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Absolute Continuity of Joint Laws of Multiple Stable Stochastic Integrals

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We are interested in the laws of multiple stable stochastic integrals defined by LePage series representation in references[3,10,11]. We continue the study started in Ref. 3 and give conditions ensuring absolute continuity of joint laws of stable integrals. To this end, we apply a stratification method on the Skorohod space on which we first take back the problem.

KEY WORDS: Multiple stable stochastic integrals; random measure; LePage representation; absolute continuity; stratification method; Skorohod space.

1. INTRODUCTION

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and for $0 < \alpha < 2$, let M be an α-stable random measure on $([0,1], \mathscr{B}(0,1), \lambda)$ with control measure the Lebesgue measure λ and skewness intensity $\beta : [0,1] \to [-1,1]$ with respect to the terminology of Samorodnitsky–Taqqu in Ref. 12, Section 3

$$M(A) \sim S_\alpha \left(\lambda(A)^{1/\alpha}, \frac{\int_A \beta(s)ds}{\lambda(A)}, 0 \right), \quad A \in \mathscr{B}([0,1]),$$

where $S_\alpha(\sigma, \beta, \mu)$ represents the α-stable law with scaled parameter $\sigma \geqslant 0$, skewness parameter $\beta \in [-1,1]$ and translation parameter $\mu \in \mathbb{R}$.

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The simple α-stable integrals $I_1(f) := \int_{[0,1]} f\,dM$ are well known for $f \in L^\alpha([0,1])$, they have the following stable law:

$$S_\alpha\left(\left(\int_0^1 |f(t)|^\alpha dt\right)^{1/\alpha}, \frac{\int_0^1 |f(s)|^\alpha \operatorname{sgn}(f(s))\beta(s)ds}{\int_0^1 |f(t)|^\alpha dt}, \mu_f\right), \tag{1.1}$$

where $\mu_f = \begin{cases} -\frac{2}{\pi}\int_0^1 f(s)\beta(s)\log|f(s)|ds & \text{if } \alpha = 1 \\ 0 & \text{else} \end{cases}$ (see Ref. 12, Section 3).

The case of d-multiple α-stable integrals for $0 < \alpha < 2$

$$I_d(f) = \int_{[0,1]^d} f\,dM^d$$

has been studied via different approaches: by Krakowiak–Szulga in Ref. 7 using product stable random measure, by Rosiński–Woyczyński in Ref. 9 with iterated integration for $1 \leqslant \alpha < 2$ (see also Ref. 8 for $0 < \alpha < 2$), by Surgailis in Ref. 14 for $1 < \alpha < 2$ using interpolation in Lorentz space.

We have given in Ref. 3 an alternative construction based on the generalization of the LePage representation, which is well known in the case $d = 1$ (see also Samorodnitsky–Szulga and Samorodnitsky–Taqqu[10,11] and references in Ref. 3).

This generalization consist in introducing the random multiple series

$$S_d(f) = C_\alpha^{d/\alpha} \sum_{i_1,\ldots,i_d > 0} \gamma_{i_1}\cdots\gamma_{i_d}\Gamma_{i_1}^{-1/\alpha}\cdots\Gamma_{i_d}^{-1/\alpha} f(V_{i_1},\ldots,V_{i_d}), \tag{1.2}$$

where C_α is a normalization factor equals to $(\int_0^\infty x^{-\alpha}\sin x\,dx)^{-1}$, and $\{\Gamma_i\}_{i>0}$ is the sequence of arrival time of a standard Poisson process independent of $(V_i,\gamma_i)_i$ i.i.d random vectors with V_i uniformly distributed on [0,1] and $\gamma_i = \pm 1$ with conditional laws

$$\mathbb{P}\{\gamma_i = -1 \mid V_i\} = \frac{1-\beta(V_i)}{2}, \quad \mathbb{P}\{\gamma_i = +1 \mid V_i\} = \frac{1+\beta(V_i)}{2}.$$

We can construct the multiple stable integral using this series for integrand

$$f \in L^\alpha(\log_+)^{d-1}([0,1]^d) := \left\{ f : [0,1]^d \to \mathbb{R} \,\middle|\, \int_{[0,1]^d} |f|^\alpha (1 + \log_+ |f|)^{d-1} d\lambda \right\}, \tag{1.3}$$

where $\log_+ x := \log(x \vee 1)$. In Ref. 3, we state

Theorem 1.1 (theorem 3.2, Ref. 3). Let M be an α-stable measure, $\alpha \in (0, 2)$. When $\alpha \geqslant 1$, assume additionally that M is symmetric ($\beta \equiv 0$). Then for $f \in L^\alpha(\log_+)^{d-1}([0,1]^d)$, the series (1.2) is $\mathbb{P}$-a.s. convergent and we can construct $I_d(f)$ with the following representation property:

$$I_d(f) \stackrel{\mathscr{L}}{=} S_d(f). \tag{1.4}$$

Such a construction, whose main condition (1.3) is close to those of Refs. 10 and 14, affords new ways to study the law of $I_d(f)$. We have proved in Ref. 3, theorem 5.1 the absolute continuity of these law $\mathscr{L}(I_d(f))$ with respect to Lebesgue measure λ if and only if $f \not\equiv 0$. The goal here is to generalize this result for the joint laws of multiple stable integrals

$$(I_{d_1}(f_1), \dots, I_{d_p}(f_p)) \tag{1.5}$$

for integrands

$$f_1 \in L^\alpha(\log_+)^{d_1-1}([0,1]^{d_1}), \dots, f_p \in L^\alpha(\log_+)^{d_p-1}([0,1]^{d_p}). \tag{1.6}$$

The paper is organized as follows. We start by stating in Section 2 the main result (Theorem 2.1) giving absolute continuity of joint laws of (1.5) under a condition **(H)**, that we propose. We give different examples of applications where condition **(H)** is easily satisfied.

The global scheme of the proof is the same as in Ref. 3, theorem 5.1: the purpose is to deal with the counterparts of (1.5) for series (1.2) using Representation Theorem 1.1. After reducing the problem by approximation and localization in Section 3, we use the stratification method in Section 4 to end the proof. The study of joint law requires to apply stratification to more intricate transformations than in Ref. 3 and demands preliminary studies of their properties. We refer to Ref. 2 for a short exposition of the proof.

In the whole sequel a.s. stands for *almost surely*, a.e. for *almost everywhere*, i.i.d. for *independent and identically distributed*, := means a definition, C a finite and positive generic constant, and $\square$ ends a proof.

2. ABSOLUTE CONTINUITY OF JOINT LAWS

2.1. Main Result

Given $d_1, \dots, d_p$, the dimensions of p multiple stable integrals, we are interested in the law of (1.5). In order to state the result, we introduce the following notations that will be used throughout the paper:

- for $i=1,\dots,p$, $\quad N_i=d_1+\cdots+d_i$, $\quad N=N_p$,
- $a^i=(a^i_0,\dots,a^i_p)\in\mathbb{N}^{p+1}$ a $(p+1)$-partition of $d_i: d_i=|a^i|=a^i_0+\cdots+a^i_p$,
- $a=(a^1,\dots,a^p)\in(\mathbb{N}^{p+1})^p$,
- $M_a=(a^i_j)_{i\leqslant i,j\leqslant p}\in\mathscr{M}_p(\mathbb{R})$, $\quad d_i=\sum_{k=0}^p a^i_k, 1\leqslant i\leqslant p$, $\quad b_k=\sum_{i=1}^p a^i_k$, $0\leqslant k\leqslant p$,
- for $b=(b_1,\dots,b_p)\in\mathbb{N}^p$.

$$E(b)=\left\{a=(a^1,\dots,a^p)\,\middle|\,a^i_1+\cdots+a^i_p=d_i,\ \sum_{i=1}^p a^i_k=b_k,\ k=1,\dots,p\right\},\tag{2.1}$$

- σ_a the permutation of $\{1,\dots,N\}$ that sends

$$j=\sum_{u=1}^{k-1}b_u+\sum_{s=1}^{i-1}a^s_k+l,\quad l=1,\dots,a^i_k,$$

to

$$\sigma_a(j)=\sum_{v=1}^{i-1}d_i+\sum_{s=1}^{k-1}a^i_s+l,\tag{2.2}$$

- $U_a:\mathbb{R}^N\longrightarrow\mathbb{R}^N$ associated to σ_a by $U_a(t_1,\dots,t_N)=(t_{\sigma_a(1)},\dots,t_{\sigma_a(N)})$,
- $\phi(t)=\phi(t_1,\dots,t_N)=f_1(t_1,\dots,t_{N_1})\cdots f_p(t_{N_{p-1}+1},\dots,t_{N_p})$,
- $\phi_b(t)=\phi_b(t_1,\dots,t_N)=\sum_{a\in E(b)}\prod_{i=1}^p\frac{d_i!}{a^i_0!\cdots a^i_p!}\det M_a\phi(U_a(t))$,
- denoting $\Pi_{b_1,\dots,b_d}$ the sub-group of Π_n of permutations preserving the following so-called "b-blocks": $(1,\dots,b_1),(b_1+1,\dots,b_1+b_2),\dots,(b_1+b_2+\cdots b_{p-1}+1,\dots,b_1+b_2+\cdots+b_p=N)$:

$$S_{b_1,\dots,b_d}\phi(t)=\frac{b_1!\cdots b_d!}{N!}\sum_{\sigma\in\Pi_{b_1,\dots,b_d}}\phi(t_{\sigma(1)},\dots,t_{\sigma(N)}),\tag{2.3}$$

- $\bar{\phi}_b=S_{b_1,\dots,b_d}\phi_b$ symmetrized of ϕ in each b-blocks.

We can then state the main result.

Theorem 2.1. Let $f_1,\dots,f_p$ be given by (1.6) and satisfying hypothesis **(H)**: $\bar{\phi}_b=S_{b_1,\dots,b_d}\phi_b\neq 0$ a.e. on $[0,1]^N$ for some $b=(b_1,\dots,b_p)\in(\mathbb{N}^*)^p$ with

$$|b|=N=d_1+\cdots+d_p.$$

Then $\mathscr{L}(I_{d_1}(f_1), \dots, I_{d_p}(f_p)) \ll \lambda^p$.

Remark 2.1. Hypothesis **(H)** is to be compared to its counterpart of Theorem 5 in Ref. 5 about the absolute continuity of joint laws of multiple Wiener–Itô integrals. Condition **(H)** is not easy to understand, it deals in a sense with how overlapped are the f_i's. We illustrate it in the next Section in several cases.

As observed in Ref. 3, Section 3.2, there is no restriction in assuming the f_i's are symmetric, this will be assumed in the sequel.

In the following, we also use multi indicial notations:

$$\mathbf{i} = (i_1, i_2, \dots, i_d), \quad \mathbf{V_i} = (V_{i_1}, V_{i_2}, \dots, V_{i_d}),$$
$$[\mathbf{i}] = i_1 i_2 \dots i_d, \quad [\mathbf{a_i}] = a_{i_1} a_{i_2} \dots a_{i_d}.$$

We thus rewrite (1.2) in the compact form:

$$S_d(f) = C_\alpha^{d/\alpha} \sum_{i>0} [\gamma_i][\Gamma_{\mathbf{i}}]^{-1/\alpha} f(\mathbf{V_i}). \tag{2.4}$$

2.2. Examples for (H)

In order to apply Theorem 2.1 to multiple stable stochastic integrals of functions $f_1, \dots, f_p$, we have to check condition **(H)**: we give in this Section several cases where **(H)** is easily satisfied. Moreover, we note also that the conclusion of Theorem 2.1 fails when **(H)** does not hold.

(1) Case $p=1, d_1=1$ with $b=1$.
We have $E(b)=\{1\}$ and $\sigma_1 = id$. It is easy to see that $\bar{\phi}(t) = \phi(t) = f(t)$. Condition **(H)** is satisfied if $f \not\equiv 0$ a.e. This is well known since a simple stable integral has law given by (1.1). In particular, it is not degenerated if and only if its scaled parameter $\sigma_f = \left(\int_{[0,1]} |f|^\alpha d\lambda\right)^{1/\alpha}$ is not zero that is $f \not\equiv 0$ a.e.
Conversely, if **(H)** is not satisfied, the law of $I_d(f)$ is degenerated.

(2) Case $p > 1, d_1 = \dots = d_p = 1$ with $b = (1, \dots, 1)$
We have

$$E(b) = \left\{ a = (a_{i,j})_{i \leqslant, j \leqslant p} \,\middle|\, \sum_{j=1}^p a_{i,j} = 1 \ \forall i = 1, \dots, p, \right.$$
$$\left. \sum_{i=1}^p a_{i,j} = 1 \,\forall j = 1, \dots, p \right\}.$$

It is easy to see that card $E(b) = p!$. For $\sigma \in \Pi_p$, consider the matrix $a_\sigma = (a^\sigma_{i,j})_{1 \leqslant i,j \leqslant p}$ associated to σ by

$$a^\sigma_{i,j} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{else.} \end{cases}$$

Clearly, $a_\sigma \in E(b)$, moreover $\sigma_{a_\sigma} = \sigma$, since from (2.2), we have

$$\sigma_{a_\sigma}(n) = \sum_{i=1}^{\sigma(n)-1} 1 + \sum_{s=1}^{n-1} a^\sigma_{\sigma(n),s} + 1 = \sigma(n).$$

We deduce $E(b) \simeq \Pi_p$ and for each $a \in E(b)$, we have $\prod_{i=1}^{p} (d_i!/(a^i_0! \cdots a^i_p!)) = 1$ and $\det a_\sigma = \epsilon(\sigma)$, signature of the permutation σ. It follows:

$$\phi_b(t) = \sum_{\sigma \in \Pi_p} \epsilon(\sigma) \times 1 \times f_1(t_{\sigma(1)}) \cdots f_p(t_{\sigma(p)}) = \det\left\{(f_i(t_j))_{1 \leqslant i,j \leqslant p}\right\}.$$

We have also $S_b \phi_b = \phi_b$ and condition **(H)** is thus satisfied if

$$\det\left\{(f_i(t_j))_{1 \leqslant i,j \leqslant p}\right\} \not\equiv 0 \text{ a.e.}$$

Conversely in the particular case $p = 3$ and $f_3 = f_1 + f_2$, it is easy to see $\det\{f_i(t_j)\} \equiv 0$: condition **(H)** is not satisfied; and the law of

$$(I_1(f_1), I_1(f_2), I_1(f_3)) = (I_1(f_1), I_1(f_2), I_1(f_1) + I_1(f_2))$$

is not absolutely continuous since $(I_1(f_1), I_1(f_2), I_1(f_3))$ belongs to the hyperplan $\mathscr{P}$ of $\mathbb{R}^3$ with equation $x + y - z = 0$, thus

$$\mathbb{P}\{(I_1(f_1), I_1(f_2), I_1(f_3)) \in \mathscr{P}\} = 1, \quad \lambda^3(\mathscr{P}) = 0.$$

(3) Case $p = 1, d_1 = d > 1$ with $b = d$.
We have $E(b) = \{d\}$ and $\sigma_d = id$. We easily see $\bar{\phi}(t) = \phi(t) = f(t)$ because f is symmetric. Condition **(H)** is satisfied if $f \not\equiv 0$ a.e.: we re-find the necessary and sufficient condition for absolute continuity of the law of multiple stable integral obtained in Ref. 3.

(4) Case $p = 2, d_1 = d_2 = 2$ with $b = (2, 2)$.
We have first

$$E(b) = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \right\}.$$

We associate by (2.2) the following permutations to the non-degenerated first and third matrices

$$\begin{pmatrix}1\ 2\ 3\ 4\end{pmatrix}, \quad \begin{pmatrix}3\ 4\ 1\ 2\end{pmatrix}.$$

Since $\prod_{i=1}^{p} d_i!/(a_0^i!\cdots a_p^i!)=1$, we have

$$S_b\phi_b(t)=\phi_b(t)=4f_1(t_1,t_2)f_2(t_3,t_4)-4f_1(t_3,t_4)f_2(t_1,t_2).$$

Condition **(H)** is thus satisfied if there are no reals c_1, c_2 such that

$$c_1 f_1 = c_2 f_2 \quad \text{a.e.}$$

Conversely, if f_1, f_2 are proportional, the joint law of their double stable integrals is degenerated.

(5) Case $p=2, d_1=1, d_2=d$ with $b=(1,d)$.
We have first

$$E(b)=\left\{\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & d-1\end{pmatrix}\right\}.$$

The corresponding permutations are, respectively,

$$\begin{pmatrix}1\ 2\ 3\ \cdots\ d\end{pmatrix}, \begin{pmatrix}2\ 1\ 3\ \cdots\ d\end{pmatrix}.$$

We deduce

$$\phi_b(t)=d(f_1(t_1)f_2(t_2,t_3,\dots,t_{d+1})-f_1(t_2)f_2(t_1,t_3,\dots,t_{d+1}))$$

and

$$S_b\phi_b(t)=d\ f_1(t_1)f_2(t_2,t_3,\dots,t_{d+1})-\sum_{i=2}^{d+1} f_1(t_i)\ \underbrace{f_2(t_2,\dots,t_{d+1})}_{\text{with } t_1 \text{ in } i\text{th position}}.$$

Condition **(H)** holds if

$$f_1(t_1)f_2(t_2,t_3,\dots,t_{d+1})\not\equiv\frac{1}{d}\sum_{i=2}^{d+1} f_1(t_i)\ \underbrace{f_2(t_2,\dots,t_{d+1})}_{\text{with } t_1 \text{ in } i\text{th position}} \quad \text{a.e.}$$

For example for $d=2$, **(H)** is satisfied if

$$f_1(t_1)f_2(t_2,t_3)\not\equiv\frac{1}{2}(f_1(t_2)f_2(t_1,t_3)+f_1(t_3)f_2(t_2,t_1)) \quad \text{a.e.}$$

(6) Case $p=3, d_1=1, d_2=1, d_3=2$ with $b=(1,1,2)$. We obtain

$$E(b) = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&2 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&0&1\\0&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&0\\1&0&0\\0&0&2 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&1\\1&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&1&0\\1&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1\\1&0&0\\0&1&1 \end{pmatrix} \right\}.$$

The permutations we associate are, respectively,

$$(1\ 2\ 3\ 4), (1\ 3\ 2\ 4), (2\ 1\ 3\ 4), (3\ 1\ 2\ 4), (3\ 2\ 1\ 4), (2\ 3\ 1\ 4).$$

After some technical computations, we derive **(H)** is satisfied if

$$S_b\phi_b(t) = \begin{vmatrix} f_1(t_1) & f_2(t_1) & f_3(t_1,t_4) \\ f_1(t_2) & f_2(t_2) & f_3(t_2,t_4) \\ f_1(t_3) & f_2(t_3) & f_3(t_3,t_4) \end{vmatrix} + \begin{vmatrix} f_1(t_1) & f_2(t_1) & f_3(t_1,t_3) \\ f_1(t_2) & f_2(t_2) & f_3(t_2,t_3) \\ f_1(t_4) & f_2(t_4) & f_3(t_4,t_3) \end{vmatrix} \not\equiv 0 \quad \text{a.e.}$$

2.3. Representation of Multiple Stable Integrals

In order to prove Theorem 2.1, we first take back the study to random multiple LePage series

$$(S_{d_1}(f_1), S_{d_2}(f_2), \ldots, S_{d_p}(f_p))$$

applying Representation Theorem 1.1. In the sequel, we consider the stable process η given by

$$\eta_t = M([0,t]), \quad t \in [0,1].$$

The sample paths of η live in $\mathbb{D} := \mathbb{D}([0,1])$, the Skorohod space of *cadlag* functions on [0,1]; we denote by P its law. It is easy to see, applying Theorem 1.1 in the one-dimensional case ($d=1$) to the function $f=\mathbf{1}_{[0,t]}$ that

$$\eta_t = \int_{[0,1]} \mathbf{1}_{[0,t]} dM \stackrel{\mathscr{L}}{=} C_\alpha^{1/\alpha} \sum_{i>0} \gamma_i \Gamma_i^{-1/\alpha} \mathbf{1}_{[0,t]}(V_i).$$

We infer the following interpretations, yet used in Ref. 3:

- $(V_i)_{i>0}$ are the times of the jumps of the stable process η, that will be called its jump-times,
- $C_\alpha^{1/\alpha}\Gamma_i^{-1/\alpha}$ is the modulus of the jump at V_i, decreasingly ordered,
- γ_i indicates the direction of the ith jump.

Proposition 2.1. We have the following equality of joint laws:

$$(S_{d_1}(f_1),\dots,S_{d_p}(f_p))\overset{\mathcal{L}}{=}(I_{d_1}(f_1),\dots,I_{d_p}(f_p)). \tag{2.5}$$

Remark 2.2. Consequently, to prove Theorem 2.1, we shall replace the multiple stable integrals $I_{d_i}(f_i)$ by their multiple LePage representation counterparts $S_{d_i}(f_i)$.

Proof. To see (2.5), we have to show for any reals $\theta_1,\dots,\theta_p$, the equality

$$\theta_1 S_{d_1}(f_1)+\cdots+\theta_p S_{d_p}(f_p)\overset{\mathcal{L}}{=}\theta_1 I_{d_1}(f_1)+\cdots+\theta_p I_{d_p}(f_p). \tag{2.6}$$

Let start with the case of simple functions

$$f_1=\sum_{k=1}^{n_1} a_{1,k}\mathbf{1}_{\Delta_{1,k}},\quad f_2=\sum_{k=1}^{n_2} a_{2,k}\mathbf{1}_{\Delta_{2,k}},\dots,\quad f_p=\sum_{k=1}^{n_p} a_{p,k}\mathbf{1}_{\Delta_{p,k}},$$

where for each $1\leqslant j\leqslant p$ and $1\leqslant k\leqslant n_j: \Delta_{j,k}=\Delta_{j,k}^1\times\cdots\times\Delta_{j,k}^{d_j}$. Since for $1\leqslant j\leqslant p$ and $1\leqslant k\leqslant n_j$

$$\begin{aligned} S_{d_j}\left(\mathbf{1}_{\Delta_{j,k}}\right) &= C_\alpha^{d_j/\alpha}\sum_{i>0}[\gamma_i][\Gamma_i]^{-1/\alpha}\mathbf{1}_{\Delta_{j,k}}(\mathbf{V}_i)\\ &=\prod_{l=1}^{d_j} C_\alpha^{1/\alpha}\sum_{i>0}\gamma_i\Gamma_i^{-1/\alpha}\mathbf{1}_{\Delta_{j,k}^l}(V_i)\\ &=\prod_{l=1}^{d_j} S_1\left(\mathbf{1}_{\Delta_{j,k}^l}\right), \end{aligned}$$

we have

$$\begin{aligned} \theta_1 S_{d_1}(f_1)+\cdots+\theta_p S_{d_p}(f_p) &= \sum_{i=1}^{n_1}\theta_1 a_{1,i}\prod_{k=1}^{d_1} S_1\left(\mathbf{1}_{\Delta_{1,i}^k}\right)+\cdots\\ &\quad+\sum_{i=1}^{n_p}\theta_p a_{p,i}\prod_{k=1}^{d_p} S_1\left(\mathbf{1}_{\Delta_{p,i}^k}\right). \end{aligned} \tag{2.7}$$

But obviously by linearity and by (1.4), we have the following equality in law:

$$
\begin{aligned}
&\Big(S_1\left(\mathbf{1}_{\Delta^1_{1,1}}\right),\dots,S_1\left(\mathbf{1}_{\Delta^{d_1}_{1,1}}\right),\dots,S_1\left(\mathbf{1}_{\Delta^1_{1,n_1}}\right),\dots,S_1\left(\mathbf{1}_{\Delta^{d_1}_{1,n_1}}\right),\dots,\\
&\qquad S_1\left(\mathbf{1}_{\Delta^1_{p,n_p}}\right),\dots,S_1\left(\mathbf{1}_{\Delta^{d_p}_{p,n_p}}\right)\Big)\\
&\overset{\mathcal{L}}{=}\Big(I_1\left(\mathbf{1}_{\Delta^1_{1,1}}\right),\dots,I_1\left(\mathbf{1}_{\Delta^{d_1}_{1,1}}\right),\dots,\\
&\qquad I_1\left(\mathbf{1}_{\Delta^1_{1,n_1}}\right),\dots,I_1\left(\mathbf{1}_{\Delta^{d_1}_{1,n_1}}\right),\dots,I_1\left(\mathbf{1}_{\Delta^1_{p,n_p}}\right),\dots,I_1\left(\mathbf{1}_{\Delta^{d_p}_{p,n_p}}\right)\Big).
\end{aligned}
\tag{2.8}
$$

And since the following function from $\mathbb{R}^{\Sigma_{i=1}^p n_i d_i}$ to $\mathbb{R}$:

$$
\begin{aligned}
&\left(x^{(1)}_{1,1},\dots,x^{(d_1)}_{1,1},\dots,x^{(1)}_{1,n_1},\dots,x^{(d_1)}_{1,n_1},\dots,x^{(1)}_{p,1},\dots,x^{(d_p)}_{p,1},\dots,x^{(1)}_{p,n_p},\dots,x^{(d_p)}_{p,n_p}\right)\\
&\longmapsto \sum_{i=1}^{n_1}\theta_1 a_{1,i}\prod_{k=1}^{d_1}x^{(k)}_{1,i}+\dots+\sum_{i=1}^{n_p}\theta_p a_{p,1}\prod_{k=1}^{d_p}x^{(k)}_{p,i}
\end{aligned}
$$

is continuous, it follows with (2.7), (2.8) and the counterpart of (2.7) with I instead of S_1 that

$$
\begin{aligned}
&\theta_1 S_{d_1}(f_1)+\dots+\theta_p S_{d_p}(f_p)\\
&\quad\overset{\mathcal{L}}{=}\sum_{i=1}^{n_1}\theta_1 a_{1,i}\prod_{k=1}^{d_1}I_1\left(1_{\Delta^k_{1,i}}\right)+\dots+\sum_{i=1}^{n_p}\theta_p a_{p,i}\prod_{k=1}^{d_p}I_1\left(1_{\Delta^k_{p,i}}\right)\\
&\quad\overset{\mathcal{L}}{=}\theta_1 I_{d_1}(f_1)+\dots+\theta_p I_{d_p}(f_p).
\end{aligned}
$$

We thus obtain first (2.6) for simple functions.
Consider now

$$f_1\in L^\alpha(\log_+)^{d_1-1}([0,1]^{d_1}),\dots,f_p\in L^\alpha(\log_+)^{d_p-1}([0,1]^{d_p}).$$

Let, for each $i\leqslant p$, $(f_{n,i})_n$ be a sequence of simple functions with $f_{n,i}\to f_i$ in $L^\alpha(\log_+)^{d_i-1}([0,1]^{d_i})$ as $n\to\infty$. Using $\xrightarrow{\mathbb{P}}$ to mean convergence in Probability, from the properties of continuity in Probability of I_d and S_d stated in Ref. 3 (Section 4.1.2 and 4.2.3), we have

$$S_{d_i}(f_{n,i})\xrightarrow{\mathbb{P}}S_{d_i}(f_i),\quad I_{d_i}(f_{n,i})\xrightarrow{\mathbb{P}}I_{d_i}(f_i).$$

So

$$\theta_1 S_{d_1}(f_{n,1}) + \cdots + \theta_p S_{d_p}(f_{n,p}) \xrightarrow{\mathbb{P}} \theta_1 S_{d_1}(f_1) + \cdots + \theta_p S_{d_p}(f_p),$$
$$\theta_1 I_{d_1}(f_{n,1}) + \cdots + \theta_p I_{d_p}(f_{n,p}) \xrightarrow{\mathbb{P}} \theta_1 I_{d_1}(f_1) + \cdots + \theta_p I_{d_p}(f_p)$$

with moreover equality (2.6) for sequences $f_{n,1}, \ldots, f_{n,p}$. We thus deduce (2.6) for $f_1 \in L^\alpha(\log_+)^{d_1-1}([0,1]^{d_1}), \ldots, f_p \in L^\alpha(\log_+)^{d_p-1}([0,1]^{d_p})$, which proves Proposition 2.1. □

3. REDUCTION OF THE PROBLEM

Thank to Proposition 2.1, we study the joint law of $(S_{d_1}(f_1), \ldots, S_{d_p}(f_p))$. To this end, consider $F = (F_1, \ldots, F_p)$ with $F_i : \mathbb{D} \to \mathbb{R}$ given by

$$F_i(x) = \sum_{t_1,\ldots,t_{d_i}} \delta_x(t_1)\cdots\delta_x(t_{d_i}) f_i(t_1, \ldots, t_{d_i}),$$

where $\delta_x(t)$ stands for the jump of $x \in \mathbb{D}$ at t and $(t_i)_{i>0}$ is the list of its jump-times. The interpretations given in Section 2.3. via LePage Representation (1.4), ensure

$$\begin{aligned} F_i(\eta(\omega)) &= C_\alpha^{d_i/\alpha} \sum_{k_1,\ldots,k_{d_i}>0} \left(\gamma_{k_1}\Gamma_{k_1}^{-1/\alpha}\right)\cdots\left(\gamma_{k_{d_i}}\Gamma_{k_{d_i}}^{-1/\alpha}\right) f_i\left(V_{k_1}, \ldots, V_{k_{d_i}}\right) \\ &= S_{d_i}(f_i)(\omega). \end{aligned}$$

So that

$$F(\eta) \stackrel{\mathscr{L}}{=} (S_{d_1}(f_1), \ldots, S_{d_p}(f_p))$$

and we study the absolute continuity of PF^{-1}.

To this end, we apply the same general argument as for Theorem 5.1 in Ref. 3. First, it is enough to see for all $\varepsilon > 0$, there is χ_ε measurable in $\chi = \mathbb{D}$ with $P(\chi_\varepsilon) > 1 - \varepsilon$ and

$$P_{\chi_\varepsilon} F^{-1} \ll \lambda^p, \tag{3.1}$$

where P_A stands for the restriction of P to the measurable set A. This step will be called the approximation procedure.

Next using separability of χ_ε, we show (3.1) exhibiting for all $x \in \chi_\varepsilon$ some neighborhood $V(x)$ of x such that

$$P_{V(x)} F^{-1} \ll \lambda^p. \tag{3.2}$$

This step will be called the localization procedure.

Finally, (3.2) is justified using the stratification method. As in Ref. 3, these preliminary procedures of approximation and of localization are necessary in order to apply successfully the stratification method (for a general description of this method, we refer to Refs. 4 and 6. The course is now however much more intricate.

We start first by giving approximating sets χ_ε and neighbourhood $V(x)$ for $x \in \chi_\varepsilon$ and the stratification method is used in Section 4.

Approximation

Let b given by hypothesis **(H)** of Theorem 2.1: the set

$$A_{\bar{\phi}_b} = \left\{ x \in \mathbb{R}^N \, |\bar{\phi}_b(x) \neq 0 \right\} \in \mathscr{B}(\mathbb{R}^N)$$

has a positive Lebesgue measure.

Denote $t = (t_1, \ldots, t_N)$ a Lebesgue point in $A_{\bar{\phi}_b}$. By density of such points, we can select t with its coordinates all distinct $(t_i \neq t_j, i \neq j)$. Let $\varepsilon > 0$ be fixed, there is a product neighbourhood $V_\varepsilon = U_1^\varepsilon \times \cdots \times U_N^\varepsilon$ of t satisfying

$$U_i^\varepsilon \cap U_j^\varepsilon = \emptyset, \quad i \neq j \text{ and } \frac{\lambda^N(V_\varepsilon \cap A_{\bar{\phi}_b})}{\lambda^N(V_\varepsilon)} \geqslant 1 - \varepsilon. \tag{3.3}$$

Consider the following sets:

$\chi_{0,\varepsilon} = \{x \in \chi \mid$ for $i = 1, 2, \ldots, d$, x has at least one jump at an instant in U_i^ε, the maximal modulus of these jumps being reached only once$\}$,

$\chi_\varepsilon = \Big\{ x \in \chi_{0,\varepsilon} | x$ has a single maximal jump on each U_i^ε at $T_{U_i^\varepsilon}(x)$ with $T_\varepsilon(x) := \left(T_{U_1^\varepsilon}(x), \ldots, T_{U_N^\varepsilon}(x) \right) \in A_{\bar{\phi}_b} \Big\}$.

We study first $T_{U_i^\varepsilon}(x)$: in order to lighten notations, suppose for the moment $U_i^\varepsilon = (a, b)$. LePage Representation in the one-dimensional case gives

$$\eta_t \stackrel{\mathscr{L}}{=} C_\alpha^{1/\alpha} \sum_{k>0} \gamma_k \Gamma_k^{-1/\alpha} \mathbf{1}_{[0,t]}(V_k).$$

The largest jump of η on (a, b) is $C_\alpha^{1/\alpha} \Gamma_p^{-1/\alpha}$ and it takes place at V_p with $p = \inf\{k | V_k \in (a, b)\}$. Denoting $A_k = \{V_i \notin (a, b) \forall i < k,\ V_k \in (a, b)\}$, we have

$$T_{(a,b)}(\eta) = \sum_{k \geqslant 1} V_k \mathbf{1}_{A_k}.$$

For $A \in \mathscr{B}([a,b])$

$$\begin{aligned} P\left\{T_{(a,b)}(\eta) \in A\right\} &= P\left\{\sum_{k \geqslant 1} V_k \mathbf{1}_{A_k} \in A\right\} \\ &= \sum_{l \geqslant 1} P\left\{A_l, V_l \in A\right\} \\ &= \sum_{l \geqslant 1} P\left\{V_1 \notin (a,b)\right\}^{l-1} P\left\{V_l \in A\right\} \\ &= \sum_{l \geqslant 1} (1-\lambda(a,b))^{l-1} \lambda(A) \\ &= \frac{\lambda(A)}{\lambda\{(a,b)\}}. \end{aligned}$$

$T_{(a,b)}(\eta)$ is thus uniformly distributed on (a,b); that is the law of $T_{U_i^{\varepsilon}}(\eta)$ is uniform on U_i^{ε}. Since for $i \neq j, U_i^{\varepsilon} \cap U_j^{\varepsilon} = \emptyset$, by independence of increments of η, the random variables $T_{U_i^{\varepsilon}}(\eta)$ and $T_{U_j^{\varepsilon}}(\eta)$ are independent and it follows:

$$\mathscr{L}(T_{\varepsilon}(\eta)) = \mathscr{L}\left(T_{U_1^{\varepsilon}}(\eta), \ldots, T_{U_d^{\varepsilon}}(\eta)\right) = \bigotimes_{j=1}^{d} \mathscr{L}\left(T_{U_j^{\varepsilon}}(\eta)\right).$$

The random variables $T_{\varepsilon}(\eta)$ is thus uniform on V_{ε} and

$$\begin{aligned} P(\chi_{\varepsilon}) &= P\left\{x, T_{\varepsilon}(x) \in A_f\right\} \\ &= P_{\chi_{0,\varepsilon}} T_{\varepsilon}^{-1}(A_f). \end{aligned}$$

Since $P_{\chi_{0,\varepsilon}} T_{\varepsilon}^{-1}$ is concentrated and is uniformly distributed on V_{ε}, we have

$$P_{\chi_{0,\varepsilon}} T_{\varepsilon}^{-1}(\cdot) = \frac{\lambda^N(V_{\varepsilon} \cap \cdot)}{\lambda^N(V_{\varepsilon})}.$$

So that by (3.3)

$$P_{\chi_{0,\varepsilon}} T_{\varepsilon}^{-1}\left(A_f^c\right) = \frac{\lambda^N\left(V_{\varepsilon} \cap A_f^c\right)}{\lambda^N(V_{\varepsilon})} \leqslant \varepsilon$$

and finally

$$P(\chi_{\varepsilon}) = P_{\chi_{0,\varepsilon}} T_{\varepsilon}^{-1}\left(A_f\right) \geqslant 1-\varepsilon.$$

We obtain an approximation of $\chi = \mathbb{D}$ as expected and we are thus bring back to the study of (3.1) for each $\varepsilon > 0$.

As explained previously, using separability, it is enough now to exhibit a neighborhood $V(x)$ for each $x \in \chi_\varepsilon$ with the absolute continuity stated in (3.2).

Localization

Let $x \in \chi_\varepsilon$ be fixed and denote for $i = 1, \dots, N$

- $t_i = T_{U_i^\varepsilon}(x)$ the time of the largest jump of x in U_i^ε,
- t_i' the time of the second largest jump of x in U_i^ε, $|\delta_x(t_i')| < |\delta_x(t_i)|$,
- $\varepsilon_0 = (1/2) \min_{i=1,\dots,N} |\delta_x(t_i)|$.

By finiteness of the number of jumps of x larger than $\varepsilon_0/2$, select $\delta_1 > 0$ such that t_i is the single time of $\Delta_i' = (t_i - \delta_1, t_i + \delta_1) \subset U_i^\varepsilon$ where a jump larger than $\varepsilon_0/2$ in modulus occurs. Let the following technical conditions be fulfilled:

- $\varepsilon_0/2 < \varepsilon_1 < \varepsilon_2 < \cdots \varepsilon_p < \varepsilon_0,$ (3.4)
- $\delta_2 < \frac{1}{4} \min \left\{ \varepsilon_0, 2\delta_1, \inf_{i=1,\dots,N} \{|\delta_x(t_i)| - |\delta_x(t_i')|\}, 2\varepsilon_1 - \varepsilon_0 \right\},$ (3.5)
- $\beta = \delta_1 - \delta_2 \quad (\delta_2 \leqslant \beta \leqslant \delta_1),$
- $\Delta_i = (t_i - \beta, t_i + \beta) \subset \Delta_i' \subset U_i^\varepsilon.$

We can now define the tools useful to apply successfully in the sequel the stratification method.

Definition 3.1 (local field). Let $m \in \mathbb{N}^*$, $\Delta_1, \Delta_2, \dots, \Delta_m$ disjoint open sub-intervals of [0,1], $\tau_1, \tau_2, \dots, \tau_m$ reals and $\varepsilon > 0$ called parameters of the field. For $s, t \in [0, 1]$, let

$$\varphi_s(t) = \sum_{i=1}^{m} \tau_i \mathbf{1}_{\Delta_i}(s) \mathbf{1}_{[s,\infty)}(t). \tag{3.6}$$

We define the local field $\{l_x, x \in \mathbb{D}\}$ by

$$l_x = \sum_{s \in J_x^+(\varepsilon)} \varphi_s^+ - \sum_{s \in J_x^-(\varepsilon)} \varphi_s^-, \tag{3.7}$$

where φ_s^+ *et* φ_s^- are, respectively, the positive and negative part of φ_s and $J_x^+(\varepsilon) = \{s \in [0, 1] | \delta_x(s) \geqslant \varepsilon\}$, $\quad J_x^-(\varepsilon) = \{s \in [0, 1] | \delta_x(s) \leqslant -\varepsilon\}$.

With

$$\omega_x(t) = \begin{cases} \tau_i & \text{if } t \in \Delta_i, \quad |\delta_x(t)| > \varepsilon, \quad \delta_x(t)\tau_i > 0, \\ 0 & \text{else} \end{cases} \tag{3.8}$$

the jumps of x and $x + cl_x$ are linked by

$$\delta_{x+cl_x}(t) = \delta_x(t) + c\omega_x(t). \tag{3.9}$$

Let associate to a local field l, the following set $A(l)$, denoting $\Delta_i = (a_i, b_i)$:

- $A(l)^+$ the set of $x \in \mathbb{D}$ such that for i with $\tau_i > 0, x$ does not have jumps of length exactly ε on $\Delta_i, \delta_x(a_i) < \varepsilon, \delta_x(b_i) < \varepsilon$, and x has at least a jump larger than ε on Δ_i,
- $A(l)^-$ the set of $x \in \mathbb{D}$ such that for i with $\tau_i < 0, x$ does not have jumps of length exactly $-\varepsilon$ on $\Delta_i, \delta_x(a_i) > -\varepsilon, \delta_x(b_i) > -\varepsilon$, and x has at least a jump lower than $-\varepsilon$ on Δ_i,
- $$A(l) = A(l)^+ \cap A(l)^-. \tag{3.10}$$

Then set $A(l)$ is suitable to study the local field l, we dispose of the following properties whose proofs are deferred to Appendixes A.1 and A.2.

Lemma 3.1. The set $A(l)$ is open in $\mathbb{D}$.

Lemma 3.2. The local field l is continuous on $A(l)$.

In order to apply the stratification method, consider in the sequel p local fields $l^i, 1 \leqslant i \leqslant p$, and their open set $A(l^i)$, given by (3.10). We select them with the following parameters: for $i = 1, \ldots, p$,

- ε_i given by (3.4),
- $m_i = b_i$, given by hypothesis **(H)**,
- $\Delta^i_j = \Delta_{b_1+\cdots+b_{i-1}+j}$ for $j = 1, \ldots, b_i$,
- τ^i_j with the same sign as $\delta_x(t_i)$ with constant modulus $\tau > 0$.

We have $x \in A(l) = \cap_{i=1}^p A(l^i)$, open set. We thus consider the following neighbourhood $V(x)$ for localization:

$$V(x) = B(x, \delta_2) \cap A(l) \cap \chi_\varepsilon, \tag{3.11}$$

where δ_2 is given in (3.5).

4. STRATIFICATION IN $\mathbb{D}$

4.1. General Scheme

We use stratification method in the neighbourhood $V(x)$ of x. For a deep description of the method we refer to Ref. 6. The general argument is the following: considering an admissible semigroup $\{G_c\}$ (that is adapted to the study of P, see below for the exact definition), we define a partition Γ of the space. We can then see P as a mixture of conditional measures $\{P_\gamma, \gamma \in \chi/\Gamma\}$. In that way, we express

$$PF^{-1} = \int_{\chi/\Gamma} P_\gamma F^{-1} P_\Gamma(d\gamma)$$

and we study the conditional distributions $P_\gamma F^{-1}, \gamma \in X/\Gamma$.

In order to apply this scheme, we first define the family of transformations $\{G_c\}_c$ for $c = (c_1, \dots, c_p) \in (\mathbb{R}^+)^p$ associated to local fields $l^i, i = 1, \dots, p$, by

$$G_{c_1,\dots,c_p} : \begin{cases} A(l) & \longrightarrow A(l), \\ x & \longmapsto x + c_1 l_x^1 + \cdots + c_p l_x^p. \end{cases} \tag{4.1}$$

Remark 4.1. The open set $A(l)$ is stable by $G_{c_1,\dots,c_p}$, roughly speaking because each term l_x^i emphasizes the membership of $A(l^i)$ and do not alter conditions to belong to $A(l^j)$ for $j \neq i$. A more detailed justification is contained as a by-product in the proof of Lemma A.3 in Appendix A.4.

With ω^i associated to l^i as in (3.8), we have for $c \in (\mathbb{R}^+)^p$,

$$\delta_{G_c(x)}(t) = \delta_x(t) + c_1\omega_x^1(t) + \cdots + c_p\omega_x^p(t). \tag{4.2}$$

The following lemma, whose proof is deferred to Appendix A.3, justifies we can apply stratification (see Ref. 6, Section 4).

Lemma 4.1. $\{G_c\}_{c\in(\mathbb{R}^+)^p}$ defines an admissible semigroup on $A(l)$, that is:

(i) $G_0 = id$,
(ii) $G_{c_1} \circ G_{c_2} = G_{c_1+c_2}$,
(iii) $\forall c \in (\mathbb{R}^+)^p$, G_c is one-to-one and admissible,
(iv) $\forall x \in A(l), c \mapsto G_c x$ is one-to-one.

Since $\{G_c\}_c$ introduced in (4.1) satisfies Lemma 4.1, we can apply the scheme of stratification: define an equivalence relation $\sim$ on $A(l)$ by

$$x_1 \sim x_2 \text{ if and only if there are } c_1, c_2 \in (\mathbb{R}^+)^p \text{ with } G_{c_1}x_1 = G_{c_2}x_2. \tag{4.3}$$

Let Γ be the partition given by $\sim$ in strata γ, typically

$$\gamma = \left\{ x + \sum_{i=1}^{p} c_i l_x^i, \quad (c_1, \dots, c_p) \in (\mathbb{R}^+)^p \right\}.$$

Let $\pi : A(l) \to A(l)/\Gamma$ the canonical projection and P_Γ the quotient measure. The equivalent classes $\pi^{-1}(\gamma)$ will be called orbits; from Ref. 6, Proposition 4.2, they are p-dimensional and we equip them with a p-dimensional Lebesgue measure λ_γ^p.

We dispose of the following result guaranteing the conditional measures have good properties:

Theorem 4.1 (theorem 4.1, Ref. 6). Let the partition Γ of a complete separable metric space χ into orbits of an admissible semigroup $\{G_c\}_{c \in \mathbb{R}_+^p}$ be measurable (that is Γ is a partition into the preimages of points for some measurable mapping of χ to a complete separable metric space).

Then for P_Γ-almost all γ, the conditional measure P_γ exists, $P_\gamma\{\pi^{-1}(\gamma)\} = 1$ and P_γ is absolutely continuous with respect to λ_γ^p.

Theorem 4.1 applies if Γ defined in (4.3) is measurable; to show this, define

$$S_i^+(x) = J_x^+(\varepsilon_i) \cap \left\{ \cup_{j|\tau_j^i > 0} \Delta_j^i \right\}, \tag{4.4}$$

$$S_i^-(x) = J_x^-(\varepsilon_i) \cap \left\{ \cup_{j|\tau_j^i < 0} \Delta_j^i \right\}, \tag{4.5}$$

$$S_i(x) = S_i^+(x) \cup S_i^-(x). \tag{4.6}$$

$S_i(x)$ is the set of jumps of x larger in modulus than ε_i in some interval Δ_j^i and with the same sign as τ_j^i. Roughly speaking, $S_i(x)$ represents the set of "transformable" jump-times of x for l^i, that is to say such that $|\delta_x(t)| > \varepsilon_i$ and $\delta_x(t)\tau_j^i > 0$. Consider $j_i(s)$ the index such that for $s \in S_i(x), s \in \Delta_{j_i(s)}^i$ and let

$$c_i(x) = \min_{s \in S_i(x)} \left\{ \frac{|\delta_x(s)| - \varepsilon_i}{|\tau_{j_i(s)}^i|} \right\}. \tag{4.7}$$

We define now a function that sends $x \in A(l)$ to the "beginning" of its orbits.

Proposition 4.1. Let $f : A(l) \to \mathbb{D}$ given by

$$f(x) = x - c_1(x) l_x^1 - \cdots - c_p(x) l_x^p. \tag{4.8}$$

f is continuous and generates the partition Γ, which is thus measurable.

The proof of this proposition is deferred to Appendix A.4. Finally, Theorem 4.1 can be applied so that conditional measures $\{P_\gamma\}_\gamma$ exist with

$$P_\gamma\left\{\pi^{-1}(\gamma)\right\}=1, \quad P_\gamma \ll \lambda_\gamma^p. \tag{4.9}$$

We can express $P_{V(x)}F^{-1}$ as a mixture of conditional distributions

$$P_{V(x)}F^{-1}=\int_{V(x)/\Gamma} P_\gamma F^{-1} P_\Gamma(d\gamma). \tag{4.10}$$

To prove $P_{V(x)}F^{-1} \ll \lambda^p$, it remains to show that for P_Γ-almost all γ with $\pi^{-1}(\gamma)\cap V(x)\neq\emptyset$, we have $P_\gamma F^{-1}\ll\lambda^p$. Because of properties (4.9), it is enough to study the restriction F_γ of F on trace of strata $\pi^{-1}(\gamma)\cap V(x)$ on the neighbourhood of x. This is the goal of the forthcoming section.

4.2. Study of Conditional Functionals

The conditional functional $F_\gamma:\mathbb{R}^p\to\mathbb{R}^p$ is given by

$$\begin{aligned} F_\gamma(c) &= \left(F_{1,\gamma}(c),\ldots,F_{p,\gamma}(c)\right) \\ &= F\left(x+c_1 l_x^1+\cdots+c_p l_x^p\right), \quad c=(c_1,\ldots,c_p). \end{aligned}$$

Let note in the sequel

$$l=(l^1,\ldots,l^p) \quad \text{and} \quad <c,l_x>=c_1 l_x^1+\cdots+c_p l_x^p.$$

Since (4.2) can also be rewritten $\delta_{x+<c,l_x>}(t)=\delta_x(t)+<c,\omega_x(t)>$, we rewrite

$$\begin{aligned} F_{i,\gamma}(c) &= F_i(x+<c,l_x>) \\ &= \sum_{t_1,\ldots,t_{d_i}}\left(\prod_{j=1}^{d_i}\left(\delta_x(t_j)+<c,\omega_x(t_j)>\right)\right) f_i(t_1,\ldots,t_{d_i}). \end{aligned}$$

We obtain a polynomial in $c_1,\ldots,c_p$. We express its coefficient computing the inner product as follows:

$$\sum_{\substack{I_0,\ldots,I_p \text{ partition of} \\ \{1,\ldots,d_i\},\ \mathrm{card}\, I_k=a_k \\ a_0+\cdots+a_p=d_i}} \left(\prod_{j\in I_0}\delta_x(t_j)\right)\left(\prod_{j\in I_1}\omega_x^1(t_j)\right)\cdots\left(\prod_{j\in I_p}\omega_x^p(t_j)\right)1^{a_0}c_1^{a_1}\cdots c_p^{a_p}.$$

Whence for all $i=1,\ldots,p$, permutating summations

$$F_{i,\gamma}(c) = \sum_{\substack{a=(a_0,\ldots,a_p)\\|a|=d_i}} \sum_{\substack{\{I_k\}\text{ partition of}\\\{1,\ldots,d_i\},\text{ card }I_k=a_k}} \sum_{t_1,\ldots,t_{d_i}} \left(\prod_{j\in I_0}\delta_x(t_j)\right)\left(\prod_{j\in I_1}\omega_x^1(t_j)\right)$$
$$\ldots\left(\prod_{j\in I_p}\omega_x^p(t_j)\right) f_i(t_1,\ldots,t_{d_i})c_1^{a_1}\cdots c_p^{a_p}$$
$$= \sum_{\substack{a^i=(a_0^i,\ldots,a_p^i)\\|a^i|=d_i}} B_{a^i}^i c^{a^i}$$

with in order to simplify heavy notations,

- $c^{a^i} = 1^{a_0^i} c_1^{a_1^i}\cdots c_p^{a_p^i}$ for $a^i = \left(a_0^i, a_1^i,\ldots,a_p^i\right)$,
- $B_{a^i}^i = \sum_{\substack{\{I_k\}\text{ partition of}\\\{1,\ldots,d_i\},\text{ card }I_k=a_k^i}} \sum_{t_1,\ldots,t_{d_i}} \left(\prod_{I_0}\right)\left(\prod_{I_1}\right)\cdots\left(\prod_{I_p}\right) f_i(t_1,\ldots,t_{d_i}).$

We obtain an explicit expression for $F_\gamma(c)$. As $P_\gamma \ll \lambda_\gamma^p$, it is enough to show the Jacobian of F_γ is not zero to derive $P_\gamma F^{-1} \ll \lambda^p$. This is the goal of the forthcoming computations (where we do not precise inner products which are the same as before).

$$\frac{\partial F_{i,\gamma}}{\partial c_j}(c) = \sum_{\substack{a=(a_0,\ldots,a_p)\\|a|=d_i}} \sum_{\substack{\{I_k\}\text{ partition of}\\\{1,\ldots,d_i\},\text{ card }I_k=a_k}} \sum_{t_1,\ldots,t_{d_i}} \left(\prod_{I_0}\cdots\right)\left(\prod_{I_1}\cdots\right)$$
$$\cdots\left(\prod_{I_p}\cdots\right) f_i(t_1,\ldots,t_{d_i})a_j\frac{c_1^{a_1}\ldots c_p^{a_p}}{c_j}$$
$$= \sum_{\substack{a^i=(a_0^i,\ldots,a_p^i)\\|a^i|=d_i}} B_{a^i}^i a_j^i \frac{c^{a^i}}{c_j}.$$

We compute now the Jacobian $J_\gamma(c) := \left|\left(\frac{\partial F_{i,\gamma}}{\partial c_j}(c)\right)_{1\leqslant i,j\leqslant p}\right|$

$$
\begin{aligned}
J_\gamma(c) &= \sum_{\sigma\in\Pi_p}\epsilon(\sigma)\prod_{i=1}^{p}\sum_{|a^i|=d_i} B^i_{a^i}, a^i_{\sigma(i)}\frac{c^{a^i}}{c_{\sigma(i)}}\\
&= \sum_{\sigma\in\Pi_p}\epsilon(\sigma)\sum_{\substack{|a^i|=d_i\\ i=1,\dots,p}}\left(\prod_{i=1}^{p}B^i_{a^i}\right)\left(\prod_{i=1}^{p}a^i_{\sigma(i)}\right)\left(\prod_{i=1}^{p}\frac{c^{a^i}}{c_{\sigma(i)}}\right).
\end{aligned}
$$

But

$$
\prod_{i=1}^{p}\frac{c^{a^i}}{c_{\sigma(i)}} = \frac{\prod_{i=1}^{p}c^{a^i}}{c_1\cdots c_p} = \prod_{k=1}^{p}c_k^{(\Sigma_{i=1}^{p}a_k^i)-1}.
$$

Hence

$$
J_\gamma(c) = \sum_{\substack{|a^i|=d_i\\ i=1,\dots,p}}\left(\prod_{i=1}^{p}B^i_{a^i}\right)\left(\prod_{k=1}^{p}c_k^{(\Sigma_{i=1}^{p}a_k^i)-1}\right)\sum_{\sigma\in\Pi_p}\epsilon(\sigma)\left(\prod_{i=1}^{p}a^i_{\sigma(i)}\right).
$$

Use now notations given in Section 2.1: for $a^i = \left(a_0^i, \dots, a_p^i\right) \in \mathbb{R}^{p+1}, i = 1, \dots, p, M_a = \left(a_j^i\right)_{1\leqslant i,j\leqslant p} \in \mathscr{M}_p(\mathbb{R}), M_a^0 = \left(a_j^i\right)_{\substack{0\leqslant i\leqslant p\\ 1\leqslant j\leqslant p}} \in \mathscr{M}_{p+1,p}(\mathbb{R}), d_i = \Sigma_{k=0}^{p}a_k^i$ and $b_k = \Sigma_{i=1}^{p}a_k^i$.

Remark 4.2. There is no restriction in supposing $b_k \geqslant 1$ for $k \geqslant 1$, because if $b_k = \Sigma_{i=1}^{p}a_k^i = 0, M_a$ a has a null column, each a_k^i being zero, and $\det M_a = 0$; the term relative to such a b_k is thus zero.

We have

$$
J_\gamma(c) = \sum_{\substack{|a^i|=d_i\\ i=1,\dots,p}}\left(\prod_{i=1}^{p}B^i_{a^i}\right)\left(\prod_{k=1}^{p}c_k^{b_{k-1}}\right)\det M_a.
$$

This Jacobian is a polynomial in $c_1, \dots, c_p$. We show now it is not degenerated exhibiting a non-zero coefficient. For $b = (b_1, \dots, b_p) \in (\mathbb{N}^*)^p$, the coefficient relative to monomial $c_1^{b_1-1}\cdots c_p^{b_p-1}$ is

$$
A_{\gamma,b} = \sum_{a\in E(b)}\left(\prod_{i=1}^{p}B^i_{a^i}\right)\det M_a,
$$

where

$$E(b) = \Big\{ a = (a^1, \ldots, a^p) \in (\mathbb{N}^{p+1})^p \big| a_1^i + \cdots + a_p^i = d_i, \\ \sum_{i=1}^{p} a_k^i = b_k \text{ for } k = 1, \ldots, p \Big\}.$$

We study $A_{\gamma,b}$ using the following developed expression

$$A_{\gamma,b} = \sum_{a \in E(b)} \left(\prod_{i=1}^{p} \sum_{\substack{\{I_k\}\text{partition of } \{1,\ldots,d_i\},\\ \text{card } I_k = a_k^i}} \sum_{t_1,\ldots,t_{d_i}} \left(\prod_{I_0}\right)\left(\prod_{I_1}\right)\cdots\left(\prod_{I_p}\right) f_i(t_1,\ldots,t_{d_i}) \right) \det M_a. \tag{4.11}$$

Develop the former outer product $\prod_{i=1}^{p}(\cdots)$ using beforehand the symmetry of the f_i's (see Remark 2.1)

$$\sum_{\substack{\{I_k\}\text{partition of } \{1,\ldots,d_i\},\\ \text{card } I_k = a_k^i}} \sum_{t_1,\ldots,t_{d_i}} \left(\prod_{I_0}\right)\left(\prod_{I_1}\right)\cdots\left(\prod_{I_p}\right) f_i(t_1,\ldots,t_{d_i})$$

$$= C(a^i) \sum_{t_1,\ldots,t_{d_i}} \prod_{j=1}^{a_0^i} \delta_x(t_j) \prod_{j=a_0^i+1}^{a_0^i+a_1^i} \omega_x^1(t_j) \cdots \prod_{j=a_0^i+\cdots+a_{p-1}^i+1}^{a_0^i+\cdots+a_p^i} \omega_x^p(t_j) f_i(t_1,\ldots,t_{d_i})$$

with

$$C(a)^i = \text{card } \Big\{ \text{partition of } \{1,\ldots,d_i\} \text{ in } \{I_k\}_{k=1,\ldots,p} \text{with card} I_k = a_k^i \Big\} \\ = \frac{d_i!}{a_0^i! \cdots a_p^i!}.$$

With $\alpha_k^i = \sum_{j \leqslant k} a_j^i$, the ith factor in the outer product (4.11) is

$$\frac{d_i!}{a_0^i! \cdots a_p^i!} \sum_{t_1,\ldots,t_{d_i}} \prod_{j=1}^{\alpha_0^i} \delta_x(t_j) \prod_{j=\alpha_0^i+1}^{\alpha_1^i} \omega_x^1(t_j) \cdots \prod_{j=\alpha_{p-1}^i+1}^{\alpha_p^i} \omega_x^p(t_j) f_i(t_1,\ldots,t_{d_i}).$$

With $N_i = d_1 + \cdots + d_i$, $N = N_p$, split

$$t_1, \ldots, t_{d_1}, t_{d_1+1}, \ldots, t_{d_1+d_2}, \ldots, t_{d_1+d_2+\cdots+d_{p-1}+1}, \ldots, t_{d_1+d_2+\cdots+d_p}$$

in blocks of length d_i

$$t_{N_{i-1}+1}, \ldots, t_{N_i}, \quad i = 1, \ldots, p.$$

In each of them, consider the partition, in $a_0^i, \ldots, a_p^i$ parts

$$t_{N_{i-1}+1}, \ldots, t_{N_{i-1}+\alpha_0^i}, t_{N_{i-1}+\alpha_0^i+1}, \ldots, t_{N_{i-1}+\alpha_1^i}, \ldots,$$
$$t_{N_{i-1}+\alpha_{p-1}^i+1}, \ldots, t_{N_{i-1}+\alpha_p^i} = t_{N_i}.$$

With $k_j^i = N_{i-1} + \alpha_j^i$, $k_p^i = N_{i-1} + \alpha_p^i = N_i$, the i^{th} block is made up of the following sub-blocks:

$$\underbrace{t_{k_p^{i-1}+1}, \ldots, t_{k_0^i},}_{\text{1st sub-block}} \underbrace{t_{k_0^i+1}, \ldots, t_{k_1^i},}_{\text{2nd sub-block}} \underbrace{t_{k_1^i+1}, \ldots, t_{k_2^i},}_{\text{3rd sub-block}} \ldots, \underbrace{t_{k_{p-1}^i+1}, \ldots, t_{k_p^i}}_{(p+1)\text{th sub-block}}.$$

We obtain the explicit formula

$$A_{\gamma,b} = \sum_{a \in E(b)} \left(\prod_{i=1}^{p} \frac{d_i!}{a_0^i! \cdots a_p^i!} \right) \det M_a \prod_{i=1}^{p} \sum_{t_{N_{i-1}+1, \ldots t_{N_i}}} \prod_{j=k_p^{i-1}+1}^{k_0^i} \delta_x(t_j)$$
$$\times \prod_{j=k_0^i+1}^{k_1^i} \omega_x^1(t_j) \cdots \prod_{j=k_{p-1}^i+1}^{k_p^i} \omega_x^p(t_j) f_i(t_{N_{i-1}+1, \ldots, t_{Ni}}).$$

Using the following compact additional notations:

- $< g_1, \delta_x > = \sum_t \delta_x(t) g_1(t)$,
- $< g_1, \omega_x > = \sum_t \omega_x(t) g_1(t)$,
- $< g_2, \omega_x^1 \otimes \omega_x^2 > = \sum_{t_1, t_2} \omega_x^1(t_1) \omega_x^2(t_2) g_2(t_1, t_2)$

the inner preceding sum may be rewritten with $a_0^i + a_1^i + \cdots + a_p^i = d_i = N_i - N_{i-1}$

$$< f_i, \delta_x^{\otimes a_0^i} \otimes \omega_x^{1, \otimes a_1^i} \otimes \cdots \otimes \omega_x^{p, \otimes a_p^i} > .$$

The next step consists in a change of variable: to this end consider for $a \in E(b)$ the permutation σ_a of $\{1, \ldots, N\}$ defined in (2.2) and the mapping $U_a : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$U_a(t_1, \ldots, t_N) = (t_{\sigma_a(1)}, \ldots, t_{\sigma_a(N)}).$$

The change of variable given by U_a transforms

$$A_{\gamma,b}=\sum_{a\in E(b)}\left(\prod_{i=1}^{p} c(a^i)\right)\det M_a\left\langle f_1\otimes\cdots\otimes f_p,\otimes_{i=1}^{p}\left(\delta_x^{\otimes a_0^i}\otimes\omega_x^{1,\otimes a_1^i}\otimes\omega_x^{p,\otimes a_p^i}\right)\right\rangle$$

into

$$A_{\gamma,b}=<\phi_b, G_b> \tag{4.12}$$

with

- $\phi(t)=f_1(t_1,\dots,t_{N_1})\cdots f_p(t_{N_{p-1}+1},\dots,t_{N_p})$,
- $\phi_b(t)=\sum_{a\in E(b)}\prod_{i=1}^{p} C(a^i)\det M_a\phi(U_a(t))$,
- $G_b=\delta_x^{\otimes b_0}\otimes\omega_x^{1,\otimes b_1}\otimes\cdots\otimes\omega_x^{p,\otimes b_p}$.

Since G_b is an invariant function by a permutation preserving each block $(t_1,\dots,t_{b_1}),\dots,(t_{b_1+\dots+b_{d-1}+1},\dots,t_N)$, we can beforehand symmetrize: for $b_1,\dots,b_d\geqslant 1$ with $b_1+\cdots+b_d=N$, define

$$S_{b_1,\dots,b_d}f(t)=\frac{b_1!\cdots b_d!}{N!}\sum_{\sigma\in\prod_{b_1,\dots,b_d}} f(t_{\sigma(1),\dots,t_{\sigma(N)}}),$$

where $\prod_{b_1,\dots,b_d}$ is the sub-group of $\prod_N$ of permutations preserving the so-called "b-blocks":

$$\begin{aligned}&\{1,\dots,b_1\},\{b_1+1,\dots,b_1+b_2\},\\&\quad\{b_1+b_2+\cdots b_{p-1}+1,\dots,b_{p-1}+1,\dots,b_1+b_2+\cdots b_p=N\}.\end{aligned}$$

Coefficient $A_{\gamma,b}$ of (4.12) can be rewritten finally

$$A_{\gamma,b}=\frac{N!}{\prod_{i=1}^{d} b_i!}<S_{b_1,\dots,b_d}\phi_b, G_b>.$$

We achieve the proof in the next section studying $A_{\gamma,b}$ for b given by (**H**) (for which relative coefficient b_0 is 0).

4.3. Coefficient $A_{\gamma,b}$

Before studying $A_{\gamma,b}$ for an arbitrary orbit γ satisfying $\pi^{-1}(\gamma)\cap V(x)\neq\emptyset$, we deal with the orbit γ_x of x. We are thus first concerned with the non-nullity of

$$< \bar{\phi}_b, G_b > \; = \; < \bar{\phi}_b, \delta_x^{\otimes b_0} \otimes \omega_x^{1,\otimes b_1} \otimes \cdots \otimes \omega_x^{p,\otimes b_p} >$$
$$= \sum_{s_1,\dots,s_N} \prod_{i=1}^{b_1} \omega_x^1(s_1) \cdots \prod_{i=b_1+\cdots b_{p-1}+1}^{b_1+\cdots b_p} \omega_x^p(s_i) \bar{\phi}_b(s_1,\dots,s_N). \tag{4.13}$$

As in Section 3, $t_1,\dots,t_N$ stands for jump-times of x in $U_i^\varepsilon, i=1,\dots,N$, let $\{t_k\}_{k>N}$ be its other jump-times. From definition of w_x^i in (3.8), we have

- $\omega_x^i(t_k)=0$ if $k>N$ because either $t_k \notin \cup_j \Delta_i^j$ or $|\delta_x(t_k)|<\varepsilon_0/2<\varepsilon_i$,
- $\omega_x^i(t_k)=0$ if $k \notin \{b_1+\cdots+b_{i-1}+1,\dots,b_1+\cdots+b_i\}$, the i^{th} block,
- $\omega_x^i(t_k) = \tau_j^i$ if $t_k = t_i^j = t_{b_1+\cdots+b_{i-1}+j}$ because then $t_k = t_j^i \in \Delta_j^i, |\delta_x(t_k)|>\varepsilon_0>\varepsilon_i$ and $\delta_x(t_k)$ has the same sigh as τ_j^i.

From these considerations, the non-null terms in the outer sum of (4.13) are the ones with

- for $1 \leqslant j \leqslant b_1, s_j$ in the first block,
- ......
- for $b_1+\cdots+b_{p-1}+1 \leqslant j \leqslant b_1+\cdots+b_p, s_j$ in the p^{th} block.

With possibly a permutation in each "b-block", we ask in fact for

$$(s_1,\dots,s_N)=(t_1,\dots,t_N)$$

and since $\bar{\phi}_b$ is invariant by permutations preserving "b-blocks", we derive

$$A_{\gamma_x,b} = \pm \frac{N!}{\prod_{i=1}^d b_i!} \tau^N \bar{\phi}_b(t_1,\dots,t_N) \neq 0$$

because from the choice of t_i in section 3,

$$(t_1,\dots,t_N)=T_\varepsilon(x) \in A_{\bar{\phi}_b} = \{x \in \mathbb{R}^N | \bar{\phi}_b \neq 0\}.$$

We study now $A_{\gamma,b}$ for a fixed orbit γ with $\pi^{-1}(\gamma) \cap V(x) \neq \emptyset$ where we recall that $V(x)$ is the neighbourhood of fixed x given in (3.11).

Let $y \in V(x)$ represent $\gamma := \gamma_y$ and $(s_k)_{k>0}$ be its jump-times. From the definition of Skorohod's topology (see Ref. 1), let $\rho \in \Lambda$, the set of increasing continuous bijections of [0,1], with

$$\sup_{t\in[0,1]} |x(\rho(t)) - y(t)| < \delta_2 \quad \text{and} \quad \sup_{t\in[0,1]} |\rho(t)-t| < \delta_2,$$

where δ_2 is given in (3.5). We have

$$\delta_x(\rho(t)) - 2\delta_2 < \delta_y(t) < \delta_x(\rho(t)) + 2\delta_2,$$
$$|\delta_x(\rho(t))| - 2\delta_2 < |\delta_y(t)| < |\delta_x(\rho(t))| + 2\delta_2.$$

First $\rho^{-1}(t_i) \in \Delta_i = (t_i - \beta, t_i + \beta)$ because $|\rho(t_i) - t_i| < \delta_2$ and $\delta_2 < \beta$. Moreover $|\delta_y(\rho^{-1}(t_i))| > |\delta_x(t_i)| - 2\delta_2 \geqslant 2\varepsilon_0 - \frac{1}{2}\varepsilon_0 = \frac{3}{2}\varepsilon_0 > \varepsilon_0 > \varepsilon_i$.
If $t \in \Delta_i \backslash \{\rho^{-1}(t_i)\}$, we have also $\rho(t) \in \Delta_i' = (t_i - \delta_1, t_i + \delta_1)$ because $|\rho(t) - t| < \delta_2$ and $\beta = \delta_1 - \delta_2$.
Whence, since

- $|\delta_x(\rho(t))| \leqslant \varepsilon_0/2$ because $\rho(t) \neq t_i$ single instant in Δ_i' when occurs a jumps of x larger than $\varepsilon_0/2$,
- $2\delta_2 < \varepsilon_1 - \varepsilon_0/2$ by choice (3.5) of δ_2

we have

$$|\delta_y(t)| < |\delta_x(\rho(t))| + 2\delta_2 \leqslant \frac{\varepsilon_0}{2} + 2\delta_2 < \varepsilon_1 \leqslant \varepsilon_i.$$

Consequence. For $t \in \Delta_i$,

- if $t = \rho^{-1}(t_i)$ then $t \in \Delta_i, |\delta_y(t)| > \varepsilon_i, \delta_y(t)$ has the same sign as $\delta_x(t_i)$,
- if $t \neq \rho^{-1}(t_i)$ then $|\delta_y(t)| < \varepsilon_i$.

Observe moreover that for $t \in U_i^\varepsilon, t \neq \rho^{-1}(t_i)$

- $$|\delta_y(t)| \leqslant |\delta_x(\rho^{-1}(t))| + 2\delta_2 < |\delta_x(t_i')| + 2\delta_2 \tag{4.14}$$
  because $\rho^{-1}(t) \neq t_i$ implies $|\delta_x(\rho^{-1}(t))| < |\delta_x(t_i')|$,
- $$|\delta_y(\rho^{-1}(t_i))| > |\delta_x(t_i)| - 2\delta_2. \tag{4.15}$$

By choice (3.5): $\delta_2 < (1/4) \min_{i=1,\dots,N} \{|\delta_x(t_i)| - |\delta_x(t_i')|\}$, we deduce from (4.14) and (4.15) $|\delta_y(t)| < |\delta_y(\rho^{-1}(t_i))|$. Then we have $\rho^{-1}(t_i) = T_{U_i^\varepsilon}(y)$ and

$$(\rho^{-1}(t_1), \dots, \rho^{-1}(t_N)) = T_\varepsilon(y).$$

Writing $(s_i)_{i>0}$ for the jump-times of y, the preceding considerations lead to

- $\omega_y^i(s_k) = 0 \quad \text{if} \quad s_k \notin \cup_{j=1}^{b_i} \Delta_j^i,$

- $\omega_y^i(s_k)\neq 0$ if $s_k\notin\cup_{j=1}^{b_i}\Delta_j^i$ and $s_k=\rho^{-1}(t_j^i)$.

We estimate now coefficient $A_{\gamma_y,b}$ for $y\in V(x)$ representing orbit $\gamma=\gamma_y$ such that $\pi^{-1}(\gamma_y)\cap V(x)\neq\emptyset$

$$A_{\gamma_y,b}=\sum_{s_1,\dots,s_N}\prod_{i=1}^{b_1}\omega_y^1(s_i)\dots\prod_{i=b_1+\cdots b_{p-1}}^{b_1+\cdots b_p}\omega_y^p(s_i)\bar{\phi}_b(s_1,\dots,s_N).$$

The jump-times $s_1,\dots,s_N$, or possibly their images by a permutation in the "b-blocks", are equal to $\rho^{-1}(t_1),\dots,\rho^{-1}(t_N)$; by invariance of $\bar{\phi}_b$ by a permutation preserving "b-blocks", we derive

$$A_{\gamma_y,b}=\pm\frac{N!}{\prod_{i=1}^d b_i!}\tau^N\bar{\phi}_b(s_i,\dots,s_N)\neq 0$$

because $(s_1,\dots,s_N)=T_\varepsilon(y)\in A_{\bar{\phi}_b}$ and $y\in\chi_\varepsilon$. The coefficient $A_{\gamma_y,b}$ is thus not zero.

4.4. Conclusion

For any γ with $\pi^{-1}(\gamma)\cap V(x)\neq\emptyset$, the restriction $F_{\gamma,V(x)}$ of F to trace of orbit $\pi^{-1}(\gamma)\cap V(x)$, has a non-degenerated Jacobian (a polynomial with at least one coefficient non-zero). Since moreover $P_\gamma\ll\lambda_\gamma^p$, we have finally

$$P_{\gamma,V(x)}F^{-1}\ll\lambda^p.$$

Chaining arguments

$$P_{V(x)}F^{-1}\ll\lambda^p\overset{localization}{\Rightarrow}P\chi_\varepsilon F^{-1}\ll\lambda^p\overset{approximation}{\Rightarrow}PF^{-1}\ll\lambda^p,$$

one obtains absolute continuity of $\mathscr{L}(S_{d_1}(f_1),\dots,S_{d_p}(f_p))$ with respect to λ^p and so those of $\mathscr{L}(I_{d_1}(f_1),\dots,I_{d_p}(f_p))$ by Representation Theorem 1.1. This ends the proof of Theorem 2.1. □

A. APPENDIXES

We proove now Lemmas 3.1, 3.2, 4.1 and Proposition 4.1, all about local fields.

A.1. Proof of Lemma 3.1. The set $A(l)$ defined in (3.10) is open in $\mathbb{D}$ Since the case of $A(l)^+$ and $A(l)^-$ are similar, we restrict ourselves to $m=1$ and $A(l)=A(l)^+$. Supppose l is defined by parameters $\varepsilon>0$, interval $\Delta=(a,b)$ and $\tau>0$. Consider $x\in A(l)$ and introduce

$\alpha_- = \inf\{s \in (a,b) | \delta_x(s) > \varepsilon\}$, $\alpha_+ = \sup\{s \in (a,b) | \delta_x(s) > \varepsilon\}$,
$\beta_- = \sup\{s < a | \delta_x(s) \geqslant \varepsilon\}$, $\beta_- = \inf\{s > b | \delta_x(s) \geqslant \varepsilon\}$,
$\gamma_- = \sup\{\delta_x(s) | s \in (\beta_-, \beta_+), \delta_x(s) < \varepsilon\}$, $\gamma_+ = \inf\{\delta_x(s) | s \in (\beta_-, \beta_+), \delta_x(s) < \varepsilon\}$.

Since $\delta_x(a) < \varepsilon, \delta_x(b) < \varepsilon$ we have $a < \alpha_- \leqslant \alpha_+ < b$ because inf are min by finiteness of jumps larger than ε. Similarly, $\beta_- < a < b < \beta_+$ and $\gamma_- < \varepsilon < \gamma_+$. Consider so far

$$r < \frac{1}{2} \min\{\gamma_+ - \varepsilon, \varepsilon - \gamma_-, a - \beta_-, b - \beta_+, \alpha_- - a, b - \alpha_+\}. \tag{A.1}$$

Let $V(x) = B(x, r)$ be a neighbourhood of x in $\mathbb{D}$ equipped with the Skorohod topology. Consider $y \in V(x)$, there is $\rho \in \Lambda$ with

$$\sup_{t \in [0,1]} |x(t) - y(\rho(t))| < r \quad \text{and} \quad \sup_{t \in [0,1]} |\rho(t) - t| < r. \tag{A.2}$$

We have

$$\delta_x(t) - 2r < \delta_y(\rho(t)) < \delta_x(t) + 2r.$$

First, it is easy to deduce from the second part of (A.2),

$$(a,b) \subset \rho\{a - r, b + r)\}, \quad \{a\} \subset \rho\{(a - r, a + r)\} \quad \text{and} \quad \{b\} \subset \rho\{(b - r, b + r)\}.$$

For $t \in (a - r, b + r) \subset (\beta_-, \beta_+)$, we have $\delta_x(t) \notin (\gamma_-, \gamma_+)$ and since $|\delta_y(\rho(t)) - \delta_x(t)| \leqslant 2r$, we obtain $\delta_y(\rho(t)) \neq \varepsilon$ that is: $\forall s \in (a,b), \delta_y(s) \neq \varepsilon$. For $t \in (a - r, a + r) \subset (\beta_-, \alpha_-)$, we have $\delta_x(t) < \varepsilon$, from the definition of γ_-, we even have $\delta_x(t) \leqslant \gamma_-$, and $\delta_y(\rho(t)) < \delta_x(t) + 2r \leqslant \gamma_- + 2r < \varepsilon$.

Let $t \in (a,b)$ be a "transformable" jump-time of x by l. From the definitions of α_-, α_+, we have $\alpha_- < t < \alpha_+$ and it follows $\rho(t) \in (\alpha_- - r, \alpha_+ + r) \subset (a,b)$. Moreover

$$\delta_y(\rho(t)) > \delta_x(t) - 2r \geqslant \gamma_+ - 2r > \varepsilon.$$

Finally, at time $\rho(t)$, y has also a "transformable" jump. It follows $y \in A(l)$ and $V(x) \subset A(l)$, so that $A(l)$ is open.

If the local field is more complex, with $m \geqslant 2$, start by defining $\alpha^i_-, \alpha^i_+, \beta^i_-, \beta^i_+, \gamma^i_-, \gamma^i_+$ relatively to each subinterval $\Delta_i := (a_i, b_i)$ as previously. In order to avoid overlapping, taking $\eta < \frac{1}{4} \min_{i \leqslant m-1} |a_{i+1} - b_i|$, we demand to β^i_-, β^i_+ to satisfy moreover

$$a_i - \eta < \beta^i_-, \ \ \beta^i_+ < b_i + \eta.$$

Associating r_i to each Δ_i by (A.1), take $r < \min_{i \leqslant m}\{r_i, \eta\}$. It is easy to see from the case $m = 1$ that for $x \in A(l)$, we still have $B(x, r) \subset A(l)$. □

Remark A.1. As the preceding proof of Lemma 3.1 suggests, we can connect "transformable" jumps for x and y. More precisely,

- if $\delta_x(t) > \varepsilon, t \in \Delta_i$ then $t \in (\alpha_-^i, \alpha_+^i)$ and $\delta_x(t) \geqslant \gamma_+^i$ so that $\rho(t) \in \Delta_i, \delta_y(\rho(t)) > \varepsilon : y$ has at $\rho(t)$ a transformable jump,
- if $\delta_x(t) < \varepsilon, t \in (\beta_-^i, \beta_+^i)$ then $\delta_x(t) \leqslant \gamma_-^i$, so that $\delta_y(\rho(t)) < \varepsilon$: the jump of y at $\rho(t)$ is not transformable,
- if $t \notin \cup_i (\beta_-^i, \beta_+^i)$ then $\rho(t) \notin \cup_i \Delta_i$: the jump of y at $\rho(t)$ is not transformable,
- if $\delta_x(t) > \varepsilon$ and $t \notin \cup_i \Delta_i$ then $t \notin \cup_i (\beta_-^i, \beta_+^i)$ and so $\rho(t) \notin \cup_i \Delta_i$, the jump of y at $\rho(t)$ is not transformable.

By symmetry, we have in fact a bijection between transformable jumps for x and for y. This bijection is given by $\rho \in \Lambda$ realizing $d(x, y)$ as in (A.2).

A.2. Proof of Lemma 3.2. The local field l is continuous on $A(l)$
From (3.6), $\varphi_{i,s}^+ \equiv 0$ if $s \in J_x^+(\varepsilon_i) \setminus \cup_j \Delta_j^i$ and Definition (3.7) of l can be rewritten

$$l_x = \sum_{i \mid \tau_i > 0} \tau_i \sum_{s \in J_x^+(\varepsilon) \cap \Delta_i} \mathbf{1}_{t \geqslant s} - \sum_{i \mid \tau_i < 0} \tau_i \sum_{s \in J_x^-(\varepsilon) \cap \Delta_i} \mathbf{1}_{t \geqslant s}.$$

Let $x \in A(l), B(x, r) \subset A(l)$ neighbourhood of $x, y \in B(x, r)$ and $\rho \in \Lambda$ such that $d(x, y) < r$ (like in (A.2)). Note $s_1^i, \ldots, s_{p_i}^i$ for the transformable jump-times of x in Δ_i those of y are thus $\rho(s_j^i)$, $j \leqslant p_i$ (see Remark A.1). We have

$$\begin{aligned} l_x &= \sum_{i \mid \tau_i > 0} \tau_i \sum_{j \leqslant p_i} \mathbf{1}_{t \geqslant s_j^i} - \sum_{i \mid \tau_i < 0} \tau_i \sum_{j \leqslant p_i} \mathbf{1}_{t \geqslant s_j^i}, \\ l_y &= \sum_{i \mid \tau_i > 0} \tau_i \sum_{j \leqslant p_i} \mathbf{1}_{t \geqslant \rho\left(s_j^i\right)} - \sum_{i \mid \tau_i < 0} \tau_i \sum_{j \leqslant p_i} \mathbf{1}_{t \geqslant \rho\left(s_j^i\right)}. \end{aligned}$$

And it easily follows:

$$l_y \circ \rho = l_x. \tag{A.3}$$

Whence

$$d(l_x, l_y) \leqslant \sup_{t \in [0,1]} |l_x(t) - l_y(\rho(t))| + \sup_{t \in [0,1]} |\rho(t) - t| = \sup_{t \in [0,1]} |\rho(t) - t| \leqslant r.$$

The lemma is thus proved. □

A.3. Proof of Lemma 4.1.

Lemma 4.1. $\{G_c\}_{c\in(\mathbb{R}^+)^p}$ in (4.1) defines a admissible semigroup on $A(l)$:

(i) $G_0 = id$,
(ii) $G_{c_1} \circ G_{c_2} = G_{c_1+c_2}$,
(iii) $\forall c \in (\mathbb{R}^+)^p$, G_c is one-to-one and admissible,
(iv) $\forall x \in A(l), c \mapsto G_c x$ is one-to-one.

Proof. We check points **(ii)–(iv), (i)** being clear
(ii) We intend to see $G_{c+d}(x) = G_c(G_d(x))$. Since

$$G_c(G_d(x)) = G_d(x) + \sum_{i=1}^{p} c_i l^i_{G_d(x)},$$

it is enough to show $l^i_{G_d(x)} = l^i_x$ for $1 \leqslant i \leqslant p$.

Let see more generally that if there are $c, d \in (\mathbb{R}^+)^p$ such that $G_c(x) = G_d(y)$ then $l^i_x = l^i_y$ for all $i = 1, \ldots, p$. To this end, use notations (4.4)–(4.6) for the ith local field l_i. Since from (3.6), $\varphi^+_{i,s} \equiv 0$ if $s \in J^+_x(\varepsilon_i) \setminus \cup_{j=1}^{m_i} \Delta^i_j$, (3.7) can be rewritten

$$l^i_x = \sum_{s\in S^+_i(x)} (\varphi_{i,s})^+ - \sum_{s\in S^-_i(x)} (\varphi_{i,s})^-. \tag{A.4}$$

Thus, it is enough to see that for x, y with $G_c(x) = G_d(y)$

$$S^-_i(x) = S^-_i(y), \quad S^+_i(x) = S^+_i(y). \tag{A.5}$$

But

$$x + c_1 l^1_x + \cdots + c_p l^p_x = y + d_1 l^1_y + \cdots + d_p l^p_y.$$

Let $s \in S^+_i(x)$, for example $s \in \Delta^i_j$, in a neighbourhood of s, for $k \neq i$ and $t \in \cup_{k\neq i} S_k(x)$, $\varphi_{k,t}$ is clearly constant and so are l^k_x, l^k_y. Then

$$\delta_{G_d(y)}(s) = \delta_{G_c(x)}(s) = \delta_x(s) + c_i \tau^i_j > \varepsilon_i.$$

So it must be $\delta_y(s) > \varepsilon_i$, and since $s \in \Delta^i_j$, $\tau^i_j > 0$, we have

$$s \in J_y(\varepsilon_i) \cap \left\{\cup_{j|\tau^i_j>0} \Delta^i_j\right\} = S^+_i(y).$$

It follows $S^+_i(x) \subset S^+_i(y)$ and by symmetry the second part of (A.5). Doing the same for $S^-_i(x), S^-_i(y)$ completes (A.5).

Then from (A.4), we derive $l_x^i = l_y^i$ and in particular $l_{G_d(x)}^i = l_x^i$ so that $G_c(G_d(x)) = G_{c+d}(x)$.
(iii) For any $c = (c_1, \dots, c_p)$, $G_c x = x + c_1 l_x^1 + \dots + c_p l_x^p$ is a transformation $x \mapsto x + \tilde{c}\tilde{l}_x$ for some local field $\tilde{l}$ and $\tilde{c} \in \mathbb{R}$. Admissibility is thus due to Ref. 6, theorem 21.1, which asserts admissibility of such transformations. To see $x \longmapsto G_c(x)$ is one-to-one, consider x, y such that $G_c(x) = G_c(y)$:

$$x + \sum_{i=1}^{p} c_i l_x^i = y + \sum_{i=1}^{p} c_i l_y^i.$$

If $t \notin \cup_{i,j} \Delta_j^i$, since the local fields are constant close to t, we have $\delta_x(t) = \delta_y(t)$. Else, for example $t \in \Delta_j^i$, and

- if $|\delta_x(t)| > \varepsilon_i$, we have $|\delta_{G_c(x)}(t)| = |\delta_x(t) + c_i \omega_x^i(t)| > \varepsilon_i$. So $|\delta_{G_c(y)}(t)| > \varepsilon_i$ and necessarily, $|\delta_y(t)| > \varepsilon_i$, with $\delta_y(t)$ with the same sign as $\delta_x(t)$, so that $\delta_{G_c(y)}(t) = \delta_y(t) + c_i \omega_y^i(t)$ with $\omega_x^i(t) = \omega_x^i(t) = \tau_j^i$, thus
$$\delta_x(t) = \delta_y(t).$$
- if $|\delta_x(t)| \leqslant \varepsilon_i$, by the local constancy of local fields
$$\delta_x(t) = \delta_{G_c(x)}(t) = \delta_{G_c(y)}(t) = \delta_y(t).$$

Since moreover $x(0) = y(0)$, we derive $x = y$.
(iv) Let $x \in A(l)$ be fixed, $c \mapsto G_c(x)$ is one-to-one indeed:
For c, d such that $G_c(x) = G_d(x)$ and $t \in \Delta_j^i$ a "l^i-transformable" jump-time of x (that is with $|\delta_x(t)| > \varepsilon_i$), we have

$$\delta_{G_c(x)}(t) = \delta_x(t) + c_i \omega_x^i(t), \quad \delta_{G_d(x)}(t) = \delta_x(t) + d_i \omega_x^i(t)$$

since $\omega_x^i(t) = \tau_j^i \neq 0$, we derive $c_i = d_i$. Doing the same for the other coordinates, we obtain $c = d$. □

A.4. Proof of Proposition 4.1

Proposition 4.1. Let $f : A(l) \to \mathbb{D}$ be given by

$$f(x) = x - c_1(x) l_x^1 - \dots - c_p(x) l_x^p.$$

f is continuous and generates the partition Γ defined in (4.3), which is thus measurable.

The proof consists in the following three lemmas, the first one is a by-product of the proof of **(ii)** in Lemma 4.1

Lemma A.1.

$$x \sim y \Longrightarrow \left(l_x^1, \dots, l_x^p\right) = \left(l_y^1, \dots, l_y^p\right).$$

Lemma A.2.

$$x \sim y \Longleftrightarrow f(x) = f(y).$$

Proof.
(1) $x \sim y \Longrightarrow f(x) = f(y)$:
From Definition (4.8) of f and Lemma A.1, it remains to connect $c_i(x)$ and $c_i(y)$. We still have (A.5). Consider first index p, in a neighbourhood of $s \in S_p^+(x) = S_p^+(y)$, l_y^i, l_x^i are constant for $i < p$. Let $c, d \in (\mathbb{R}^+)^p$ such that $G_c(x) = G_d(y)$. We have

$$\delta_{G_c x}(s) = \delta_{G_d y}(s) \Longleftrightarrow \delta_x(s) + c_p \tau_{i_p(s)}^p = \delta_y(s) + d_p \tau_{i_p(s)}^p.$$

So that

$$\frac{\delta_x(s) - \varepsilon_p}{\tau_{i_p(s)}^p} = \frac{\delta_y(s) + (d_p - c_p)\tau_{i_p(s)}^p - \varepsilon_p}{\tau_{i_p(s)}^p} = \frac{\delta_y(s) - \varepsilon_p}{\tau_{i_p(s)}^p} + d_p - c_p.$$

In the same way for $s \in S_p^-(x)$

$$\frac{|\delta_x(s)| - \varepsilon_p}{|\tau_{i_p(s)}^p|} = \frac{|\delta_y(s)| - \varepsilon_p}{|\tau_{i_p(s)}^p|} + d_p - c_p.$$

Taking minimum for $s \in S_p(x) = S_p^-(x) \cup S_p^+(x) = S_p^-(y) \cup S_p^+(y) = S_p(y)$, we obtain

$$\begin{aligned} c_p(x) &= c_p(y) + d_p - c_p, \\ c_p(x) + c_p &= c_p(y) + d_p. \end{aligned}$$

Doing the same relatively to the other local fields, we obtain with Lemma A.1

$$\begin{aligned} x \sim y &\Longleftrightarrow x + c_1 l_x^1 + \cdots + c_p l_x^p = y + d_1 l_y^1 + \cdots + d_p l_y^p \text{ for some } c, d \in (\mathbb{R}^+)^p \\ &\Longrightarrow x + (c_1 - d_1) l_x^1 + \cdots + (c_p - d_p) l_x^p = y \\ &\Longrightarrow x + (c_1(y) - c_1(x)) l_x^1 + \cdots + (c_p(y) - c_p(x)) l_x^p = y \\ &\Longrightarrow x - c_1(x) l_x^1 - \cdots - c_p(x) l_x^p = y - c_1(y) l_y^1 - \cdots - c_p(y) l_y^p \\ &\Longrightarrow f(x) = f(y). \end{aligned}$$

(2) $f(x)=f(y)\Rightarrow x\sim y$
Let $s\in S_p^+(x):\delta_x(s)>\varepsilon_p, s\in\Delta^p_{i_p(s)}, \tau^p_{i_p(s)}>0$.
In a neighbourhood of s, since local fields l_x^i $i\leqslant p-1$ are constant, we have by definition of $c_p(x)$

$$\delta_{f(y)}(s)=\delta_{f(x)}(s)=\delta_x(s)-c_p(x)\tau^p_{i_p(s)}\geqslant\varepsilon_p.$$

But in a neighbouthood of $s\in\Delta^p_{i_p(s)}$, we have also l_y^i constant for $i\leqslant p-1$, so that $\delta_{f(y)}(s)=\delta_y(s)-c_p(y)\tau^p_{i_p(s)}\geqslant\varepsilon_p>0$ and

$$\delta_y(s)\geqslant\varepsilon_p+c_p(y)\tau^p_{i_p(s)}\geqslant\varepsilon_p. \tag{A.6}$$

Since $y\in A(l)\subset A(l^p)$, y does not have jumps of length exactly ε_p in $\cup_j\Delta_j^p$ and the right-hand side of (A.6) must be a strict inequality. Then since $s\in\Delta^p_{i_p(s)}$ and $\tau^p_{i_p(s)}>0$, we obtain

$$s\in J_y^+(\varepsilon_p)\cap\left\{\cup_j\Delta_j^p\right\}=S_p^+(y).$$

So $S_p^+(x)\subset S_p^+(y)$ and equality by symmetry. We can also do the same to derive $S_p^-(x)=S_p^-(y)$ and $S_p(x)=S_p(y)$. From (A.4), we deduce $l_x^p=l_y^p$ and arguing the same for $i<p$, we obtain $l_x^i=l_y^i, i\leqslant p$. Finally

$$\begin{aligned}
f(x)=f(y) &\Longleftrightarrow x-c_1(x)l_x^1-\cdots-c_p(x)l_x^p=y-c_1(y)l_y^1-\cdots-c_p(y)l_y^p\\
&\Longrightarrow x+c_1(y)l_x^1-\cdots+c_p(y)l_x^p=y+c_1(x)l_y^1+\cdots+c_p(x)l_y^p\\
&\Longrightarrow G_{c(y)}x=G_{c(x)}(y)\\
&\Longrightarrow x\sim y.
\end{aligned}$$

□

Lemma A.3. The function f defined in (4.8) is continuous on $A(l)$.

Proof. First, consider the case of a transformation G_c defined like in (4.1) but from a single local field ($p=1$ and $G_c(x)=x+cl_x$). Still consider $S^+(x), S^-(x), S(x)$ as in (4.4)–(4.6) and for $s\in S(x), i(s)$ the index such that $s\in\Delta_{i(s)}$. Finally introduce $c(x)$ and f as in (4.7) and (4.8).
Let $x\in A(l)$ and $B(x,r)$ neighbourhood of x as in Appendix A.1.
Let for some $\epsilon>0, y\in B(x,r), d(x,y)\leqslant\epsilon$, where d is the Skorohod metric; there is $\rho\in\Lambda$ with

$$\sup_{t\in[0,1]}|x(t)-y(\rho(t))|<\epsilon,\qquad \sup_{t\in[0,1]}|\rho(t)-t|<\epsilon$$

and with also (A.3) holding true from the proof of Lemma 3.2 in Appendix A.2. We have seen in Remark A.1, there is a bijection between "transformable" jump-times in Δ_i of x and those of y given by ρ. Let $s \in S^+(x)$, we have

$$\delta_x(s) - 2\epsilon < \delta_y(\rho(s)) < \delta_x(s) + 2\epsilon,$$

whence

$$\frac{|\delta_x(s) - \varepsilon|}{|\tau_{i(s)}|} - \frac{2\epsilon}{|\tau_{i(s)}|} < \frac{|\delta_y(\rho(s))| - \varepsilon}{|\tau_{i(s)}|} < \frac{|\delta_x(s)| - \varepsilon}{|\tau_{i(s)}|} + \frac{2\epsilon}{|\tau_{i(s)}|}.$$

Doing the same for $s \in S^-(x)$ and using the bijection between $S(x)$ and $S(y)$, we have taking minimum for $s \in S(x)$

$$c(x) - \frac{2\epsilon}{\tau_+} < c(y) < c(x) + \frac{2\epsilon}{\tau_-} \tag{A.7}$$

with $\tau_- = \min_i |\tau_i|$, $\tau_+ = \max_i |\tau_i|$. Next, we have

$$\begin{aligned} &|f(y)(\rho(t)) - f(x)(t)| \\ &\quad = |y(\rho(t)) - c(y)l_y(\rho(t)) - x(t) + c(x)l_x(t)| \\ &\quad \leqslant |y(\rho(t)) - x(t)| + |c(y)l_y(\rho(t)) - c(y)l_x(t)| + |c(y)l_x(t) - c(x)l_x(t)|. \end{aligned}$$

Denoting $\|\cdot\|$ the uniform norm on [0,1], we derive

$$\|f(x) - f(y)\circ\rho\| \leqslant \|x - y\circ\rho\| + |c(y)|\|l_x - l_y\circ\rho\| + \|l_x\||c(y) - c(x)|.$$

But

$$\|l_x\| \leqslant \tau_+, \quad |c(y) - c(x)| \leqslant 2\epsilon/\tau_-, \quad \|x - y\circ\rho\| \leqslant \epsilon,$$

so that with (A.3)

$$d(f(x), f(y)) \leqslant 2(1 + \tau_+/\tau_-)\epsilon.$$

We derive continuity of f in this case.

Consider now the general case of $\{G_c\}_c$ defined by p local fields $l^1, \dots, l^p$ like in (4.1) and satisfying the conditions given at the end of Section 3, with moreover

$$\varepsilon_i < \varepsilon_{i+1}, \quad \left\{\cup_{j\leqslant m_i}\Delta^i_j\right\} \cap \left\{\cup_{j\leqslant m_{i'}}\Delta^{i'}_j\right\} = \emptyset, \quad i \neq i'.$$

If $x \in A(l^i)$ then $x + c_i l^i_x \in A(l^i)$, but also $x + c_j l^j_x \in A(l^i)$ since l^j acts only on jumps of x in $\cup_{k\leqslant m_j}\Delta^j_k$, so that conditions for being in $A(l^i)$, involving only jumps in $\cup_{k\leqslant m_i}\Delta^i_k$, are not altered. Thus $x + c_j l^j_x \in A(l^i)$ for all $1 \leqslant j \leqslant p$.

Both Lemma 3.2 and (A.7) give continuity of $x \longmapsto c_i(x)l_x^i$ on $A(l^i)$. We are here interested in f given by (4.8). Define

$$f_i : \begin{cases} A(l^i) & \longrightarrow \mathbb{D} \\ x & \longmapsto x - c_i(x)l_x^i. \end{cases}$$

For $x \in A(l)$, the preceding considerations ensure f_i is continuous on $A(l) \subset A(l^i)$, moreover $f_1(x) = x - c_1(x)l_x^1 \in A(l^2)$ because conditions for being in $A(l^2)$ are not altered by l^1. Moreover, we have

$$f_2 \circ f_1(x) = f_1(x) - c_1(f_1(x))l^2_{f_1(x)}$$

but since f_1 does not act on jumps outside of $\cup_{j \leqslant m_1} \Delta_j^1$, we deduce $l^2_{f_1(x)} = l_x^2$. Similarly

$$c_2(f_1(x)) = \min_{s \in S_2(f_1(x))} \left\{ \frac{|\delta_{f_1(x)}(s)| - \varepsilon_2}{\tau^2_{i(s)}} \right\}.$$

But $S_2(f_1(x)) = S_2(x)$ and for $s \in S_2(x)$, we have $\delta_{f_1(x)}(s) = \delta_x(s)$, then $c_2(f_1(x)) = c_2(x)$ and finally

$$f_2 \circ f_1(x) = f_1(x) - c_2(x)l_x^2 = x - c_1(x)l_x^1 - c_2(x)l_x^2.$$

Moreover $f_2 \circ f_1(x) \in A(l^3)$. Thus, we infer by an immediate induction

$$f_p \circ \cdots \circ f_1(x) = x - c_1(x)l_x^1 - \cdots - c_p(x)l_x^p = f(x).$$

Finally since each f_i is continuous on $A(l) \subset A(l^i)$ we obtain the continuity of f on $A(l)$. $\square$

From Lemma A.2, f generates a partition Γ and is continuous by Lemma A.3.

The partition Γ is thus measurable and Proposition 4.1 is thereby proved.

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