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Confidence intervals for the Hurst parameter of a fractional Brownian motion based on concentration inequalities

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Abstract

In this paper, we propose, for finite sample size, exact confidence intervals of the Hurst index parametrizing a fractional Brownian by using concentration inequalities. Both cases where the scaling parameter of the fractional Brownian motion is known or unknown are considered. These intervals are obtained by observing a single discretized sample path of a fractional Brownian motion and without any assumption on the parameter H.

Keywords: concentration inequalities, confidence intervals, fractional Brownian motion, Hurst parameter

Contents

1 Introduction

Since the pioneer work of Mandelbrot and Ness (1968), the fractional Brownian motion (fBm) has become widely popular as well as in a theoretical context as in applications. Fractional Brownian motion can be defined as the only centered Gaussian process, denoted by $(B_H(t))_{t\in\mathbb{R}}$, with stationary increments and with variance function $v(\cdot)$, given by $v(t) = C^2|t|^{2H}$ for all $t \in \mathbb{R}$. The parameter $H \in (0,1)$ (resp. $C > 0$) is referred to as the Hurst parameter (resp. the scaling coefficient). In particular, when $H = 1/2$, it is the standard Brownian motion. In general, the fractional Brownian motion is an H-self-similar process, that is for all $\delta > 0$, $(B_H(\delta t))_{t\in\mathbb{R}} \stackrel{d}{=} \delta^H (B_H(t))_{t\in\mathbb{R}}$ (where $\stackrel{d}{=}$ means equal in finite-dimensional distributions) with autocovariance function behaving like $O(|k|^{2H-2})$ as $|k| \to +\infty$. Thus, the discretized increments of the fractional Brownian motion (called the fractional Gaussian noise) constitute a short-range dependent process, when $H < 1/2$, and a long-range dependent process, when $H > 1/2$. The index H characterizes also the path regularity since the fractal dimension of the fractional Brownian motion is equal to $D = 2 - H$. General references on self-similar processes and long-memory processes are given in Beran (1994) or Doukhan et al. (2003).

The aim of this paper is to construct confidence intervals for the Hurst parameter based on a single observation of a discretized sample path of the interval $[0,1]$ of a fractional Brownian motion. To do so, the most popular strategy consists in using the *asymptotic normality* of some estimators of the Hurst parameter, see Coeurjolly (2000) for a survey on the estimation of the self-similarity or Shen et al. (2007) and Coeurjolly (2008) for more recent discussions in a robust context. Recently, a new strategy based on concentration inequalities for Gaussian processes obtained by Nourdin and Viens (2009) has been proposed by Breton et al. (2009). In this case, the confidence intervals are *non-asymptotic* and they appear to be very interesting when the sample size is moderate. Our contribution is to improve this direction both from a theoretical and practical point of view. In order to present our different contributions, let us first recall the confidence interval proposed by Breton et al. (2009).

Proposition 1 *Assume that one observes a fractional Brownian motion at times i/n for $i = 0,\dots,n+1$ with scaling coefficient $C = 1$ and with Hurst parameter satisfying $H \le H^\star$ for some known $H^\star \in (0,1)$. Fix $\alpha \in (0,1)$, then for all n large enough satisfying $q_n(\alpha) < (4 - 4^{H^\star})\sqrt{n}$, where $q_n(\alpha) := \frac{1}{2}\left(b(\alpha) + \sqrt{b(\alpha)^2 + 852\log\left(\frac{2}{\alpha}\right)}\right)$ with $b(\alpha) := \frac{71}{\sqrt{n}}\log\left(\frac{2}{\alpha}\right)$, we have*

$$\mathbb{P}\left(H \in \left[\max\left(0, \widetilde{H}_n^{inf}(q_n(\alpha))\right), \widetilde{H}_n^{sup}(q_n(\alpha))\right]\right) \ge 1 - \alpha, \tag{1}$$

where for $t > 0$

$$\begin{aligned} g_n\left(\widetilde{H}_n^{inf}(t)\right) &:= \frac{1}{2} - \frac{\log(S_n)}{2\log(n)} + \frac{\log\left(1 - \frac{t}{(4-4^{H^\star})\sqrt{n}}\right)}{2\log(n)} \\ g_n\left(\widetilde{H}_n^{sup}(t)\right) &:= \frac{1}{2} - \frac{\log(S_n)}{2\log(n)} + \frac{\log\left(1 + \frac{t}{(4-4^{H^\star})\sqrt{n}}\right)}{2\log(n)} \end{aligned}$$

where g_n is the function defined by $g_n(x) = x - \frac{\log(4-4^x)}{2\log(n)}$ and S_n is the following statistic

$$S_n := \frac{1}{n}\sum_{i=1}^{n}\left(B_H\left(\frac{i+1}{n}\right) - 2B_H\left(\frac{i}{n}\right) + B_H\left(\frac{i-1}{n}\right)\right)^2. \tag{2}$$

Let us give some general comments on this result. First, note that this procedure cannot be applied to a fractional Brownian motion whose scaling coefficient C is unknown. Secondly, important drawbacks of this procedure rely upon the assumptions made on $H^\star$ and n, which exclude the possibility to use this confidence interval when the sample size is small:

- Given α and $H^\star$, the following table presents the minimal value of the sample size n in order to ensure that $q_n(\alpha) < (4 - 4^{H^\star})\sqrt{n}$.

	$H^\star$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\alpha = 1\%$	271	298	335	388	471	611	886	1592	4936
$\alpha = 5\%$	189	208	233	270	328	425	617	1108	3437
$\alpha = 10\%$	154	169	190	220	266	346	501	900	2791

- The following table exhibits the maximal value of $H^\star$, denoted by $\widetilde{H}^\star$, required in order to ensure $q_n(\alpha) < (4-4^{H^\star})\sqrt{n}$ in terms of α and n. Note that $\widetilde{H}^\star = \log\left(\max\left(1, 4 - q_n(\alpha)/\sqrt{n}\right)\right) / \log(4)$, which means that, given α and n, a confidence interval is only available for $H \in (0, \widetilde{H}^\star)$.

	n					
	50	100	200	500	10000	10000
$\alpha = 1\%$	0.00	0.00	0.00	0.53	0.93	0.93
$\alpha = 5\%$	0.00	0.00	0.17	0.65	0.94	0.94
$\alpha = 10\%$	0.00	0.00	0.34	0.70	0.95	0.95

We are now in position to specify our different contributions:

- We slightly improve the bounds of the concentration inequality obtained by Nourdin and Viens (2009), see Section 2 and Proposition 2 for more details. Note in particular that, in contrast to Nourdin and Viens (2009) and Breton et al. (2009), we are tracing the constant to optimize numerically our bounds.

- In the case where the scaling parameter C is known, we propose a new confidence interval without any preliminary assumption on the Hurst parameter H and with a very slight condition on the sample size. For instance, in comparison to the previous tables, our confidence interval is computable as soon as $n \geq 3$. Furthermore, by using ideas similar in Coeurjolly (2001) for the problem of the estimation of the Hurst parameter, we also propose a confidence interval when the scaling parameter C is unknown. This new confidence interval has the nice property to be independent of C and independent of the discretization step. It is remarkable that, in the both cases (C known or unknown), the lengths of the confidence intervals we propose behave asymptotically like the ones derived in an asymptotic approach, that is they behave like $1/\sqrt{n}\log(n)$ when C is known and $1/\sqrt{n}$ when C is unknown.

- As suggested by the expression of the statistic (2), the procedure described in Proposition 1 is based on the increments of order 2 of the discretized sample path of the fractional Brownian motion. Taking the increments of order 2 is a special case of filter to work with and it is known that discrete filtering has been proposed and used in an estimation context, see Istas and Lang (1997), Kent and Wood (1997) and Coeurjolly (2001). Recall that the main interest in filtering the fractional Browian motion is that the action of filtering destroys the correlation. In particular, the increments of order 2 of the fractional

Brownian motion constitute a short-range dependent process (*i.e.* its correlation function is absolutely summable) and this is indeed required to obtain an efficient concentration inequality. In this paper, we propose to construct confidence intervals not only based on the increments of order 2 but on more general filters such as, for instance, increments of larger order or the Daubechies wavelet filters... Finally, let us also underline that a crucial step consists in obtaining an upper-bound of the supremum on the interval $(0, 1)$ of the ℓ^1-norm of the correlation function of the discrete filtered series of the fractional Brownian motion. When considering the increments of order 2, Breton et al. (2009) have obtained the bound $17.75/(4-4^{H^\star})$. We have widely improved this point since we compute explicitly this supremum for a large class of filters (including increments of order 2). As an example, for the increments of order 2, this gives the explicit value 8/3.

- Based on a large simulation study, we assess the efficiency of the different procedures that we propose and we compare them with ones based on an asymptotic scheme. We discuss and comment these (contrasted) results.

The rest of this paper is organized as follows. In Section 2, we give the concentration inequalities specially designed for our purposes. The filtering setting is introduced in Section 3 where the bounds for the ℓ^1-norm of the correlation function of the filtered series are also obtained. Our confidence intervals for the Hurst parameter are proposed and proved in Section 4, both when the scaling parameter is known or unknown. Our results are discussed and compared to the literature in Section 5. Finally, computations expliciting some bounds for some special filters are given in Appendix A.

2 Concentration inequalities

Proposition 1 above is based on concentration inequalities proposed by Nourdin and Viens (2009) (see Proposition 3) for smooth enough random variables with respect to Malliavin calculus (see Theorem 4.1-*i*)). By applying such inequalities to the random variables $\sqrt{n}V_n$ where $V_n = \frac{1}{n}\sum_{i=1}^n H_2(X_i)$, $H_2(t) = t^2 - 1$ is the second Hermite polynomial, and $X = \{X_i\}_{1\le i\le n}$ is a stationary Gaussian process with variance 1 and correlation function ρ, we obtain concentration inequalities for H_2-variations of stationary Gaussian processes. In the sequel, for a sequence $(u_i)_{i\in\mathbb{Z}}$, we set $\|u\|_{\ell^1_n} := \sum_{|i|\le n}|u_i|$.

Proposition 2 *Let $\kappa_n = 2\|\rho\|_{\ell^1_n}$. Then, for all $t > 0$, we have:*

$$\mathbb{P}\left(\sqrt{n}V_n \ge t\right) \quad \le \quad \varphi_{r,n}(t;\kappa_n) := e^{-\frac{t\sqrt{n}}{\kappa_n}}\left(1+\frac{t}{\sqrt{n}}\right)^{\frac{n}{\kappa_n}} \tag{3}$$

$$\mathbb{P}\left(\sqrt{n}V_n \le -t\right) \quad \le \quad \varphi_{l,n}(t;\kappa_n) := e^{\frac{t\sqrt{n}}{\kappa_n}}\left(1-\frac{t}{\sqrt{n}}\right)^{\frac{n}{\kappa_n}}\mathbf{1}_{[0,\sqrt{n}]}(t). \tag{4}$$

Note that Proposition 2 can be applied to short-memory as well as to long-memory stationary Gaussian processes (as soon as n remains finite). In order to derive Proposition 2 below, we shall briefly (and freely) use some notions of Malliavin calculus. We refer to Breton et al. (2009) and references therein for any details. We stress that, once Proposition 2 is derived, only basic probability tools will be used. Without restriction, we assume the Gaussian random variables X_i have the form $X_i = X(h_i)$ where $X(\aleph) = \{X(h) : h \in \aleph\}$ is an isonormal Gaussian process over $\aleph = \mathbb{R}^{n+1}$ and $\{h_i : i = 0, \ldots, n\}$ is a finite subset of $\aleph$ verifying

$\mathbb{E}[X(h_i)X(h_j)] = \rho(i-j) = \langle h_i, h_j\rangle_{\aleph}$. With such a representation, V_n can be seen as a double Wiener-Itô integral $V_n = I_2\left(\frac{1}{n}\sum_{i=0}^{n} h_i \otimes h_i\right)$, so that, to make easier the presentation, we rewrite Th. 4.1 in Nourdin and Viens (2009) only for such random variables. Moreover, in order to optimize our forthcoming results, we slightly improve the bounds of Th. 4.1:

Proposition 3 *Let $Z = I_2(f)$ satisfying*

$$\|DZ\|_{\aleph}^2 \leq aZ + b \tag{5}$$

for some constants $a \geq 0$ and $b > 0$. Then, for all $t > 0$

$$\begin{aligned}
\mathbb{P}(Z \geq t) &\leq \varphi_r(t;a,b) := e^{-\frac{2t}{a}}\left(1+\frac{at}{b}\right)^{\frac{2b}{a^2}} \\
\mathbb{P}(Z \leq -t) &\leq \varphi_l(t;a,b) := e^{\frac{2t}{a}}\left(1-\frac{at}{b}\right)^{\frac{2b}{a^2}} \mathbf{1}_{[0,b/a]}(t).
\end{aligned}$$

Proof: The proof is a slight improvement of the bounds in (Nourdin and Viens, 2009, Theorem 4.1) obtained by a careful reading of the proof (with the following correspondance with the notation therein: $g_Z(Z) = \frac{1}{2}\|DZ\|_{\aleph}^2$, $\alpha = a/2$ and $\beta = b/2$). Denoting by ρ the density of Z, the argument of (Nourdin and Viens, 2009, Theorem 4.1) is based on the following key formula (see (3.16) in Nourdin and Viens (2009))

$$\|DZ\|_{\aleph}^2 = \frac{2\int_Z^{+\infty} y\rho(y)dy}{\rho(Z)}. \tag{6}$$

For the sake of self-containess, we sketch the main steps of the argument. For any $A > 0$, define $m_A : [0,+\infty) \to \mathbb{R}$ by $m_A(\theta) = \mathbb{E}\left[e^{\theta Z}\mathbf{1}_{\{Z\leq A\}}\right]$. We have $m_A'(\theta) = \mathbb{E}\left[Ze^{\theta Z}\mathbf{1}_{\{Z\leq A\}}\right]$ and integration by part yields

$$\begin{aligned}
m_A'(\theta) &= \int_{-\infty}^{A} xe^{\theta x}\rho(x)dx \\
&\leq \theta\int_{-\infty}^{A} e^{\theta x}\left(\int_z^{+\infty} y\rho(y)dy\right)dx \qquad (7)\\
&\leq \frac{\theta}{2}\mathbb{E}\left[\|DZ\|_{\aleph}^2 e^{\theta Z}\mathbf{1}_{\{Z\leq A\}}\right]. \qquad (8)
\end{aligned}$$

where (7) comes from $\int_A^{+\infty} y\rho(y)dy \geq 0$ since $\mathbb{E}[Z] = 0$, and (8) comes from (6). Because of (5), we obtain for any $\theta \in (0, 2/a)$:

$$m_A'(\theta) \leq \frac{\theta b}{2-\theta a} m_A(\theta). \tag{9}$$

Solving (9), using $m_A(0) = \mathbb{P}(Z \leq A) \leq 1$ and applying Fatou's Lemma ($A \to +\infty$) yield the following bound for the Laplace transform and any $\theta \in (0, 2/a)$:

$$\mathbb{E}[e^{\theta Z}] \leq \exp\left(-\frac{b}{a}\theta - \frac{2b}{a^2}\ln\left(1-\frac{a\theta}{2}\right)\right).$$

The Chebychev inequality together with a standard minimization entail:

$$\mathbb{P}(Z \geq t) \leq \exp\left(\min_{\theta\in(0,2/a)}\left\{-\left(t+\frac{b}{a}\right)\theta - \frac{2b}{a^2}\ln\left(1-\frac{a\theta}{2}\right)\right\}\right)$$

The minimization is achieved in $\widetilde{\theta} = (2t)/(at+b)$ and gives the first bound in Proposition 3. Applying the same argument to $Y = -Z$, satisfying $\|DY\|^2_{\aleph} \le -aY + b$, we derive similarly the second bound. Note in particular that condition 5 implies that $Z \ge -b/a$ so that the left tail only makes sense for $t \in (-b/a, 0)$. $\square$

Remark 1 *Nourdin and Viens (2009) have obtained the bounds*

$$\phi_l(t;a,b) = \exp\left(-\frac{t^2}{b}\right) \qquad \textit{and} \qquad \phi_r(t;a,b) = \exp\left(-\frac{t^2}{at+b}\right).$$

Table 1 proposes a comparison of these bounds with ours through the comparisons of the values of their reciprocal functions since these quantities are of great interest for the considered problem. Observe that the most important differences occur when n is moderate. The example $a = 4/\sqrt{n}$ and $b = 4$ corresponds approximately to the choices of parameters that will be used in the next sections.

		$\alpha = 1\%$		$\alpha = 2.5\%$		$\alpha = 5\%$		$\alpha = 10\%$	
		$\varphi_l^{-1}(\alpha)$	$\varphi_r^{-1}(\alpha)$	$\varphi_l^{-1}(\alpha)$	$\varphi_r^{-1}(\alpha)$	$\varphi_l^{-1}(\alpha)$	$\varphi_r^{-1}(\alpha)$	$\varphi_l^{-1}(\alpha)$	$\varphi_r^{-1}(\alpha)$
$n = 50$	NV	6.0697	9.2102	5.4324	7.9062	4.8955	6.8751	4.2919	5.7878
	BC	4.4720	7.1547	4.1398	6.9040	3.8372	6.0847	3.4712	5.2008
$n = 100$	NV	6.0697	8.1851	5.4324	7.1048	4.8955	6.2383	4.2919	5.3107
	BC	4.9090	7.3551	4.4966	6.4575	4.1314	5.7249	3.7012	4.9267
$n = 500$	NV	6.0697	6.9492	5.4324	6.1322	4.8955	5.4606	4.2919	4.7235
	BC	5.5334	6.6309	5.0017	5.8810	4.5449	5.2591	4.0218	4.5708
$n = 1000$	NV	6.0697	6.6801	5.4324	5.9190	4.8955	5.2891	4.2919	4.5930
	BC	5.6877	6.4641	5.1259	5.7478	4.6462	5.1513	4.1000	4.4883
$n = 10000$	NV	6.0697	6.2567	5.4324	5.5819	4.8955	5.0168	4.2919	4.3850
	BC	5.9475	6.1931	5.3345	5.5312	4.8159	4.9757	4.2308	4.3536

Table 1: Computations of the quantities $\varphi_l^{-1}(\alpha)$ and $\varphi_r^{-1}(\alpha)$ for the bounds obtained by Nourdin and Viens (2009) (NV) and ours (BC) (see Remark 1 and Proposition 3) for different values of n and α and for the particular case where $a = 4/\sqrt{n}$ and $b = 4$.

Remark 2 *Note that $\varphi_r(\cdot;a,b)$ (resp. $\varphi_l(\cdot;a,b)$) is a bijective function from $(0,+\infty)$ (resp. $(0,b/a)$) to $(0,1)$. Obviously, the index l in φ_l (resp. r in φ_r) indicates we consider the left (resp. right) tails.*

We explain now how Proposition 2 derives from Proposition 3: standard Malliavin calculus shows that, for $Z = \sqrt{n}V_n$, $\|DZ\|^2_{\aleph} = \frac{1}{n}\sum_{i,j=1}^{n} X(i)X(j)\rho(j-i)$, see Theorem 2.1 in Breton et al. (2009). The following lemma ensures that condition (5) in Proposition 3 holds true with $a = 2\kappa_n/\sqrt{n}$ and $b = 2\kappa_n$.

Lemma 4 *For $Z = \sqrt{n}V_n$, we have $\|DZ\|^2_{\aleph} \le \kappa_n\left(\frac{1}{\sqrt{n}}Z + 1\right)$.*

The proof of Lemma 4 is a very slight modification of the first part of the proof of Theorem 3.1 in Breton et al. (2009) to which we refer. Finally, Proposition 3 applies and entails Proposition 2.

3 Applications to quadratic variations of fractional Brownian motion

3.1 Notation

From now on, B_H stands for a fBm with Hurst parameter $H \in (0,1)$ and with scaling coefficient $C > 0$ and $\mathbf{B}_H$ is the vector of observations at times i/n for $i = 0, \dots, n-1$. We consider a filter a of length $\ell + 1$ and order p, that is a vector with $\ell + 1$ real components a_i, $0 \le i \le \ell$, satisfying

$$\sum_{q=0}^{\ell} q^j a_q = 0 \text{ for } j = 0, \dots, p-1 \text{ and } \sum_{q=0}^{\ell} q^p a_q \neq 0. \tag{10}$$

For instance, we shall consider the following filters: Increments 1 ($a = \{-1, 1\}$ with $\ell = 1$, $p = 1$), Increments 2 ($a = \{1, -2, 1\}$ with $\ell = 2$, $p = 2$), Daublets 4 ($a = \{-0.09150635, -0.15849365, 0.59150635, -0.34150635\}$ with $\ell = 3$, $p = 2$), Coiflets 6 ($a = \{-0.05142973, -0.23892973, 0.60285946, -0.27214054, -0.05142973, 0.01107027\}$ with $\ell = 5$, $p = 2$), see *e.g.* Daubechies (2006) and Percival and Walden (2000) for more details. Let $\mathbf{V}^a$ denote the vector $\mathbf{B}_H$ filtered with a and given for $i = \ell, \dots, n-1$ by

$$V^a\left(\frac{i}{n}\right) := \sum_{q=0}^{\ell} a_q B_H\left(\frac{i-q}{n}\right).$$

Let us denote by $\pi_H^a(\cdot)$ and $\rho_H^a(\cdot)$ the covariance and the correlation functions of the filtered series given by (see Coeurjolly (2001))

$$\mathbb{E}[V^a(k)V^a(k+j)] = C^2 \times \pi_H^a(j) \quad \text{with} \quad \pi_H^a(j) = -\frac{1}{2}\sum_{q,r=0}^{\ell} a_q a_r |q - r + j|^{2H} \tag{11}$$

and $\rho_H^a(\cdot) := \pi_H^a(\cdot)/\pi_H^a(0)$ which is independent of C. Finally, define S_n^a and V_n^a as

$$S_n^a := \frac{1}{n-\ell}\sum_{i=\ell}^{n-1} V^a\left(\frac{i}{n}\right)^2$$

and

$$V_n^a := \frac{n^{2H}}{C^2\pi_H^a(0)} S_n^a - 1 = \frac{1}{n-\ell}\sum_{i=\ell}^{n-1}\left(\frac{n^{2H}}{C^2\pi_H^a(0)} \times V^a\left(\frac{i}{n}\right)^2 - 1\right).$$

Note that $V_n^a \stackrel{d}{=} \frac{1}{n-\ell}\sum_{i=\ell}^{n-1} H_2(X_i^a)$ where $H_2(t) = t^2 - 1$ is the second Hermite polynomial and X^a is a stationary Gaussian process with variance 1 and with correlation function ρ_H^a. Applying Proposition 2 with these notation, we obtain fo all $s, t \ge 0$:

$$\mathbb{P}\left(-s \le \sqrt{n-\ell}\, V_n^a \le t\right) \ge 1 - \varphi_{r,n-\ell}(t; \kappa_{n,H}^a) - \varphi_{l,n-\ell}(s; \kappa_{n,H}^a) \tag{12}$$

where $\kappa_{n,H}^a = 2\sum_{|i|\le n} |\rho_H^a(i)|$. As previously explained, the action of filtering a discretized sample path of a fBm destroys the correlation of the increments. More precisely, it is proved that, for some explicit k_H, $\rho_H^a(i) \sim k_H |i|^{2H-2p}$, see *e.g.* Coeurjolly (2001). Thus, $\rho_H^a(\cdot)$ is summable if $p > H + 1/2$, *i.e.* $\rho_H^a(\cdot)$ is summable for all $H \in (0,1)$ for $p \ge 2$ and only for $H \in (0, 1/2]$ if $p = 1$ (in the case $H = 1/2$, observe that $\rho_{1/2}^a(k) = 0$ for all $|k| \ge \ell$).

One of the aim is to obtain bounds in (12) independently of H and easily computable. Since $\varphi_{l,n}(t,\cdot)$ and $\varphi_{r,n}(t,\cdot)$ are non-decreasing, the bound (12) remains true with $\kappa^a := 2\sup_{H\in(0,\tau)} \|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ replacing $\kappa_{n,H}$. Here, and in the sequel, we set $\tau = 1/2$ when $p = 1$ and $\tau = 1$ when $p \ge 2$. The following section will prove (among other things) that this quantity is finite.

3.2 Bounds of $\|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ independent of H

In this section, we show that $\kappa^a = \sup_{H\in(0,\tau)} \kappa_H^a$ is finite for a large class of filters, including the collection of dilated filters $(a^m)_{m\geq 1}$ of a filter a that will be used in the next section. Recall that a^m is the filter of length $m\ell+1$ with same order p as a and defined for $i=0,\dots,m\ell$ by

$$a_i^m = \begin{cases} a_{i/m} & \text{if } i/m \text{ is an integer} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

As a typical example, if $a := a^1 = \{1,-2,1\}$, then $a^2 := \{1,0,-2,0,1\}$.

Since $\pi_H^a(0) \neq 0$, observe that, for a fixed $i \in \mathbb{Z}$, the functions $H \mapsto \pi_H^a(i)$ and $H \mapsto \rho_H^a(i)$ are continuous respectively on $[0,1]$ and on $(0,1)$. Moreover, since for any filter a,

$$\pi_0^a(0) = -\frac{1}{2}\sum_{q,r=0,q\neq r}^{\ell} a_q a_r = -\frac{1}{2}\sum_{q,r=0}^{\ell} a_q a_r + \frac{1}{2}\sum_{q=0}^{\ell} a_q^2 = \frac{1}{2}\sum_{q=0}^{\ell} a_q^2 > 0, \tag{14}$$

the function $H \mapsto \rho_H^a(i)$ is continuous in 0. In particular, this ensures that for $p=1$, $\|\rho_\cdot^a\|_{\ell^1(\mathbb{Z})}$ is continuous on $[0,1/2)$. Actually, this may be not continuous in $1/2$ but nevertheless $\kappa^a = 2\sup_{H\in[0,1/2]} \|\rho_H^a\|_{\ell^1(\mathbb{Z})} < +\infty$ for instance $\kappa^{\{-1,1\}} = 4$ and $\kappa^{\{-1,1\}^2} = 8$. We refer to Appendix A for the computation of the exact values and to Table 3 for the estimation of some other similar constants.

For any filter of order $p \geq 2$, observe that $\pi_1^a(i) = 0$ for all i. Let us consider the following assumption on the filter a, denoted $\mathbf{H}^a$:

$$\tau^a := \sum_{q,r=0}^{\ell} a_q a_r (q-r)^2 \log(|q-r|) \neq 0, \tag{15}$$

with the convention $0\log(0)=0$. Tab. 2 below shows that Assumption $\mathbf{H^a}$ is satisfied for a large class of filters. Then, from the rule of l'Hospital,

$$\lim_{H\to 1^-} \rho_H^a(i) = \frac{\sum_{q,r=0}^{\ell} a_q a_r (q-r+i)^2 \log(|q-r+i|)}{\sum_{q,r=0}^{\ell} a_q a_r (q-r)^2 \log(|q-r|)} < +\infty.$$

Therefore, under $\mathbf{H^a}$, $\rho_H^a(i)$ is a continuous function of $H \in [0,1]$. Actually, the same is true for the ℓ^1-norm of a filter of order $p \geq 2$ as stated in Proposition 5 below.

		m				
	a	1	2	3	4	5
$p=2$	Increments 2	5.55	22.18	49.91	88.72	138.63
	Daublets 4	0.62	2.47	5.56	9.89	15.45
	Coiflets 6	0.61	2.42	5.45	9.69	15.15
$p=3$	Increments 3	13.50	53.98	121.46	215.94	337.40
	Daublets 6	0.49	1.98	4.45	7.90	12.35
$p=4$	Increments 4	41.43	165.70	372.84	662.82	1035.66
	Daublets 8	0.45	1.81	4.08	7.25	11.32
	Symmlets 8	0.45	1.81	4.08	7.25	11.32
	Coiflets 12	0.45	1.79	4.03	7.16	11.19

Table 2: Computations of τ^{a^m} for different filters a and its dilatation a^m for $m=1,\dots,5$.

Proposition 5 *Let a be a filter of order $p \geq 2$ satisfying $\mathbf{H}^a$ in (15). Then $\|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ is a continuous function of $H \in [0,1]$.*

Proof: From (11), we have

$$\rho_H^a(j) = \frac{|j|^{2H}}{\sum_{q,r=0}^{\ell} a_q a_r |q-r|^{2H}} \sum_{q,r=0}^{\ell} a_q a_r \left|1 + \frac{q-r}{j}\right|^{2H}.$$

For $|j| \geq \ell + 1$, we have $q - r + j \geq 0$ for $0 \leq q, r \leq \ell$, so that:

$$\begin{aligned}
\rho_H^a(j) &= \frac{|j|^{2H}}{\sum_{q,r=0}^{\ell} a_q a_r |q-r|^{2H}} \sum_{q,r=0}^{\ell} a_q a_r \left(1 + \frac{q-r}{j}\right)^{2H} \\
&= \frac{|j|^{2H}}{\sum_{q,r=0}^{\ell} a_q a_r |q-r|^{2H}} \sum_{q,r=0}^{\ell} a_q a_r \sum_{k=0}^{+\infty} \frac{(2H)(2H-1)\dots(2H-k+1)}{k!} \left(\frac{q-r}{j}\right)^k \\
&= \frac{|j|^{2H}}{\sum_{q,r=0}^{\ell} a_q a_r |q-r|^{2H}} \sum_{k=2p}^{+\infty} \frac{(2H)(2H-1)\dots(2H-k+1)}{k! j^k} \sum_{q,r=0}^{\ell} a_q a_r (q-r)^k. \quad (16)
\end{aligned}$$

Observe that in (16), the outer sum starts at $k = 2p$. This is due to the property (10) of the filter a of order p which implies the following remark:

$$\begin{aligned}
\sum_{q,r=0}^{\ell} a_q a_r (q-r)^k &= \sum_{q,r=0}^{\ell} a_q a_r \sum_{i=0}^{k} \binom{k}{i} q^i (-r)^{k-i} \\
&= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \left(\sum_{q=0}^{\ell} a_q q^i \sum_{r=0}^{\ell} a_r r^{k-i} \right) \\
&= 0 \text{ if } k \leq 2p - 1.
\end{aligned}$$

As a consequence, for $p \geq 2$, each summand in the outer sum (16) contains the factor $2H - 2$ in the product $(2H)(2H-1)\dots(2H-k+1)$. Observe that under $\mathbf{H}^a$ in (15), the rule of l'Hospital ensures that the function $\theta_a(H) = (2-2H)/(\sum_{q \neq r} a_q a_r |q-r|^{2H})$ is bounded at $H = 1^-$. Since moreover this function is continuous in H, we derive, under $\mathbf{H}^a$, that $\|\theta_a\|_\infty := \sup_{H \in [0,1)} |\theta_a(H)| < +\infty$.

Now, from (16), we have

$$\begin{aligned}
&|\rho_H^a(j)| \\
&= \left| \theta_a(H) |j|^{2H-2p} \sum_{k=0}^{+\infty} \frac{(2H)(2H-1)(2H-3)\dots(2H-2p-k+1)}{(2p+k)! j^k} \sum_{q,r=0}^{\ell} a_q a_r (q-r)^{k+2p} \right| \\
&\leq |\theta_a(H)| |j|^{2H-2p} \sum_{k=0}^{+\infty} \frac{(2p+k-1)!}{(2p+k)! j^k} \sum_{q,r=0}^{\ell} |a_q||a_r||q-r|^{k+2p} \\
&\leq \|\theta_a\|_\infty |j|^{2H-2p} \sum_{q,r=0}^{\ell} |a_q||a_r||q-r|^{2p} \sum_{k=0}^{+\infty} \frac{1}{(k+1)} \left(\frac{|q-r|}{\ell+1}\right)^k \\
&\leq C(a) |j|^{2H-2p} \quad (17)
\end{aligned}$$

where

$$C(a) \quad = \quad \|\theta_a\|_\infty \sum_{q,r=0}^{\ell} |a_q||a_r||q-r|^{2p} \left(\frac{(\ell+1)\ln(\ell+1)}{\ell} \right) < +\infty.$$

When $p \geq 2$, the bound (17) ensures that the convergence of the series $\sum_{i\in\mathbb{Z}} |\rho_H^a(i)|$ is uniform in $H \in [0,1]$ and thus $H \mapsto \|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ is continuous on $[0,1]$. □

Proposition 5 proves the following bound is finite for a filter a of order $p \geq 2$ satisfying $\mathbf{H^a}$:

$$\kappa^a = 2 \sup_{H\in(0,1)} \kappa_H^a = 2 \sup_{H\in(0,1)} \|\rho_H^a\|_{\ell^1(\mathbb{Z})} = 2 \sup_{H\in[0,1]} \|\rho_H^a\|_{\ell^1(\mathbb{Z})} < +\infty. \tag{18}$$

As a consequence of this result, this means that the constant κ^a can be obtained by optimizing the function $H \mapsto \|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ on the interval $[0,1]$. See Tab. 3 below for the computation of such constants for different typical filters.

For dilated increment-type filters, we manage to compute the exact value of $\|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ (see Appendix A for more details)

$$\|\rho_H^a\|_{\ell^1(\mathbb{Z})} = 1 + \sum_{k=1}^{\ell-1} \frac{\left| \sum_{j=-\ell}^{\ell} \alpha_j |j+k|^{2H} \right|}{-\sum_{j=1}^{\ell} \alpha_j j^{2H}} + (-1)^{p+1} \epsilon(2H-1) \frac{\sum_{k=-\ell+1}^{\ell} \alpha_k S_{\ell+k-1}^H}{-\sum_{j=1}^{\ell} \alpha_j j^{2H}},$$

where $\alpha_j = \sum_{\substack{q,r=0 \\ q-r=j}}^{\ell} a_q a_r$, $\epsilon(2H-1) := sign(2H-1)$ and where $S_k^H = \sum_{j=0}^{k} j^{2H}$. For the dilated double increments filter $a = \{1,-2,1\}^m$ for example, this leads to $\kappa^{\{1,-2,1\}} = 2 \times 8/3 = 16/3$ and $\kappa^{\{1,-2,1\}^2} = 2 \times \left(2 + \frac{25\log(5) - 27\log(3)}{8\log(2)}\right) \simeq 7.813554$.

	a	m 1	2	3	4	5
$p=1$	Increments 1	2	4	6	8	10
$p=2$	Increments 2	2.667	3.907	5.745	7.565	9.376
	Daublets 4	2.250	4.356	6.641	8.906	11.162
	Coiflets 6	2.259	4.327	6.582	8.816	11.042
$p=3$	Increments 3	3.200	3.783	5.396	7.406	9.200
	Daublets 6	2.429	4.516	6.688	8.833	10.966
$p=4$	Increments 4	3.657	4.304	6.364	8.514	10.350
	Daublets 8	2.648	5.026	7.349	9.648	12.044
	Coiflets 12	2.701	5.112	7.459	9.775	12.229

Table 3: Computation of $\sup_{H\in I} \|\rho_H^{a^m}\|_{\ell^1}$ for different filters a and for $m = 1,\ldots,5$. Note that $I = [0, 0.5]$ for $p = 1$ and $I = [0,1]$ for $p > 1$.

4 Confidence intervals of the Hurst parameter

For any $\alpha \in (0,1)$, denote by $q_{\bullet,n}^a(\alpha) := (\varphi_{\bullet,n})^{-1}(\alpha;\kappa^a)$ for $\bullet = l,r$. In order to make easier the presentation, define also

$$x_{l,n-\ell}^a(\alpha) := 1 - \frac{q_{l,n-\ell}^a(\alpha)}{\sqrt{n-\ell}} \quad \text{and} \quad x_{r,n-\ell}^a(\alpha) := 1 + \frac{q_{r,n-\ell}^a(\alpha)}{\sqrt{n-\ell}}.$$

Note that Remark 2 above ensures that for any $\alpha \in (0,1)$ and for all $n > \ell$, $x^a_{l,n-\ell}(\alpha) > 0$. For further reference, observe that for $\bullet = l, r$ and $n \to +\infty$:

$$q^a_{\bullet,n-\ell}(\alpha) \sim q^a(\alpha) := \sqrt{2\kappa^a \log(1/\alpha)}. \tag{19}$$

In the sequel, we restrict ourselves, to filters of order $p \geq 2$ which allows us to make no assumption on H. Taking a filter of order $p = 1$ would have constrained us to assume that $H \leq 1/2$.

4.1 Scaling parameter C known

In this section, we assume, without loss of generality, that $C = 1$. Our confidence interval in Proposition 6 below is expressed in terms of the reciprocal function of $g_n(x) := 2x\log(n) - \log(\pi^a_x(0))$, $x \in (0,1)$. In order to ensure that g_n is indeed invertible, we assume that

$$n \geq \exp\left(\sup_{x\in(0,1)} \frac{\sum_{q,r=0}^{\ell} a_q a_r \log(|q-r|)|q-r|^{2x}}{\sum_{q,r=0}^{\ell} a_q a_r |q-r|^{2x}}\right). \tag{20}$$

In this case, the function g_n is a strictly increasing bijection from $(0,1)$ to $\big(-\log(\pi^a_0(0)), +\infty\big)$. Moreover recall that a filter of length $\ell+1$ requires a sample size $n \geq \ell+1$. Obviously, condition (20) only makes sense if the filter a satisfies:

$$\sup_{x\in(0,1)} \frac{\sum_{q,r=0}^{\ell} a_q a_r \log(|q-r|)|q-r|^{2x}}{\sum_{q,r=0}^{\ell} a_q a_r |q-r|^{2x}} < +\infty.$$

Since $\lim_{x\to 1^-} \sum_{q,r=0}^{\ell} a_q a_r |q-r|^{2x} = 0^-$ (we stress that this function vanishes with non-positive values of because it is continuous, negative in $x = 0$, see (14), and does not vanish), the previous condition is equivalent to the more explicit following one

$$\sum_{q,r=0}^{\ell} a_q a_r \log(|q-r|)(q-r)^2 \geq 0. \tag{21}$$

Table 4 exhibits the minimal sample size n required to satisfy (20) for different filters a^m (for $m = 1,\ldots,5$) with different order $p = 2,3,4$. Obviously, condition (21) is in force for all these filters.

We state now our main result when the scaling parameter is known:

Proposition 6 *Let $\alpha \in (0,1)$ be fixed and a be a filter satisfying $\mathbf{H}^a$ in (15)*

1. *For $n \geq \ell+1$, we have:*

$$\mathbb{P}\Big(\log\big(x^a_{l,n-\ell}(\alpha/2)\big) - \log(S^a_n) \leq g_n(H) \leq \log\big(x^a_{r,n-\ell}(\alpha/2)\big) - \log(S^a_n)\Big) \geq 1-\alpha. \tag{22}$$

2. *Moreover if the filter a satisfies (21) and $n \geq \ell+1$ satisfies (20), we have:*

$$\mathbb{P}\left(H \in \left[\widetilde{H}^{\inf}_n(\alpha), \widetilde{H}^{\sup}_n(\alpha)\right]\right) \geq 1-\alpha, \tag{23}$$

where

$$\begin{aligned}
\widetilde{H}^{\inf}_n(\alpha) &:= \max\Big(0, g_n^{-1}\left(\log\big(x^a_{l,n-\ell}(\alpha/2)\big) - \log(S^a_n)\right)\Big)\\
\widetilde{H}^{\sup}_n(\alpha) &:= \min\Big(\tau, g_n^{-1}\left(\log\big(x^a_{r,n-\ell}(\alpha/2)\big) - \log(S^a_n)\right)\Big).
\end{aligned}$$

	a	m 1	2	3	4	5
$p=2$	Increments 2	3	4	6	9	11
	Daublets 4	4	6	10	13	15
	Coiflets 6	6	11	15	21	26
$p=3$	Increments 3	4	6	10	13	15
	Daublets 6	6	11	15	21	26
$p=4$	Increments 4	4	9	13	17	21
	Daublets 8	7	15	22	29	36
	Symmlets 8	7	15	22	29	36
	Coiflets 12	12	23	34	44	56

Table 4: Minimal sample size n required to satisfy (20) for different dilated filters a^m of different orders p.

3. As $n \to +\infty$, the proposed confidence interval in (23) *satisfies almost surely*

$$\left[\widetilde{H}_n^{\inf}(\alpha), \widetilde{H}_n^{\sup}(\alpha)\right] \to \{H\}$$

and the length μ_n of the confidence interval satisfies

$$\mu_n \sim \frac{2q^a(\alpha/2)}{\sqrt{n}} \frac{1}{g_n'(H)} \sim \frac{q^a(\alpha/2)}{\sqrt{n}\log(n)},$$

where q^a is defined above in (19).

Remark 3 *Proposition 6 generalizes Proposition 1 derived from Breton et al. (2009). The scaling parameter is still assumed to be known. However, we do not need to know an upper-bound of H and our condition on n is much sharper than the one required in Proposition 1. As an example, for $a = (1, -2, 1)$, condition* (20) *is satisfied for all $n \geq 3$, whereas the minimal sample size allowing to derive a confidence interval from Proposition 1 is* 1108 *for $\alpha = 5\%$ and $H^\star = 0.8$.*

Proof: Consider the set

$$A := \left\{-q^a_{l,n-\ell}(\alpha/2) \leq \sqrt{n-\ell}V_n^a \leq q^a_{r,n-\ell}(\alpha/2)\right\}.$$

The bound (12) entails $\mathbb{P}(A) \geq 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$. It is now sufficient to notice that

$$\begin{aligned}
A &= \left\{x^a_{l,n-\ell}(\alpha/2) \leq 1 + V_n^a \leq x^a_{r,n-\ell}(\alpha/2)\right\} \\
&= \left\{x^a_{l,n-\ell}(\alpha/2) \leq \frac{n^{2H}}{\pi_H^a(0)} S_n^a \leq x^a_{r,n-\ell}(\alpha/2)\right\} \\
&= \left\{\log\left(\frac{x^a_{l,n-\ell}(\alpha/2)}{S_n^a}\right) \leq g_n(H) \leq \log\left(\frac{x^a_{r,n-\ell}(\alpha/2)}{S_n^a}\right)\right\}
\end{aligned}$$

which proves (22). Next, since under (20) and (21), g_n is an increasing bijection, (23) comes immediately from (22). Finally, from (19), we have

$$\log\left(x^a_{l,n-\ell}(\alpha/2)\right) \sim -\frac{q^a(\alpha/2)}{\sqrt{n}} \quad \text{and} \quad \log\left(x^a_{r,n-\ell}(\alpha/2)\right) \sim \frac{q^a(\alpha/2)}{\sqrt{n}}$$

as $n \to +\infty$. Moreover, since $1 + V_n^a = \frac{n^{2H}}{\pi_H^a(0)} S_n^a = S_n^a e^{g_n(H)}$, we have almost surely

$$-\log\left(S_n^a\right) = -\log\left(1 + V_n^a\right) + g_n(H) = g_n(H) - V_n^a(1 + o(1)) \sim g_n(H).$$

It is proved in Coeurjolly (2001) (Proposition 1) that V_n^a converges almost surely towards 0 for any filter and for all $H \in (0,1)$ which implies the almost sure convergence of the confidence interval and the asymptotic behavior of the length μ_n of the confidence interval. $\square$

4.2 Scaling parameter C unknown

The idea to construct confidence intervals when the scaling coefficient C is unknown consists in using the collection of the dilated filters a^m defined in (13).

Let us first introduce some specific notation: let $M \geq 2$ and consider a vector $\mathbf{d} = (d_1, \dots, d_M)^T$ with non zero real components such that $\sum_{i=1}^M d_i = 0$ and such that $\mathbf{d}^T \mathbf{L_M} > 0$, where $\mathbf{L_M} = (\log(m))_{m=1,\dots,M}$. Denote by I^- and I^+ the subsets of $\{1, \dots, M\}$ defined by

$$I^- = \{i \in \{1, \dots, M\} : d_i < 0\} \quad \text{and} \quad I^+ = \{i \in \{1, \dots, M\} : d_i > 0\}.$$

The following confidence interval is expressed in terms of $\mathbf{L_{S_n}} := \left(\log\left(S_n^{a^m}\right)\right)_{m=1,\dots,M}$.

Proposition 7 *Let $\alpha \in (0,1)$ be fixed and denote by $\mathbf{L_{X_n^{\inf}}}$ and $\mathbf{L_{X_n^{\sup}}}$ the two following vectors with components*

$$\left(\mathbf{L_{X_n^{\inf}}}\right)_m = \begin{cases} \log\left(x_{l,n-m\ell}^{a^m}(\alpha/2M)\right) & \text{if } m \in I^- \\ \log\left(x_{r,n-m\ell}^{a^m}(\alpha/2M)\right) & \text{if } m \in I^+ \end{cases}, \quad \left(\mathbf{L_{X_n^{\sup}}}\right)_m = \begin{cases} \log\left(x_{r,n-m\ell}^{a^m}(\alpha/2M)\right) & \text{if } m \in I^- \\ \log\left(x_{l,n-m\ell}^{a^m}(\alpha/2M)\right) & \text{if } m \in I^+. \end{cases}$$

1. *Let $n \geq M\ell + 1$. Then we have*

$$\mathbb{P}\left(H \in \left[\widetilde{H}_n^{\inf}(\alpha), \widetilde{H}_n^{\sup}(\alpha)\right]\right) \geq 1 - \alpha \tag{24}$$

where

$$\begin{aligned} \widetilde{H}_n^{\inf}(\alpha) &= \max\left(0, \frac{1}{2\mathbf{d}^T \mathbf{L_M}}\left(\mathbf{d}^T \mathbf{L_{S_n}} - \mathbf{d}^T \mathbf{L_{X_n^{\inf}}}\right)\right) \\ \widetilde{H}_n^{\sup}(\alpha) &= \min\left(1, \frac{1}{2\mathbf{d}^T \mathbf{L_M}}\left(\mathbf{d}^T \mathbf{L_{S_n}} - \mathbf{d}^T \mathbf{L_{X_n^{\sup}}}\right)\right). \end{aligned}$$

2. *As $n \to +\infty$, the proposed confidence interval in* (24) *satisfies almost surely*

$$\left[\widetilde{H}_n^{\inf}(\alpha), \widetilde{H}_n^{\sup}(\alpha)\right] \to \{H\}$$

and its length μ_n satisfies

$$\mu_n := \frac{\mathbf{d}^T\left(\mathbf{L_{X_n^{\inf}}} - \mathbf{L_{X_n^{\sup}}}\right)}{2\mathbf{d}^T \mathbf{L_M}} \sim \frac{1}{\sqrt{n}} \frac{\mathbf{d}^T \mathbf{q_M}(\alpha/2M)}{\mathbf{d}^T \mathbf{L_M}}$$

where $\mathbf{q_M}(\alpha/2M)$ is the vector of length M with components defined by

$$\left(\mathbf{q_M}(\alpha/2M)\right)_m := \begin{cases} -q^{a^m}(\alpha/2M) & \text{if } m \in I^- \\ q^{a^m}(\alpha/2M) & \text{if } m \in I^+ \end{cases}$$

with q^{a^m} defined in (19).

Remark 4 *Proposition 7 generalizes Proposition 6 since this new confidence interval does not assume that the scaling parameter, C is known. More specifically, note that the definition of the interval does not depend on C. Note also, that if $\mathbf{B_H}$ were not observed on $[0,1)$ but with a dilatation factor, then the confidence interval would remain unchanged.*

Proof: For $m=1,\ldots,M$, we consider the following event

$$A_m \quad := \quad \left\{ x_{l,n-m\ell}^{a^m}(\alpha/2M) \leq 1+V_n^{a^m} \leq x_{r,n-m\ell}^{a^m}(\alpha/2M) \right\}.$$

The bounds (12) entails that $\mathbb{P}(A_m) \geq 1-\frac{\alpha}{2M}-\frac{\alpha}{2M} = 1-\frac{\alpha}{M}$. First, recall that

$$V_n^{a^m} = \frac{n^{2H}}{C^2\pi_H^{a^m}(0)}\, S_n^{a^m} - 1 = \gamma \times \frac{1}{m^{2H}}\, S_n^{a^m} - 1 \quad \text{with } \gamma := \gamma_{C,H,n} = \frac{n^{2H}}{C^2\pi_H^{a}(0)}.$$

The crucial point in the definition of the confidence interval relies on the fact that γ is independent of m. Second, note that for $m=1,\ldots,M$:

$$\begin{aligned}
& A_m \\
= \quad & \left\{ \log\left(x_{l,n-m\ell}^{a^m}(\alpha/2M)\right) \leq \log\left(1+V_n^{a^m}\right) \leq \log\left(x_{r,n-m\ell}^{a^m}(\alpha/2M)\right)\right\} \\
= \quad & \Big\{ \log\left(x_{l,n-m\ell}^{a^m}(\alpha/2M)\right) - \log(\gamma) \leq \log\left(S_n^{a^m}\right) - 2H\log(m) \\
& \qquad\qquad\qquad\qquad \leq \log\left((x_{r,n-m\ell}^{a^m}(\alpha/2M)\right) - \log(\gamma) \Big\} \\
= \quad & \Big\{ \log\left(S_n^{a^m}\right) - \log\left(x_{r,n-m\ell}^{a^m}(\alpha/2M)\right) + \log(\gamma) \leq 2H\log(m) \\
& \qquad\qquad\qquad\qquad \leq \log\left(S_n^{a^m}\right) - \log\left((x_{l,n-m\ell}^{a^m}(\alpha/2M)\right) + \log(\gamma) \Big\} \\
= \quad & \left\{ d_m\left((\mathbf{L_{S_n}})_m - (\mathbf{L_{X_n^{\inf}}})_m + \log(\gamma)\right) \leq 2d_m H(\mathbf{L_M})_m \leq d_m\left((\mathbf{L_{S_n}})_m - (\mathbf{L_{X_n^{\sup}}})_m + \log(\gamma)\right)\right\}.
\end{aligned}$$

Next, we consider the following event

$$\begin{aligned}
B \quad := \quad & \left\{ \mathbf{d}^T\mathbf{L_{S_n}} - \mathbf{d}^T\mathbf{L_{X_n^{\inf}}} + \mathbf{d}^T\mathbf{1}\log(\gamma) \leq 2H\mathbf{d}^T\mathbf{L_M} \leq \mathbf{d}^T\mathbf{L_{S_n}} - \mathbf{d}^T\mathbf{L_{X_n^{\sup}}} + \mathbf{d}^T\mathbf{1}\log(\gamma)\right\} \\
= \quad & \left\{ \mathbf{d}^T\mathbf{L_{S_n}} - \mathbf{d}^T\mathbf{L_{X_n^{\inf}}} \leq 2H\mathbf{d}^T\mathbf{L_M} \leq \mathbf{d}^T\mathbf{L_{S_n}} - \mathbf{d}^T\mathbf{L_{X_n^{\sup}}}\right\} \\
= \quad & \left\{ H \in \left[\widetilde{H}_n^{\inf}(\alpha), \widetilde{H}_n^{\sup}(\alpha)\right]\right\}
\end{aligned}$$

where $\mathbf{1}=(1,\ldots,1)^T$. Since $A_1\cap A_2\cap\ldots\cap A_M \subset B$, setting $A^c=\Omega\setminus A$, we have

$$\begin{aligned}
\mathbb{P}(B) \quad \geq \quad & \mathbb{P}(A_1\cap\ldots\cap A_M) = 1-\mathbb{P}\left((A_1\cap\ldots\cap A_M)^c\right) = 1-\mathbb{P}(A_1^c\cup\ldots\cup A_M^c) \\
\geq \quad & 1-\sum_{m=1}^{M}\mathbb{P}(A_m^c) = \sum_{m=1}^{M}\mathbb{P}(A_m) - (M-1) \\
\geq \quad & M\left(1-\frac{\alpha}{M}\right) - (M-1) = 1-\alpha,
\end{aligned}$$

which ends the proof of (24). Next, as $n\to+\infty$, the following estimate holds almost surely

$$\begin{aligned}
\log\left(S_n^{a^m}\right) \quad & = \quad 2H\log(m) - \log(\gamma) + \log\left(1+V_n^{a^m}\right) \\
& = \quad 2H\log(m) - \log(\gamma) + V_n^{a^m}(1+o(1)),
\end{aligned}$$

and implies that almost surely, when $n \to +\infty$,

$$\begin{aligned}
\mathbf{d}^T \mathbf{L}_{\mathbf{S_n}} &= 2H\mathbf{d}^T\mathbf{L_M} - \mathbf{d}^T\mathbf{1}\log(\gamma) + \mathbf{d}^T\left(V_n^{a^m}\right)_{m=1,\dots,M}(1+o(1)) \\
&= 2H\mathbf{d}^T\mathbf{L_M} + \mathbf{d}^T\left(V_n^{a^m}\right)_{m=1,\dots,M}(1+o(1)) \\
&\to 2H\mathbf{d}^T\mathbf{L_M}.
\end{aligned}$$

From (19), one has also the following estimates as $n \to +\infty$:

$$\left(\mathbf{L}_{\mathbf{X}_\mathbf{n}^{\inf}}\right)_m \sim \frac{1}{\sqrt{n}} \times \begin{cases} -q^{a^m}(\alpha/2M) & \text{if } m \in I^- \\ q^{a^m}(\alpha/2M) & \text{if } m \in I^+ \end{cases}, \quad \left(\mathbf{L}_{\mathbf{X}_\mathbf{n}^{\sup}}\right)_m \sim \frac{1}{\sqrt{n}} \times \begin{cases} q^{a^m}(\alpha/2M) & \text{if } m \in I^- \\ -q^{a^m}(\alpha/2M) & \text{if } m \in I^+. \end{cases}$$

These different results imply the almost sure convergence of the confidence interval towards $\{H\}$. For the asymptotic of the length μ_n of the confidence interval, it is sufficient to note that $(\mathbf{L}_{\mathbf{X}^{\inf}} - \mathbf{L}_{\mathbf{X}^{\sup}}) \sim \frac{1}{\sqrt{n}}\mathbf{q_M}(\alpha/2M)$. $\square$

5 Simulations and discussion

5.1 Confidence intervals based on the central limit theorem

5.1.1 Methodology

There exists a very wide litterature on the estimation of the Hurst parameter, see *e.g.* Coeurjolly (2000) and references therein. For all of the available procedures, the confidence interval comes from a limit theorem so that it is of asymptotic very nature. In contrast, our confidence intervals in (23) and (24) are non-asymptotic since they are based on concentration inequalities. In order to compare our procedures, we choose to focus only on one of these procedures which has several similarities with this paper. These procedures are based on discrete filtering and are presented in detail in Coeurjolly (2001). For the sake of self-containess, we first summarize them:

- **Scaling parameter C known.** The procedure is based on the fact that almost surely $\frac{n^{2H}}{\pi_H^a(0)} S_n^a \to 1$, $n \to +\infty$. With the same function $g_n(x) = 2x\log(n) - \log(\pi_x^a(0))$ as the one used to derive the confidence interval in Proposition 6, this yields the estimator:

  $$\widehat{H}_n^{std}(a) := g_n^{-1}(-\log(S_n^a)).$$

  Note that the confidence interval (23) is very close to this estimator. In particular, the middle of the interval (23) behaves asymptotically as $\widehat{H}_n^{std}(a)$.

- **Scaling parameter C unknown.** The idea of Coeurjolly (2000) in this context is to use the following property of quadratic variations of dilated filters $\mathbb{E}[S_n^{a^m}] = m^{2H}\gamma$ with $\gamma := \frac{C^2\pi_H^a(0)}{n^{2H}}$ and the almost sure convergence of $S_n^{a^m}/\mathbb{E}[S_n^{a^m}]$ towards 1 for all m. The idea is then to estimate H via a simple linear regression of $\mathbf{L}_{\mathbf{S_n}}$ on $2\mathbf{L_M}$ for M dilated filters. Here, the notation $\mathbf{L}_{\mathbf{S_n}}$ and $\mathbf{L_M}$ are the same as the ones in Proposition 7. This leads to the estimator

  $$\widehat{H}_n^{gen}(a, M) := \frac{\mathbf{A}^T\mathbf{L}_{\mathbf{S_n}}}{2\|\mathbf{A}\|^2},$$

  where $\mathbf{A} = \left(\log(m) - \frac{1}{M}\sum_{m=1}^M \log(m)\right)_{m=1,\dots,M}$. There is again an analogy between this estimator and our confidence interval in Proposition 7. Indeed, with $\mathbf{d} = \mathbf{A}$, the interval in (24) rewrites

$$\left[\max\left(0,\frac{\mathbf{A}^T\left(\mathbf{L}_{\mathbf{S_n}}-\mathbf{L}_{\mathbf{X}_\mathbf{n}^{\inf}}\right)}{2\|\mathbf{A}\|^2}\right),\min\left(1,\frac{\mathbf{A}^T\left(\mathbf{L}_{\mathbf{S_n}}-\mathbf{L}_{\mathbf{X}_\mathbf{n}^{\sup}}\right)}{2\|\mathbf{A}\|^2}\right)\right],$$

since $\mathbf{d}^T\mathbf{L}_\mathbf{M}=\mathbf{A}^T\mathbf{A}=\|A\|^2$. Again, the middle of this interval behaves asymptotically as $\widehat{H}_n^{gen}(a,M)$. In the particular case $M=2$ the estimator $\widehat{H}_n^{gen}(a,2)$ takes the simple following form

$$\widehat{H}_n^{gen}(a,2):=\frac{1}{2\log 2}\log\left(\frac{S_n^{a^2}}{S_n^{a^1}}\right)$$

and the bounds of the interval in (24) rewrite as

$$\begin{aligned}\widetilde{H}_n^{\inf}(\alpha) &:= \max\left(0,\frac{1}{2\log 2}\left(\log\left(\frac{S_n^{a^2}}{S_n^{a^1}}\right)-\log\left(\frac{x^{a^2}_{r,n-2\ell}(\alpha/4)}{x^{a^1}_{l,n-\ell}(\alpha/4)}\right)\right)\right)\\ \widetilde{H}_n^{\sup}(\alpha) &:= \min\left(1,\frac{1}{2\log 2}\left(\log\left(\frac{S_n^{a^2}}{S_n^{a^1}}\right)-\log\left(\frac{x^{a^2}_{l,n-2\ell}(\alpha/4)}{x^{a^1}_{r,n-\ell}(\alpha/4)}\right)\right)\right).\end{aligned}$$

5.1.2 Asymptotic confidence intervals

We refer the reader to Coeurjolly (2001) where the following central limit theorems (CLT) are proved for $\widehat{H}_n^{std}(a)$ and $\widehat{H}_n^{gen}(a,M)$

$$\sqrt{n}\log(n)\frac{\widehat{H}_n^{std}(a)-H}{\sigma_{std}(\widehat{H}_n^{std})}\xrightarrow{d}\mathcal{N}(0,1),\quad n\to+\infty \tag{25}$$

where $\xrightarrow{d}$ stands for the convergence in distribution, $\mathcal{N}(0,1)$ is the normal standard distribution and $\sigma^2_{std}(H):=\frac{1}{2}\|\rho_H^a\|_{\ell^2(\mathbb{Z})}$, and

$$\sqrt{n}\frac{\widehat{H}_n^{gen}(a,M)-H}{\sigma_{std}(\widehat{H}_n^{gen},M)}\xrightarrow{d}\mathcal{N}(0,1),\quad n\to+\infty \tag{26}$$

where $\sigma^2_{gen}(H,M):=\frac{\mathbf{A}^T\underline{\mathbf{G}}\mathbf{A}}{4\|\mathbf{A}\|^4}$ where $\underline{\mathbf{G}}$ is the $(M\times M)$-matrix defined by $G_{m_1,m_2}=\left\|\rho_H^{a^{m_1},a^{m_2}}\right\|^2_{\ell^2(\mathbb{Z})}$ for $m_1,m_2=1,\dots,M$, and for all $i\in\mathbb{Z}$

$$\rho_H^{a^{m_1},a^{m_2}}(i)=\frac{-\frac{1}{2}\sum_{q,r=0}^{\ell}a_a a_r|m_1q-m_2r+i|^{2H}}{\sqrt{\pi_H^{a^{m_1}}(0)\pi_H^{a^{m_2}}(0)}}.$$

Note that in the special case where $M=2$, the constant $\sigma^2_{gen}(H,2)$ takes the simple form

$$\sigma^2_{gen}(H,2)=\frac{1}{2(\log 2)^2}\left(\left\|\rho_H^{a^1}\right\|^2_{\ell^2(\mathbb{Z})}+\left\|\rho_H^{a^2}\right\|^2_{\ell^2(\mathbb{Z})}-2\left\|\rho_H^{a^1,a^2}\right\|^2_{\ell^2(\mathbb{Z})}\right).$$

Thanks to the CLTs, (25) and (26) an asymptotic confidence interval to the level $1-\alpha$, $\alpha\in(0,1)$, can be easily constructed

$$IC^{clt}_\bullet(\alpha)=\left[\max\left(0,\widehat{H}_n^\bullet-\Phi^{-1}(1-\alpha/2)\times\frac{\widehat{\sigma^\bullet}}{v_n^\bullet}\right),\min\left(1,\widehat{H}_n^\bullet+\Phi^{-1}(1-\alpha/2)\times\frac{\widehat{\sigma^\bullet}}{v_n^\bullet}\right)\right] \tag{27}$$

where $\bullet=std,gen$, $v_n^{std}=\sqrt{n}\log(n)$, $v_n^{gen}=\sqrt{n}$ and Φ is the cumulative distribution function of a standard Gaussian random variable.

5.2 Comparisons of approaches

In the following tables, we compare, via Monte-Carlo experiments, the confidence intervals based on concentration inequalities (23), (24) and on central limit theorems (27). The fractional Brownian motions have been generated by using the circulant matrix method (*e.g.* Kent and Wood (1997), Coeurjolly (2000)). We have realized a very large simulation study. The "best" results (in terms of choices of the filters a, of the maximum dilatation factor M) are summarized in Table 5 for the standard fractional Brownian motion (*i.e.* $C = 1$) and in Table 6 for the general one (*i.e.* C unknown).

In Figure 1, we also compare, in terms of H, the asymptotic lengths of the confidence intervals obtained by each approach.

			$H = 0.2$			$H = 0.5$			$H = 0.8$	
		Cover.	Length	$\widehat{H}$	Cover.	Length	$\widehat{H}$	Cover.	Length	$\widehat{H}$
$n = 50$	CI[i2]	100.0	0.2191	0.1875	100.0	0.2029	0.4832	100.0	0.1553	0.7824
	CLT[i2]	95.2	0.1330	0.2058	97.0	0.1227	0.5013	99.6	0.1125	0.8003
	CI[d4]	100.0	0.2086	0.1886	100.0	0.1941	0.4841	100.0	0.1482	0.7834
	CLT[d4]	94.6	0.1217	0.2050	97.2	0.1133	0.5004	99.2	0.1076	0.7999
$n = 100$	CI[i2]	100.0	0.1298	0.1936	100.0	0.1212	0.4946	100.0	0.0952	0.7931
	CLT[i2]	95.0	0.0800	0.2009	97.6	0.0737	0.5017	99.8	0.0676	0.8003
	CI[d4]	100.0	0.1224	0.1941	100.0	0.1149	0.4949	100.0	0.0902	0.7933
	CLT[d4]	95.6	0.0732	0.2005	96.4	0.0680	0.5012	99.6	0.0646	0.7997
$n = 500$	CI[i2]	99.6	0.0430	0.1994	100.0	0.0408	0.4988	99.8	0.0336	0.7988
	CLT[i2]	94.4	0.0265	0.2004	96.4	0.0244	0.4998	98.8	0.0224	0.7998
	CI[d4]	99.6	0.0402	0.1995	100.0	0.0383	0.4990	99.8	0.0316	0.7989
	CLT[d4]	95.4	0.0243	0.2003	96.0	0.0225	0.4999	98.4	0.0214	0.7998
$n = 1000$	CI[i2]	100.0	0.0274	0.1998	100.0	0.0262	0.4996	100.0	0.0219	0.7997
	CLT[i2]	96.6	0.0169	0.2003	97.6	0.0155	0.5000	99.2	0.0142	0.8001
	CI[d4]	100.0	0.0256	0.1998	100.0	0.0245	0.4996	100.0	0.0205	0.7998
	CLT[d4]	96.4	0.0154	0.2002	97.2	0.0143	0.5000	98.8	0.0136	0.8001
$n = 10000$	CI[i2]	99.8	0.0066	0.2000	100.0	0.0063	0.4999	100.0	0.0055	0.8000
	CLT[i2]	94.2	0.0040	0.2000	96.2	0.0037	0.5000	98.4	0.0034	0.8000
	CI[d4]	99.8	0.0061	0.2000	99.8	0.0059	0.5000	100.0	0.0051	0.8000
	CLT[d4]	94.4	0.0037	0.2000	95.0	0.0034	0.5000	98.2	0.0032	0.8000

Table 5: Monte-carlo experiment based on 500 replications of a standard fractional Brownian motion with Hurst parameter $H = 0.2, 0.5, 0.8$ and for different values of the sample size n. The filters i2 and d4 denote respectively the filter of Increments of order 2 and the Daublets 4.

		$H = 0.2$			$H = 0.5$			$H = 0.8$		
		Cover.	Length	$\widehat{H}$	Cover.	Length	$\widehat{H}$	Cover.	Length	$\widehat{H}$
$n = 50$	CLT[i2,2]	95.4	0.5970	0.3225	92.2	0.6776	0.5064	97.2	0.5422	0.7062
	CI[i2,2]	100.0	1.0000	0.5000	100.0	1.0000	0.5000	100.0	1.0000	0.5000
	CLT[i2,5]	89.4	0.3706	0.2121	88.2	0.5083	0.4838	94.2	0.4595	0.7265
	CI[i2,5]	100.0	1.0000	0.5000	100.0	1.0000	0.5000	100.0	1.0000	0.5000
	CLT[d4,2]	98.0	0.4899	0.2685	92.2	0.5817	0.4966	94.4	0.4836	0.7228
	CI[d4,2]	100.0	1.0000	0.5000	100.0	1.0000	0.5000	100.0	1.0000	0.5000
	CLT[d4,5]	86.8	0.3477	0.2064	88.2	0.4848	0.4739	91.8	0.4564	0.7183
	CI[d4,5]	100.0	1.0000	0.5000	100.0	1.0000	0.5000	100.0	1.0000	0.5000
$n = 100$	CLT[i2,2]	97.0	0.4689	0.2628	94.0	0.5232	0.4939	98.0	0.4143	0.7604
	CI[i2,2]	100.0	0.9997	0.4999	100.0	1.0000	0.5000	100.0	1.0000	0.5000
	CLT[i2,5]	92.4	0.2907	0.1999	91.2	0.3670	0.4911	91.0	0.3521	0.7682
	CI[i2,5]	100.0	0.9998	0.4999	100.0	0.9992	0.5004	100.0	0.9078	0.5461
	CLT[d4,2]	97.6	0.3865	0.2299	93.6	0.4259	0.4900	93.8	0.3704	0.7690
	CI[d4,2]	100.0	1.0000	0.5000	100.0	1.0000	0.5000	100.0	1.0000	0.5000
	CLT[d4,5]	90.2	0.2691	0.1965	89.4	0.3509	0.4882	90.4	0.3486	0.7655
	CI[d4,5]	100.0	1.0000	0.5000	100.0	0.9993	0.5003	100.0	0.9026	0.5487
$n = 500$	CLT[i2,2]	95.8	0.2540	0.2057	92.8	0.2365	0.4997	94.0	0.2095	0.7983
	CI[i2,2]	100.0	0.6990	0.3495	100.0	0.9399	0.5028	100.0	0.6864	0.6568
	CLT[i2,5]	95.0	0.1363	0.2004	93.6	0.1657	0.4980	93.8	0.1712	0.7983
	CI[i2,5]	100.0	0.5772	0.2886	100.0	0.7113	0.5192	100.0	0.5361	0.7319
	CLT[d4,2]	95.2	0.1965	0.2032	93.8	0.1908	0.4987	94.2	0.1820	0.7982
	CI[d4,2]	100.0	0.7002	0.3501	100.0	0.9459	0.5048	100.0	0.6806	0.6597
	CLT[d4,5]	93.6	0.1250	0.1997	93.6	0.1586	0.4977	94.2	0.1700	0.7967
	CI[d4,5]	100.0	0.5972	0.2986	100.0	0.7272	0.5316	100.0	0.5329	0.7335
$n = 1000$	CLT[i2,2]	95.4	0.1829	0.2019	93.8	0.1673	0.4988	94.4	0.1485	0.7988
	CI[i2,2]	100.0	0.5500	0.2750	100.0	0.6912	0.5015	100.0	0.5441	0.7279
	CLT[i2,5]	95.0	0.0963	0.1990	92.2	0.1173	0.4992	94.0	0.1211	0.7972
	CI[i2,5]	100.0	0.4596	0.2302	100.0	0.5022	0.5092	100.0	0.4434	0.7779
	CLT[d4,2]	94.6	0.1392	0.2009	93.2	0.1350	0.4981	93.8	0.1287	0.7979
	CI[d4,2]	100.0	0.5491	0.2745	100.0	0.6873	0.5026	100.0	0.5412	0.7294
	CLT[d4,5]	96.0	0.0884	0.1993	92.8	0.1123	0.4998	94.4	0.1203	0.7974
	CI[d4,5]	100.0	0.4725	0.2365	100.0	0.5130	0.5168	100.0	0.4419	0.7790
$n = 10000$	CLT[i2,2]	95.0	0.0579	0.2001	95.2	0.0529	0.5010	95.4	0.0469	0.8007
	CI[i2,2]	100.0	0.2179	0.2004	100.0	0.2179	0.5012	100.0	0.2179	0.8009
	CLT[i2,5]	94.4	0.0305	0.2001	94.8	0.0371	0.5002	96.4	0.0383	0.8006
	CI[i2,5]	100.0	0.1594	0.2008	100.0	0.1594	0.5009	100.0	0.1594	0.8013
	CLT[d4,2]	95.0	0.0440	0.2001	95.2	0.0427	0.5006	95.6	0.0407	0.8007
	CI[d4,2]	100.0	0.2165	0.2006	100.0	0.2165	0.5011	100.0	0.2165	0.8011
	CLT[d4,5]	94.4	0.0280	0.2001	94.0	0.0355	0.5001	97.0	0.0381	0.8004
	CI[d4,5]	100.0	0.1633	0.2020	100.0	0.1633	0.5020	100.0	0.1633	0.8023

Table 6: Monte-carlo experiment based on 500 replications of a standard fractional Brownian motion with Hurst parameter $H = 0.2, 0.5, 0.8$, for $M = 2, 5$ and for different values of the sample size. The filters i2 and d4 denote respectively the filter of Increments of order 2 and the Daublets 4. For these simulations the vector $\mathbf{d}$ has been fixed to the vector $\mathbf{A}$.

5.3 Discussion

Recall that the main objectives of this paper are the following:

1. to propose non-asymptotic confidence intervals for the Hurst parameter of a standard or non-standard fBm based on concentration inequalities that are in particular computable for small sample size;
2. to compare these procedures with approaches based on central limit theorems.

The first point is achieved with several theoretical improvements:

- When the scaling parameter C is known, we have refined the confidence interval proposed in Breton et al. (2009): the upper bound $H \leq H^\star < 1$ is relaxed, the condition on the sample size n is sharper and our new confidence intervals are valid for a large class of filter a.
- As a by-product in our way to optimize the numeric bounds, we have slightly improved the bounds obtained by Nourdin and Viens (2009) in the general concentration inequality (see Proposition 2).
- The case where C is unknown has never been considered with concentration inequalities before Proposition 7.
- The asymptotic properties are similar to that of confidence intervals based on central limit theorems. More specifically, the length of the confidence intervals derived by concentration inequalities behaves asymptotically as the ones of confidence intervals based on central limit theorems, that is $1/(\sqrt{n}\log(n))$ when C is known and $1/\sqrt{n}$ when C is unknown.

The second point is not conclusive: while the Monte-Carlo experiments are decent when C is known (in terms of coverage rate and of lengths of the confidence intervals), they are not good when C is unknown: the lengths equal often 1, *i.e.* the intervals correspond to $(0,1)$, when the sample size is small and are about five times larger when n is large. In fact, the confidence intervals derived from concentration inequalities are too much "sympathetic": the coverage rate is rather far from $1-\alpha$ (based on 500 replications, it is even often equal to 100%). This is the main reason why the length of the confidence interval is sometimes much larger than the ones based on central limit theorems.

As a conclusion, this work is the first attempt to define *computable* confidence intervals for the Hurst parameter H of a standard and a non-standard fractional Brownian motion with another approach than the classical one based on central limit theorems (at the very exception of Breton et al. (2009) where the first non-asymptotic confidence intervals were derived for the standard fBM with a more theoretical motivation). We did not get around the question of the numerical performances via Monte-Carlo experiments. The conclusion is that the proposed procedure (at least when C is unknown) appears to be underperforming. However, this approach is innovative, ambitious and provides encouraging signs via its research perspectives.

A Exact computations of ℓ_1-norm for filtered fBm

In this section, we describe how explicit exact bound can be obtained for the correlation of a filtered fBm. Let a be a filter of order p and length ℓ. Its covariance function is given by

$$\pi_H^a(k) = -\frac{1}{2}\sum_{q,r=0}^{\ell} a_q a_r |q-r+k|^{2H} = -\frac{1}{2}\sum_{j=-\ell}^{\ell} \alpha_j |j+k|^{2H}$$

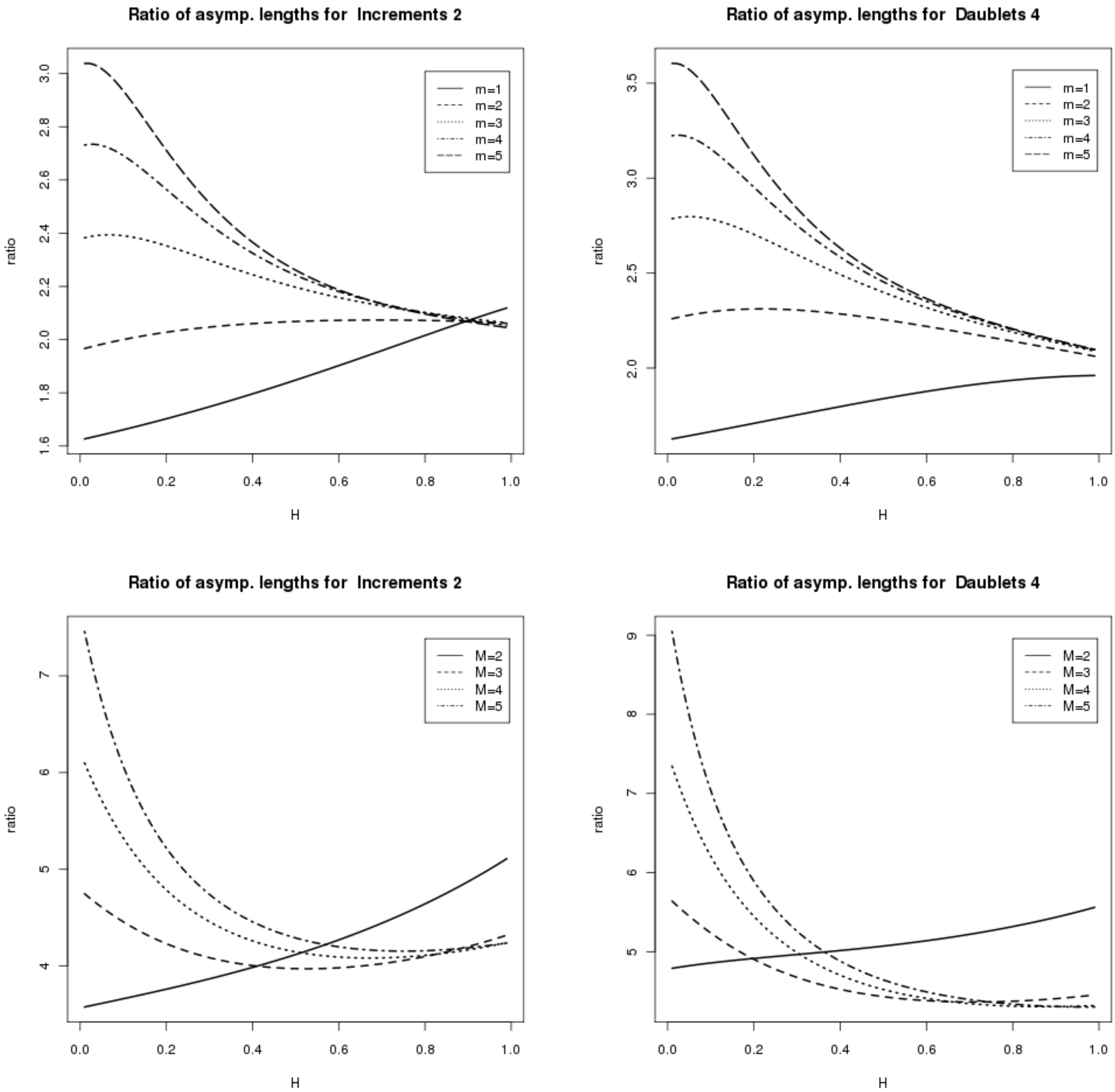

Figure 1: Ratio of asymptotic lengths of confidence intervals of procedures derived by concentration inequalities and central limit theorem when the scaling parameter C is known (top) and unknown (bottom). The confidence level equals $1-\alpha = 95\%$. For the general procedure, the vector $\mathbf{d}$ has been fixed to $\mathbf{d} := \mathbf{L_M} - \overline{\mathbf{L_M}}$

where $\alpha_j = \sum_{\substack{q,r=0\\q-r=j}}^{\ell} a_q a_r$. Note that

- $\alpha_j = \alpha_{-j}$, in particular $\pi_H^a(0) = -\sum_{j=1}^{\ell} \alpha_j j^{2H}$;
- $\sum_{j=-\ell}^{\ell} \alpha_j = \sum_{q,r}^{\ell} a_q a_r = 0$,
- for all $h \le 2p-1$, we have

$$\begin{aligned}
\sum_{j=-\ell}^{\ell} j^h \alpha_j &= \sum_{j=-\ell}^{\ell} j^h \sum_{q-r=j} a_q a_r = \sum_{j=-\ell}^{\ell} \sum_{q-r=j} (q-r)^h a_q a_r = \sum_{q,r=0}^{\ell} (q-r)^h a_q a_r \\
&= \sum_{q,r=0}^{\ell} \sum_{k=0}^{h} \binom{h}{k} q^k (-r)^{h-k} a_q a_r \\
&= \sum_{k=0}^{h} \left((-1)^{h-k} \binom{h}{k} \left(\sum_{q=0}^{\ell} q^k a_q \right) \left(\sum_{q=0}^{\ell} r^{h-k} a_r \right) \right) \\
&= 0. \qquad (28)
\end{aligned}$$

- $\sum_{j\neq 0} \alpha_j = -\alpha_0 = -\sum_{q=0}^{\ell} a_q^2 < 0$, $\alpha_\ell = a_0 a_\ell$.

A crucial observation is that, at least for $|k|$ large enough, all the $\pi_H^a(k)$, and thus all the $\rho^a(k)$, have the same sign. Indeed, using (28), we have for $|k| \ge \ell$:

$$\begin{aligned}
\pi_H^a(k) &= -\frac{1}{2} \sum_{j=1}^{\ell} \alpha_j \left(|k+j|^{2H} + |k-j|^{2H} - 2|k|^{2H} \right) \\
&= -\frac{|k|^{2H}}{2} \sum_{j=1}^{\ell} \alpha_j \left((1+j/k)^{2H} + (1-j/k)^{2H} - 2 \right) \\
&= -|k|^{2H} \sum_{i=p}^{+\infty} \left(\frac{(2H)(2H-1)\dots(2H-2i+1)}{(2i)!k^{2i}} \left(\sum_{j=1}^{\ell} \alpha_j j^{2i} \right) \right) \\
&\sim -|k|^{2H-2p} \frac{(2H)(2H-1)\dots(2H-2p+1)}{(2p)!} \left(\sum_{j=1}^{\ell} \alpha_j j^{2p} \right).
\end{aligned}$$

This observation allows to reduce the computation of the ℓ^1-norm $\|\rho_H^a\|_{\ell^1(\mathbb{Z})}$, which is an infinite sum with modulus, to an infinite sum of correlations but without modulus plus some finite sum (with modulus remaining). Essentially, it remains to compute the sum of correlation without modulus. This is done below. But observe first that if there exists some $k(H,a) \in \mathbb{N}$ so that the correlations $\rho_H^a(k)$ have all the same sign for $|k| \ge k(H,a)$ large enough. The value $k(H,a)$ is not known in general. However for some family of filters (including increment-type filters in and their dilatations $(in)^m$, $n, m \ge 1$), $k(H,a)$ is known and explicit computations are tractable:

Proposition 8 *For a dilated increment-type filter $a \in \{(in)^m : n, m \ge 1\}$, we have $k(H,a) = \ell$, i.e. the following property holds true:*

$$\textit{for all } |j| \ge \ell,\ \pi_H^a(j) \textit{ is of the same sign as } (-1)^{p+1}(2H-1). \qquad (29)$$

Proof: Let $\theta_m(f)(x) = f(x+m) - 2f(x) + f(x-m)$. Observe that if f is a convex (resp. concave) function, then $\theta_m(f)(x) \geq 0$ (resp. $\theta_m(f)(x) \leq 0$). For the $i1$ filter, we have $\pi_H^{i1}(x) = \frac{1}{2}\theta_1(|x|^{2H})$, for the $i2$ filter, we have $\pi_H^{i2}(x) = -\frac{1}{2}\theta_1^{\circ 2}(|x|^{2H})$ and more generally for the m-dilatation of the in filter, we have $\pi_H^{(in)^m}(x) = \frac{(-1)^{n+1}}{2}\theta_m^{\circ n}(|x|^{2H})$.

Observe also that the function $|x|^{2H}$ and all its iterated derivatives $(|x|^{2H})^{(2p)}$ of even order are convex if $H \geq 1/2$, concave if $H \leq 1/2$. By an immediate induction on n, we show that the same holds true for all $\theta_m^{\circ n}(|x|^{2H})$. In particular for $|j| \geq \ell m$, we obtain that $\pi_H^{(in)^m}(j)$ is of the same sign as $(-1)^{n+1}(2H-1)$. $\square$

Obviously, the property (29) does not hold true for any filter (consider for instance $\{1,-4,5,-2\}$). In order to make easier our following explicit computation to derive exact value for $\|\rho^a\|_{\ell^1(\mathbb{Z})}$, we consider a filter a satisfying (29) but we stress that for each particular filter the same strategy applies with some specific $k(H,a)$. First, for all $N \geq \ell$, we have:

$$\begin{aligned}
-2\sum_{j=\ell}^{N} \pi_H^a(j) &= \sum_{j=\ell}^{N}\sum_{k=-\ell}^{\ell} \alpha_k |j+k|^{2H} \\
&= \sum_{k=-\ell}^{\ell} \alpha_k \sum_{j=\ell}^{N} |j+k|^{2H} = \sum_{k=-\ell}^{\ell} \alpha_k \sum_{j=\ell+k}^{N+k} |j|^{2H} \\
&= \alpha_{-\ell} S_{N-\ell}^H + \sum_{k=-\ell+1}^{\ell} \alpha_k \left(S_{N+k}^H - S_{\ell+k-1}^H\right) \\
&= \alpha_{-\ell} S_{N-\ell}^H + \sum_{k=-\ell+1}^{\ell} \alpha_k \left(S_{N-l}^H + \sum_{j=N-\ell+1}^{N+k} |j|^{2H} - S_{\ell+k-1}^H\right) \\
&= \left(\sum_{k=-\ell}^{\ell} \alpha_k\right) S_{N-l}^H + \left(\sum_{k=-\ell+1}^{\ell} \alpha_k \sum_{j=N-\ell+1}^{N+k} |j|^{2H}\right) - \left(\sum_{k=-\ell+1}^{\ell} \alpha_k S_{\ell+k-1}^H\right) \\
&= x_N - \sum_{k=-\ell+1}^{\ell} \alpha_k S_{\ell+k-1}^H
\end{aligned}$$

where $S_k^H = \sum_{j=0}^{k} j^{2H}$ and

$$\begin{aligned}
x_N &= \sum_{j=N-\ell+1}^{N+\ell} \left(|j|^{2H} \sum_{k=j-N}^{\ell} \alpha_k\right) \\
&= |N+\ell|^{2H} \sum_{i=0}^{2\ell-1} \left(\left(1 - \frac{i}{N+\ell}\right)^{2H} \sum_{k=\ell-i}^{\ell} \alpha_k\right) \\
&= |N+\ell|^{2H} \sum_{i=0}^{2\ell-1} \left(\left(1 - \frac{2Hi}{N+\ell} + \frac{2H(2H-1)i^2}{2(N+\ell)^2} + O\left(\frac{1}{(N+\ell)^3}\right)\right) \sum_{k=\ell-i}^{\ell} \alpha_k\right) \qquad (30)
\end{aligned}$$

But

$$\sum_{i=0}^{2\ell-1}\sum_{k=\ell-i}^{\ell} \alpha_k = \sum_{k=-\ell+1}^{\ell} (\ell+k)\alpha_k = \sum_{k=-\ell}^{\ell} (\ell+k)\alpha_k = 0$$

and

$$\begin{aligned}\sum_{i=0}^{2\ell-1}\left(i\sum_{k=\ell-i}^{\ell}\alpha_k\right) &= \sum_{k=-\ell+1}^{\ell}\left(\alpha_k\sum_{i=\ell-k}^{2\ell-1} i\right)=\sum_{k=-\ell}^{\ell}\left(\alpha_k\sum_{i=\ell-k}^{2\ell-1} i\right)\\ &= \sum_{k=-\ell}^{\ell}\alpha_k\left(\frac{2\ell(2\ell-1)}{2}-\frac{(\ell-k)(\ell-k-1)}{2}\right)=0\end{aligned}$$

because of (28). We obtain $x_N=O\big((N+\ell)^{2H-2}\big)\to 0$, $N\to+\infty$. Actually, expanding $(1-i/(N+\ell))^{2H}$ to the $(2p-1)$-th order in (30), and since $\sum_{i=1}^{N} i^k$ is a polynomial in N of degree $k+1$, (28) shows that $x_N=O\big((N+\ell)^{2H-2p+1}\big)$. Finally with the property (29), we have:

$$2\sum_{j=\ell}^{+\infty}|\pi_H^a(j)|=(-1)^{p+1}\epsilon((2H-1)\sum_{k=-\ell+1}^{\ell}\alpha_k S_{\ell+k-1}^H$$

and

$$\begin{aligned}\|\rho_H^a\|_{\ell^1(\mathbb{Z})} &= 1+2\sum_{k=1}^{\ell-1}|\rho_H^a(k)|+2\sum_{k=\ell}^{+\infty}|\rho_H^a(k)|\\ &= 1+\sum_{k=1}^{\ell-1}\left|\frac{\sum_{j=-\ell}^{\ell}\alpha_j|j+k|^{2H}}{\sum_{j=1}^{\ell}\alpha_j j^{2H}}\right|+(-1)^{p+1}\epsilon(2H-1)\frac{\sum_{k=-\ell+1}^{\ell}\alpha_k S_{\ell+k-1}^H}{|\sum_{j=1}^{\ell}\alpha_j j^{2H}|}\\ &= 1+\sum_{k=1}^{\ell-1}\frac{\left|\sum_{j=-\ell}^{\ell}\alpha_j|j+k|^{2H}\right|}{-\sum_{j=1}^{\ell}\alpha_j j^{2H}}+(-1)^{p+1}\epsilon(2H-1)\frac{\sum_{k=-\ell+1}^{\ell}\alpha_k S_{\ell+k-1}^H}{-\sum_{j=1}^{\ell}\alpha_j j^{2H}},\end{aligned}\tag{31}$$

where we recall that $\epsilon(2H-1)=sign(2H-1)$. First, note that the modulus has been removed in the denominator of (31) according to the following observation:

$$\sum_{j=1}^{\ell}\alpha_j j^{2H}=\frac{1}{2}\sum_{j=-\ell}^{\ell}\alpha_j j^{2H}\to_{H\to 0}\frac{1}{2}\sum_{j\neq 0}\alpha_j=\sum_{j=-\ell}^{\ell}\alpha_j-\alpha_0=-\alpha_0<0.$$

Since we assume moreover $\pi_H^a(0)\neq 0$, this means that $\pi_H^a(0)>0$ and that $\left|\sum_{j=1}^{\ell}\alpha_j j^{2H}\right|=-\sum_{j=1}^{\ell}\alpha_j j^{2H}$.

Next, note that (31) is an explicit expression involving only finite sums and can be easily explicitely optimized for $H\in(0,1)$ for every given a satisfying $\mathbf{H}^a$. Note that, for $p\geq 2$, when $H\to 1$, right-hand side of (31) remains well defined. Observe first that since for any fixed k, $\lim_{H\to 1}S_k^H=S_k^1=\frac{k(k+1)(2k-1)}{6}$, we have using (28)

$$\lim_{H\to 1}\sum_{k=-\ell+1}^{\ell}\alpha_k S_{\ell+k-1}^H=\frac{1}{6}\sum_{k=-\ell+1}^{\ell}\alpha_k(\ell+k-1)(\ell+k)(2\ell+2k-1)=0.$$

The same holds true for $\sum_{j=-\ell}^{\ell}\alpha_j|j+k|^{2H}$ and $\sum_{j=1}^{\ell}\alpha_j j^{2H}$, but under $\mathbf{H}^a$ in (15), the rule of l'Hospital entails $\lim_{H\to 1^-}\|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ exists and is finite. Since obviously, $\|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ is a continuous function of $H\in[0,1)$, this ensures the continuity of $\|\rho_H^a\|_{\ell^1(\mathbb{Z})}$ on $[0,1]$ and the constant κ^a in our confidence interval is obtained by maximazing the explicit function in (31).

Dilated simple increments $(i1)^m = \{-1,1\}^m$**.** In this case, $\ell = m$, $p = 1$, $\alpha_j = 0$ for $1 < j < m$ and $\alpha_0 = 2$, $\alpha_{\pm m} = -1$ so that (31) rewrites:

$$\left\|\rho_H^{\{-1,1\}^m}\right\|_{\ell^1(\mathbb{Z})} = 1 + \sum_{j=1}^{m-1} \frac{\left||j+m|^{2H} - 2|j|^{2H} + |j-m|^{2H}\right|}{m^{2H}} + \frac{S_{2m-1}^H - 2S_{m-1}^H}{m^{2H}}. \tag{32}$$

For instance for $m = 1$, $\left\|\rho_H^{\{-1,1\}}\right\|_{\ell^1(\mathbb{Z})} = 2$ and for $m = 2$, $\left\|\rho_H^{\{-1,1\}}\right\|_{\ell^1(\mathbb{Z})} = 2\frac{4^H+9^H-1}{4^H}$, so that $\kappa^{i1} = 4$ and $\kappa^{(i1)^2} = 8$ (recall that in this case, we optimize for $H \in (0, 1/2]$).

In general, since the right-hand side of (32) is a continuous function of H, and since for all $k \geq 1$, $S_k^{1/2} = \frac{k(k+1)}{2}$, we have $\lim_{H\to(1/2)^-} \left\|\rho_H^{\{-1,1\}^m}\right\|_{\ell^1(\mathbb{Z})} = 2m$ while $\left\|\rho_{1/2}^{\{-1,1\}^m}\right\|_{\ell^1(\mathbb{Z})} = m$, exhibiting a discontinuity of the ℓ^1-norm for the dilated i^1 filters.

Dilated double increments $(i2)^m = \{1,-2,1\}^m$**.** In this case, $\ell = 2m$, $p = 2$ and $\alpha_0 = 6$, $\alpha_{\pm m} = -4$, $\alpha_{\pm 2m} = 1$, $\alpha_j = 0$, $j \neq 0, \pm m, \pm 2m$, so that (31) rewrites:

$$\begin{aligned}\|\rho_H^{\{1,-2,1\}^m}\|_{\ell^1(\mathbb{Z})} &= 1 + \sum_{k=1}^{2m-1} \frac{\left||k-2m|^{2H} - 4|k-m|^{2H} + 6|k|^{2H} - 4|k+m|^{2H} + |k+2m|^{2H}\right|}{m^{2H}(4-4^H)} \\ &\quad + \epsilon(1-2H)\frac{-4S_{m-1}^H + 6S_{2m-1}^H - 4S_{3m-1}^H + S_{4m-1}^H}{m^{2H}(4-4^H)}.\end{aligned}$$

In order to obtain explicit values, we focus on the cases $m = 1$ and $m = 2$. First, for $m = 1$, (31) reduces to

$$\|\rho_H^{i2}\|_{\ell^1(\mathbb{Z})} = \begin{cases} 1 + \frac{10-7\times 4^H+2\times 9^H}{4-4^H}, & H \leq 1/2 \\ 2, & H \geq 1/2 \end{cases}$$

and elementary computations entail:

$$\kappa^{i2} = 2 \times \lim_{H\to 0^+} \|\rho_H^{i2}\|_{\ell^1(\mathbb{Z})} = 2\left(1 + \frac{5}{3}\right) = \frac{16}{3}.$$

Next, for $m = 2$, since

$$\begin{aligned} 2\pi_H^{(i2)^2}(1) &= -2 + 3\times 9^H - 25^H & \geq 0 \quad \forall H \in (0,1) \\ 2\pi_H^{(i2)^2}(2) &= -7\times 4^H + 4\times 16^H - 36^H & \leq 0 \quad \forall H \in (0,1) \\ 2\pi_H^{(i2)^2}(3) &= 3 - 6\times 9^H + 4\times 25^H - 49^H & \leq 0 \quad \forall H \in (0,1) \end{aligned}$$

expression (31) reduces to

$$\left\|\rho_H^{(i2)^2}\right\|_{\ell^1(\mathbb{Z})} = \begin{cases} 1 + \frac{-6+10\times 4^H+12\times 9^H-7\times 16^H-8\times 25^H+2\times 36^H+2\times 49^H}{4^H(4-4^H)} & \text{for } H \leq 1/2 \\ 1 + \frac{-4+4\times 4^H+6\times 9^H-16^H-2\times 25^H}{4^H(4-4^H)} & \text{for } H \geq 1/2. \end{cases}$$

An elementary study of this function, together with the rule of l'Hospital, entails that

$$\kappa^{(i2)^2} = 2\times \sup_{H\in[0,1]} \left\|\rho_H^{(i2)^2}\right\|_{\ell^1(\mathbb{Z})} = 2\times \lim_{H\to 1-} \left\|\rho_H^{(i2)^2}\right\|_{\ell^1(\mathbb{Z})} = 2\left(1 + \frac{25\log(5) - 27\log(3)}{8\log(2)}\right) \simeq 7.813554.$$

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