA 9. *Let X_n and X be B-valued random variables, where B is a Polish space. If $X_n \xrightarrow{\mathbb{P}} X$ in B and $f\colon B \longrightarrow \mathbb{R}$ is continuous, then $f(X_n) \xrightarrow{\mathbb{P}} f(X)$.*

From Lemmas 7 and 9, we derive that

$$\operatorname{argmax} G_{n,c}\zeta_n \xrightarrow{\lambda\otimes\mathbb{P}} \operatorname{argmax} G_{\infty,c}W_\Phi^0$$

as $n \to +\infty$. Since Φ' is continuous, we also have

$$\Phi'\big(\operatorname{argmax}(G_{n,c}\zeta_n)\big) \xrightarrow{\lambda\otimes\mathbb{P}} \Phi'\big(\operatorname{argmax}(G_{\infty,c}W_\Phi^0)\big), \quad n \to +\infty,$$

that is, $\varphi'_{n,\zeta_n} \longrightarrow \varphi'_{n,W_\Phi^0}$ in measure $\lambda \otimes \mathbb{P}$.

Finally, we can apply Proposition 2 to get a weakened version of (4.4):

$$\Big\| \lambda_{[0,\delta]}\varphi_{n,\zeta_n}^{-1} - \lambda_{[0,\delta]}\varphi_{\infty,W_\Phi^0}^{-1} \Big\| \xrightarrow{\mathbb{P}_{\Omega(x)}} 0, \quad n \to +\infty, \tag{4.15}$$

where $\mathbb{P}_{\Omega(x)}$ is the restriction of $\mathbb{P}$ to $\Omega(x)$.

4.3. Conclusion on condition (iv)

Gathering convergences (4.9) and (4.15), we prove condition (iv) of Theorem A. By the dominated convergence, from (4.9) one easily derives that

$$E\Big[\big\|\lambda\varphi_{\infty,\zeta_n}^{-1} - \lambda\varphi_{\infty,W_\Phi^0}^{-1}\big\|\mathbf{1}_{\Omega(x)}\Big] \to 0, \quad n \to +\infty.$$

Next, since

$$E\Big[\big\|\lambda\varphi_{n,\zeta_n}^{-1} - \lambda\varphi_{\infty,W_\Phi^0}^{-1}\big\|\mathbf{1}_{\Omega(x)}\Big] \leqslant \mathrm{e} + 2\mathbb{P}\Big\{\omega \in \Omega(x) \mid \big\|\lambda\varphi_{n,\zeta_n}^{-1} - \lambda\varphi_{\infty,W_\Phi^0}^{-1}\big\| > \mathrm{e}\Big\},$$

from (4.15) we derive that

$$\overline{\lim}_n E\Big[\big\|\lambda\varphi_{n,\zeta_n}^{-1} - \lambda\varphi_{\infty,W_\Phi^0}^{-1}\big\|\mathbf{1}_{\Omega(x)}\Big] \leqslant \mathrm{e}.$$

Then, letting $\varepsilon \to 0$, we get

$$\lim_{n\to+\infty} E\Big[\big\|\lambda\varphi_{n,\zeta_n}^{-1} - \lambda\varphi_{\infty,W_\Phi^0}^{-1}\big\|\mathbf{1}_{\Omega(x)}\Big] = 0.$$

Finally,

$$\begin{aligned}\overline{\lim}_n E\Big[\big\|\lambda\varphi_{n,\zeta_n}^{-1} - \lambda\varphi_{\infty,\zeta_n}^{-1}\big\|\mathbf{1}_{\Omega(x)}\Big] &\leqslant \overline{\lim}_n E\Big[\big\|\lambda\varphi_{n,\zeta_n}^{-1} - \lambda\varphi_{\infty,\zeta_n}^{-1}\big\|\mathbf{1}_{\Omega(x)}\Big] \\ &\quad + \overline{\lim}_n E\Big[\big\|\lambda\varphi_{\infty,\zeta_n}^{-1} - \lambda\varphi_{\infty,W_\Phi^0}^{-1}\big\|\mathbf{1}_{\Omega(x)}\Big] \\ &= 0.\end{aligned}$$

Condition (iv) of Theorem A is finally fulfilled.

5. CONDITION (v) OF THEOREM A

The purpose of this section is to prove the continuity of the mapping

$$z \in V(x) \longmapsto \lambda_{[0,\delta]}\varphi_{\infty,z}^{-1}$$

P_∞-almost everywhere. Letting $z_n \to z$, we will apply once more Proposition 1 to

$$\varphi_{\infty,z_n}(c) = \sup_{t\in\mathbb{R}}\big(z_n(t) - c\Phi'(t)\big), \qquad \varphi_{\infty,z}(c) = \sup_{t\in\mathbb{R}}\big(z(t) - c\Phi'(t)\big).$$

First of all, in order to calculate, as in Section 4.1.3, the derivatives

$$\varphi'_{\infty,z_n}(c) = -\Phi'\big(\operatorname{argmax}(z_n - c\Phi')\big), \qquad \varphi'_{\infty,z}(c) = -\Phi'\big(\operatorname{argmax}(z - c\Phi')\big) \tag{5.1}$$

and to be able to apply either Proposition 1 or Proposition 2 for deriving the convergence in variation of related image measures, we have to choose z_n and z such that, for almost all c,

$$\#\operatorname{argmax}(z_n - c\Phi') = \#\operatorname{argmax}(z - c\Phi') = 1.$$

Here the condition $\#\operatorname{Argmax}(z_n) = 1$ cannot be directly verified. However, we can avoid this problem by replacing condition (v) of Theorem A by the following weaker one:

(v′) *the application $z \in \mathcal{X}_0 \mapsto \lambda\varphi_{\infty,z}^{-1}$ is continuous for some measurable $\mathcal{X}_0 \subset \mathcal{X}$ such that* $P_n(\mathcal{X}_0) = P_\infty(\mathcal{X}_0) = 1$.

Indeed, a carefull reading of the proof of Theorem A allows one to replace condition (v) in Theorem A by (v′).

Here we take $\mathcal{X}_0$ to be the subset of functions from $\mathbb{D}(\mathbb{R})$ that reach their supremum only once. We have $P_n(\mathcal{X}_0) = P_\infty(\mathcal{X}_0) = 1$ and we now prove (v′) applying Proposition 1:

- Since $z_n \to z$ in $\mathbb{D}(\mathbb{R})$, the first condition is easily satisfied.
- In order to obtain the convergence of the derivatives (5.1) in $L^1([0,\delta])$, it suffices to prove that, for c such that $\#\operatorname{argmax}\{z - c\Phi'\} = 1$,

  $$\operatorname{argmax}\{z_n - c\Phi'\} \to \operatorname{argmax}(z - c\Phi'), \quad n \to +\infty.$$

  However, since P_∞-almost all z are continuous and since z_n converge to z in $\mathbb{D}$, the convergence $z_n \to z$ also holds uniformly. Since the argmax of z is unique, the convergence of the argmax's follows from Lemma 7.

– $\varphi'_{\infty,z}(c)$ is given by (5.1) and is thus nonzero, since Φ' vanishes only at $\pm\infty$, whereas argmax $(z - c\Phi')$ is necessarily finite for $z \in V(x)$.

Finally, Proposition 1 can be applied, yielding

$$\lambda_{[0,\delta]}\varphi^{-1}_{\infty,z_n} \xrightarrow{var} \lambda_{[0,\delta]}\varphi^{-1}_{\infty,z}$$

whenever $z_n \to z$ in $\mathcal{X}_0$. Condition (v') is thus satisfied.

All five conditons of Theorem A are proved, and this proves Theorem 1.

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