Dimension Free and Infinite Variance Tail Estimates on Poisson Space

**Jean-Christophe Breton · Christian Houdré ·
Nicolas Privault**

Abstract Concentration inequalities are obtained on Poisson space, for random functionals with finite or infinite variance. In particular, dimension free tail estimates and exponential integrability results are given for the Euclidean norm of vectors of independent functionals. In the finite variance case these results are applied to infinitely divisible random variables such as quadratic Wiener functionals, including Lévy's stochastic area and the square norm of Brownian paths. In the infinite variance case, various tail estimates such as stable ones are also presented.

Keywords Concentration · Infinite divisibility · Stable laws · Poisson space ·
Ornstein-Uhlenbeck semi-group · Quadratic Wiener functionals · Large deviations

Mathematics Subject Classification (2000) 60F99 · 60E07 · 60G57

1 Introduction and Notation

Let Ω^X denote the set of Radon measures

$$\Omega^X = \left\{ \omega = \sum_{i=1}^{N} \epsilon_{t_i} : (t_i)_{i=1}^{i=N} \subset X, \ t_i \neq t_j, \ \forall i \neq j, \ N \in \mathbb{N} \cup \{\infty\} \right\}$$

N. Privault (✉)
Laboratoire de Mathématiques, Université de Poitiers, Téléport 2-BP30179,
86962 Chasseneuil, France
e-mail: nicolas.privault@math.univ-poitiers.fr

J.-C. Breton
Laboratoire de Mathématiques, Université de la Rochelle, Avenue Michel Crépeau,
17042 La Rochelle, France
e-mail: jcbreton@univ-lr.fr

C. Houdré
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA
e-mail: houdre@math.gatech.edu

where X is a σ-compact metric space with distance d_X, and ϵ_t denotes the Dirac measure at $t \in X$. Let ν be a diffuse Radon measure on X, and let P be the Poisson measure with intensity ν on Ω^X. Let the linear, closable, finite difference operator

$$D : L^2(\Omega^X, P) \longrightarrow L^2(\Omega^X \times X, P \otimes \nu)$$

be defined via

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad dP \times \nu(d\omega, dx)\text{-a.e.}$$

where as a convention we identify $\omega \in \Omega^X$ with its support, cf. e.g. [20, 22, 23].

In [1, 8, 27], Poisson tail estimates are obtained under the hypothesis

$$DF \leq K, \quad P \otimes \nu\text{-a.e.,} \quad \text{and} \quad \|DF\|_{L^\infty(\Omega^X, L^2(X, \nu))} \leq \tilde{\alpha} < \infty,$$

for some $K \geq 0$. While (modified) logarithmic Sobolev inequalities and the Herbst method are used in [1] and [27], the methods of [8] rely on covariance representations ([4, 6]). Recently the results of [6] have further led in [7] to estimates for Lipschitz functions of stable random vectors. Even more recently, dimension free concentration is obtained in [9] for the Euclidean norm as well as for various classes of functions of independent infinitely divisible vectors having finite exponential moments.

In the present paper we first obtain new deviation inequalities on Poisson space via the covariance method. Then, by replacing the bounds on DF and on $\|DF\|_{L^\infty(\Omega^X, L^2(X, \nu))}$ by growth conditions, deviation results for Poisson functionals with infinite variance are given.

Let us briefly describe the content of the paper. In Sect. 2 we deal with Lévy measures with finite variance, using the covariance representation method involving the Ornstein-Uhlenbeck semi-group. This leads to general deviation results for Poisson functionals having finite exponential moments. In Sect. 3 we obtain dimension free deviation estimates and exponential integrability properties for random vectors of such Poisson functionals. Since an infinitely divisible random vector can be represented as a vector of Poisson stochastic integrals, these results are then applied to derive deviation inequalities for Lipschitz functions of infinitely divisible vectors. In Sect. 4, we study the particular case of quadratic Wiener functionals, including the square norm of Brownian path, the sample variance of Brownian motion and Lévy's stochastic area. For such i.i.d. vectors, this also gives dimension free inequalities in Euclidean norm, and large deviation estimates in ℓ^p-norm, $p \in [1, \infty]$, recovering tail estimates of [2] for non-decoupled Gaussian chaos of degree 2. In Sect. 5 we adapt the method of [7] to prove other tail estimates under weaker hypothesis on the gradient. For example, if ν is the Lévy measure of an α-stable vector, the bounds on D can be replaced by the growth conditions

$$\sup_{x \in B_X(0, R)} |D_x F| \leq C' R \quad \text{and} \quad \|DF\|^2_{L^\infty(\Omega^X, L^2(B_X(0, R)))} \leq C R^{2-\alpha}, \quad R \geq R_0 \qquad (1.1)$$

where $B_X(0, R) = \{x \in X : d_X(0, x) \leq R\}$ is the ball of radius R in X. Here, 0 denotes a fixed arbitrary center in X, whose choice has no influence on the growth conditions (1.1). This leads to an estimate of stable type for the deviation of F from one of its medians.

Let us now introduce some notation which will be used throughout the paper. The multiple Poisson stochastic integral $I_n(f_n)$ is defined for $n \geq 1$ as

$$I_n(f_n)(\omega) = \int_{\Delta_n} f_n(y_1, \ldots, y_n)(\omega(dy_1) - \nu(dy_1)) \cdots (\omega(dy_n) - \nu(dy_n)),$$

for every square-integrable symmetric function $f_n \in L^2(X, \nu)^{\circ n}$, where

$$\Delta_n = \big\{ (x_1, \ldots, x_n) \in X^n : x_i \neq x_j, \ \forall i \neq j \big\}.$$

For $n = 0$ we let $I_0(f_0) = f_0$ for any real constant f_0. Recall the isometry formula

$$E[I_n(f_n) I_m(g_m)] = n! \mathbf{1}_{\{n=m\}} \langle f_n, g_m \rangle_{L^2(X,\nu)^{\circ n}},$$

see [21], and recall also that every square-integrable random variable $F \in L^2(\Omega^X, P)$ admits the Wiener-Poisson decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

The operator D defined above is such that

$$D_x I_n(f_n)(\omega) = n I_{n-1}(f_n(*, x))(\omega), \quad P(d\omega) \otimes \nu(dx)\text{-a.e.}, \quad n \in \mathbb{N},$$

and in particular,

$$D_x I_1(f)(\omega) = f(x), \quad \nu(dx)\text{-a.e.}$$

We denote by $\mathrm{Dom}(D)$ the domain of D, i.e. the space of functionals $F \in L^2(\Omega^X, P)$ such that $DF \in L^2(\Omega^X \times X, P \otimes \nu)$. Recall also that the Ornstein-Uhlenbeck semi-group $(P_t)_{t \in \mathbb{R}_+}$ is defined via

$$P_t I_n(f_n) = e^{-nt} I_n(f_n), \quad f_n \in L^2(X, \nu)^{\circ n}, \ n \in \mathbb{N}.$$

In the sequel we also use the integral representation of the Ornstein-Uhlenbeck semi-group $(P_t)_{t \in \mathbb{R}_+}$ in terms of a probability kernel $p_t(\omega, d\tilde{\omega}, d\hat{\omega})$, cf. e.g. [26]:

$$P_t F(\omega) = \int_{\Omega^X \times \Omega^X} F(\tilde{\omega} \cup \hat{\omega}) p_t(\omega, d\tilde{\omega}, d\hat{\omega}). \tag{1.2}$$

When $X = \mathbb{R}^n$, $|\cdot|_p$ denotes the ℓ^p-norm on $\mathbb{R}^n$, $p \geq 1$. Assuming that

$$\int_{\mathbb{R}^n} 1 \wedge |y|_2^2 \nu(dy) < \infty,$$

any n-dimensional infinitely divisible (ID) random vector $F = (F_1, \ldots, F_n)$ without Gaussian component and with Lévy measure ν can be represented as the vector of single Poisson stochastic integrals

$$F = \left(\int_{\{|y|_2 \leq 1\}} y_k (\omega(dy) - \nu(dy)) + \int_{\{|y|_2 > 1\}} y_k \omega(dy) + b_k \right)_{1 \leq k \leq n} \tag{1.3}$$

for some $b \in \mathbb{R}^n$. Indeed, the characteristic function of F is given by

$$\varphi_F(u) = E\big[e^{i \langle F, u \rangle}\big] = \exp\left(i \langle b, u \rangle + \int_{\mathbb{R}^n} \big(e^{i \langle y, u \rangle} - 1 - i \langle y, u \rangle \mathbf{1}_{\{|y|_2 \leq 1\}} \big) \nu(dy) \right),$$

$u \in \mathbb{R}^n$.

2 Deviation Results from the Ornstein-Uhlenbeck Semi-group

As in [8], we need the following covariance identity on Poisson space, which is obtained from the Ornstein-Uhlenbeck semi-group.

Lemma 2.1 *Let* $F, G \in \text{Dom}(D)$, *then*

$$\text{Cov}(F, G) = E\left[\int_0^\infty e^{-s} \int_X D_y F P_s D_y G \nu(dy) ds\right]. \tag{2.1}$$

Proof By orthogonality of multiple integrals of different orders and continuity of P_s, $s \in \mathbb{R}_+$, on $L^2(\Omega^X, P)$, it suffices to prove the identity for $F = I_n(f_n)$ and $G = I_n(g_n)$:

$$
\begin{aligned}
E[I_n(f_n)I_n(g_n)] &= n! \langle f_n, g_n \rangle_{L^2(X,\nu)^{\circ n}} = n! \int_{X^n} f_n g_n d\nu^{\otimes n} \\
&= n! \int_X \int_{X^{(n-1)}} f_n(x, y) g_n(x, y) \nu^{\otimes(n-1)}(dx) \nu(dy) \\
&= n \int_X E[I_{n-1}(f_n(\cdot, y)) I_{n-1}(g_n(\cdot, y))] \nu(dy) \\
&= \frac{1}{n} E\left[\int_X D_y I_n(f_n) D_y I_n(g_n) \nu(dy)\right] \\
&= E\left[\int_0^\infty e^{-ns} \int_X D_y I_n(f_n) D_y I_n(g_n) \nu(dy) ds\right] \\
&= E\left[\int_0^\infty e^{-s} \int_X D_y I_n(f_n) P_s D_y I_n(g_n) \nu(dy) ds\right]. \qquad \square
\end{aligned}
$$

Using the covariance identity (2.1) and the representation (1.2) we first state a general deviation result which slightly improves the one presented in [8]. In particular it will be applied, in Sect. 3, to obtain deviation inequalities on product spaces for vectors of random functionals. In this proposition and the following ones, the supremum on Ω^X can be taken as an essential supremum with respect to P.

Proposition 2.2 *Let* $F \in \text{Dom}(D)$ *be such that* $e^{sF} \in \text{Dom}(D)$, $0 \le s \le t_0$, *for some* $t_0 > 0$. *Then*

$$P(F - E[F] \ge x) \le \exp\left(\min_{0 < t < t_0} \int_0^t h(s)ds - tx\right), \quad x > 0$$

where

$$h(s) = \sup_{(\omega,\omega') \in \Omega^X \times \Omega^X} \left|\int_X \left(e^{sD_y F(\omega)} - 1\right) D_y F(\omega') \nu(dy)\right|, \quad s \in [0, t_0). \tag{2.2}$$

If moreover h *is nondecreasing and finite on* $[0, t_0)$ *then*

$$P(F - E[F] \ge x) \le \exp\left(-\int_0^x h^{-1}(s)ds\right), \quad 0 < x < h(t_0^-) \tag{2.3}$$

where h^{-1} is the left-continuous inverse of h:

$$h^{-1}(x) = \inf\{t > 0 : h(t) \geq x\}, \quad 0 < x < h(t_0^-).$$

Proof We start by deriving the following inequality for a centered random variable F:

$$E[Fe^{sF}] \leq h(s)E[e^{sF}], \quad 0 \leq s \leq t_0. \tag{2.4}$$

This follows from (2.1). Indeed, using the integral representation (1.2) of the Ornstein-Uhlenbeck semi-group $(P_t)_t$ for $P_v D_y F(\omega)$, we have

$$E[Fe^{sF}] = E\left[\int_0^\infty e^{-v} \int_X D_y e^{sF} P_v D_y F v(dy) dv\right]$$

$$= \int_{\Omega^X} \int_0^\infty e^{-v} \int_X \left(e^{sD_y F(\omega)} - 1\right) e^{sF(\omega)}$$

$$\times \int_{\Omega^X \times \Omega^X} D_y F(\tilde{\omega} \cup \hat{\omega}) p_v(\omega, d\tilde{\omega}, d\hat{\omega}) v(dy) dv P(d\omega)$$

$$\leq \int_{\Omega^X} \int_0^\infty e^{-v} e^{sF(\omega)}$$

$$\times \int_{\Omega^X \times \Omega^X} \left|\int_X \left(e^{sD_y F(\omega)} - 1\right) D_y F(\tilde{\omega} \cup \hat{\omega}) v(dy)\right| p_v(\omega, d\tilde{\omega}, d\hat{\omega}) dv P(d\omega)$$

$$\leq \sup_{(\omega, \omega') \in \Omega^X \times \Omega^X} \left|\int_X \left(e^{sD_y F(\omega)} - 1\right) D_y F(\omega') v(dy)\right| E\left[e^{sF} \int_0^\infty e^{-v} dv\right]$$

$$= \sup_{(\omega, \omega') \in \Omega^X \times \Omega^X} \left|\int_X \left(e^{sD_y F(\omega)} - 1\right) D_y F(\omega') v(dy)\right| E[e^{sF}]$$

which yields (2.4). In the general case, we let $L(s) = E[e^{s(F - E[F])}]$ and obtain:

$$\frac{L'(s)}{L(s)} \leq h(s), \quad 0 \leq s \leq t_0$$

which using Chebychev's inequality gives:

$$P(F - E[F] \geq x) \leq \exp\left(-tx + \int_0^t h(s) ds\right). \tag{2.5}$$

Using the relation $\frac{d}{dt}(\int_0^t h(s) ds - tx) = h(t) - x$, we can then optimize as follows:

$$\min_{0 < t < t_0} \int_0^t h(s) ds - tx = \int_0^{h^{-1}(x)} h(s) ds - xh^{-1}(x)$$

$$= \int_0^x s \, dh^{-1}(s) - xh^{-1}(x)$$

$$= -\int_0^x h^{-1}(s) ds, \tag{2.6}$$

hence

$$P(F - E[F] \geq x) \leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \quad 0 < x < h(t_0^-). \qquad \square$$

If h_1 is a non-decreasing function defined by (2.2) and h_2 is another non-decreasing functions such that $h_1(t) \leq h_2(t)$, $t \in [0, t_0]$, then clearly we have

$$P(F - E[F] \geq x) \leq \exp\left(-\int_0^x h_1^{-1}(s)ds\right)$$

$$\leq \exp\left(-\int_0^x h_2^{-1}(s)ds\right), \quad 0 < x < h_1(t_0^-).$$

In the sequel we derive several corollaries of Proposition 2.2 and discuss possible choices for the function h, in particular for vectors of random functionals. For simplicity of notation we will drop the indices in h_1, h_2 in general, and $h(t)$ will denote any function satisfying (2.3).

Note that since

$$h(t) \leq \int_X \|D_y F\|_\infty \left\| e^{t|D_y F|} - 1 \right\|_\infty \nu(dy),$$

Proposition 2.2 recovers Proposition 3.3 in [8], which is obtained via a covariance identity relying on the Clark formula. In the next proposition and following [9], we obtain a better result by applying Proposition 2.2 with more careful bounds.

Proposition 2.3 *Let $F \in L^1(\Omega^X, P)$ and let $K : X \to \mathbb{R}_+$ be a non-negative function such that*

$$D_y F(\omega) \leq K(y), \quad y \in X, \omega \in \Omega^X. \tag{2.7}$$

Then

$$P(F - E[F] \geq x) \leq \exp\left(\min_{t>0} \int_0^t h(s)ds - tx\right), \quad x > 0$$

where

$$h(t) = \sup_{\omega \in \Omega^X} \int_X \frac{e^{tK(y)} - 1}{K(y)} |D_y F(\omega)|^2 \nu(dy), \quad t > 0, \tag{2.8}$$

with

$$\frac{e^{tK} - 1}{K} = t$$

if $K = 0$. If moreover h is finite on $[0, t_0)$ then

$$P(F - E[F] \geq x) \leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \quad 0 < x < h(t_0^-). \tag{2.9}$$

If $K(y) \equiv 0$, $y \in X$, we have:

$$P(F - E[F] \geq x) \leq \exp\left(-\frac{x^2}{2\tilde{\alpha}^2}\right), \quad x > 0,$$

with

$$\tilde{\alpha}^2 = \sup_{\omega \in \Omega^X} \int_X |D_y F(\omega)|^2 \nu(dy).$$

Proof Letting $F_n = \max(-n, \min(F, n))$ we have $e^{sF_n} \in \text{Dom}(D)$, hence we may apply Proposition 2.2 to F_n, $n \geq 1$. Note moreover that since K is $\mathbb{R}_+$-valued, the condition $D_y F_n(\omega) \leq K(y)$, $\omega \in \Omega^X$, $y \in X$ is satisfied. We have

$$h_n(t) := \sup_{(\omega,\omega') \in \Omega^X \times \Omega^X} \left| \int_X \frac{e^{tD_y F_n(\omega)} - 1}{D_y F_n(\omega)} D_y F_n(\omega) D_y F_n(\omega') \nu(dy) \right|$$

$$\leq \sup_{(\omega,\omega') \in \Omega^X \times \Omega^X} \int_X \frac{e^{tK(y)} - 1}{K(y)} |D_y F_n(\omega)| |D_y F_n(\omega')| \nu(dy)$$

$$\leq \frac{1}{2} \sup_{(\omega,\omega') \in \Omega^X \times \Omega^X} \int_X \frac{e^{tK(y)} - 1}{K(y)} \left(|D_y F_n(\omega)|^2 + |D_y F_n(\omega')|^2 \right) \nu(dy)$$

$$\leq \sup_{\omega \in \Omega^X} \int_X \frac{e^{tK(y)} - 1}{K(y)} |D_y F_n(\omega)|^2 \nu(dy)$$

$$\leq \sup_{\omega \in \Omega^X} \int_X \frac{e^{tK(y)} - 1}{K(y)} |D_y F(\omega)|^2 \nu(dy)$$

which allows us to conclude first for $F_n - E[F_n]$. As n goes to infinity, F_n converges to F a.s. and $E[F_n]$ tends to $E[F]$ by dominated convergence, which proves the result for $F - E[F]$. $\qquad\square$

Note that if $K : X \to \mathbb{R}$ in (2.7) is not necessarily positive and $F, e^{sF} \in \text{Dom}(D)$, $0 \leq s \leq t_0$, for some $t_0 > 0$, then applying Proposition 2.2 and the above argument directly to F yields:

$$P(F - E[F] \geq x) \leq \exp\left(\min_{0 < t < t_0} \int_0^t h(s)ds - tx \right), \quad x > 0,$$

and (2.9) also holds provided h is finite on $[0, t_0)$.

Part of the next corollary recovers a result of [27] (see also [8]). This result is used in Corollary 2.6 below as well as in the infinite variance case in Sect. 5.

Corollary 2.4 *Let $F \in L^2(\Omega^X, P)$ be such that $DF \leq K$, $P \otimes \nu$-a.e., for some $K \in \mathbb{R}$, and $\|DF\|_{L^\infty(\Omega^X, L^2(X, \nu))} \leq \tilde{\alpha}$. We have for $K > 0$:*

$$P(F - E[F] \geq x) \leq e^{x/K} \left(1 + \frac{xK}{\tilde{\alpha}^2} \right)^{-\frac{x}{K} - \frac{\tilde{\alpha}^2}{K^2}}, \quad x > 0, \tag{2.10}$$

and for $K = 0$:

$$P(F - E[F] \geq x) \leq \exp\left(-\frac{x^2}{2\tilde{\alpha}^2} \right), \quad x > 0. \tag{2.11}$$

Proof If $K \geq 0$, let us first assume that F is a bounded random variable. The function h in (2.8) is such that

$$h(t) \leq \frac{e^{tK} - 1}{K} \|DF\|^2_{L^\infty(\Omega^X, L^2(X, \nu))} \leq \tilde{\alpha}^2 \frac{e^{tK} - 1}{K}, \quad t > 0.$$

Applying (2.5) with $\tilde{\alpha}^2 (e^{tK} - 1)/K$ gives

$$P(F - E[F] \geq x) \leq \exp\left(-tx + \frac{\tilde{\alpha}^2}{K^2} (e^{tK} - tK - 1)\right).$$

Optimizing in t with $t = K^{-1} \log(1 + Kx/\tilde{\alpha}^2)$ (or using directly (2.3) with the inverse $K^{-1} \log(1 + Kt/\tilde{\alpha}^2)$) we have

$$P(F - E[F] \geq x) \leq \exp\left(\frac{x}{K} - \left(\frac{x}{K} + \frac{\tilde{\alpha}^2}{K^2}\right) \log\left(1 + \frac{xK}{\tilde{\alpha}^2}\right)\right)$$

which yields (2.12), (2.11) and (2.10), depending on the value of K. For unbounded F, apply the above to $F_n = \max(-n, \min(F, n))$ with $|DF_n| \leq |DF|$, $n \geq 1$. Then (2.10) follows since, as n goes to infinity, F_n converges to F in $L^2(\Omega^X)$, DF_n converges to DF in $L^2(\Omega^X, L^2(X, \nu))$, and $DF_n \leq K$, $n \geq 1$. The same argument applies if $K = 0$. $\square$

In case $K < 0$ and $e^{tF} \in \mathrm{Dom}(D)$ for all $t > 0$, Proposition 2.3 yields in a similar way:

$$P(F - E[F] \geq x) \leq e^{x/K} \left(1 + \frac{xK}{\tilde{\alpha}^2}\right)^{-\frac{x}{K} - \frac{\tilde{\alpha}^2}{K^2}}, \quad 0 < x < -\frac{\tilde{\alpha}^2}{K}. \tag{2.12}$$

If F is an infinitely divisible random variable in $\mathbb{R}^n$, without Gaussian component and with Lévy measure ν, the representation (1.3) shows that for any Lipschitz(c) function $f : \mathbb{R}^n \to \mathbb{R}$,

$$|D_x f(F)(\omega)| = |f(F(\omega \cup \{x\})) - f(F(\omega))|$$
$$\leq c \|F(\omega \cup \{x\}) - F(\omega)\|$$
$$= c \|x\| \tag{2.13}$$

where $\|\cdot\|$ is any norm in $\mathbb{R}^n$. Hence when $X = \mathbb{R}^n$ and ν has bounded support, Corollary 2.4 also recovers Corollary 1 of [6] with

$$K = \inf\{r > 0 : \nu(\{x \in X : \|x\| > r\}) = 0\},$$

and $\tilde{\alpha}^2 = \int_{\mathbb{R}^n} \|y\|^2 \nu(dy)$, i.e.

$$P(f(F) - E[f(F)] \geq x) \leq e^{\frac{x}{cK}} \left(1 + \frac{xK}{c\tilde{\alpha}^2}\right)^{-\frac{x}{cK} - \frac{\tilde{\alpha}^2}{K^2}}, \quad x > 0.$$

On a product $X = \{1, \ldots, n\} \times Y$, where Y is a $|\cdot|_Y$-normed linear space, we have the identification

$$\Omega^X \simeq \Omega^Y \times \cdots \times \Omega^Y, \qquad \omega = (\omega_1, \ldots, \omega_n) \in \Omega^X,$$

and

$$D_{(i,y)}F(\omega) = \sum_{j=1}^{n} 1_{\{i=j\}}\big(F(\omega_1,\ldots,\omega_{i-1},\omega_i \cup \{y\},\omega_{i+1},\ldots,\omega_n) - F(\omega_1,\ldots,\omega_n)\big),$$

$i = 1,\ldots,n$, $y \in Y$. Proposition 2.2 can be directly applied with $d\nu(i,y) = d\nu_i(y)$, $i = 1,\ldots,n$, $y \in Y$, and

$$h(t) = \left\| \sum_{i=1}^{n} \int_Y \|D_{(i,y)}F\|_\infty \big(e^{t|D_{(i,y)}F|} - 1\big)\nu_i(dy) \right\|_\infty, \tag{2.14}$$

or if $\nu_1 = \cdots = \nu_n$, with

$$h(t) = \tilde{\beta} \int_Y |y|_Y \big(e^{t\beta|y|_Y} - 1\big)\nu_1(dy)$$

where

$$\tilde{\beta} = \sup_{\omega,y\neq 0} \sum_{i=1}^{n} \frac{|D_{(i,y)}F(\omega)|}{|y|_Y}, \quad \text{and} \quad \beta = \sup_{i,\omega,y\neq 0} \frac{|D_{(i,y)}F(\omega)|}{|y|_Y}.$$

In fact, a stronger result can be obtained as a corollary of Proposition 2.3.

Corollary 2.5 *Let $X = \{1,\ldots,n\} \times Y$, where Y is a $|\cdot|_Y$-normed linear space and $d\nu(i,y) = d\nu_i(y)$, $i = 1,\ldots,n$, $y \in Y$. Let $F : \Omega^X \to \mathbb{R}$ and let $\beta_i \geq 0$, $i = 1,\ldots,n$, be such that*

$$D_{(i,y)}F(\omega) \leq \beta_i|y|_Y, \quad i = 1,\ldots,n, \ y \in Y, \ \omega \in \Omega^X.$$

Then

$$P(F - E[F] \geq x) \leq \exp\left(\min_{t>0} \int_0^t h(s)ds - tx\right), \quad x > 0$$

where

$$h(t) = \sup_{\omega\in\Omega^X} \sum_{i=1}^{n} \int_Y \frac{e^{t\beta_i|y|_Y} - 1}{\beta_i|y|_Y}(D_{(i,y)}F(\omega))^2\nu_i(dy), \quad t > 0. \tag{2.15}$$

If moreover h is finite on $[0, t_0)$ then

$$P(F - E[F] \geq x) \leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \quad 0 < x < h(t_0^-).$$

If $\beta_i = 0$, $i = 1,\ldots,n$, i.e. for decreasing functionals, we have:

$$P(F - E[F] \geq x) \leq \exp\left(-\frac{x^2}{2\tilde{\alpha}^2}\right), \quad x > 0,$$

with

$$\tilde{\alpha}^2 = \sup_{\omega\in\Omega^X} \sum_{i=1}^{n} \int_Y (D_{(i,y)}F(\omega))^2\nu_i(dy).$$

Proof Apply Proposition 2.3 with $K(i, y) = \beta_i |y|_Y$, $1 \le i \le n$, $y \in Y$. $\qquad\square$

As a consequence of (2.15), and if $v_1 = \cdots = v_n$, one can take:

$$h(t) = \frac{\tilde{\alpha}^2}{\beta} \int_Y |y|_Y \left(e^{t\beta|y|_Y} - 1\right) v_1(dy), \quad t \in [0, t_0], \tag{2.16}$$

with

$$\tilde{\alpha}^2 = \sup_{\omega \in \Omega^X, \ y \ne 0} \sum_{i=1}^n \frac{|D_{(i,y)} F(\omega)|^2}{|y|_Y^2}, \quad \text{and} \quad \beta = \sup_{i, \omega, y \ne 0} \frac{|D_{(i,y)} F(\omega)|}{|y|_Y}.$$

Taking

$$h(t) = \sum_{i=1}^n \beta_i \int_Y |y|_Y \left(e^{t\beta_i|y|_Y} - 1\right) v_i(dy), \quad t \in [0, t_0], \tag{2.17}$$

allows us to recover the bound implied by (2.14) in this case.

For example, if $n = 1$ and

$$F_1(\omega) = \int_Y u_1(y)(\omega(dy) - v_1(dy)), \ldots, F_m(\omega) = \int_Y u_m(y)(\omega(dy) - v_1(dy))$$

are m (not necessarily independent) single Poisson stochastic integrals and $F = g(F_1, \ldots, F_m)$, we have

$$\beta \le \sup_{x_1, \ldots, x_m, y \ne 0} \frac{|g(x_1 + u_1(y), \ldots, x_m + u_m(y)) - g(x_1, \ldots, x_m)|}{|y|_Y}.$$

The following statement is obtained from Corollary 2.4 on a product space, in the same way as Corollary 2.5 is obtained from Proposition 2.3.

Corollary 2.6 *Let* $X = \{1, \ldots, n\} \times Y$ *with* $dv(i, y) = dv_i(y)$, $i = 1, \ldots, n$, $y \in Y$. *Let* F *be such that* $D_{(i,y)} F(\omega) \le K$, $P \otimes v_i$-*a.e.*, $i = 1, \ldots, n$, *for some* $K \in \mathbb{R}$ *and*

$$\left\| \sum_{i=1}^n \|D_{(i,\cdot)} F\|_{L^2(Y; v_i)}^2 \right\|_{L^\infty(\Omega^X, P)} \le \tilde{\alpha}^2.$$

We have for $K > 0$:

$$P(F - E[F] \ge x) \le e^{x/K} \left(1 + \frac{xK}{\tilde{\alpha}^2}\right)^{-\frac{x}{K} - \frac{\tilde{\alpha}^2}{K^2}}, \quad x > 0, \tag{2.18}$$

and for $K = 0$:

$$P(F - E[F] \ge x) \le \exp\left(-\frac{x^2}{2\tilde{\alpha}^2}\right), \quad x > 0. \tag{2.19}$$

Moreover if $K < 0$ *and* $e^{tF} \in \text{Dom}(D)$ *for all* $t > 0$, *then*

$$P(F - E[F] \ge x) \le e^{x/K} \left(1 + \frac{xK}{\tilde{\alpha}^2}\right)^{-\frac{x}{K} - \frac{\tilde{\alpha}^2}{K^2}}, \quad 0 < x < -\frac{\tilde{\alpha}^2}{K}. \tag{2.20}$$

3 Application to Random Vectors

We start by applying Corollary 2.5 and Corollary 2.6 to random vectors $(F_1, \ldots, F_n)$ on the product space $\Omega^X \simeq \Omega^Y \times \cdots \times \Omega^Y$ where $X = \{1, \ldots, n\} \times Y$ and Y is a $|\cdot|_Y$-normed linear space. Corollary 2.5 yields

$$P(g(F_1, \ldots, F_n) - E[g(F_1, \ldots, F_n)] \geq x) \leq \exp\left(-\int_0^x h^{-1}(s)\, ds\right),$$

$0 < x < h(t_0^-)$, where $g : \mathbb{R}^n \to \mathbb{R}$, provided the function

$$h(t) = \left\| \sum_{i=1}^n \int_Y (D_{(i,y)} g(F_1(\omega), \ldots, F_n(\omega)))^2 \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y} \nu_i(dy) \right\|_{L^\infty(\Omega^X, P)}, \quad t \in [0, t_0],$$

is finite on $[0, t_0)$, with β_i as in Corollary 2.5. Several particular cases are now presented.

3.1 Random Vectors with Independent Components

If $F_1, \ldots, F_n$ are n independent random variables defined on $\Omega^X = \Omega^Y \times \cdots \times \Omega^Y$ with $F_i = F_i(\omega_i)$, $i = 1, \ldots, n$, and $g : \mathbb{R}^n \to \mathbb{R}$, an ℓ^1-Lipschitz(c) function, we have

$$|D_{(i,y)} g(F_1, \ldots, F_n)(\omega)|$$
$$= |g(F_1(\omega_1), \ldots, F_i(\omega_i \cup \{y\}), \ldots, F_n(\omega_n)) - g(F_1(\omega_1), \ldots, F_n(\omega_n))|$$
$$\leq c|F_i(\omega_i \cup \{y\}) - F_i(\omega_i)|$$
$$\leq c|D_y F_i(\omega)|.$$

Now we can take in Corollary 2.5:

$$h(t) \leq c^2 \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_Y \frac{e^{ct\beta_i |y|_Y} - 1}{c\beta_i |y|_Y} (D_y F_i(\omega))^2 \nu_i(dy), \quad t \in [0, t_0],$$

with

$$\beta_i = \sup_{y \in Y, \, \omega_i \in \Omega^Y} \frac{|D_y F_i(\omega_i)|}{|y|_Y}.$$

Moreover when $\nu_1 = \cdots = \nu_n$, we can take in (2.16):

$$\tilde{\alpha}^2 = c^2 \sup_{\omega, y \neq 0} \sum_{i=1}^n \frac{|D_y F_i(\omega)|^2}{|y|_Y^2}, \quad \text{and} \quad \beta = c \sup_{i, \omega, y \neq 0} \frac{|D_y F_i(\omega)|}{|y|_Y}.$$

3.2 Independent Vectors of Poisson Stochastic Integrals

Assume that Y is a normed linear space and that $\int_Y 1 \wedge |y|_Y^2 \nu_i(dy) < \infty$, $i = 1, \ldots, n$. If $G = g(F_1, \ldots, F_n)$ where $g : Y^n \to \mathbb{R}$ and $F_1, \ldots, F_n$ are independent Poisson stochastic integrals of the form (1.3):

$$F_i(\omega_i) = \int_{\{|y|_Y \leq 1\}} y\, (\omega_i(dy) - \nu_i(dy)) + \int_{\{|y|_Y > 1\}} y\omega_i(dy) + b_i, \quad 1 \leq i \leq n,$$

we have

$$D_{(i,y)}g(F_1, \ldots, F_n) = g(F_1, \ldots, F_i + y, \ldots, F_n) - g(F_1, \ldots, F_n). \tag{3.1}$$

From Corollary 2.5 we have, denoting by $(e_1, \ldots, e_n)$ the canonical basis on $\mathbb{R}^n$:

$$
\begin{aligned}
h(t) &= \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_Y \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y} (g(F_1(\omega_1), \ldots, F_i(\omega_i) + y, \ldots, F_n(\omega_i)) \\
&\quad - g(F_1(\omega_1), \ldots, F_i(\omega_i), \ldots, F_n(\omega_n)))^2 v_i(dy) \\
&\leq \sup_{x \in \mathbb{R}^n} \sum_{i=1}^n \int_Y \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y} (g(x + ye_i) - g(x))^2 v_i(dy), \tag{3.2}
\end{aligned}
$$

which recovers Theorem 1 of [9] and (3) therein as a particular case. We may also take

$$\tilde{\alpha}^2 = \sup_{x, y \neq 0} \sum_{i=1}^n \frac{|g(x + ye_i) - g(x)|^2}{|y|_Y^2}, \quad \text{and} \quad \beta = \sup_{i, x, y \neq 0} \frac{|g(x + ye_i) - g(x)|}{|y|_Y}, \tag{3.3}$$

in (2.16). If $g : Y^n \to \mathbb{R}$ is ℓ^1-Lipschitz(c), then $\beta = c$, and so (2.17) gives:

$$h(t) = c \sum_{i=1}^n \int_Y |y|_Y \left(e^{tc|y|_Y} - 1\right) v_i(dy), \quad t \in [0, t_0]. \tag{3.4}$$

For $g(x) = \sup(x_1, \ldots, x_n)$, with $Y = \mathbb{R}$, $\beta = 1$ and

$$D_{(i,y)}g(F_1, \ldots, F_n) = \begin{cases} 0, & y \leq \sup(F_1, \ldots, F_n) - F_i, \\ F_i + y - \sup(F_1, \ldots, F_n), & y > \sup(F_1, \ldots, F_n) - F_i, \end{cases} \tag{3.5}$$

$i = 1, \ldots, n$, $y \in \mathbb{R}$. Hence (2.15) leads to

$$
\begin{aligned}
h(t) &= \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_{\mathbb{R}} \frac{e^{t|y|} - 1}{|y|} (g(F_1(\omega_1), \ldots, F_i(\omega_i) + y, \ldots, F_n(\omega_i)) \\
&\quad - g(F_1(\omega_1), \ldots, F_i(\omega_i), \ldots, F_n(\omega_n)))^2 v_i(dy) \\
&= \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_{\sup(F_1, \ldots, F_n) - F_i}^{\infty} \frac{e^{t|y|} - 1}{|y|} (F_i + y - \sup(F_1, \ldots, F_n))^2 v_i(dy) \\
&\leq \sum_{i=1}^n \int_0^{\infty} y \left(e^{ty} - 1\right) v_i(dy). \tag{3.6}
\end{aligned}
$$

Note that in (3.3) the constants $\tilde{\alpha}^2$ and β can be computed in terms of the Lipschitz constant of g with respect to the ℓ^1-norm. This however does not lead to dimension free estimates. Next, we show, using (3.2), that dimension free estimates can be obtained when g is the Euclidean norm on $\mathbb{R}^n$. The other results of [9] can similarly be generalized to the present framework.

3.3 Dimension Free Inequalities for Random Vectors

Dimension free inequalities for ℓ^2-Lipschitz functions of independent infinitely divisible random vectors with finite exponential moments have been obtained in Corollary 4 of [9]. In the next proposition we extend this result to Poisson random functionals.

Proposition 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be ℓ^2-Lipschitz(c), and let $F = (F_1, \ldots, F_n)$ be a vector of independent random functionals. Let*

$$\beta_i = \sup_{y \in Y,\ \omega \in \Omega^X} \frac{|D_y F_i(\omega)|}{|y|_Y}, \quad i = 1, \ldots, n,$$

and assume that

$$h(t) = 8 \max_{i=1,\ldots,n} \beta_i \int_Y |y|_Y \left(e^{t\beta_i |y|_Y} - 1\right) v_i(dy) \tag{3.7}$$

$$+ \frac{2n}{E[|F - E[F]|_2^2]} \max_{i=1,\ldots,n} \beta_i^3 \int_Y |y|_Y^3 \left(e^{t\beta_i |y|_Y} - 1\right) v_i(dy)$$

is finite in $t \in [0, t_0)$. Then

$$P\left(f(F_1, \ldots, F_n) \geq E[f(F_1, \ldots, F_n)] + c \sqrt{2 \sum_{i=1}^n \operatorname{Var} F_i + cx}\right)$$

$$\leq \exp\left(-\int_0^x h^{-1}(s)\,ds\right), \quad 0 < x < h(t_0^-). \tag{3.8}$$

Proof Define $\phi : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi(x) = \sqrt{E[|x - G|_2^2]}$$

where $|x|_2$ is the Euclidean norm of $x \in \mathbb{R}^n$ and G is an independent copy of F. As in the proof of Corollary 4 in [9], we have

$$|\phi(x + u e_i) - \phi(x)|^2 \leq \frac{8u^2 E[(x_i - G_i)^2]}{E[|x - G|^2]} + \frac{2u^4}{\sum_{k=1}^n \operatorname{Var} G_k}, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}.$$

Hence for $\phi(F)$, Corollary 2.5 applies with

$$h_\phi(t) = \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_Y \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y} (D_{(i,y)} \phi(F(\omega)))^2 v_i(dy)$$

$$\leq \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_Y \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y}$$

$$\times (\phi(F_1, \ldots, F_i + D_y F_i, \ldots, F_n)(\omega) - \phi(F_1, \ldots, F_n)(\omega))^2 v_i(dy)$$

$$\leq \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_Y \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y} \left(8|D_y F_i(\omega)|^2 \frac{E_G[(F_i(\omega) - G_i)^2]}{E_G[|F(\omega) - G|^2]} + \frac{2|D_y F_i(\omega)|^4}{\sum_{k=1}^n \operatorname{Var} G_k}\right) v_i(dy)$$

$$\leq 8 \sup_{\omega \in \Omega^X} \max_{i=1,\ldots,n} \int_Y \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y} |D_y F_i(\omega)|^2 \nu_i(dy)$$

$$+ \frac{2}{\sum_{k=1}^n \text{Var} \, G_k} \sup_{\omega \in \Omega^X} \sum_{i=1}^n \int_Y \frac{e^{t\beta_i |y|_Y} - 1}{\beta_i |y|_Y} |D_y F_i(\omega)|^4 \nu_i(dy)$$

$$\leq 8 \max_{i=1,\ldots,n} \beta_i \int_Y |y|_Y \left(e^{t\beta_i |y|_Y} - 1\right) \nu_i(dy)$$

$$+ \frac{1}{2\sum_{k=1}^n \text{Var} \, G_k} \sum_{i=1}^n \beta_i^3 \int_Y |y|_Y^3 \left(e^{t\beta_i |y|_Y} - 1\right) \nu_i(dy), \quad t \in [0, t_0]$$

$$\leq h(t).$$

Finally, Corollary 2.5 with the bounds

$$E[\phi(F)] \leq \sqrt{2 \sum_{k=1}^n \text{Var} \, G_k},$$

and $|f(x) - E[f(F)]| \leq c\phi(x)$, yields (3.8). $\qquad\square$

Note that since for any $F_1, \ldots, F_n$,

$$n \min_{1 \leq k \leq n} (E[|F_k|])^2 \leq E[|F|_2^2] \leq n \max_{1 \leq k \leq n} E[|F_k|^2],$$

the function h in (3.7) is bounded independently of the dimension n when $F_1, \ldots, F_n$ are i.i.d.

For the Euclidean norm of independent infinitely divisible random vectors with finite exponential moments, better results have been obtained in Corollary 3 of [9]. In the next proposition we extend this result to Poisson random functionals.

Proposition 3.2 *Let $F = (F_1, \ldots, F_n)$ be a vector of independent random functionals, and let*

$$\beta_i = \sup_{y \in Y, \, \omega \in \Omega^X} \frac{|D_y F_i(\omega)|}{|y|_Y}, \quad i = 1, \ldots, n, \tag{3.9}$$

and assume that

$$h(t) = 8 \max_{i=1,\ldots,n} \beta_i \int_Y |y|_Y \left(e^{t\beta_i |y|_Y} - 1\right) \nu_i(dy)$$

$$+ \frac{2n}{E[|F|_2^2]} \max_{i=1,\ldots,n} \beta_i^3 \int_Y |y|_Y^3 \left(e^{t\beta_i |y|_Y} - 1\right) \nu_i(dy)$$

is finite in $t \in [0, t_0)$. Then

$$P(|(F_1, \ldots, F_n)|_2 \geq 2E[|(F_1, \ldots, F_n)|_2] + x) \leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \tag{3.10}$$

$0 < x < h(t_0^-)$.

Proof Let $f(x) = (|x|_2 - E[|F|_2])^+$, $x \in \mathbb{R}^n$. From [9] we have the inequality

$$|f(x + ue_i) - f(x)|^2 \leq 8|u|^2 \frac{|x_i|^2}{|x|_2^2} + \frac{2|u|^4}{E[|F|_2^2]}, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}.$$

Hence for $f(F)$, repeating the bounds in the proof of Proposition 3.1 we get

$$h(t) \leq 8 \max_{i=1,\ldots,n} \beta_i \int_Y |y|_Y \left(e^{t\beta_i |y|_Y} - 1\right) v_i(dy)$$

$$+ \frac{2}{E[|F|_2^2]} \sum_{i=1}^n \beta_i^3 \int_Y |y|_Y^3 \left(e^{t\beta_i |y|_Y} - 1\right) v_i(dy), \quad t \in [0, t_0].$$

Finally, using $|x|_2 - E[|F|_2] \leq (|x|_2 - E[|F|_2])^+$ and $E[(|F|_2 - E[|F|_2])^+] \leq E[|F|_2]$ gives (3.10) for $g(F) = |F|_2$. $\qquad \square$

Similarly to Proposition 3.1, the deviation result of (3.10) is dimension free if $F_1, \ldots, F_n$ are i.i.d.

Next, we obtain a dimension free deviation for the Euclidean norm of a vector of n i.i.d. random functionals with bounded support. The non-identically distributed case is done similarly, while for single integrals it is in [9].

Corollary 3.3 *Let $v_1 = \cdots = v_n$ have bounded support in $B_Y(0, R)$, let $F = (F_1, \ldots, F_n)$ be an i.i.d. vector and let $\beta = \beta_1 = \cdots = \beta_n$ with the notations in (3.9). Then, for all $x > 0$,*

$$P(|F|_2 \geq x + 2E[|F|_2]) \leq \exp\left(\frac{x}{\beta R} - \left(\frac{x}{\beta R} + \frac{\tilde{\alpha}_R^2}{\beta^2 R^2}\right) \log\left(1 + \frac{x\beta R}{\tilde{\alpha}_R^2}\right)\right) \qquad (3.11)$$

where

$$\tilde{\alpha}_R^2 = 8\beta^2 R \int_Y |y|_Y v_1(dy) + \frac{2\beta^4 R^2}{(E[|F_1|])^2} \int_Y |y|_Y^2 v_1(dy).$$

Proof Apply Proposition 3.2 with

$$h(t) \leq \left(8\beta \int_Y |y|_Y v_1(dy) + \frac{2n\beta^3 R}{(E[|F|_2])^2} \int_Y |y|_Y^2 v_1(dy)\right) \left(e^{t\beta R} - 1\right) \leq \tilde{\alpha}_R^2 \frac{e^{t\beta R} - 1}{\beta R},$$

and compute explicitly the right hand side of (3.10). $\qquad \square$

The following result yields an exponential integrability property, independent of n for the ℓ^2-norm of infinitely divisible random vector whose Lévy measures have bounded supports. The non identically distributed case is similar. For independent infinitely divisible random variables an analog result is obtained in [9].

Corollary 3.4 *Let $F = (F_1, \ldots, F_n)$ be as in Corollary 3.3 then for all λ, with $0 < \lambda < \beta^2 R^2/(e\tilde{\alpha}_R^2)$, we have:*

$$E\left[\exp\left(\frac{|F|_2}{\beta R} \log_+ \frac{\lambda |F|_2}{\beta R}\right)\right] < \infty, \qquad (3.12)$$

with $\log_+ x = \max(\log x, 0)$, $x > 0$.

Since $\tilde{\alpha}_R^2$ given in Corollary 3.3 does not depend on the dimension, the condition on λ in the above corollary is also dimension free.

Proof This is a direct application of Lemma 3.5 below to (3.11), with

$$a = \frac{1}{\beta R}, \qquad b = \frac{\tilde{\alpha}_R^2}{\beta^2 R^2}, \qquad c = \frac{\beta R}{\tilde{\alpha}_R^2}, \qquad A = 2E[|F|_2]. \qquad \square$$

The next lemma is classical.

Lemma 3.5 *Let X be a random variable such that for some $a, b, c > 0$ and $A < \infty$:*

$$P(X \geq x + A) \leq \exp(ax - (ax + b)\log(1 + cx)), \quad x > 0, \qquad (3.13)$$

then for all $\lambda < b/e$ we have

$$E[\exp(aX \log_+(\lambda X))] < +\infty.$$

Proof Let $\lambda < c/e$. We have, using (3.13)

$$E[\exp(aX \log_+(\lambda X))]$$

$$= \int_0^\infty P(\exp(aX \log_+(\lambda X)) \geq t)dt$$

$$= \int_{-\infty}^\infty P(aX \log_+(\lambda X) \geq y)e^y dy$$

$$\leq 1 + \int_0^\infty P(aX \log_+(\lambda X) \geq y)e^y dy$$

$$\leq 1 + \int_{1/\lambda}^\infty P(aX \log_+(\lambda X) \geq ax \log_+(\lambda x))a(1 + \log(\lambda x))e^{ax \log(\lambda x)} dx$$

$$\leq 1 + a \int_{1/\lambda}^\infty P(X \geq x)e^{ax \log(\lambda x)}(1 + \log(\lambda x))dx$$

$$\leq C + a \int_{\max(1/\lambda, A)}^\infty \exp(a(x - A) - (a(x - A) + b)\log(1 + c(x - A)))$$

$$\times \exp(ax \log(\lambda x))(1 + \log(\lambda x))dx$$

$$\leq C + a \int_{\max(1/\lambda - A, 0)}^\infty \exp(au + a(u + A)\log(\lambda(u + A)))$$

$$\times \exp(-(au + b)\log(1 + cu))(1 + \log(\lambda(u + A)))du.$$

It then suffices to study the dominant term in the above integral:

$$\int_{\max(1/\lambda - A, 0)}^\infty \exp(au - au \log(1 + cu)) \exp(au \log(\lambda(u + A)))du$$

$$= \int_{\max(1/\lambda - A, 0)}^\infty \exp\left(au - au \log\left(\frac{1 + cu}{\lambda(u + A)}\right)\right)du.$$

Since $\lim_{u \to \infty} \log(\frac{1+cu}{\lambda(u+A)}) = \log \frac{c}{\lambda} > 1$, for $\lambda < c/e$, the convergence of the integral follows from

$$\int_{\max(1/\lambda - A, 0)}^{\infty} \exp(au - au \log(c/\lambda)) du < \infty. \qquad \square$$

3.4 Random Vectors with Non-independent Components

First, we obtain the following from Corollary 2.6:

Corollary 3.6 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be* ℓ^2*-Lipschitz(c) and let* $F = (F_1, \ldots, F_n)$ *such that* $\sum_{j=1}^n |D_{(i,y)} F_j(\omega)|^2 \le K^2$, $P \otimes \nu_i(d\omega, dy)$*-a.e.,* $i = 1, \ldots, n$, *for some* $K \ge 0$ *and*

$$\left\| \sum_{i,j=1}^n \| D_{(i,\cdot)} F_j(\omega) \|_{L^2(Y;\nu_i)}^2 \right\|_{L^\infty(\Omega^X, P)} \le \tilde{\alpha}^2.$$

Then

$$P(f(F) - E[f(F)] \ge x) \le e^{\frac{x}{cK}} \left(1 + \frac{xK}{c\tilde{\alpha}^2} \right)^{-\frac{x}{cK} - \frac{\tilde{\alpha}^2}{K^2}}, \quad x > 0. \qquad (3.14)$$

Proof Note that since f is ℓ^2-Lipschitz(c) we have for $G = f(F)$:

$$(D_{(i,y)} G(\omega))^2 = (f(F_1(\omega \cup \{(i, y)\}), \ldots, F_i(\omega \cup \{(i, y)\}), \ldots, F_n(\omega \cup \{(i, y)\}))$$
$$- f(F_1(\omega), \ldots, F_i(\omega), \ldots F_n(\omega)))^2$$
$$\le c^2 \sum_{j=1}^n |F_j(\omega \cup \{(i, y)\}) - F_j(\omega)|^2$$
$$= c^2 \sum_{j=1}^n |D_{(i,y)} F_j(\omega)|^2.$$

So that $|D_{(i,y)} G(\omega)| \le cK$ and $\sum_{i=1}^n \| D_{(i,\cdot)} G(\omega) \|_{L^2(Y;\nu_i)}^2 \le c^2 \tilde{\alpha}^2$, $P(d\omega)$-a.s., and Corollary 2.6 applies to G. $\qquad \square$

From Corollary 3.6 we can derive an exponential integrability result for the Euclidean norm of a vector of arbitrary functionals on $X = \{1, \ldots, n\} \times Y$, provided $\nu_1 = \cdots = \nu_n$ has support in $B_Y(0, R)$. This completes the sharper result stated in Corollary 3.4 in the case of independent components. However, in the infinitely divisible case, it is slightly less sharp as Corollary 3 of [24].

Corollary 3.7 *Let* $\nu_1 = \cdots = \nu_n$ *have bounded support in* $B_Y(0, R)$, *and let* $F = (F_1, \ldots, F_n)$ *be a vector of* n (*non-necessarily independent*) *random functionals. Assume that*

$$\sum_{j=1}^n \frac{|D_{(i,y)} F_j(\omega)|^2}{|y|_Y^2} \le \tilde{\alpha}^2 < \infty, \quad P \otimes \nu_i(d\omega, dy)\text{-a.e.,} \quad i = 1, \ldots, n,$$

and

$$\left\| \sum_{i,j=1}^{n} \|D_{(i,y)}F_j(\omega)\|_{L^2(Y;v_i)}^2 \right\|_{L^\infty(\Omega^X,P)} \leq \hat{\alpha}^2 < \infty.$$

Then (3.12) *holds for* $0 < \lambda < \tilde{\alpha}^2 R^2/\hat{\alpha}^2$:

$$E\left[\exp\left(\frac{|F|_2}{\tilde{\alpha}R}\log_+\frac{\lambda|F|_2}{\tilde{\alpha}R}\right)\right] < \infty. \tag{3.15}$$

Proof First, note that $|D_{(i,y)}F_j| \leq \tilde{\alpha}|y|_Y \leq \tilde{\alpha}R$, since $D_{(i,y)}F_j$ is zero for $|y|_Y > R$, v_1 being supported on $B_Y(0, R)$. We can thus apply Corollary 3.6 and get

$$P(f(F) - E[f(F)] \geq x) \leq e^{\frac{x}{\tilde{\alpha}R}}\left(1 + \frac{x\tilde{\alpha}R}{\hat{\alpha}^2}\right)^{-\frac{x}{\tilde{\alpha}R} - \frac{\hat{\alpha}^2}{\tilde{\alpha}^2 R^2}}, \quad x > 0 \tag{3.16}$$

which is (3.14) with $K = \tilde{\alpha}R$ and $c = 1$. Finally (3.15) follows from (3.16) as (3.12) follows from (3.11) in Corollary 3.4. $\qquad\square$

In the previous corollary, $\hat{\alpha}$ is dimension dependent, unlike Corollary 3.4, so that the exponential integrability is not dimension free in the dependent case. As an application of Corollary 3.6 we obtain an upper large deviation bound in the dependent case, for random functionals with bounded support.

Corollary 3.8 *Let* $v_1 = \cdots = v_n$ *have bounded support in* $B_Y(0, R)$ *and let* $F = (F_1, \ldots, F_n)$ *be a vector of* n (*non-necessarily independent*) *random functionals. Assume that*

$$\sum_{j=1}^{n} \frac{|D_{(i,y)}F_j(\omega)|^2}{|y|_Y^2} \leq \tilde{\alpha}^2 < \infty, \quad P \otimes v_i(d\omega, dy)\text{-a.e.}, \ i = 1, \ldots, n,$$

and

$$\left\| \sum_{i,j=1}^{n} \|D_{(i,y)}F_j(\omega)\|_{L^2(Y;v_i)}^2 \right\|_{L^\infty(\Omega^X,P)} < \infty.$$

Then for any ℓ^2-*Lipschitz(c) function* $f : \mathbb{R}^n \to \mathbb{R}$, *we have*

$$\limsup_{x \to \infty} \frac{\log P(|f(F)| \geq x)}{x \log x} \leq -\frac{1}{c\tilde{\alpha}R}.$$

When restricted to single Poisson integrals, the previous result recovers the upper estimate of Corollary 4 in [24] in which a deviation result is obtained for the norm of infinitely divisible vector with Lévy measure having a bounded support. See also [12] for related results in the framework of large deviations for Poisson stochastic integrals.

4 Quadratic Wiener Functionals

The results of the previous section apply in particular to quadratic Wiener functionals since they have infinitely divisible laws, cf. [17], and can be represented as Poisson stochastic

integrals with finite variance. Note that exact estimates for the tail probabilities of (quadratic) Wiener functionals have been obtained in [11], see also [5, 13, 18, 19]. Here we present dimension free results for norms of vectors of independent quadratic functionals. In this section we take $X = \mathbb{R}$.

4.1 Second Order Wiener Integrals

It is well-known (see e.g. [17]) that every centered quadratic Wiener functional can be determined by a symmetric Hilbert-Schmidt operator $A : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with eigenvalues $(a_k)_{k \in \mathbb{N}}$ and a complete orthonormal basis of eigenvectors $(h_k)_{k \in \mathbb{N}}$ in $L^2(\mathbb{R}_+)$. In particular it can be expressed as a second order Wiener integral $J_2(f_2)$ with respect to a standard Brownian motion $(B_t)_{t \in \mathbb{R}_+}$, with

$$J_2(f_2) = \frac{1}{2} \sum_{k=0}^{\infty} a_k \left(\left(\int_0^{\infty} h_k(t) dB_t \right)^2 - 1 \right)$$

where the series converges in $L^2(\Omega^X)$, and f_2 has the decomposition

$$f_2 = \frac{1}{2} \sum_{k=0}^{\infty} a_k h_k \otimes h_k,$$

converging in $L^2(\mathbb{R}_+^2)$. Note that $J_2(f_2)$ is distinct from the double Poisson stochastic integral $I_2(f_2)$. The variance of $J_2(f_2)$ is

$$\text{Var}[J_2(f_2)] = \|f_2\|^2_{L^2(\mathbb{R}_+^2)} = \frac{1}{2} \sum_{k=0}^{\infty} a_k^2.$$

In the sequel we consider a vector $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ of mutually independent second order Wiener integrals of $f_2^1, \ldots, f_2^n \in L^2(\mathbb{R}_+^2)$ with respect to possibly different Brownian motions. Denote also by $(a_k^i)_{k \in \mathbb{N}}$ the eigenvalues associated to $J_2^i(f_2^i)$, $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, $J_2^i(f_2^i)$ is infinitely divisible, integrable, and centered with Lévy measure

$$\nu_i(dy) = \mathbf{1}_{\{y>0\}} \sum_{k; a_k^i > 0} \frac{1}{2|y|} e^{-y/a_k^i} dy + \mathbf{1}_{\{y<0\}} \sum_{k; a_k^i < 0} \frac{1}{2|y|} e^{-y/a_k^i} dy, \tag{4.1}$$

cf. Theorem 2 of [17]. Hence from (1.3), $J_2^i(f_2^i)$ has the representation

$$J_2^i(f_2^i)(\omega_i) = \int_{-\infty}^{\infty} y(\omega_i(dy) - \nu_i(dy)), \tag{4.2}$$

as a single Poisson stochastic integral. Denote by

$$a_+ = \max_{1 \le i \le n} \max_{k, a_k^i > 0} a_k^i, \qquad a_- = \max_{1 \le i \le n} \max_{k, a_k^i < 0} \left(-a_k^i \right), \qquad a = \max_{1 \le i \le n} \max_{k \in \mathbb{N}} \left| a_k^i \right|,$$

the maxima of the spectral radii associated to $J_2^1(f_2^1), \ldots, J_2^n(f_2^n)$.

In the next proposition we apply Corollary 2.5 to obtain a deviation result for ℓ^1-Lipschitz functions of quadratic Wiener functionals. Note that Corollary 4 of [9] (or Proposition 3.1 applied to Poisson stochastic integrals) would yield dimension free deviation results when g is ℓ^2-Lipschitz, however with an additional range condition.

Proposition 4.1 *Let $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ be a vector of independent second order Wiener integrals. For any ℓ^1-Lipschitz(c) function g:*

$$P\big(g\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big) - E\big[g\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big)\big] \geq x\big) \leq \exp\left(-\int_0^x h^{-1}(s)ds\right),$$

$x > 0$, where h^{-1} is the inverse of the function

$$h(t) = \frac{1}{2} \sum_{i=1}^n \sum_{k=0}^\infty \frac{tc^2(a_k^i)^2}{1 - ct|a_k^i|}, \quad t \in \big[0, (ca)^{-1}\big).$$

Moreover,

$$P\big(g\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big) - E\big[g\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big)\big] \geq x\big)$$

$$\leq \exp\left(-\frac{x}{ac} + \frac{2\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}{a^2} \log\left(1 + \frac{ax}{2c\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}\right)\right) \quad (4.3)$$

$$\leq \exp\left(-\frac{1}{c}\left(1 - \frac{\log 3}{2}\right) \min\left(\frac{x}{a}, \frac{x^2}{4c\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}\right)\right), \quad x > 0. \quad (4.4)$$

Proof From Corollary 2.5 and (4.2), (4.1), (3.4), we have

$$h(t) \leq c \sum_{i=1}^n \int_{-\infty}^\infty |y|\big(e^{tc|y|} - 1\big)\nu_i(dy)$$

$$= \frac{c}{2} \sum_{i=1}^n \sum_{\substack{k=0 \\ a_k^i > 0}}^\infty \int_0^\infty \big(e^{tcy} - 1\big)e^{-y/a_k^i}dy + \frac{c}{2} \sum_{i=1}^n \sum_{\substack{k=0 \\ a_k^i < 0}}^\infty \int_{-\infty}^0 \big(e^{-tcy} - 1\big)e^{-y/a_k^i}dy$$

$$= \frac{1}{2} \sum_{i=1}^n \sum_{\substack{k=0 \\ a_k^i > 0}}^\infty \frac{tc^2(a_k^i)^2}{1 - cta_k^i} + \frac{1}{2} \sum_{i=1}^n \sum_{\substack{k=0 \\ a_k^i < 0}}^\infty \frac{tc^2(a_k^i)^2}{1 + cta_k^i}$$

$$= \frac{1}{2} \sum_{i=1}^n \sum_{k=0}^\infty \frac{tc^2(a_k^i)^2}{1 - ct|a_k^i|}.$$

For the second part of the Proposition, since

$$h(t) \leq \frac{1}{2} \sum_{i=1}^n \sum_{k=0}^\infty \frac{tc^2(a_k^i)^2}{1 - cta} \leq \frac{2c^2t}{(1 - cta)} \sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2, \quad t \in \big[0, (ca)^{-1}\big),$$

one can take

$$h^{-1}(t) = \frac{t}{cat + 2c^2 \sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2},$$

from which (4.3) follows with explicit computations. Finally (4.4) comes from the following elementary inequality:

$$\left(1 - \frac{\log 3}{2}\right) \min\left(Ax, \frac{A^2 x}{4B}\right) \le Ax - B \log\left(1 + \frac{Ax}{B}\right)$$

for all $A, B > 0$. □

Alternatively, and since $\lim_{t \to \infty} h^{-1}(t) = 1/(ca)$, we have for any $\varepsilon > 0$

$$P\left(g\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right) - E\left[g\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right)\right] \ge x\right)$$
$$\le C_1(a, c, n, \varepsilon) \exp\left(-x\left((ca)^{-1} - \varepsilon\right)\right), \tag{4.5}$$

$x > 0$, for some constant $C_1(a, c, n, \varepsilon)$ depending on c, n and ε. It follows that there is a constant $C_2(a, c, n, \lambda)$ such that

$$E\left[e^{\lambda |g(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))|}\right] < C_2(a, c, n, \lambda) < \infty, \tag{4.6}$$

for all $\lambda < 1/(ac)$, and every ℓ^1-Lipschitz(c) function $g : \mathbb{R}^n \to \mathbb{R}$. In fact, (4.3) implies that for all $\lambda < -1 - \frac{2}{a^2 c} \sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2$,

$$E\left[\exp\left(\frac{1}{ac}\left|g\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right)\right| + \lambda \log\left(1 + \left|g\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right)\right|\right)\right)\right]$$
$$< C_3(a, c, n, \lambda),$$

for some $C_3(a, c, n, \lambda) < \infty$. For the supremum of $J_2^1(f_2^1), \ldots, J_2^n(f_2^n)$, which is a Lipschitz function with respect to the ℓ^∞-norm, hence with respect to the ℓ^1-norm, the previous corollary can be strengthened by making use of (3.5).

Proposition 4.2 *Let $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ be a vector of independent second order Wiener integrals. Then,*

$$P\left(\sup\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right) - E\left[\sup\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right)\right] \ge x\right)$$
$$\le \exp\left(-\int_0^x h^{-1}(s) ds\right),$$

$x > 0$, *where h^{-1} is the inverse of the function*

$$h(t) = \frac{1}{2} \sum_{i=1}^n \sum_{\substack{k=0 \\ a_k^i > 0}}^\infty \frac{t(a_k^i)^2}{1 - t a_k^i}, \quad t \in [0, 1/a_+).$$

Moreover,

$$P\left(\sup\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right) - E\left[\sup\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right)\right] \ge x\right)$$
$$\le \exp\left(-\frac{x}{a_+} + \frac{2\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}{(a_+)^2} \log\left(1 + \frac{a_+ x}{2\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}\right)\right)$$
$$\le \exp\left(-\left(1 - \frac{\log 3}{2}\right) \min\left(\frac{x}{a_+}, \frac{x^2}{4\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}\right)\right), \quad x > 0. \tag{4.7}$$

Proof Follow the lines of the proof of Proposition 4.1 starting from (3.6) instead of (3.4). □

Note that in dimension one and for second order Wiener integrals, (4.7) above implies the upper deviation bound of [18] (Example 5.1), since

$$a_+ = 2 \sup_{\|h\|_{L^2(\mathbb{R}_+)}=1} \langle f_2, h \otimes h \rangle_{L^2(\mathbb{R}_+^2)}.$$

Counterparts of (4.5), (4.6) for $\sup(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ can be derived in the same way. Our next result is a first lower bound.

Proposition 4.3 *Let* $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ *be a vector of* (*centered*) *mutually independent quadratic Wiener functionals. For any* $b \in (0, 1)$, *there exists* $x_b > 0$ *such that*

$$P\left(\left|\left(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\right)\right|_\infty \ge x\right) \ge \frac{1-b}{2x} \sum_{i=1}^n a^i e^{-x/a^i}, \quad x > x_b,$$

with $a^i = \max_{k \in \mathbb{N}} |a_k^i|$, $1 \le i \le n$.

Proof Let $F_1, \ldots, F_n$ be n independent random variables with respective distribution $ID(m_1, 0, \nu_1), \ldots, ID(m_n, 0, \nu_n)$. We may and do assume that $\{k : a_k^i \ne 0\}$ is finite for all $i = 1, \ldots, n$, since the general conclusion can be obtained by convergence in distribution. We have for $x > 0$:

$$P(|(F_1, \ldots, F_n)|_\infty \ge x) \ge P(\exists i \in \{1, \ldots, n\} : |F_i| \ge x)$$

$$= 1 - P(|F_i| < x, \ 1 \le i \le n)$$

$$= 1 - \prod_{i=1}^n P(|F_i| < x).$$

Writing

$$F_i = F_i^+ + F_i^- - m_i$$

with

$$F_i^+(\omega_i) = \int_0^\infty y\omega_i(dy), \qquad F_i^-(\omega_i) = \int_{-\infty}^0 y\omega_i(dy), \qquad m_i = \int_{-\infty}^\infty y\nu_i(dy),$$

we have

$$P(F_i < x) = \int_0^\infty P\left(F_i^+ < x + m_i + y\right) dP(F_i^- = -y)$$

$$\le \int_0^\infty P(\omega_i([x + m_i + y, \infty)) = 0) dP(F_i^- = -y)$$

$$= \int_0^\infty \exp(-\nu_i([x + m_i + y, \infty))) dP(F_i^- = -y).$$

Here,

$$v_i([x, \infty)) = \int_x^\infty \sum_{k, a_k^i > 0} \frac{1}{2|y|} e^{-y/a_k^i} dy \underset{x \to \infty}{\sim} N_+ \int_x^\infty \frac{e^{-y/a_+^i}}{2y} dy \underset{x \to \infty}{\sim} a_+^i N_+^i \frac{e^{-x/a_+^i}}{2x}$$

where $f(x) \sim_{x \to \infty} g(x)$ means that $\lim_{x \to +\infty} f(x)/g(x) = 1$, and $a_+^i = \max_{k, a_k^i > 0} |a_k^i|$, $N_+^i = \#\{k, \ a_k^i = a_+^i\}$. Hence for all $b' \in (0, 1)$ there exists $x_{i,b'} > 0$ such that

$$P(F_i < x) \le \exp\left(-a_+^i N_+^i \frac{e^{-x/a_+^i}}{2x} (1 - b')\right), \quad x > x_{i,b'}.$$

Similarly we have

$$v_i((-\infty, -x]) = \int_{-\infty}^{-x} \sum_{k, a_k^i < 0} \frac{1}{2|y|} e^{-y/a_k^i} dy \underset{x \to \infty}{\sim} a_-^i N_-^i \frac{e^{-x/a_-^i}}{2x},$$

with $a_-^i = \max_{k, a_k^i < 0} |a_k^i|$ and $N_-^i = \#\{k, \ a_k^i = -a_-^i\}$. Hence $x_{i,b'}$ can be chosen such that

$$P(F_i > -x) \le \exp\left(-a_-^i N_-^i \frac{e^{-x/a_-^i}}{2x} (1 - b')\right), \quad x > x_{i,b'},$$

thus

$$P(|F_i| < x) \le \exp\left(-a^i \frac{e^{-x/a^i}}{2x} (1 - b')\right), \quad x > x_{i,b'}.$$

For $x > \max\{x_{1,b'}, \ldots, x_{n,b'}\}$. It follows that

$$P(|(F_1, \ldots, F_n)|_\infty \ge x) \ge 1 - \exp\left(-\sum_{i=1}^n a^i \frac{e^{-x/a^i}}{2x} (1 - b')\right),$$

and so for any $b \in (0, 1)$, there exists $x_b > 0$ such that

$$P(|(F_1, \ldots, F_n)|_\infty \ge x) \ge \frac{1 - b}{2x} \sum_{i=1}^n a^i e^{-x/a^i}, \quad x > x_b. \qquad \square$$

Note that without the independence assumption on $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$, a similar argument leads to the estimate

$$P\left(\left|(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))\right|_\infty \ge x\right) \ge a(1 - b) \frac{e^{-x/a}}{2x},$$

for any $b \in (0, 1)$ and x large enough. A version of Proposition 4.3 can also be stated for $\sup(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$. For $n = 1$, and for second order Wiener integrals, this also implies the lower deviation bound obtained in Example 5.1 in [18].

Proposition 4.4 *Let $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ be a vector of independent quadratic Wiener functionals. For any $b \in (0, 1)$, there exists $x_b > 0$ such that*

$$P\big(\sup\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big) \geq x\big) \geq \frac{1-b}{2x} \sum_{i=1}^n a_+^i e^{-x/a_+^i}, \quad x > x_b,$$

with $a_+^i = \max_{k \in \mathbb{N}, a_k^i > 0} a_k^i$, $1 \leq i \leq n$.

Proof We follow the lines of proof of Proposition 4.3. Let $F_1, \ldots, F_n$ be n independent random variables with respective distribution $ID(m_1, 0, \nu_1), \ldots, ID(m_n, 0, \nu_n)$. We have for $x > 0$:

$$P(\sup(F_1, \ldots, F_n) \geq x) \geq 1 - \prod_{i=1}^n P(F_i < x),$$

which leads to

$$P(\sup(F_1, \ldots, F_n) \geq x) \geq 1 - \exp\left(-\sum_{i=1}^n a_+^i \frac{e^{-x/a_+^i}}{2x}(1 - b')\right),$$

for x sufficiently large. Hence, for any $b \in (0, 1)$, there exists $x_b > 0$ such that

$$P(\sup(F_1, \ldots, F_n) \geq x) \geq \frac{1-b}{2x} \sum_{i=1}^n a_+^i e^{-x/a_+^i}, \quad x > x_b. \qquad \square$$

Without the independence assumption on $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ we get

$$P\big(\sup\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big) \geq x\big) \geq a_+(1 - b)\frac{e^{-x/a_+}}{2x}, \tag{4.8}$$

for any $b \in (0, 1)$ and x large enough. In the next corollary we derive an exact tail estimate for the ℓ_p-norm of vectors of independent quadratic Wiener functionals, recovering, in the special case of second order integrals, the result obtained in [2] for non-decoupled Gaussian chaos, see also [14, Corollary 3.9].

Corollary 4.5 *Let $p \in [1, \infty]$, and let $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ be a vector of independent quadratic Wiener functionals. Then*

$$\lim_{x \to +\infty} \frac{\log P(|(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))|_p \geq x)}{x} = -\frac{1}{a}. \tag{4.9}$$

Proof For any $b \in (0, 1)$, from Proposition 4.3 and Proposition 4.1, there exists $x_b > 0$ such that

$$(1 - b)\frac{e^{-x/a}}{2x} \sum_{i=1}^n a^i \leq P\big(\big|\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big)\big|_\infty \geq x\big)$$

$$\leq P\big(\big|\big(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)\big)\big|_p \geq x\big)$$

$$\leq \exp\left(-\frac{x - M}{a} + \frac{2\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}{a^2} \log\left(1 + \frac{a(x - M)}{2\sum_{i=1}^n \|f_2^i\|_{L^2(\mathbb{R}_+^2)}^2}\right)\right),$$

$x > \max(x_b, M)$, with $M = E[|(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))|_1]$. $\qquad \square$

Note that for $n = 1$ and for second order Wiener integrals, the above result coincides with Theorem 2.2 of [5] (see also [13] and [19]), since $a/2$ is also the strong operator norm of the linear map canonically associated to f_2, i.e.

$$a = 2 \sup_{\|h\|_{L^2(\mathbb{R}_+)}=1} |\langle f_2, h \otimes h \rangle_{L^2(\mathbb{R}_+^2)}|.$$

A result of [5] states that

$$\lim_{x \to +\infty} \frac{\log P(\sup_{t \in \mathbb{R}_+} |J_m(f_m^t)| \geq x)}{x} = -\frac{1}{2 \sup_{t \in \mathbb{R}_+} \|f_m^t\|_{L^2(\mathbb{R}_+^m)}}, \tag{4.10}$$

provided $(J_m(f_m^t))_{t \in \mathbb{R}_+}$ is a process of m-th order integrals with a.s. continuous sample paths (see also Remark 4.3 in [18]). It is clear that for $n = 1$, $m = 2$ and $p = +\infty$, (4.9) and (4.10) coincide. However, (4.10) does not imply (4.9), since as is well known the process $(J_m(f_m^t))_{t \in \mathbb{R}_+}$ cannot be jointly measurable and have independent components. For the supremum of $J_2^1(f_2^1), \ldots, J_2^n(f_2^n)$ we similarly have:

Corollary 4.6 *Let $(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))$ be a vector of independent quadratic Wiener functionals, then*

$$\lim_{x \to +\infty} \frac{\log P(\sup(J_2^1(f_2^1), \ldots, J_2^n(f_2^n)) \geq x)}{x} = -\frac{1}{a_+}.$$

Proof Apply Proposition 4.2 and Proposition 4.4. □

A left deviation estimate for $\sup(J_2^1(f_2), \ldots, J_2^n(f_2))$ can be independently obtained from

$$P\left(\sup(J_2^1(f_2), \ldots, J_2^n(f_2)) \leq x\right) = \prod_{k=1}^{n} P(J_2^k(f_2) \leq x),$$

which can then be estimated from Proposition 4.4 or Proposition 9.17 of [11]. Counterparts of (4.7) in Proposition 4.2, as well as Proposition 4.4, Corollary 4.6 and (4.8) can also be stated for the left deviation of $\inf(J_2^1(f_2), \ldots, J_2^n(f_2))$, replacing a_+^i by a_-^i, $i = 1, \ldots, n$, and a_+ by a_-. Since

$$a_- = -2 \inf_{\|h\|_{L^2(\mathbb{R}_+)}=1} \langle f_2, h \otimes h \rangle_{L^2(\mathbb{R}_+^2)},$$

this will imply the one-dimensional left tails of [18] (Example 5.1). For an arbitrary norm $\| \cdot \|$ on $\mathbb{R}^n$, we have

$$\|x\| = \left\| \sum_{i=1}^{i=n} x_i e_i \right\| \leq |x|_1 \max_{1 \leq i \leq n} \|e_i\|.$$

Hence,

$$\limsup_{x \to +\infty} \frac{\log P(\|(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))\| \geq x)}{x} \leq -\frac{1}{a \max_{1 \leq i \leq n} \|e_i\|}.$$

Similarly, since $\|x\| \geq C_{\|\cdot\|}|x|_\infty$ for some constant $C_{\|\cdot\|} > 0$ depending on the norm $\|\cdot\|$,

$$\liminf_{x \to +\infty} \frac{\log P(\|(J_2^1(f_2^1), \ldots, J_2^n(f_2^n))\| \geq x)}{x} \geq -\frac{1}{aC_{\|\cdot\|}}.$$

For the Euclidean norm, we also have the following dimension free deviation inequality obtained from Proposition 3.2 for an i.i.d. vector. The independent but non identically distributed case is similar with more notation.

Proposition 4.7 *Let* $(J_2^1(f_2), \ldots, J_2^n(f_2))$ *be an i.i.d. vector of second order Wiener integrals, and let* $b \in (0, 1)$. *Then,*

$$P\big(\big|(J_2^1(f_2), \ldots, J_2^n(f_2))\big|_2 - 2E\big[\big|(J_2^1(f_2), \ldots, J_2^n(f_2))\big|_2\big] \geq x\big)$$
$$\leq e^{-(1-b)\frac{x}{a} + K_b}, \quad x > 0, \tag{4.11}$$

and

$$P\big(\big|(J_2^1(f_2), \ldots, J_2^n(f_2))\big|_2 - 2E\big[\big|(J_2^1(f_2), \ldots, J_2^n(f_2))\big|_2\big] \geq x\big)$$
$$\leq e^{-(1-b)\frac{x}{a}}, \quad x \geq \frac{2a}{b} K_{b/2} \tag{4.12}$$

where

$$K_b = -\frac{16\|f_2\|^2_{L^2(\mathbb{R}^2_+)}}{a^2} \log b - 8\|f_2\|^2_{L^2(\mathbb{R}^2_+)}\left(\frac{2}{a^2} + \frac{1}{(E[|J_2(f_2)|])^2}\right)(1-b)$$
$$+ \frac{4\|f_2\|^2_{L^2(\mathbb{R}^2_+)}}{(E[|J_2(f_2)|])^2}\left(\frac{1-b^2}{b^2}\right).$$

Proof Applying Proposition 3.2 with $\beta = 1$ gives

$$P\big(\big|(J_2^1(f_2), \ldots, J_2^n(f_2))\big|_2 - 2E\big[\big|d(J_2^1(f_2), \ldots, J_2^n(f_2))\big|_2\big] \geq x\big)$$
$$\leq \exp\left(-\int_0^x h^{-1}(s)ds\right) = \min_{0<t<1/a} \exp\left(-tx + \int_0^t h(s)ds\right), \quad x > 0 \tag{4.13}$$

where

$$h(t) = 4\sum_{\substack{k=0 \\ a_k>0}}^\infty \int_0^\infty (e^{ty} - 1)e^{-y/a_k}dy + \frac{1}{(E[|J_2(f_2)|])^2}\sum_{\substack{k=0 \\ a_k>0}}^\infty \int_0^\infty y^2(e^{ty} - 1)e^{-y/a_k}dy$$

$$+ 4\sum_{\substack{k=0 \\ a_k<0}}^\infty \int_{-\infty}^0 (e^{-ty} - 1)e^{-y/a_k}dy + \frac{1}{(E[|J_2(f_2)|])^2}\sum_{\substack{k=0 \\ a_k<0}}^\infty \int_{-\infty}^0 y^2(e^{-ty} - 1)e^{-y/a_k}dy$$

$$= 4\sum_{k=0}^\infty \frac{ta_k^2}{1 - t|a_k|} + \frac{2}{(E[|J_2(f_2)|])^2}\sum_{k=0}^\infty |a_k|^3\left(\frac{1}{(1 - t|a_k|)^3} - 1\right)$$

$$\leq \frac{16\|f_2\|^2_{L^2(\mathbb{R}^2_+)}}{a}\frac{ta}{1 - ta} + \frac{8a\|f_2\|^2_{L^2(\mathbb{R}^2_+)}}{(E[|J_2(f_2)|])^2}\left(\frac{1}{(1 - ta)^3} - 1\right).$$

Letting

$$A = \frac{16\|f_2\|^2_{L^2(\mathbb{R}^2_+)}}{a^2} \quad \text{and} \quad B = \frac{8\|f_2\|^2_{L^2(\mathbb{R}^2_+)}}{(E[|J_2(f_2)|])^2},$$

we have

$$\int_0^t h(s)ds \le -A(\log(1-ta)+ta) + B\left(\frac{1}{2(1-ta)^2} - ta - \frac{1}{2}\right)$$

$$= -A\log(1-ta) - (A+B)ta + B\frac{ta}{(1-ta)^2} - \frac{B}{2}\frac{(ta)^2}{(1-ta)^2}.$$

Taking $t = (1-b)/a$, the min in (4.13) is bounded by

$$\min_{0<t<1/a} \exp\left(-tx + \int_0^t h(s)ds\right) \le \exp\left(-(1-b)\frac{x}{a} + K_b\right)$$

where

$$K_b = \int_0^{\frac{1-b}{a}} h(s)ds = -A\log b - (A+B)(1-b) + B\frac{1-b}{b^2} - B\frac{(1-b)^2}{2b^2},$$

and (4.11) follows. Taking $t = (1-b/2)/a$ in (4.13) yields

$$\min_{0<t<1/a} \exp\left(-tx + \int_0^t h(s)ds\right) \le \exp\left(-(1-b/2)\frac{x}{a} + K_{b/2}\right) \le \exp\left(-(1-b)\frac{x}{a}\right),$$

$x \ge \frac{2a}{b}K_{b/2}$, and (4.12) follows. $\qquad\square$

Note that the growth of K_b is in $1/b^2$, as $b \to 0$.

4.2 Square Norm of Brownian Paths on $[0, T]$

An example of quadratic Wiener functional for which the coefficients $(a_k)_{k\in\mathbb{N}}$ can be explicitly computed is given by the (compensated) integrated squared Brownian motion

$$\mathfrak{h}_T = \int_0^T (B(t))^2 dt - \frac{T^2}{2}$$

on the interval $[0, T]$. In this case, from [10] or §3.1.1 of [17], we have $a_k = \frac{4T^2}{(2k+1)^2\pi^2}, k \ge 0$, and the above results apply with $a = \frac{4T^2}{\pi^2}$ and

$$\sum_{k=0}^{\infty} a_k^2 = \frac{T^2}{2}.$$

Letting $(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n)$ be a vector of i.i.d. copies of $\mathfrak{h}_T$, Proposition 4.1 states in this case that for any ℓ^1-Lipschitz(c) function $g : \mathbb{R}^n \to \mathbb{R}$:

$$P\big(g\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big) - E\big[g\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big)\big] \geq x\big)$$

$$\leq \exp\left(-\frac{\pi^2 x}{4cT^2} + \frac{n\pi^4}{32T^2}\log\left(1 + \frac{8x}{cn\pi^2}\right)\right)$$

$$\leq \exp\left(-\left(1 - \frac{\log 3}{2}\right)\min\left(\frac{\pi^2 x}{4cT^2}, \frac{x^2}{nc^2 T^2}\right)\right), \quad x > 0,$$

and for all $\varepsilon > 0$:

$$P\big(g\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big) - E\big[g\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big)\big] \geq x\big) \leq C_1(c, n, T, \varepsilon) \exp\big(-x\big(\pi^2/(4cT^2) - \varepsilon\big)\big),$$

$x > 0$. It follows that

$$E\big[e^{\lambda|g(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n)|}\big] < C_2(c, n, T, \lambda),$$

for all $\lambda < \pi^2/(4cT^2)$ and some constant $C_2(c, n, T, \lambda) < \infty$. For any $b \in (0, 1)$ we also get the lower bound

$$P\big(\big|\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big)\big|_\infty \geq x\big) \geq 2n(1 - b)\frac{T^2}{x\pi^2}e^{-x\pi^2/(4T^2)},$$

for x sufficiently large, and from Corollary 4.5:

$$\lim_{x \to +\infty} \frac{\log P(|(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n)|_p \geq x)}{x} = -\frac{\pi^2}{4T^2}, \quad p \in [1, \infty].$$

For the Euclidean norm and from Proposition 4.7, we again obtain dimension free deviation inequalities: for any $b \in (0, 1)$,

$$P\big(\big|\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big)\big|_2 - 2E\big[\big|\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big)\big|_2\big] \geq x\big) \leq \exp\left(-\frac{\pi^2(1-b)}{4T^2}x + K_b\right), \quad x \geq 0,$$

and

$$P\big(\big|\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big)\big|_2 - 2E\big[\big|\big(\mathfrak{h}_T^1, \ldots, \mathfrak{h}_T^n\big)\big|_2\big] \geq x\big) \leq \exp\left(-\frac{\pi^2(1-b)}{4T^2}x\right), \quad x \geq \frac{8T^2}{\pi^2 b}K_{b/2}$$

where

$$K_b = -\frac{\pi^4}{8T^2}\log b - T^2\left(\frac{\pi^4}{8T^4} + \frac{1}{(E[|\mathfrak{h}_T|])^2}\right)(1-b) + \frac{T^2}{2(E[|\mathfrak{h}_T|])^2}\frac{1-b^2}{b^2}.$$

4.3 Sample Variance of Brownian Motion on $[0, T]$

A second example is given by the (compensated) sample variance of Brownian motion

$$\mathfrak{v}_T = \int_0^T \left(B(t) - \frac{1}{T}\int_0^T B(s)ds\right)^2 dt - \frac{T^2}{6}$$

on the interval $[0, T]$. From [10], or §3.3.1 of [17], $a_k = \frac{T^2}{k^2\pi^2}$, $k \geq 1$, $a_0 = 0$, $a = \frac{T^2}{\pi^2}$ and

$$\sum_{k=1}^{\infty} a_k^2 = \frac{T^2}{6}.$$

Letting $(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)$ be a vector of i.i.d. copies of $\mathfrak{v}_T$, we have from Proposition 4.1 that for every ℓ^1-Lipschitz(c) function $g : \mathbb{R}^n \to \mathbb{R}$:

$$P\big(g\big(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n\big) - E\big[g\big(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n\big)\big] \geq x\big)$$

$$\leq \exp\left(-\frac{\pi^2 x}{cT^2} + \frac{n\pi^4}{6T^2}\log\left(1 + \frac{6x}{cn\pi^2}\right)\right)$$

$$\leq \exp\left(-\left(1 - \frac{\log 3}{2}\right)\min\left(\frac{\pi^2 x}{cT^2}, \frac{6x^2}{2nc^2 T^2}\right)\right), \quad x > 0,$$

and for all $\varepsilon > 0$:

$$P\big(g\big(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n\big) - E\big[g\big(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n\big)\big] \geq x\big) \leq C_1(c, n, T, \varepsilon)\exp\big(-x\big(\pi^2/(cT^2) - \varepsilon\big)\big),$$

$x > 0$. It follows that

$$E\big[e^{\lambda|g(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)|}\big] < C_2(c, n, T, \lambda),$$

for all $\lambda < \pi^2/(cT^2)$ and some $C_2(c, n, T, \lambda) < \infty$. For any $b \in (0, 1)$ we also get the lower bound

$$P\big(\big|(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)\big|_\infty \geq x\big) \geq n(1 - b)\frac{T^2}{2x\pi^2}e^{-x\pi^2/T^2},$$

for x sufficiently large, and from Corollary 4.5:

$$\lim_{x \to +\infty} \frac{\log P\big(|(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)|_p \geq x\big)}{x} = -\frac{\pi^2}{T^2}, \quad p \in [1, \infty].$$

From Proposition 4.7, for any $b \in (0, 1)$:

$$P\big(\big|(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)\big|_2 - 2E\big[\big|(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)\big|_2\big] \geq x\big) \leq \exp\left(-\frac{\pi^2(1 - b)}{T^2}x + K_b\right), \quad x \geq 0,$$

and

$$P\big(\big|(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)\big|_2 - 2E\big[\big|(\mathfrak{v}_T^1, \ldots, \mathfrak{v}_T^n)\big|_2\big] \geq x\big) \leq \exp\left(-\frac{\pi^2(1 - b)}{T^2}x\right), \quad x \geq \frac{2T^2}{\pi^2 b}K_{b/2}$$

where

$$K_b = -\frac{2}{3}\frac{\pi^4}{T^2}\log b - \frac{T^2}{3}\left(\frac{2\pi^4}{T^4} + \frac{1}{(E[|\mathfrak{v}_T|])^2}\right)(1 - b) + \frac{T^2}{6(E[|\mathfrak{v}_T|])^2}\frac{1 - b^2}{b^2}.$$

4.4 Lévy's Stochastic Area

Let $(B^1(t), B^2(t))_{t \in \mathbb{R}_+}$, be a two-dimensional Brownian motion. Lévy's stochastic area S_T on $[0, T]$ is

$$S_T = \frac{1}{2}\int_0^T \big(B^1(t)dB^2(t) - B^2(t)dB^1(t)\big),$$

cf. [15]. From Sect. 3.2.1 in [17], we have $a_k = \frac{T}{(2k+1)\pi}$, $k \in \mathbb{Z}$, $a = T/\pi$ and $\sum_k a_k^2 = T^2/2$. Proposition 4.1 applies and gives

$$
P\big(g\big(S_T^1, \ldots, S_T^n\big) - E\big[g\big(S_T^1, \ldots, S_T^n\big)\big] \geq x\big) \leq \left(1 + \frac{2x}{ncT\pi}\right)^{\frac{n\pi^2}{2}} \exp\left(-\frac{\pi x}{cT}\right), \quad (4.14)
$$

$x > 0$. But since the Lévy measure of S_T has the analytic expression

$$
\nu_1(dy) = \frac{1}{2y \sinh \frac{\pi y}{T}} dy \qquad (4.15)
$$

(cf. page 175 of [15], §3.2.1 of [17] or Example 15.15 of [25]), direct computations can be done and we have

Proposition 4.8 *Let $g : \mathbb{R}^n \to \mathbb{R}$ be ℓ^1-Lipschitz(c), and let $(S_T^1, \ldots, S_T^n)$ be an i.i.d. vector of Lévy's stochastic areas on $[0, T]$. Then,*

$$
P\big(g\big(S_T^1, \ldots, S_T^n\big) - E\big[g\big(S_T^1, \ldots, S_T^n\big)\big] \geq x\big) \leq \left(1 + \frac{\pi x}{4ncT}\right)^{4n} \exp\left(-\frac{\pi x}{cT}\right).
$$

The bounds in (4.14) and in (4.16) are not comparable, but they both converge to 1 as n goes to infinity.

Proof Using the representation of S_T as the compensated Poisson stochastic integral

$$
\int_{-\infty}^{\infty} y(\omega(dy) - \nu_1(dy)) \qquad (4.16)
$$

and (3.4) derived from Corollary 2.5, we have

$$
\begin{aligned}
h(t) &= nc \int_{-\infty}^{\infty} |y|_Y \big(e^{tc|y|_Y} - 1\big)\nu_1(dy) \\
&= nc \int_{-\infty}^{\infty} |y|_Y \big(e^{tc|y|_Y} - 1\big)\frac{1}{2y \sinh \frac{\pi y}{T}} dy \\
&= 2nc \int_{0}^{\infty} \frac{e^{tcy} - 1}{e^{\pi y/T} - e^{-\pi y/T}} dy \\
&\leq 2nc \int_{0}^{\infty} \frac{e^{tcy} - 1}{e^{\pi y/T} - 1} dy \\
&\leq \frac{2nc^2 tT}{\pi} \int_{0}^{\infty} e^{-\frac{y}{2}(\frac{\pi}{T} - ct)} dy \\
&\leq \frac{4nc^2 tT}{\pi(\frac{\pi}{T} - ct)}, \quad 0 < t < \pi/(cT),
\end{aligned}
$$

using the inequality

$$
\frac{e^{ux} - 1}{e^{vx} - 1} \leq \frac{u}{v} e^{(u-v)x/2}, \quad x > 0, \qquad (4.17)
$$

for $0 < u < v$. Hence

$$-\int_0^x h^{-1}(s)ds \le -\frac{\pi^2}{T}\int_0^x \frac{s}{4nc^2T + c\pi s}ds = -\frac{\pi}{cT}x + 4n\log\left(1 + \frac{\pi x}{4ncT}\right). \qquad \square$$

The above result can also be obtained from Theorem 1 in [6] in place of (3.4). Alternatively, we have $\lim_{t\to\infty} h^{-1}(t) = \pi(cT)^{-1}$ since $\lim_{t\to\pi(cT)^{-1}} h(t) = +\infty$. Hence for all $\varepsilon > 0$, we also derive as in (4.5)

$$P\big(g\big(S_T^1, \dots, S_T^n\big) - E\big[g\big(S_T^1, \dots, S_T^n\big)\big] \ge x\big) \le C_1(c, n, T, \varepsilon)\exp\big(-x\big(\pi(cT)^{-1} - \varepsilon\big)\big),$$

$x > 0$, for some constant $C_1(c, n, T, \varepsilon)$ depending on T, c, n and ε. This last inequality is not dimension free. Nevertheless it yields

$$E\big[e^{\lambda|g(S_T^1, \dots, S_T^n)|}\big] < C_2(c, n, T, \lambda) < \infty, \tag{4.18}$$

for all $\lambda < \pi/(cT)$, and every ℓ^1-Lipschitz(c) function $g : \mathbb{R}^n \to \mathbb{R}$. For any $b \in (0, 1)$, we also get the lower bound from Proposition 4.3:

$$P\big(\big|(S_T^1, \dots, S_T^n)\big|_\infty \ge x\big) \ge (1 - b)\frac{nTe^{-\pi x/T}}{2\pi x},$$

for x sufficiently large. And from Corollary 4.5, for $p \in [1, \infty]$, we have

$$\lim_{x\to+\infty} \frac{\log P(|(S_T^1, \dots, S_T^n)|_p \ge x)}{x} = -\frac{\pi}{T}$$

where we replaced T by $\max_{1\le k\le n} T_k$ for non identical variables $S_{T_1}^1, \dots, S_{T_n}^n$. It recovers a univariate result of [3] and extends it to ℓ^p-norms of i.i.d. random vectors, independently of their dimension.

For the Euclidean norm and from Proposition 4.7, we again obtain dimension free deviation inequalities: for any $b \in (0, 1)$, we have

$$P\big(\big|(S_T^1, \dots, S_T^n)\big|_2 - 2E\big[\big|(S_T^1, \dots, S_T^n)\big|_2\big] \ge x\big) \le \exp\left(-\frac{(1-b)\pi}{T}x + K_b\right), \quad x > 0, \tag{4.19}$$

and

$$P\big(\big|(S_T^1, \dots, S_T^n)\big|_2 - 2E\big[\big|(S_T^1, \dots, S_T^n)\big|_2\big] \ge x\big) \le \exp\left(-\frac{(1-b)\pi}{T}x\right), \quad x \ge \frac{2T}{\pi b}K_{b/2} \tag{4.20}$$

where

$$K_b = -32\log b - 32(1 - b) + \frac{16T^2}{\pi^2(E[|S_T^1|])^2}\frac{(1-b)^2}{b^2}.$$

5 The Infinite Variance Case

In [7], deviation results have been derived for Lipschitz functions of stable random vectors. In this section, we extend these results to general Poisson functionals under arbitrary intensity measures. Deviations are now given with respect to a median rather than to the mean

(which may not exist). For A in $\mathcal{B}(X)$ (the Borel σ-field of X), let $\nu_R(A) = \nu(A \cap B_X(0, R))$, where 0 denotes an arbitrary fixed point in X. The proofs of the forthcoming results are inspired by that of Theorem 1 in [7]: configurations are truncated and we will use the following notation on the configuration space. For a fixed $R > 0$ and any $\omega \in \Omega^X$, let

$$\omega_R = \omega \cap B_X(0, R), \qquad \omega_R^c = \omega \cap B_X(0, R)^c = \{x \in \omega : d_X(0, x) > R\}.$$

Given a stochastic functional F on the configuration space, we also set

$$F_R(\omega) = F(\omega_R) = F(\omega \cap B_X(0, R)),$$

and denote by γ a non-negative and non-increasing function such that

$$P\left(\left\{\omega \in \Omega^X : \omega \cap B_X(0, R)^c \neq \emptyset\right\}\right) \leq \gamma(R),$$

for all R large enough. The next lemma will be used in the sequel. It allows us to control $m(F_R) - m(F)$ as in [7].

Lemma 5.1 *Let F be a stochastic functional on the configuration space such that there exists a non-negative and non-decreasing function $\tilde{\beta}$ (resp. non-increasing function $\tilde{\gamma}$) defined on $\mathbb{R}_+$, such that for all R greater than a given R_0:*

$$P(F_R - m(F_R) \geq \tilde{\beta}(R)) \leq \tilde{\gamma}(R). \tag{5.1}$$

Then we have

$$m(F_R) - m(F) \leq \tilde{\beta}(R), \tag{5.2}$$

for all R such that

$$R \geq \max\left(R_0, \inf_{0 < \delta < 1/2} \max\left(\gamma^{-1}(\delta), \tilde{\gamma}^{-1}\left(\frac{1}{2} - \delta\right)\right)\right). \tag{5.3}$$

Proof The case $m(F) \geq m(F_R)$ being trivial, we consider henceforth $m(F_R) \geq m(F)$. Let $0 < \delta < 1/2$ and assume

$$R \geq \gamma^{-1}(\delta).$$

We have

$$0 < \frac{1}{2} - \delta \leq \frac{1}{2} - \gamma(R)$$

$$\leq P(F \leq m(F)) - P\left(\left\{\omega \in \Omega^X : \omega_R^c \neq \emptyset\right\}\right)$$

$$= E[\mathbf{1}_{\{F(\omega_R \cup \omega_R^c) \leq m(F)\}} - \mathbf{1}_{\{\omega_R^c \neq \emptyset\}}]$$

$$\leq E[\mathbf{1}_{\{F(\omega_R) \leq m(F)\}}]$$

$$= P(F_R \leq m(F))$$

$$= P(F_R - m(F_R) \leq m(F) - m(F_R))$$

$$= P(-F_R - m(-F_R) \geq m(F_R) - m(F))$$

where we used the fact that $-m(F_R)$ is a median of $-F_R$. Consider the decreasing function

$$H_R(x) = P(-F_R - m(-F_R) \geq x), \quad x \in \mathbb{R},$$

and let $I_R(y) = \sup\{z \geq 0, H_R(z) \geq y\}$ denote its inverse. We have

$$m(F_R) - m(F) \leq I_R\big(P(F \leq m(F)) - P(\{\omega \in \Omega^X : \omega_R^c \neq \emptyset\})\big) \leq I_R\left(\frac{1}{2} - \delta\right). \quad (5.4)$$

Assume further that

$$R \geq \tilde{\gamma}^{-1}\left(\frac{1}{2} - \delta\right).$$

From (5.1) applied to $-F_R$, we have

$$H_R(\tilde{\beta}(R)) = P(-F_R - m(-F_R) \geq \tilde{\beta}(R)) \leq \tilde{\gamma}(R) \leq \frac{1}{2} - \delta,$$

that is finally $I_R(\frac{1}{2} - \delta) \leq \tilde{\beta}(R)$, and from (5.4):

$$m(F_R) - m(F) \leq I_R\left(\frac{1}{2} - \delta\right) \leq \tilde{\beta}(R). \qquad \square$$

The next result provides a general deviation property for stochastic functionals with infinite variance on Poisson space.

Theorem 5.2 *Let F be a stochastic functional on the configuration space such that there exists a non-negative and non-decreasing function β, defined on $\mathbb{R}_+$, and a constant $C > 0$ such that for all R greater than a given R_0:*

(i) $\sup_{y \in B_X(0,R)} |D_y F(\omega)| \leq \beta(R), \ P(d\omega)$-a.s.
(ii) $\|DF\|_{L^\infty(\Omega^X, L^2(\nu_R))}^2 \leq C\beta^2(R)\gamma(R)$

Then

$$P(F - m(F) \geq x) \leq (1 + Ce)\gamma \circ \beta^{-1}(x/4),$$

for all

$$x \geq 2\beta\left(\gamma^{-1}\left(\frac{1}{2(1 + Ce)}\right)\right).$$

Proof Configurations are truncated to deal on the one hand with the functional restricted to the truncated configuration and on the other hand with the rest of the configuration which is controlled using the function γ. We have

$$P(F - m(F) \geq x) = P\big(F - m(F) \geq x, \omega_R^c = \emptyset\big) + P\big(F - m(F) \geq x, \omega_R^c \neq \emptyset\big)$$

$$\leq P(F_R - m(F) \geq x) + P\big(\{\omega \in \Omega^X : \omega_R^c \neq \emptyset\}\big). \quad (5.5)$$

For the first term, in order to apply (2.10) in Corollary 2.4 (which provides a deviation result from the mean rather than from a median), let

$$g(x) = (x - m(F_R))^+ \wedge r, \quad x \in \mathbb{R}.$$

Then $E[g(F_R)] \leq r P(F_R \geq m(F_R)) \leq r/2$. Moreover if $F_R \geq m(F_R) + r$ then $g(F_R) \geq g(m(F_R) + r) \geq r$, hence

$$\{F_R \geq m(F_R) + r\} \subset \{g(F_R) \geq r\},$$

and

$$P(F_R - m(F_R) \geq r) \leq P(g(F_R) \geq r) \leq P(g(F_R) - E[g(F_R)] \geq r/2). \qquad (5.6)$$

On the other hand, $g(F_R)$ satisfies

$$D_y g(F_R)(\omega) \leq |g(F_R(\omega \cup \{y\})) - g(F_R(\omega))| \leq |F_R(\omega \cup \{y\}) - F_R(\omega)| = |D_y F(\omega_R)|,$$

since $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz(1). Thus

$$\sup_{y \in B_X(0,R)} D_y g(F_R) \leq \beta(R) \quad \text{and} \quad \|Dg(F_R)\|_{L^\infty(\Omega^X, L^2(\nu_R))}^2 \leq C\gamma(R)\beta(R)^2,$$

and from (5.6) and Corollary 2.4 we get

$$P(F_R - m(F_R) \geq x) \leq e^{x/(2\beta(R))} \left(1 + \frac{x}{2C\gamma(R)\beta(R)}\right)^{-x/(2\beta(R))},$$

and taking $x = 2\beta(R)$ we have:

$$P(F_R - m(F_R) \geq 2\beta(R)) \leq e\left(1 + \frac{1}{C\gamma(R)}\right)^{-1} \leq eC\gamma(R), \qquad (5.7)$$

and from Lemma 5.1 with $\tilde{\beta}(R) = 2\beta(R)$, $\tilde{\gamma}(R) = Ce\gamma(R)$ and condition (5.1) given by (5.7), we get:

$$m(F_R) - m(F) \leq 2\beta(R),$$

i.e. using (5.7):

$$P(F_R - m(F) \geq 4\beta(R)) \leq P(F_R - m(F_R) \geq 2\beta(R)) \leq Ce\gamma(R),$$

i.e. for $x \geq 4\beta(R)$, we have

$$P(F_R - m(F) \geq x) \leq Ce\gamma \circ \beta^{-1}\left(\frac{x}{4}\right), \qquad (5.8)$$

under condition (5.3) which can be rewritten in terms of x as

$$x \geq \max\left(2\beta(\gamma^{-1}(\delta)), 2\beta\left(\gamma^{-1}\left(\frac{1}{Ce}\left(\frac{1}{2} - \delta\right)\right)\right)\right),$$

i.e.

$$\gamma \circ \beta^{-1}(x/2) \leq \min\left(\delta, \frac{1}{Ce}\left(\frac{1}{2} - \delta\right)\right).$$

The optimal bound with $\delta \in (0, 1/2)$, being obtained for $\delta_0 = \frac{1}{2(1+Ce)} \in (0, 1/2)$, i.e. the condition on x becomes

$$x \geq 2\beta \circ \gamma^{-1}\left(\frac{1}{2(1 + Ce)}\right). \qquad (5.9)$$

The estimate (5.8), together with

$$P\left(\{\omega \in \Omega^X : \omega_R^c \neq \emptyset\}\right) \leq \gamma(R) \leq \gamma \circ \beta^{-1}\left(\frac{x}{4}\right),$$

gives

$$P(F - m(F) \geq x) \leq (Ce + 1)\gamma \circ \beta^{-1}\left(\frac{x}{4}\right),$$

using (5.5), under the condition (5.9). □

Note that in the hypotheses of Theorem 5.2 it is sufficient to assume

$$\sup_{y \in B_X(0,R)} |D_y F(\omega_R)| \leq \beta(R), \quad P(d\omega)\text{-a.s.},$$

and

$$\|DF_R\|^2_{L^\infty(\Omega^X, L^2(\nu_R))} \leq C\beta^2(R)\gamma(R),$$

instead of (i) and (ii). The next corollary presents a particular and more tractable case of Theorem 5.2.

Corollary 5.3 *Let $F : \Omega^X \to \mathbb{R}$, and let*

$$\gamma(R) = 1 - e^{-\nu(\{y \in X : d_X(0,y) > R\})}, \quad R > 0,$$

and assume that

$$\sup_{y \in B_X(0,R)} |D_y F| \leq C'R \quad and \quad \|DF\|^2_{L^\infty(\Omega^X, L^2(\nu_R))} \leq CR^2\gamma(R),$$

for all $R \geq R_0 > 0$. Then

$$P(F - m(F) \geq x) \leq \left(1 + \frac{Ce}{(C')^2}\right)\gamma\left(\frac{x}{4C'}\right), \quad x \geq 2C'\gamma^{-1}\left(\frac{1}{2(1 + eC/(C')^2)}\right).$$

On $\mathbb{R}^n$ equipped with the Euclidean norm $|\cdot|_2$, consider an ℓ^2-Lipschitz(c) function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a n-dimensional infinitely divisible random vector $F = (F_1, \ldots, F_n)$ without Gaussian component and with Lévy measure ν. Let us apply Corollary 5.3 to the random functional $G = f(F)$, where F is given as in (1.3) by:

$$F = \left(\int_{\{|y|_2 \leq 1\}} y_k(\omega(dy) - \nu(dy)) + \int_{\{|y|_2 > 1\}} y_k\omega(dy) + b_k\right)_{1 \leq k \leq n}.$$

For the gradient, we have if $y \notin \omega$:

$$|D_y G(\omega)| = |G(\omega \cup \{y\}) - G(\omega)|$$

$$= \left|f\left(\int_{\{|u|_2 \leq 1\}} u(\omega(du) - \nu(du)) + \int_{\{|u|_2 > 1\}} \omega(du) + y + b\right) - f(F)\right|$$

$$\leq c|y|_2,$$

since f is ℓ^2-Lipschitz(c), and we obtain $|D_y G(\omega)| \le cR$, for $|y|_2 \le R$. In this case, for $G = f(F)$ the conclusion of Corollary 5.3 reads

$$P(G - m(G) \ge x) \le (1 + Ce)\left(1 - \exp\left(-\nu\left(\left\{u \in \mathbb{R}^n : |u|_2 > \frac{x}{4c}\right\}\right)\right)\right). \quad (5.10)$$

When $f(x) = |x|_2$ is the Euclidean norm on $\mathbb{R}^n$, Lemma 5.4 below also yields a lower bound on $P(|F - m|_2 \ge x)$ which has the same order as the upper bound (5.10).

Lemma 5.4 *Let F be an infinitely divisible random vector $ID(b, 0, \nu)$ in $\mathbb{R}^n$. Then for all $m \in \mathbb{R}$ we have*

$$P(\|F - m\| \ge x) \ge \frac{1}{4}\left(1 - \exp\left(-\nu(\{u \in \mathbb{R}^n : \|u\| \ge 2x\})\right)\right), \quad x > 0 \quad (5.11)$$

where $\|\cdot\|$ denotes any norm on $\mathbb{R}^n$.

Proof We start by assuming that F is symmetric. Then, since F can be taken to be the value $F(1)$ at time 1 of a Lévy process $(F(t))_{0 \le t \le 1}$ starting from $F(0) = 0$, we have from Lévy's inequality:

$$P(\|F\| \ge x) = P\left(\left\|\sum_{k=1}^{n} F\left(\frac{k}{n}\right) - F\left(\frac{k-1}{n}\right)\right\| \ge x\right)$$

$$\ge \frac{1}{2}P\left(\max_{1 \le j \le n}\left\|F\left(\frac{j}{n}\right) - F\left(\frac{j-1}{n}\right)\right\| \ge x\right).$$

Hence

$$P(\|F\| \ge x) \ge \liminf_{n \to \infty} \frac{1}{2}P\left(\max_{1 \le j \le n}\left\|F\left(\frac{j}{n}\right) - F\left(\frac{j-1}{n}\right)\right\| \ge x\right)$$

$$\ge \frac{1}{2}P\left(\liminf_{n \to \infty}\max_{1 \le j \le n}\left\|F\left(\frac{j}{n}\right) - F\left(\frac{j-1}{n}\right)\right\| > x\right)$$

$$\ge \frac{1}{2}P\left(\max_{s \in [0,1]}\|F(s) - F(s^-)\| > x\right)$$

$$\ge \frac{1}{2}\left(1 - \exp\left(-\nu\{u \in \mathbb{R}^n : \|u\| \ge x\}\right)\right) \quad (5.12)$$

where (5.12) is a n-dimensional extension of Example 22.1 in [25], which relies on the fact that if ω on $\mathbb{R}_+ \times \mathbb{R}^n$ has a jump of $\|\cdot\|$-norm greater than x, then $\max_{s \in [0,1]}\|F(s) - F(s^-)\| > x$. In the general case where F is not necessarily symmetric we apply the above to $F - G$, where G denotes an independent copy of F, and use the inequality

$$P(\|F - m\| \ge x) = \frac{1}{2}P(\|F - m\| \ge x) + \frac{1}{2}P(\|G - m\| \ge x) \ge \frac{1}{2}P(\|F - G\| \ge 2x),$$

for all $m \in \mathbb{R}$. $\qquad\square$

We now present several examples of Lévy measures ν for which the function γ can be explicitly computed, and where F has infinite variance, i.e. $\int_{\mathbb{R}^n}\|y\|^2\nu(dy) = \infty$, but where $f(F)$ satisfies the above hypothesis for f an ℓ^2-Lipschitz(c) function.

1. Let $\mathbb{R}^n \setminus \{0\}$ be equipped with the measure given for $B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$ by

$$\nu(B) = \int_{S^{n-1}} \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) \frac{|\log r|}{r^2} dr \qquad (5.13)$$

where σ is again a spherical finite measure. Since

$$\int_{\{|y|_2 \le 1\}} |y|_2^2 \nu(dy) = \sigma\left(S^{n-1}\right) < \infty$$

and

$$\nu(\{|x|_2 \ge 1\}) = \int_{S^{n-1}} \sigma(d\xi) \int_1^\infty \frac{|\log r|}{r^2} dr < \infty,$$

ν is a Lévy measure. Moreover $\int_{\{|y|_2 \ge 1\}} |y|_2^2 \nu(dy) = \infty$, hence F has infinite variance. As before:

$$P\left(\{\omega \in \Omega^X : \omega_R^c \ne \emptyset\}\right) = 1 - \exp\left(\int_{S^{n-1}} \sigma(d\xi) \int_{\{|r\xi|_2 \ge R\}} \frac{|\log r|}{r^2} dr\right)$$

$$= 1 - \exp\left(-\sigma\left(S^{n-1}\right) \int_R^\infty \frac{\log r}{r^2} dr\right)$$

$$\le \sigma\left(S^{n-1}\right) \frac{1 + \log R}{R}, \quad R > 1.$$

Thus, choose $\gamma(R) = 2\sigma\left(S^{n-1}\right) \frac{\log R}{R}$. On the other hand,

$$\|Df(F)\|_{L^\infty(\Omega^X, L^2(\nu_R))}^2 \le \int_{\{|y|_2 \le R\}} c^2 |y|_2^2 \nu(dy)$$

$$= -c^2 \int_{S^{n-1}} \sigma(d\xi) \int_0^1 \log r \, dr + c^2 \int_{S^{n-1}} \sigma(d\xi) \int_1^R \log r \, dr$$

$$= c^2 \sigma\left(S^{n-1}\right)(R \log R - R + 2)$$

$$\le c^2 R^2 \gamma(R)/2.$$

2. Let $X = \mathbb{R}^n$, with the finite (Lévy) measure ν given for $B \in \mathcal{B}(\mathbb{R}^n)$ by

$$\nu(B) = \int_{S^{n-1}} \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) \frac{e^{-1/(2r^2)}}{r^2 \sqrt{2\pi}} dr. \qquad (5.14)$$

The infinitely divisible random variable given by the Poisson stochastic integral in (1.3) is thus another example of a random variable without finite variance since

$$\int_{\{|y|_2 \ge 1\}} |y|_2^2 \nu(dy) = \infty.$$

Once more:

$$P\left(\omega_R^c \neq \emptyset\right) = 1 - \exp\left(-\int_{S^{n-1}} \sigma(d\xi) \int_{\{|r\xi|_2 \geq R\}} \frac{e^{-1/(2r^2)}}{r^2\sqrt{2\pi}} dr\right)$$

$$\leq \sigma\left(S^{n-1}\right) \int_R^\infty \frac{e^{-1/(2r^2)}}{r^2\sqrt{2\pi}} dr$$

$$= \sigma\left(S^{n-1}\right) \int_0^{1/R} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du.$$

Choose $\gamma(R) = \frac{\sigma(S^{n-1})}{\sqrt{2\pi}R}$. Moreover,

$$\|Df(F)\|_{L^\infty(\Omega^X, L^2(\nu_R))}^2 \leq c^2 \int_{B(0,R)} |y|_2^2 \nu_R(dy)$$

$$= c^2 \int_{S^{n-1}} \sigma(d\xi) \int_0^R r^2 \frac{e^{-1/(2r^2)}}{r^2\sqrt{2\pi}} dr$$

$$= c^2 \sigma\left(S^{n-1}\right) \int_0^R \frac{e^{-1/(2r^2)}}{\sqrt{2\pi}} dr$$

$$= c^2 \sigma\left(S^{n-1}\right) \int_{1/R}^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}} \frac{du}{u^2}$$

$$= c^2 \sigma\left(S^{n-1}\right) \left(\frac{Re^{-1/(2R^2)}}{\sqrt{2\pi}} - \int_{1/R}^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}} du\right)$$

$$\leq c^2 R^2 \gamma(R).$$

3. The above deviation results for $f(F)$ with F as in (1.3) a stable or an infinitely divisible random variable, and with Lévy measure either given by (6.1) or (5.13) or (5.14), continue to hold after minor changes for Hölder continuous functions of order $0 < h < 1$. Indeed, for such a function f we have

$$|D_y f(F)|_2 \leq c|y|_2^h \leq cR^h, \quad |y|_2 \leq R.$$

For instance in the case of the Lévy measure (5.13) we have

$$\|Df(F)\|_{L^\infty(\Omega^X, L^2(\nu_R))}^2 \leq \int_{S^{n-1}} \sigma(d\xi) \int_0^R c^2 r^{2h} \nu_R(dr)$$

$$= c^2 \sigma\left(S^{n-1}\right) \int_0^R r^{2h-2} |\log r| dr$$

$$= -c^2 \sigma\left(S^{n-1}\right) \int_0^1 r^{2h-2} \log r\, dr + c^2 \sigma\left(S^{n-1}\right) \int_1^R r^{2h-2} \log r\, dr$$

$$\leq c^2 \sigma\left(S^{n-1}\right) \left(\frac{R^{2h-1}\log R}{2h-1} - \frac{R^{2h-1}}{(2h-1)^2} + \frac{1}{(2h-1)^2} + \frac{1}{2h+1}\right).$$

We can thus apply Theorem 5.2 to $G = f(F)$ with (up to multiplicative constants) the functions:

$$\beta(R) = R^h \quad \text{and} \quad \gamma(R) = \frac{\log R}{R}.$$

A similar computation yields in the case of the Lévy measure (5.14):

$$\begin{aligned}
\|DF\|^2_{L^\infty(\Omega^X, L^2(\nu_R))} &\leq \int_{\{|x|_2 \leq R\}} c^2 |x|_2^{2h} \nu_R(dr) \\
&\leq \frac{c^2}{\sqrt{2\pi}} \int_{S^{n-1}} \sigma(d\xi) \int_0^R r^{2h-2} e^{-1/(2r^2)} dr \\
&\leq c^2 \sigma\left(S^{n-1}\right) \frac{R^{2h-1} e^{-1/(2R^2)}}{(2h-1)\sqrt{2\pi}}.
\end{aligned}$$

Once more, Theorem 5.2 applies here, in the Hölder continuous case, with up to multiplicative constants,

$$\beta(R) = R^h \quad \text{and} \quad \gamma(R) = \frac{e^{-1/(2R^2)}}{R\sqrt{2\pi}}.$$

Before turning to the case of stable intensity measures in the next section, we prove the following lemma for a general intensity measure ν, which is a generalization of Lemma 2 in [7].

Lemma 5.5 *Let* $F : \Omega^X \longrightarrow \mathbb{R}$ *and* $\alpha_2, \alpha_3, \alpha_4, K > 0$, *such that*

(i) $\sup_{y \in X} |D_y F(\omega)| \leq K < \infty$, $P(d\omega)$-*a.s.*
(ii) $\|DF\|^k_{L^\infty(\Omega^X, L^k(\nu))} \leq \alpha_k < \infty$, $k = 2, 3, 4$

Assume moreover $\alpha_3 \leq 2\alpha_4/K$ *and* $K^2\alpha_2/\alpha_4 > 2$. *Let* s_0 *be the* (*unique*) *positive solution of*

$$s\left(\alpha_2 - \frac{\alpha_4}{K^2}\right) = \frac{\alpha_4}{K^3}\left(e^{sK} - 1\right). \tag{5.15}$$

Let $x_0 = 3s_0(\alpha_2 - \alpha_4/K^2)$. *Then for all* $x \leq x_0$,

$$P(F - E[F] \geq x) \leq \exp\left(-\frac{x^2}{6(\alpha_2 - \alpha_4/K^2)}\right), \tag{5.16}$$

while for $x \geq x_0$,

$$P(F - E[F] \geq x) \leq K_0 \exp\left(\frac{x}{K} - \left(\frac{x}{K} + \frac{3\alpha_4}{K^4}\right)\log\left(1 + \frac{K^3 x}{3\alpha_4}\right)\right), \tag{5.17}$$

with

$$K_0 = \exp\left(-\frac{x_0}{K} + \left(\frac{x_0}{K} + \frac{3\alpha_4}{K^4}\right)\log\left(1 + \frac{K^3 x_0}{3\alpha_4}\right) - \frac{x_0^2}{6(\alpha_2 - \alpha_4/K^2)}\right). \tag{5.18}$$

Proof From Proposition 2.2 we have

$$P(F - E[F] \geq x) \leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \quad 0 < x < h(t_0^-), \tag{5.19}$$

with h given in (2.2). Using the bounds $|D_y F| \leq K$ and

$$e^{su} - 1 \leq su + \frac{s^2}{2}u^2 + \frac{e^{sK} - 1 - sK - s^2K^2/2}{K^3}u^3, \quad 0 \leq u \leq K, \ s \geq 0, \tag{5.20}$$

we have

$$h(s) \leq \sup_{\omega, \omega' \in \Omega_X} \int_X \left(s|D_y F(\omega)||D_y F(\omega')| + \frac{s^2}{2}|D_y F(\omega)|^2|D_y F(\omega')| \right.$$

$$\left. + \frac{e^{sK} - 1 - sK - s^2K^2/2}{K^3}|D_y F(\omega)|^3|D_y F(\omega')| \right) \nu(dy)$$

$$\leq s \sup_{\omega, \omega' \in \Omega_X} \int_X |D_y F(\omega)||D_y F(\omega')|\nu(dy) + \frac{s^2}{2} \sup_{\omega, \omega' \in \Omega_X} \int_X |D_y F(\omega)|^2|D_y F(\omega')|\nu(dy)$$

$$+ \frac{e^{sK} - 1 - sK - s^2K^2/2}{K^3} \sup_{\omega, \omega' \in \Omega_X} \int_X |D_y F(\omega)|^3|D_y F(\omega')|\nu(dy). \tag{5.21}$$

From Hölder's inequality, we have for $p = q = 2$:

$$\sup_{\omega, \omega' \in \Omega_X} \int_X |D_y F(\omega)||D_y F(\omega')|\nu(dy)$$

$$\leq \sup_{\omega, \omega' \in \Omega_X} \left\{ \left(\int_X |D_y F(\omega)|^2 \nu(dy) \right)^{1/2} \times \left(\int_X |D_y F(\omega')|^2 \nu(dy) \right)^{1/2} \right\} \leq \alpha_2,$$

for $p = 3/2, q = 3$:

$$\sup_{\omega, \omega' \in \Omega_X} \int_X |D_y F(\omega)|^2|D_y F(\omega')|\nu(dy)$$

$$\leq \frac{2}{3} \sup_{\omega, \omega' \in \Omega_X} \left\{ \left(\int_X |D_y F(\omega)|^3 \nu(dy) \right)^{2/3} \times \left(\int_X |D_y F(\omega')|^3 \nu(dy) \right)^{1/3} \right\} \leq \alpha_3,$$

and similarly for $p = 4/3, q = 4$:

$$\sup_{\omega, \omega' \in \Omega_X} \int_X |D_y F(\omega)|^3|D_y F(\omega')|\nu(dy)$$

$$\leq \sup_{\omega, \omega' \in \Omega_X} \left\{ \left(\int_X |D_y F(\omega)|^4 \nu(dy) \right)^{3/4} \left(\int_X |D_y F(\omega')|^4 \nu(dy) \right)^{1/4} \right\} \leq \alpha_4.$$

From (5.21) we get

$$h(s) \leq s\alpha_2 + \frac{s^2}{2}\alpha_3 + \frac{e^{sK} - 1 - sK - \frac{s^2}{2}K^2}{K^3}\alpha_4$$

$$= s\left(\alpha_2 - \frac{\alpha_4}{K^2}\right) + \frac{s^2}{2}\left(\alpha_3 - \frac{\alpha_4}{K}\right) + \frac{\alpha_4}{K^3}\left(e^{sK} - 1\right). \tag{5.22}$$

Since we assume $\alpha_3 \leq 2\alpha_4/K$, the second summand in the right-hand side of (5.22) is bounded by the third one for all $s \geq 0$. We may now end the proof as in Lemma 2 of [7]:

$$h(s) \leq 3\max\left(s\left(\alpha_2 - \frac{\alpha_4}{K^2}\right), \frac{s^2}{2}\left(\alpha_3 - \frac{\alpha_4}{K}\right), \frac{\alpha_4}{K^3}\left(e^{sK} - 1\right)\right)$$

$$= 3\max\left(s\left(\alpha_2 - \frac{\alpha_4}{K^2}\right), \frac{\alpha_4}{K^3}\left(e^{sK} - 1\right)\right)$$

$$\leq \begin{cases} 3s\left(\alpha_2 - \dfrac{\alpha_4}{K^2}\right), & 0 \leq s \leq s_0, \\[2mm] 3\dfrac{\alpha_4}{K^3}\left(e^{sK} - 1\right), & s \geq s_0 \end{cases}$$

where s_0 is the unique positive solution of (5.15) which is well defined since $K^2\alpha_2/\alpha_4 > 2$. Hence, letting $x_0 = 3s_0(\alpha_2 - \alpha_4/K^2)$ we may take

$$h^{-1}(t) = \begin{cases} \dfrac{t}{3(\alpha_2 - \alpha_4/K^2)} & \text{for } 0 \leq t \leq x_0, \\[3mm] \dfrac{1}{K}\log\left(1 + \dfrac{K^3}{3\alpha_4}t\right) & \text{for } t \geq x_0 \end{cases}$$

which yields (5.16) and (5.17) from (5.19). $\qquad\qquad\square$

Lemma 5.5 will be used in the proof of Theorem 6.1 below to obtain a deviation result under α-stable Lévy measures for all value of $\alpha \in (0, 2)$. The following lemma applies only for $\alpha \geq 1$, but will yield a slightly better range condition in Theorem 6.2, and is stated without boundedness assumption on the 4th order moment.

Lemma 5.6 *Let $F : \Omega^X \longrightarrow \mathbb{R}$ and $\alpha_2, \alpha_3, K > 0$, such that $K\alpha_2 \geq 2\alpha_3$ and*

(i) $\|DF\|^2_{L^\infty(\Omega^X, L^2(\nu))} \leq \alpha_2 < \infty$
(ii) $\|DF\|^3_{L^\infty(\Omega^X, L^3(\nu))} \leq \alpha_3 < \infty$
(iii) $\sup_{y \in X} |D_y F(\omega)| \leq K < \infty$, $P(d\omega)$-a.s.

Denote by s_0 the unique solution of

$$\frac{e^{sK} - 1}{sK} = K\frac{\alpha_2}{\alpha_3} - 1.$$

Let also $x_0 = 2s_0(\alpha_2 - \alpha_3/K)$. Then

$$P(F - E[F] \geq x) \leq \exp\left(-\frac{x^2}{4(\alpha_2 - \alpha_3/K)}\right), \quad 0 \leq x \leq x_0, \tag{5.23}$$

and

$$P(F - E[F] \geq x) \leq K_0 \exp\left(\frac{x}{K} - \left(\frac{x}{K} + 2\frac{\alpha_3}{K^3}\right) \log\left(1 + \frac{K^2 x}{2\alpha_3}\right)\right), \quad x \geq x_0, \quad (5.24)$$

with

$$K_0 = \exp\left(-x_0/K + \left(\frac{x_0}{K} + \frac{2\alpha_3}{K^3}\right) \log\left(1 + \frac{K^2 x_0}{2\alpha_3}\right) - \frac{x_0^2}{4(\alpha_2 - \alpha_3/K)}\right). \quad (5.25)$$

Proof As in the proof of Lemma 5.5, apply Proposition 2.2 with h given in (2.2) and bounded by

$$h(s) \leq s\alpha_2 + \frac{e^{sK} - 1 - sK}{K^2}\alpha_3$$

using

$$e^{su} - 1 \leq su + \frac{e^{sK} - 1 - sK}{K^2}u^2, \quad u \in [0, K],$$

instead of (5.20). We get

$$h(s) \leq 2\max\left(s\left(\alpha_2 - \frac{\alpha_3}{K}\right), \frac{e^{sK} - 1}{K^2}\alpha_3\right) \leq \begin{cases} 2s(\alpha_2 - \alpha_3/K), & s \leq s_0, \\ 2(e^{sK} - 1)\alpha_3/K^2, & s \geq s_0 \end{cases}$$

which allows us to conclude as in the proof of Lemma 5.5. $\square$

6 The Case of Stable Lévy Measures

In this section $X = \mathbb{R}^n$ and $|\cdot|_X$ stands for the usual Euclidean norm. Let $0 < \alpha < 2$ and the stable Lévy measure given by

$$\nu(B) = \int_{S^{n-1}} \sigma(d\xi) \int_0^\infty \mathbf{1}_B(r\xi) r^{-1-\alpha} dr, \quad B \in \mathcal{B}(\mathbb{R}^n) \quad (6.1)$$

where σ is a finite positive measure on S^{n-1}, the unit sphere of $\mathbb{R}^n$, called the spherical component of ν. We have

$$\begin{aligned}
P\left(\{\omega \in \Omega^X : \omega_R^c \neq \emptyset\}\right) &= 1 - P\left(\{\omega \in \Omega^X : \omega_R^c = \emptyset\}\right) \\
&= 1 - \exp\left(-\int_{\{|y|_2 > R\}} \nu(dy)\right) \\
&= 1 - \exp\left(-\int_{S^{n-1}} \sigma(d\xi) \int_{\{|r\xi|_2 \geq R\}} \frac{dr}{r^{1+\alpha}}\right) \\
&= 1 - \exp\left(-\frac{\sigma(S^{n-1})}{\alpha} R^{-\alpha}\right) \\
&\leq \frac{\sigma(S^{n-1})}{\alpha R^\alpha}. \quad (6.2)
\end{aligned}$$

Thus we can take

$$\gamma(R) = \frac{\sigma(S^{n-1})}{\alpha R^\alpha}, \quad R > 0,$$

in Theorem 5.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be ℓ^2-Lipschitz(c). In case F is a stable random variable represented by a single Poisson stochastic integral of the form (1.3), we have from (2.13):

$$
\begin{aligned}
\|Df(F)\|^2_{L^\infty(\Omega^X, L^2(\nu_R))} &\leq \int_{\{|y|_2 \leq R\}} c^2 |y|^2_2 \nu(dy) \\
&= c^2 \int_{S^{n-1}} \sigma(d\xi) \int_{\{|r\xi|_2 \geq R\}} r^{1-\alpha} dr \\
&\leq \frac{c^2 \sigma(S^{n-1})}{2 - \alpha} R^{2-\alpha} \\
&\leq \frac{2c^2}{2 - \alpha} R^2 \gamma(R),
\end{aligned}
$$

hence Theorem 1 of [7] is recovered taking $\beta(r) = cr$ and $C = 2/(2 - \alpha)$ in Theorem 5.2, i.e.

$$P(f(F) - m(f(F)) \geq x) \leq \left(1 + \frac{2e}{2 - \alpha}\right) \frac{\sigma(S^{n-1})}{\alpha} \left(\frac{x}{4c}\right)^{-\alpha}, \tag{6.3}$$

for all x such that

$$x \geq 2c\gamma^{-1}(2 - \alpha) \geq 2c\gamma^{-1}\left(\frac{1}{2(1 + 2e/(2 - \alpha))}\right)$$

where F is a stable random variable with parameter α. The constant in front of $x^{-\alpha}$ in (6.3) explodes as α goes to 0 or to 2. In fact, as noted in [7], the dependency in α^{-1} of the constant is sharp as α goes to 0 (take for example a symmetric α-stable random variable). This explosion does not occur however when α goes to 2, and the aim of the next result is to provide a deviation bound with such a non-exploding constant, for general random variables on Poisson space under α-stable intensity measures. The proof relies on Lemma 5.5, and in the particular case of stable random variables, this result also improves Theorem 2 of [7] by allowing α to be arbitrary.

Theorem 6.1 *Let $\alpha \in (0, 2)$ and $F : \Omega^X \to \mathbb{R}$ such that*

$$|D_y F(\omega)| \leq c|y|_X, \quad P(d\omega) \otimes \nu(dy)\text{-a.e.,}$$

with $c > 0$ and ν given in (6.1). Then we have

$$P(F - m(F) \geq x) \leq \sigma\left(S^{n-1}\right) \left(\frac{3}{2} e^2 + \frac{1}{\alpha}\right) \frac{(4c)^\alpha}{x^\alpha}, \tag{6.4}$$

for all

$$x \geq 4c\sigma\left(S^{n-1}\right)^{1/\alpha}$$

$$\times \left(\left(\frac{3}{2}\left(1 + \frac{4}{2 - \alpha} \log \frac{2}{2 - \alpha}\right) \log\left(1 + \frac{8}{2 - \alpha} \log \frac{2}{2 - \alpha}\right)\right) \vee \frac{4}{\alpha} \vee \left(6e^2\right)\right)^{1/\alpha}. \tag{6.5}$$

Proof Using the notation of the proof of Theorem 5.2 we have

$$D_y g(F_R)(\omega) \le |D_y F(\omega_R)| \le c|y|_X, \quad P(d\omega) \otimes \nu(dy)\text{-a.e.}$$

where $g(x) = (x - m(F_R))^+ \wedge r$. Thus

$$\sup_{y \in B_X(0,R)} D_y g(F_R) \le cR, \quad P\text{-a.s.,}$$

$$\|Dg(F_R)\|^2_{L^\infty(\Omega^X, L^2(\nu_R))} \le \frac{c^2 \sigma(S^{n-1})}{2 - \alpha} R^{2-\alpha},$$

$$\|Dg(F_R)\|^3_{L^\infty(\Omega^X, L^3(\nu_R))} \le \frac{c^3 \sigma(S^{n-1})}{3 - \alpha} R^{3-\alpha},$$

and

$$\|Dg(F_R)\|^4_{L^\infty(\Omega^X, L^4(\nu_R))} \le \frac{c^4 \sigma(S^{n-1})}{4 - \alpha} R^{4-\alpha}.$$

We now apply Lemma 5.5 to ν_R and F_R with

$$K = cR, \qquad \alpha_2 = \frac{c^2 \sigma(S^{n-1})}{2 - \alpha} R^{2-\alpha}, \qquad \alpha_3 = \frac{c^3 \sigma(S^{n-1})}{3 - \alpha} R^{3-\alpha},$$

$$\alpha_4 = \frac{c^4 \sigma(S^{n-1})}{4 - \alpha} R^{4-\alpha}.$$

Using (5.6), (5.15) reads

$$\varphi(scR) := e^{scR} - \frac{2scR}{2 - \alpha} - 1 = 0.$$

For all $\alpha \in (0, 2)$, an elementary computation shows that $\varphi(\log \frac{2}{2-\alpha}) \le 0$ and $\varphi(2\log \frac{2}{2-\alpha}) \ge 0$, hence

$$\log \frac{2}{2 - \alpha} \le s_0 cR \le 2\log \frac{2}{2 - \alpha},$$

so that for $x_0 = 3s_0(\alpha_2 - \frac{\alpha_4}{K^2}) = \frac{6c^2 \sigma(S^{n-1}) R^{2-\alpha}}{(2-\alpha)(4-\alpha)} s_0$, we have

$$c\sigma(S^{n-1}) R^{1-\alpha} \frac{3}{2(2-\alpha)} \log \frac{2}{2-\alpha} \le x_0 \le c\sigma(S^{n-1}) R^{1-\alpha} \frac{6}{2-\alpha} \log \frac{2}{2-\alpha}. \qquad (6.6)$$

For

$$r \ge 2x_0 = \frac{12c^2 \sigma(S^{n-1}) R^{2-\alpha}}{(2-\alpha)(4-\alpha)} s_0,$$

we get from Lemma 5.5:

$$P(F_R - m(F_R) \ge r) \le P(g(F_R) - E[g(F_R)] \ge r/2)$$

$$\le K_0 \exp\left(\frac{r}{2cR} - \left(\frac{r}{2cR} + 3\frac{\sigma(S^{n-1})}{(4-\alpha)R^\alpha} \right) \log\left(1 + \frac{(4-\alpha)r}{6\sigma(S^{n-1})cR^{1-\alpha}} \right) \right) \qquad (6.7)$$

with from (5.18) and (6.6):

$$K_0 = \exp\left(-\frac{x_0}{cR} + \left(\frac{x_0}{cR} + 3\frac{\sigma(S^{n-1})}{(4-\alpha)R^\alpha}\right)\log\left(1 + \frac{(4-\alpha)x_0}{3\sigma(S^{n-1})cR^{1-\alpha}}\right)\right.$$
$$\left. - \frac{(2-\alpha)(4-\alpha)x_0^2}{12\sigma(S^{n-1})c^2R^{2-\alpha}}\right)$$
$$\leq \exp\left(\frac{3\sigma(S^{n-1})}{2R^\alpha}\left(1 + \frac{4}{2-\alpha}\log\frac{2}{2-\alpha}\right)\log\left(1 + \frac{8}{2-\alpha}\log\frac{2}{2-\alpha}\right)\right).$$

Hence under the condition

$$\sigma\left(S^{n-1}\right)R^{-\alpha} \leq \frac{2}{3\left(1 + \frac{4}{2-\alpha}\log\frac{2}{2-\alpha}\right)\log\left(1 + \frac{8}{2-\alpha}\log\frac{2}{2-\alpha}\right)} \tag{6.8}$$

we get $K_0 \leq e$ and

$$2x_0 \leq 12c\sigma\left(S^{n-1}\right)R^{1-\alpha}\frac{1}{2-\alpha}\log\frac{2}{2-\alpha}$$
$$\leq 4cR\frac{\frac{2}{2-\alpha}\log\frac{2}{2-\alpha}}{\left(1 + \frac{4}{2-\alpha}\log\frac{2}{2-\alpha}\right)\log\left(1 + \frac{8}{2-\alpha}\log\frac{2}{2-\alpha}\right)}$$
$$\leq cR,$$

i.e. $r \geq 2x_0$ with $r = 2cR$. Then from (6.7) and $K_0 \leq e$ we get

$$P(F_R - m(F_R) \geq 2cR) \leq \exp\left(2 - \left(1 + \frac{3\sigma(S^{n-1})}{(4-\alpha)R^\alpha}\right)\log\left(1 + \frac{(4-\alpha)R^\alpha}{3\sigma(S^{n-1})}\right)\right)$$
$$\leq e^2\left(1 + \frac{(4-\alpha)R^\alpha}{3\sigma(S^{n-1})}\right)^{-1}$$
$$\leq \frac{3e^2\sigma(S^{n-1})}{(4-\alpha)R^\alpha}$$
$$\leq \frac{3e^2\sigma(S^{n-1})}{2R^\alpha} \tag{6.9}$$
$$= \frac{3}{2}e^2\alpha\gamma(R), \tag{6.10}$$

as long as (6.8) holds. In order to control $P(F_R - m(F) \geq x)$ from (6.9), we need to control $m(F_R) - m(F)$. For this we apply Lemma 5.1 with $\tilde{\beta}(R) = 2cR$, $\tilde{\gamma}(R) = \frac{3}{2}e^2\alpha\gamma(R)$,

$$R_0 = \left(\frac{3}{2}\sigma\left(S^{n-1}\right)\left(1 + \frac{4}{2-\alpha}\log\frac{2}{2-\alpha}\right)\log\left(1 + \frac{8}{2-\alpha}\log\frac{2}{2-\alpha}\right)\right)^{1/\alpha}$$

and (6.10). This yields, with $x = 4cR$:

$$m(F_R) - m(F) \leq x/2,$$

and

$$P(F_R - m(F) \geq x) \leq P(F_R - m(F_R) \geq x/2) \leq \frac{3}{2} e^2 \sigma\left(S^{n-1}\right)\left(\frac{4c}{x}\right)^{\alpha},$$

provided

$$R \geq \max\left(R_0, \gamma^{-1}(\delta), \gamma^{-1}\left(\frac{2}{3\alpha e^2}\left(\frac{1}{2} - \delta\right)\right)\right), \tag{6.11}$$

for any given $\delta \in (0, 1/2)$. When $x = 4cR$, this estimate together with

$$P\left(\{\omega \in \Omega^X : \omega_R^c \neq \emptyset\}\right) \leq \gamma(R) = \frac{\sigma\left(S^{n-1}\right)R^{-\alpha}}{\alpha},$$

gives, using (5.5):

$$P(F - m(F) \geq x) = P\left(F - m(F) \geq x, \omega_R^c = \emptyset\right) + P\left(F - m(F) \geq x, \omega_R^c \neq \emptyset\right)$$

$$\leq \sigma\left(S^{n-1}\right)\left(\frac{1}{\alpha} + \frac{3}{2}e^2\right)\left(\frac{x}{4c}\right)^{-\alpha}, \tag{6.12}$$

as long as (6.8) and (6.11) hold. Now, conditions (6.8) and (6.11) can be rewritten in terms of x as

$$x \geq 4c\left(\frac{3}{2}\sigma\left(S^{n-1}\right)\left(1 + \frac{4}{2-\alpha}\log\frac{2}{2-\alpha}\right)\log\left(1 + \frac{8}{2-\alpha}\log\frac{2}{2-\alpha}\right)\right)^{1/\alpha}$$

and

$$x \geq 4c\max\left(\left(\frac{\sigma\left(S^{n-1}\right)}{\alpha\delta}\right)^{1/\alpha}, \left(\frac{3\sigma\left(S^{n-1}\right)e^2}{2(1/2 - \delta)}\right)^{1/\alpha}\right).$$

When e.g. $\delta = 1/4$, the range of (6.12) can be written

$$x \geq 4c\sigma\left(S^{n-1}\right)^{1/\alpha}\left(\left(\frac{3}{2}\left(1 + \frac{4}{2-\alpha}\log\frac{2}{2-\alpha}\right)\log\left(1 + \frac{8}{2-\alpha}\log\frac{2}{2-\alpha}\right)\right)\right.$$

$$\left. \vee \frac{4}{\alpha} \vee \left(6e^2\right)\right)^{1/\alpha}. \qquad \Box$$

Using Lemma 5.6 in place of Lemma 5.5, we can state a similar deviation result under a slightly better range condition on x in case $\alpha \in [1, 2)$.

Theorem 6.2 *Assume that $\alpha \geq 1$ and let $F : \Omega^X \to \mathbb{R}$ such that*

$$|D_y F(\omega)| \leq c|y|_X, \quad P(d\omega) \otimes \nu(dy)\text{-a.e.,}$$

with $c > 0$ and ν given in (6.1). Then we have

$$P(F - m(F) \geq x) \leq \sigma\left(S^{n-1}\right)\left(1 + \frac{e^2}{2}\right)\frac{(4c)^{\alpha}}{x^{\alpha}}, \tag{6.13}$$

for all

$$x \geq 4c\sigma \left(S^{n-1}\right)^{1/\alpha} \left(\left(\left(1 + \frac{2}{2-\alpha}\log\frac{1}{2-\alpha}\right)\log\left(1 + \frac{4}{2-\alpha}\log\frac{1}{2-\alpha}\right)\right) \vee \left(4e^2\right)\right)^{1/\alpha}. \tag{6.14}$$

Proof We sketch the modifications of the proof, following the argument of Theorem 6.1 and applying Lemma 5.6 instead of Lemma 5.5 to ν_R and F_R, with

$$K = cR, \qquad \alpha_2 = \frac{c^2\sigma\left(S^{n-1}\right)}{2-\alpha}R^{2-\alpha}, \qquad \alpha_3 = \frac{c^3\sigma\left(S^{n-1}\right)}{3-\alpha}R^{3-\alpha}.$$

Under the condition

$$\sigma\left(S^{n-1}\right)R^{-\alpha} \leq \frac{1}{2\left(1 + \frac{2}{2-\alpha}\log\frac{1}{2-\alpha}\right)\log\left(1 + \frac{4}{2-\alpha}\log\frac{1}{2-\alpha}\right)} \tag{6.15}$$

and since

$$\frac{2\alpha_3}{c^2R^2}\frac{1}{2-\alpha}\log\frac{1}{2-\alpha} \leq x_0 \leq \frac{4\alpha_3}{c^2R^2}\frac{1}{2-\alpha}\log\frac{1}{2-\alpha},$$

we have $K_0 \leq e$ and also $x_0 \leq cR$. Using (5.6) with $r = 2cR \geq 2x_0$, we get applying Lemma 5.6:

$$P(F_R - m(F_R) \geq 2cR) \leq \exp\left(2 - \left(1 + 2\frac{\alpha_3}{c^3R^3}\right)\log\left(1 + \frac{2c^3R^3}{\alpha_3}\right)\right)$$

$$\leq e^2\left(1 + \frac{2c^3R^3}{\alpha_3}\right)^{-1}$$

$$\leq \frac{e^2\alpha_3}{2c^3R^3}$$

$$\leq \frac{e^2\sigma\left(S^{n-1}\right)}{2R^\alpha} \tag{6.16}$$

$$= \frac{e^2\alpha}{2}\gamma(R), \tag{6.17}$$

as long as (6.15) holds. Finally, applying Lemma 5.1 with $\tilde{\beta}(R) = 2cR$, $\tilde{\gamma}(R) = e^2\alpha\gamma(R)/2$ and condition (5.1) given by (6.17), with $x = 4cR$, derive $m(F_R) - m(F) \leq x/2$, and

$$P(F_R - m(F) \geq x) \leq P(F_R - m(F_R) \geq x/2) \leq \frac{e^2\sigma\left(S^{n-1}\right)}{2}\left(\frac{4c}{x}\right)^\alpha,$$

provided moreover for any, $0 < \delta < 1/2$,

$$R \geq \max\left(\gamma^{-1}(\delta), \gamma^{-1}\left(\frac{2}{\alpha e^2}\left(\frac{1}{2} - \delta\right)\right)\right). \tag{6.18}$$

With $x = 4cR$, this estimate together with (6.2) gives, using $\alpha \geq 1$ and (5.5):

$$P(F - m(F) \geq x) = P\left(F - m(F) \geq x, \omega_R^c = \emptyset\right) + P\left(F - m(F) \geq x, \omega_R^c \neq \emptyset\right)$$

$$\leq \sigma\left(S^{n-1}\right)\left(1 + \frac{e^2}{2}\right)\left(\frac{x}{4c}\right)^{-\alpha}, \tag{6.19}$$

as long as (6.15) and (6.18) hold. Now, conditions (6.15) and (6.18) can be rewritten in terms of x as (6.14) with e.g. $\delta = 1/4$. $\qquad\square$

Theorem 6.1 and Theorem 6.2 roughly tells us that the natural upper bound for the deviation of stable Poisson functionals is C/x^α for some constant $0 < C < +\infty$ and for x^α of order $\sigma(S^{n-1})\frac{1}{2-\alpha}(\log\frac{1}{2-\alpha})^2$. As in the finite dimensional case (see [16]), we present in the following theorem another bound of order e^{-cx^α} $(0 < c < +\infty)$ when x^α lies in an interval of order

$$\left[\frac{\sigma(S^{n-1})}{2-\alpha}, \frac{\sigma(S^{n-1})}{2-\alpha}\log\frac{1}{2-\alpha}\right]$$

as α becomes close to 2. This generalizes [16, Theorem 2] to Poisson functionals under a stable intensity measure, and is reminiscent of Gaussian deviation.

Theorem 6.3 *Let $F : \Omega^X \to \mathbb{R}$ such that for some $c > 0$,*

$$|D_y F(\omega)| \le c|y|_X, \quad P(d\omega) \otimes \nu(dy)\text{-a.e.}$$

with ν given in (6.1).

1. *Let $\varepsilon > 0$, then if α is sufficiently close to 2,*

$$P(F - m(F) \ge x) \le (\varepsilon + \sqrt{e})\exp\left(-\frac{(2-\alpha)x^\alpha}{2(4c)^\alpha\sigma(S^{n-1})}\right), \qquad (6.20)$$

   provided

$$\frac{2\sigma(S^{n-1})(4c)^\alpha}{2-\alpha}\log(4(1+\sqrt{e})) \le x^\alpha \le \frac{\sigma(S^{n-1})(4c)^\alpha}{2(2-\alpha)}\frac{\log\left(\frac{1}{2-\alpha}\right)}{3-\alpha}. \qquad (6.21)$$

2. *Let $b > 3$, $\varepsilon > 0$, and x such that $(x/4c)^\alpha = b\sigma(S^{n-1})\frac{1}{2-\alpha}\log\frac{1}{2-\alpha}$. For α close enough to 2 we have*

$$P(F - m(F) \ge x)$$
$$\le \frac{(4c)^\alpha\sigma(S^{n-1})}{x^\alpha}\left(\frac{1}{\alpha} + (2+\varepsilon)\exp\left(\frac{(2+\varepsilon)(4c)^\alpha\sigma(S^{n-1})g(2-\alpha)}{x^\alpha}\right)\right) \quad (6.22)$$

   where $g(x) = (\frac{1}{x}\log\frac{1}{x})\log(\frac{1}{x}\log\frac{1}{x})$.

Note that (2) in the Theorem 6.3 improves Theorem 6.2 when x^α is of order $\sigma(S^{n-1})\frac{1}{2-\alpha}\log\frac{1}{2-\alpha}$.

Proof We follow [16] as in the proofs of Theorem 5.2 and Proposition 6.2 above. First, using the same notation as before, we have:

$$P(F_R - m(F_R) \ge r) \le P(g(F_R) - E[g(F_R)] \ge r/2) \qquad (6.23)$$

$$\le \exp\left(-\int_0^{r/2} h_R^{-1}(s)ds\right), \quad 0 < x < h_R(t_0^-), \qquad (6.24)$$

with

$$h_R(s) \leq x\left(\alpha_2 - \frac{\alpha_3}{K}\right)s + \frac{\alpha_3}{K^2}\left(e^{sK} - 1\right)$$

$$= \frac{c^2\sigma(S^{n-1})R^{2-\alpha}}{(2-\alpha)(3-\alpha)}s + \left(e^{scR} - 1\right)\frac{c\sigma(S^{n-1})R^{1-\alpha}}{3-\alpha}$$

since again

$$\alpha_2 = \frac{c^2\sigma(S^{n-1})}{2-\alpha}R^{2-\alpha}, \qquad \alpha_3 = \frac{c^3\sigma(S^{n-1})}{3-\alpha}R^{3-\alpha}, \qquad K = cR$$

where (6.23) above comes as in (5.6) in the proof of Theorem 5.2 and (6.24) comes from the proofs of Lemma 5.6 and Proposition 2.2. Following [16], for δ, s, R satisfying

$$\frac{e^{scR} - 1}{csR} \leq \frac{\delta}{2-\alpha} \tag{6.25}$$

we have

$$h_R(s) \leq (1 + \delta)\frac{c^2\sigma(S^{n-1})R^{2-\alpha}}{(2-\alpha)(3-\alpha)}s$$

and

$$\int_0^y h_R^{-1}(t)dt \geq \frac{(3-\alpha)(2-\alpha)y^2}{2(1+\delta)c^2\sigma(S^{n-1})R^{2-\alpha}} \tag{6.26}$$

for all y such that

$$\frac{(3-\alpha)(2-\alpha)y}{(1+\delta)c^2\sigma(S^{n-1})R^{2-\alpha}} \leq s$$

where s satisfies (6.25). Taking for some $A > 0$, $R^\alpha = \frac{A\sigma(S^{n-1})}{(2-\alpha)(3-\alpha)}$ and $y = Rc$, since $scR = A/(1+\delta)$, (6.25) can be rewritten as

$$(1+\delta)\frac{e^{\frac{A}{1+\delta}} - 1}{A} \leq \frac{\delta}{2-\alpha}$$

which is satisfied whenever

$$(1+\delta)\frac{e^A}{A} \leq \frac{\delta}{2-\alpha}.$$

Choosing $\delta = \frac{e^A(2-\alpha)}{A - e^A(2-\alpha)}$ which is positive for $0 < a < A < -\log(2-\alpha)$ when α is close enough to 2, we derive from (6.26) for $a < A < -\log(2-\alpha)$

$$\exp\left(-\int_0^{cR} h_R^{-1}(t)\,dt\right) \leq e^{-\frac{A}{2}}\exp\left(\frac{e^A(2-\alpha)}{2}\right). \tag{6.27}$$

But since

$$\lim_{\alpha \to 2^-}\sup_{a < A < -\log(2-\alpha)} e^{-\frac{A}{2}}\exp\left(\frac{e^A(2-\alpha)}{2}\right)e^{\frac{A}{2(3-\alpha)}} = \sqrt{e},$$

for any $\varepsilon \in (0, 1)$ and α close to 2, from (6.24) with $r = 2cR$,

$$P(F_R - m(F_R) \ge 2cR) \le \exp\left(-\int_0^{cR} h_R^{-1}(t)dt\right) \le \left(\sqrt{e} + \frac{\varepsilon}{2}\right)e^{-\frac{A}{2(3-\alpha)}}$$

$$\le \left(\sqrt{e} + \frac{\varepsilon}{2}\right)\exp\left(-\frac{(2-\alpha)}{2\sigma(S^{n-1})}R^\alpha\right) \tag{6.28}$$

for

$$\frac{a\sigma(S^{n-1})}{(2-\alpha)(3-\alpha)} < R^\alpha < \frac{\sigma(S^{n-1})\log\frac{1}{2-\alpha}}{2(2-\alpha)(3-\alpha)}. \tag{6.29}$$

Next, control $m(F) - m(F_R)$ using Lemma 5.1 with $\tilde{\beta}(R) = 2cR$,

$$\tilde{\gamma}(R) = (1 + \sqrt{e})\exp\left(-\frac{(2-\alpha)}{2\sigma(S^{n-1})}R^\alpha\right)$$

and condition (5.1) given by (6.28) (with $\varepsilon \le 2$). This yields

$$m(F_R) - m(F) \le 2cR, \tag{6.30}$$

provided (5.3), rewritten as

$$R^\alpha \ge \max\left(\frac{\sigma(S^{n-1})}{\alpha\delta}, -\frac{2\sigma(S^{n-1})}{2-\alpha}\log\frac{1/2-\delta}{1+\sqrt{e}}\right), \tag{6.31}$$

and (6.29) above both still hold. Equations (6.28) and (6.30) yield

$$P(F_R - m(F) \ge 4cR) \le P(F_R - m(F_R) \ge 2cR)$$

$$\le \left(\sqrt{e} + \frac{\varepsilon}{2}\right)\exp\left(-\frac{(2-\alpha)}{2\sigma(S^{n-1})}R^\alpha\right) \tag{6.32}$$

provided (6.29) and (6.31) hold. Next, when (6.29) holds, (6.2) gives for α close enough to 2:

$$P\left(\omega_R^c \ne \emptyset\right) \le \frac{\sigma(S^{n-1})}{\alpha R^\alpha} \le \frac{\varepsilon}{2}\exp\left(-\frac{(2-\alpha)R^\alpha}{2\sigma(S^{n-1})}\right). \tag{6.33}$$

Finally, (5.5) together with (6.32) and (6.33) yields with $x = 4cR$,

$$P(F - E[F_R] \ge x) \le (\sqrt{e} + \varepsilon)\exp\left(-\frac{(2-\alpha)x^\alpha}{2(4c)^\alpha\sigma(S^{n-1})}\right)$$

as long as

$$a\frac{\sigma(S^{n-1})(4c)^\alpha}{(2-\alpha)(3-\alpha)} \le x^\alpha \le (4c)^\alpha\sigma\left(S^{n-1}\right)\frac{\log(1/(2-\alpha))}{2(2-\alpha)(3-\alpha)}, \tag{6.34}$$

and

$$x^\alpha \ge (4c)^\alpha\max\left(\frac{\sigma(S^{n-1})}{\alpha\delta}, -\frac{2\sigma(S^{n-1})}{2-\alpha}\log\frac{1/2-\delta}{1+\sqrt{e}}\right), \tag{6.35}$$

for any $0 < \delta < 1/2$. Taking $\delta = 1/4$, conditions (6.34) and (6.35) can be rewritten as

$$\frac{2\sigma(S^{n-1})(4c)^\alpha}{2-\alpha} \log(4(1+\sqrt{e})) \le x^\alpha \le \frac{\sigma(S^{n-1})(4c)^\alpha}{2(2-\alpha)} \frac{\log(1/(2-\alpha))}{3-\alpha}$$

which yields (6.22).

We now deal with the second part of Theorem 6.3, still following [16]. Take for some $b > 0$, $R^\alpha = \frac{b\sigma(S^{n-1})\log(1/(2-\alpha))}{2-\alpha}$ and let $A > 0$. For α close to 2 and $scR \ge \log(1/(2-\alpha)) + \log\log(1/(2-\alpha)) + A$ we have

$$\frac{e^{scR} - 1}{scR} \ge \frac{1}{(2-\alpha)(e^{-A} + \varepsilon)},$$

hence

$$h_R^{-1}(u) \ge \frac{1}{cR} \log\left(1 + \frac{(3-\alpha)u}{(e^{-A} + 1 + \varepsilon)c\sigma(S^{n-1})R^{1-\alpha}}\right) \tag{6.36}$$

whenever

$$u > u_1 = \frac{(1 + e^{-A} + \varepsilon)c\sigma(S^{n-1})e^A \log(1/(2-\alpha))}{(2-\alpha)(3-\alpha)R^{\alpha-1}}.$$

For $A > 0$ small enough and $b > 2$, we have $cR > u_1$, and integrating (6.36) over $[u_1, cR]$, we obtain

$$\int_{u_1}^{cR} h_R^{-1}(t)dt$$
$$\ge \left(\left(1 + \frac{1}{\theta cR}\right)\log(1 + \theta cR) - 1\right) - \left(\left(\frac{u_1}{cR} + \frac{1}{\theta cR}\right)\log(1 + \theta u_1) - \frac{u_1}{cR}\right) \tag{6.37}$$

with $\theta = \frac{(3-\alpha)R^{\alpha-1}}{(1+e^{-A}+\varepsilon)c\sigma(S^{n-1})}$. For α close to 2 and $A, \varepsilon > 0$ small enough, using (6.23), (6.24), (6.37), we derive

$$P(F_R - m(F_R) \ge 2cR)$$
$$\le \exp\left(-\int_0^{cR} h_R^{-1}(t)dt\right) \le \exp\left(-\int_{u_1}^{cR} h_R^{-1}(t)dt\right)$$
$$\le \frac{(2+\varepsilon)\sigma(S^{n-1})}{R^\alpha}$$
$$\times \exp\left(\frac{(2+\varepsilon)c\sigma(S^{n-1})\left(\frac{1}{2-\alpha}\log(1/(2-\alpha))\right)\log\left(\frac{1}{2-\alpha}\log(1/(2-\alpha))\right)}{R^\alpha}\right)$$
$$\le \frac{(2+\varepsilon)\sigma(S^{n-1})}{R^\alpha} \exp\left(\frac{(2+\varepsilon)\sigma(S^{n-1})g(2-\alpha)}{R^\alpha}\right), \tag{6.38}$$

with $g(x) = (\frac{1}{x}\log\frac{1}{x})\log(\frac{1}{x}\log\frac{1}{x})$ and ε some (new) positive constant. It is easy now to control $m(F) - m(F_R)$ using once more Lemma 5.1 with $\tilde{\beta}(R) = 2cR$, $\tilde{\gamma}(x) = \alpha^{-1}\sigma(S^{n-1})x^{-\alpha}$, and condition (5.1) given by (6.38). This yields

$$m(F_R) - m(F) \le 2cR \tag{6.39}$$

as long as

$$R^\alpha \geq \frac{\sigma(S^{n-1})}{\alpha\delta}, \quad \text{and} \quad \frac{3\sigma(S^{n-1})}{R^\alpha} \exp\left(\frac{3\sigma(S^{n-1})g(2-\alpha)}{R^\alpha}\right) \leq 1/2 - \delta. \qquad (6.40)$$

Then with $x = 4cR$, (6.38), (6.39) yield

$$P(F_R - m(F) \geq x) \leq \frac{(2+\varepsilon)\sigma(S^{n-1})}{R^\alpha} \exp\left(\frac{(2+\varepsilon)\sigma(S^{n-1})g(2-\alpha)}{R^\alpha}\right)$$

as long as (6.40) holds. Together with (5.5) and (6.2), this gives

$$P(F - m(F) \geq x) \leq \frac{\sigma(S^{n-1})}{R^\alpha}\left(\frac{1}{\alpha} + (2+\varepsilon)\exp\left(\frac{(2+\varepsilon)\sigma(S^{n-1})g(2-\alpha)}{R^\alpha}\right)\right)$$

for $x = 4cR$, that is (6.22) as long as (6.40) holds. This latter condition can be rewritten for $\delta \in (0, 1/2)$ and $b > 3$:

$$\frac{1}{2-\alpha} \log \frac{1}{2-\alpha} \geq \max\left(\frac{1}{\alpha\delta b}, \left(\frac{3}{b(1/2-\delta)}\right)^{\frac{b}{b-3}}\right)$$

which is obviously true for α close enough to 2 since α, b are bounded below and $\frac{b}{b-3}$ is bounded above. $\qquad \square$

References

1. Ané, C., Ledoux, M.: On logarithmic Sobolev inequalities for continuous time random walks on graphs. Probab. Theory Relat. Fields **116**(4), 573–602 (2000)
2. Arcones, M., Giné, E.: On decoupling, series expansions and tail behavior processes. J. Theor. Probab. **6**, 101–122 (1993)
3. Berthuet, R.: Loi du logarithme itéré pour certaines intégrales stochastiques. Ann. Sci. Univ. Clermont-Ferrand II Math. **19**, 9–18 (1981). Saint-Flour Probability Summer School (Saint-Flour, 1979/1980)
4. Bobkov, S., Götze, F., Houdré, C.: On Gaussian and Bernoulli covariance representations. Bernoulli **7**(3), 439–451 (2001)
5. Borell, C.: Tail probabilities in Gauss space. In: Vector Space Measures and Applications I (Proc. Conf., Univ. Dublin, Dublin, 1977), II. Lecture Notes in Math., vol. 644, pp. 73–82. Springer, Berlin (1978)
6. Houdré, C.: Remarks on deviation inequalities for functions of infinitely divisible random vectors. Ann. Probab. **30**(3), 1223–1237 (2002)
7. Houdré, C., Marchal, P.: On the concentration of measure phenomenon for stable and related random vectors. Ann. Probab. **32**(2), 1496–1508 (2004)
8. Houdré, C., Privault, N.: Concentration and deviation inequalities in infinite dimensions via covariance representations. Bernoulli **8**(6), 697–720 (2002)
9. Houdré, C., Reynaud-Bouret, P.: Concentration for infinitely divisible vectors with independent components. ArXiv Preprint math.PR/0606752 (2004)
10. Ikeda, N., Manabe, S.: Asymptotic formulae for stochastic oscillatory integrals. In: Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics (Sanda/Kyoto, 1990). Pitman Res. Notes Math. Ser., vol. 284, pp. 136–155. Longman Sci. Tech., Harlow (1993)
11. Imkeller, P.: On exact tails for limiting distributions of U-statistics in the second Gaussian chaos. In: Chaos Expansions, Multiple Wiener-Itô Integrals and Their Applications (Guanajuato, 1992). Probab. Stochastics Ser., pp. 179–204. CRC, Boca Raton (1994)
12. Jurek, Z., Wu, L.: Large deviations for some Poisson random integrals. In: Séminaire de Probabilités, XXXIV. Lecture Notes in Math., vol. 1729, pp. 185–197. Springer, Berlin (2000)
13. Ledoux, M.: A note on large deviations for Wiener chaos. In: Séminaire de Probabilités, XXIV, 1988/89. Lecture Notes in Math., vol. 1426, pp. 1–14. Springer, Berlin (1990)

14. Ledoux, M., Talagrand, M.: Probability in Banach Spaces. Springer, Berlin (1991)
15. Lévy, P.: Wiener's random function, and other Laplacian random functions. In: Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 171–187. University of California Press, Berkeley (1951)
16. Marchal, P.: Measure concentration for stable distributions with index close to 2. Electron. Commun. Probab. **10**, 29–35 (2005) (electronic)
17. Matsumoto, H., Taniguchi, S.: Wiener functionals of second order and their Lévy measures. Electron. J. Probab. **7**(14) (2002) (electronic), 30 pp.
18. Mayer-Wolf, E., Nualart, D., Pérez-Abreu, V.: Large deviations for multiple Wiener-Itô integral processes. In: Séminaire de Probabilités, XXVI. Lecture Notes in Math., vol. 1526, pp. 11–31. Springer, Berlin (1992)
19. McKean, H.P.: Wiener's theory of nonlinear noise. In: Stochastic Differential Equations (Proc. SIAM–AMS Sympos. Appl. Math., New York, 1972). SIAM–AMS Proc., vol. VI, pp. 191–209. Am. Math. Soc., Providence (1973)
20. Nualart, D., Vives, J.: Anticipative calculus for the Poisson process based on the Fock space. In: Séminaire de Probabilités XXIV. Lecture Notes in Math., vol. 1426, pp. 154–165. Springer, Berlin (1990)
21. Nualart, D., Vives, J.: A duality formula on the Poisson space and some applications. In: Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993), pp. 205–213. Birkhäuser, Basel (1995)
22. Picard, J.: Formules de dualité sur l'espace de Poisson. Ann. Inst. H. Poincaré Probab. Stat. **32**(4), 509–548 (1996)
23. Privault, N.: Equivalence of gradients on configuration spaces. Random Oper. Stoch. Equ. **7**(3), 241–262 (1999)
24. Rosiński, J.: Remarks on strong exponential integrability of vector-valued random series and triangular arrays. Ann. Probab. **23**(1), 464–473 (1995)
25. Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics, vol. 68. Cambridge University Press, Cambridge (1999)
26. Surgailis, D.: On multiple Poisson stochastic integrals and associated Markov semi-groups. Probab. Math. Stat. **3**, 217–239 (1984)
27. Wu, L.: A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. Probab. Theory Relat. Fields **118**(3), 427–438 (2000)