

Higher order paracontrolled calculus

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Abstract. We develop in this work a general version of paracontrolled calculus that allows to treat analytically within this paradigm some singular partial differential equations with the same efficiency as regularity structures, with the benefit that there is no need to introduce the algebraic apparatus inherent to the latter theory. This work deals with the analytic side of the story and offers a toolkit for the study of such equations, under the form of a number of continuity results for some operators. We illustrate the efficiency of this elementary approach on the example of the generalised parabolic Anderson model equation

$$(\partial_t + L)u = f(u)\zeta$$

for a spacial 'noise' ζ of Hölder regularity $\alpha - 2$, with $\frac{2}{5} < \alpha \leq \frac{2}{3}$, and the generalized KPZ equation

$$(\partial_t + L)u = f(u)\zeta + g(u)(\partial u)^2,$$

in the relatively mild case where $\frac{1}{2} < \alpha \leq \frac{2}{3}$.

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1 Paracontrolled calculus

1.1 Overview

Starting with T. Lyons' work on controlled differential equation [22], it is now well-understood that the construction of a robust approximation theory for continuous time stochastic systems, such as stochastic differential equations or stochastic partial differential equations, requires a twist in the notion of noise that allows to treat the resolution of such equations in a two step process.

- (a) Enhance the noise into an enriched object that lives in some space of analytic objects – this is a purely *probabilistic* step;
- (b) given *any* such object $\hat{\zeta}$ in this space, one can introduce a $\hat{\zeta}$ -dependent Banach space $\mathcal{S}(\hat{\zeta})$ such that the equation makes sense for the unknown in $\mathcal{S}(\hat{\zeta})$, and it can be solved uniquely by a *deterministic* analytic argument, such as the contraction principle, which gives the continuity of the solution as a function $\hat{\zeta}$.

These two steps are very different in nature and require totally different tools. The present work deals with the deterministic side of the story, point (b), for the study of *singular partial differential equations* (PDEs). The term *singular* refers here to the fact that the 'noise' in the equation is not regular enough for all the expressions in the equation to make sense analytically, given the expected regularity of the solution in terms of the regularity of the 'noise'. Recall that one can generically not make sense of the product of a distribution with a continuous function.

While the space of enhanced controls in Lyons' theory of controlled differential equations is universal, in the sense that it depends only on the dimension and the irregularity exponent of the control, the ground-breaking works of Hairer [16, 17] and Gubinelli-Imkeller-Perkowski [13] uncovered the fact that this space of enhanced noises, and the solution space $\mathcal{S}(\hat{\zeta})$ with it, are equation-dependent in the study of singular PDEs. Both objects will pop out naturally in our setting.

Hairer's theory of regularity structures [17] provides undoubtedly the most complete picture for the study of a whole class of singular stochastic PDEs from the above point of view – the class of the so-called singular subcritical parabolic stochastic PDEs. It comes with a very rich algebraic structure and an entirely new setting that are required to give flesh to the guiding principle that a solution should be described by the datum at each point in space-time of its high order 'jet' in a basis given by the elements of the enhanced noise. Regularity structures are introduced as a tool for describing these jets. At the same time that Hairer built his theory, Gubinelli-Imkeller-Perkowski implemented in [13] this idea of giving a local/global description of a possible solution in a different way, using the language of paraproducts and avoiding the introduction of any new setting, but providing only a first order description of the objects under study. This is what we shall call from now on the *first order paracontrolled calculus*. While this kind of approach may seem far from being as powerful as Hairer's machinery, the first order paracontrolled approach to singular stochastic PDEs has been successful in recovering and extending a number of results that can be proved within the setting of regularity structures, on the parabolic Anderson model and Burgers equations [13, 1, 2, 8], the KPZ equation [15], the scalar Φ_3^4 equation [4], the stochastic Navier-Stokes equation [24, 25, 26], or the study of the continuous Anderson Hamiltonian [7], to name but a few.

We develop in this work a high order version of paracontrolled calculus that allows to treat analytically within this paradigm some parabolic singular partial differential equations that are beyond the scope of the original formulation of the theory, with the same efficiency as regularity structures, with the benefit that there is no need to introduce the algebraic apparatus inherent to the latter theory. We refer to our setting as *paracontrolled calculus*. By a 'noise' in an equation we shall simply mean a function/distribution-valued parameter ζ – realisations of a white noise are typical examples. Within our setting, and given as input a noise ζ and some initial condition, the resolution process of a typical parabolic equation

$$(\partial_t + L)u = f(u, \zeta), \quad (1.1)$$

involves the following elementary steps. Write $\mathcal{R} := (\partial_t + L)^{-1}$ for the resolution operator, and keep in mind that we have in hands two space-time paraproducts Π and $\tilde{\Pi}$, related by the intertwining relation

$$\mathcal{R} \circ \Pi = \tilde{\Pi} \circ \mathcal{R};$$

all the objects are properly introduced below.

1. **Paracontrolled ansatz.** *The irregularity of the noise ζ , and the form of the equation, dictate the choice of a Banach solution space made up of functions/distributions of the form*

$$u = \sum_{i=1}^{k_0} \tilde{\Pi}_{u_i} Z_i + u^\sharp, \quad (1.2)$$

for some reference functions/distributions Z_i that depend formally only on ζ , to be determined later; we have for instance $Z_1 = \mathcal{R}(\zeta)$, if the equation is affine with respect to ζ . The derivatives' u_i of u also need to satisfy such a structural equation, to order $(k_0 - 1)$, and their derivatives a structural equation of order $(k_0 - 2)$, and so on. (See Proposition 21 for a justification of the name 'derivative' for the u_i .) One sees the above description (1.2) of u as a paracontrolled Taylor expansion at order k_0 for it; denote by \hat{u} the datum of u and all its derivatives.

2. **Right hand side.** *The use of a Taylor expansion formula, and continuity results for some operators, allow to rewrite the right hand side $f(u, \zeta)$ of equation (1.1) in the canonical form*

$$f(u, \zeta) = \sum_{j=1}^{k_0} \Pi_{v_j} Y_j + (\sharp)$$

where (\sharp) is some nice, in particular sufficiently regular, remainder and the distributions Y_j depend only on ζ and the Z_i .

3. **Fixed point.** *Denote by P the resolution of the free heat equation*

$$Pu_0 := (\tau, x) \mapsto (e^{-\tau L} u_0)(x).$$

Then the fixed point relation

$$\begin{aligned} u &= Pu_0 + \mathcal{R}(f(u, \zeta)) \\ &= Pu_0 + \sum_{j=1}^{k_0} \mathcal{R}(\Pi_{v_j} Y_j) + \mathcal{R}(\#) \\ &= Pu_0 + \sum_{j=1}^{k_0} \tilde{\Pi}_{v_j} Z_j + \mathcal{R}(\#), \end{aligned}$$

imposes some consistency relations on the choice of the $Z_i = \mathcal{R}(Y_i)$ that determine them uniquely as a function of ζ and Z_1 . Those expressions inside the Y_i 's that do not make sense on a purely analytical basis are precisely those elements that need to be given as components of the enhanced distribution $\hat{\zeta}$. Schauder estimates for \mathcal{R} play a role in running the fixed point argument. Note that, strictly speaking, the fixed point relation is a relation on \hat{u} rather than u . We choose to emphasize that point by rewriting the equation under the form

$$(\partial_t + L)u = f(\hat{u}, \hat{\zeta}).$$

As expected, the elements that need to be added in $\hat{\zeta}$ to ζ are those needed to make sense of the corresponding ill-defined products in the regularity structures setting. List the elements of $\hat{\zeta}$ in non-decreasing order of regularity and consider them as a basis of a finite dimensional space. A renormalisation map is a linear map of the form

$$\mathcal{M} : \hat{\zeta} \mapsto T\hat{\zeta} - \Xi,$$

for some upper triangular constant matrix T , with a unit diagonal, and some possibly space-time dependent renormalisation functions/constants Ξ .

4. Symmetry group. *The role of the extra components of $\hat{\zeta}$ in the dynamics is completely clarified by writing*

$$f(u, \zeta) = f(\hat{u}, \hat{\zeta}) = f_0(\hat{u}, \zeta) + f_1(\hat{u})\hat{\zeta}$$

as a sum of a continuous function f_0 of \hat{u} and ζ , and a continuous function f_1 of \hat{u} and $\hat{\zeta}$, that is linear with respect to $\hat{\zeta}$. If ζ is a stochastic noise and ζ^ε stands for a regularized noise, with associated canonical enhancement $\hat{\zeta}^\varepsilon$, and if a renormalisation procedure \mathcal{M}^ε provides an enhanced distribution $\mathcal{M}^\varepsilon \hat{\zeta}^\varepsilon$ converging in probability to some limit element in the space of enhanced distributions, then the solution to the well-posed equation

$$(\partial_t + L)u^\varepsilon = f(u^\varepsilon, \zeta^\varepsilon) + f_1(u^\varepsilon)(\mathcal{M}^\varepsilon - \text{Id})\hat{\zeta}^\varepsilon$$

converges in probability to the first component u of the solution to the equation

$$(\partial_t + L)u = f(\hat{u}, \hat{\zeta}). \tag{1.3}$$

Equation (1.3) makes it clear how the renormalisation group acts on the equation as a symmetry group. We shall not touch in this work on renormalisation matters, so we shall always assume that the enhancement $\hat{\zeta}$ of ζ is given. Three ingredients are used to run the above scheme in any concrete situation.

- (i) *The pair $(\Pi, \tilde{\Pi})$ of intertwined paraproducts introduced in [2]. It is crucially used to define a continuous map Φ from $\mathcal{S}(\hat{\zeta})$ to itself. The use of an ansatz solution space where Π -operators would be used in place of $\tilde{\Pi}$ -operators would not produce a map from $\mathcal{S}(\hat{\zeta})$ to itself.*
- (ii) *A high order Taylor expansion formula generalizing Bony's parilinearization formula is used to give a paracontrolled Taylor expansion of a non-linear function of u , starting from a paracontrolled function u . See section 2 for the Taylor formula.*
- (iii) **Continuity results.** The technical core of Gubinelli-Imkeller-Perkowski' seminal work [13] is a continuity result for the operator

$$\mathbf{C}(f, g; h) = \Pi(\Pi_f g, h) - f\Pi(g, h).$$

We introduce a number of other operators and prove their continuity – section 3. These operators are used crucially in analyzing the right hand side $f(u, \zeta)$ of the equation, step 2.

1.2 Setting and results

We adopt in this work essentially the same geometric and functional setting as in our previous work [2], slightly restricted so as not to bother here with the use of weighted functional spaces. All this work could be formulated in the more general geometric/functional setting of [2] (in particular the use of space-time weights allow to deal with an unbounded ambient space); we refrain from doing this as it may blur the simple ideas that we want to promote in this work. Let then (M, d, μ) be stand for a compact smooth Riemannian manifold equipped with a measure μ , and let V_1, \dots, V_{ℓ_0} stand for some smooth vector fields on M , identified with first order differential operators. Given a tuple $I = (i_1, \dots, i_k)$ in $\{1, \dots, \ell_0\}^k$, we shall set $|I| := k$ and

$$V_I := V_{i_k} \cdots V_{i_1}.$$

Set

$$L := - \sum_{i=1}^{\ell_0} V_i^2$$

and assume that L is elliptic, so that the V_i span at every point of M the whole tangent space. The operator L is then a sectorial operator in $L^2(M)$, it is injective on the quotient space of $L^2(M)$ by the space of constant functions, it has a bounded H^∞ -calculus on $L^2(M)$, and $-L$ generates a holomorphic semigroup $(e^{-tL})_{t>0}$ on $L^2(M)$. The above class of operators includes obviously the Laplacian on the flat torus. Note that under the above smoothness and ellipticity conditions, the semigroup e^{-tL} has regularity estimates at any order, by which we mean that for every tuple I , the operators $\left(t^{\frac{|I|}{2}} V_I\right) e^{-tL}$ and $e^{-tL} \left(t^{\frac{|I|}{2}} V_I\right)$ have kernels $K_t(x, y)$ satisfying the Gaussian estimate

$$\left| K_t(x, y) \right| \lesssim \frac{1}{\mu(B(x, \sqrt{t}))} e^{-c \frac{d(x, y)^2}{t}}$$

and the following regularity estimate. For $d(x, z) \leq \sqrt{t}$

$$\left| K_t(x, y) - K_t(z, y) \right| \lesssim \frac{d(y, z)}{\sqrt{t}} \frac{1}{\mu(B(x, \sqrt{t}))} e^{-c \frac{d(x, y)^2}{t}},$$

for some constants which may depend on $|I|$. Note here that we could equally well develop paracontrolled calculus in the more general setting adopted in our previous work [2]; we refrain from doing that here as it could obscure the simplicity of the ideas put forward here.

Given a finite time horizon T , we define the parabolic space \mathcal{M} as

$$\mathcal{M} := [0, T] \times M,$$

and equip it with the parabolic metric

$$\rho((\tau, x), (\sigma, y)) = \sqrt{|\tau - \sigma|} + d(x, y)$$

and the parabolic measure $\nu = \mu \otimes dt$. Then (\mathcal{M}, ρ, ν) is a doubling space (of homogeneous type). Note that for $(\tau, x) \in \mathcal{M}$ and small positive radii r , the parabolic ball $B_{\mathcal{M}}((\tau, x), r)$ has volume

$$\nu(B_{\mathcal{M}}((\tau, x), r)) \approx r^2 \mu(B_M(x, r)).$$

We shall denote by $e = (\tau, x)$ a generic element of the parabolic space \mathcal{M} .

We have chosen to work in the scale of Hölder spaces; this makes life easier, although we could equally develop paracontrolled calculus in the larger functional setting of Sobolev spaces, in the line of what we did in our previous work [1]. For a real number s , we will denote by $C^s = C^s(M)$ the Hölder space on M of order s , defined in terms of Besov spaces; and $\mathcal{C}^s = \mathcal{C}^s(\mathcal{M})$ the parabolic Hölder space. We refer the reader to the Appendix for more details on these spaces. Following our previous work [2], one can define parabolic paraproduct and resonant operators that have good continuity properties in the scale of parabolic Hölder spaces – section Appendix A.3. The high order Taylor formula and the continuity results stated in sections 2 and 3 and fully proved in Appendix B and C, make use of these operators and provide the spine of paracontrolled calculus. They are the main contributions of this work.

We illustrate our approach of the study of singular PDEs, such as described above, on the example of the generalised parabolic Anderson model equation (gPAM)

$$(\partial_t + L)u = f(u)\zeta, \tag{1.4}$$

in the case where the noise ζ has the same regularity as the 2^+ or 3 -dimensional space white noise, and on the example of the generalized KPZ equation

$$(\partial_t + L)u = f(u)\zeta + (\partial u)^2, \tag{1.5}$$

in the relatively mild case where the one-dimensional space-time noise ζ is $(\alpha - 2)$ -Hölder, with $\frac{1}{2} < \alpha \leq \frac{2}{3}$ – one dimensional space-time white noise corresponds to $\alpha < \frac{1}{2}$, by proving in both cases that one can define for each equation a solution space $\mathcal{S}(\widehat{\zeta})$ where the equation is well-posed, under the assumption that the enhancement $\widehat{\zeta}$ of the noise ζ is given. Once again, defining $\widehat{\zeta}$ in a stochastic setting is a very different question that is not studied here. We also describe explicitly the symmetry group of these equations. Along the way, we also adapt the notion of truly rough function to the present multi-dimensional setting and prove that a functions paracontrolled by a truly rough function has a uniquely determined derivative.

We have organised this work as follows. Section 2 is dedicated to our high order Taylor expansion formula. The latter provides a generalisation of Bony's paralin-earisation formula. Whereas our Taylor formula deals with the fine description of

nonlinear images of parabolic Hölder functions, we provide in section 2 simple proofs of their spatial counterpart – full proofs of the parabolic claims are given in Appendix B. A number of operators are introduced and studied in section 3; the continuity results proved here are some of our main contributions. Here again, while all the statements are about parabolic functions/distributions, we have given in this section some simple proofs of their spatial counterpart, deferring the proofs of the full statements to Appendix C. We test our paracontrolled calculus, such as described above in section 1.1, on the example of the 2^+ and 3-dimensional generalized parabolic Anderson model equation (1.4) in section 4, and on the example of the generalized KPZ equation (1.5) in section 5. Appendix A contains all the relevant details about the parabolic setting, approximation operators, Hölder spaces and paraproducts.

2 High order Taylor expansion

We explain in this section a simple procedure for getting an arbitrary high order expansion of a nonlinear map of a given Hölder function u defined on the parabolic space \mathcal{M} , in terms of its parabolic regularity properties. It provides, in the setting of Hölder spaces, a refinement over Bony's parilinearisation theorem in the form of a viable alternative to the paper [9] of Chemin; see also [10], theorem 2.5, p.18, for a more readable account of [9] in the case of a third order expansion.

In its simplest form, the classical paraproduct operator Π^0 on the d -dimensional torus is defined via Fourier analysis by modulation of the high frequencies of a given 'reference' function/distribution g by the low frequencies of another function/distribution f . For a function f on the torus, we denote by $f = \sum f_i$ its usual Littlewood-Paley representation: f_i is the dyadic bloc with Fourier coefficients only at the frequency scale 2^i . Consider the two Littlewood-Paley decompositions of two functions f, g

$$f = \sum f_i, \quad g = \sum g_j,$$

as sums of smooth functions with localized frequencies, the paraproduct of g by f is defined as

$$\Pi_f^0 g = \sum_{i < j-1} f_i g_j, \quad (2.1)$$

and the resonant part as

$$\Pi^0(f, g) = \sum_{|i-j| \leq 1} f_i g_j$$

in order that we have the product decomposition

$$fg = \Pi_g^0(f) + \Pi_f^0(g) + \Pi^0(f, g).$$

In the parabolic setting of section 1.2, one can define some paraproduct and resonant operators associated with the operator L and its semigroup, that have the same regularity properties in the scale of parabolic Hölder spaces as the operator Π^0 in the scale of spatial Hölder spaces. We denote by Π this paraproduct, introduced in [2], and whose definition is recalled in Appendix A.3. It depends implicitly on an integer-valued parameter b that is chosen once and for all, and whose precise choice is irrelevant for our purposes. It is not crucial at that stage to go into the details of the definition of Π .

The mechanics of the proof of our general Taylor expansion formula is fairly simple and better understood in the light of the proof of Bony's parilinearisation theorem given by Gubinelli, Imkeller and Perkowski in [13], which we recall first.

Theorem (Bony's Parilinearisation). *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a C_b^2 function and u be a real-valued α -Hölder function on the d -dimensional torus, with $0 < \alpha < 1$. Then*

$$f(u) = \Pi_{f'(u)}^0(u) + f(u)^\sharp$$

for some remainder $f(u)^\sharp$ of spatial Hölder regularity 2α .

Proof – This is just a copy and paste from [13]. Denote by K_i the kernels of the Fourier projectors Δ_i corresponding to the Littlewood-Paley decomposition operator, and write $K_{\leq k}$ for $\sum_{i \leq k} K_i$, with associated operator S_k . Note that by their definition we have, for any $i \geq 1$,

$$\int_{\mathbf{R}^d} K_i(y) dy = 0; \quad (2.2)$$

or more properly $\int_{\mathbf{R}^d} K_i(x, y) dy = 0$, for any $x \in \mathbf{R}^d$. The trick is then simply to write

$$f(u) - \Pi_{f'(u)}(u) = \sum \Delta_i(f(u)) - S_{i-1}(f'(u))\Delta_i(u) =: \sum \varepsilon_i$$

with

$$\varepsilon_i(x) = \int K_i(x, y) K_{\leq i-1}(x, z) \left\{ f(u(y)) - f'(u(z))u(y) \right\} dz dy,$$

and to take profit from the fact that K_i has null mean for $i \geq 1$, as put forward in identity (2.2), to see that one also has, for $i \geq 1$,

$$\varepsilon_i(x) = \int K_i(x, y) K_{\leq i-1}(x, z) \left\{ f(u(y)) - f(u(z)) - f'(u(z))(u(y) - u(z)) \right\} dz dy.$$

One thus has

$$|\varepsilon_i(x)| \lesssim \|f''\|_\infty \int |K_i(x, y) K_{\leq i-1}(x, z)| |u(y) - u(z)|^2 dz dy \lesssim 2^{-2i\alpha} \|u\|_{C^\alpha}^2,$$

which proves the claim. ▷

One can play exactly the same game and prove a general Taylor expansion result in a parabolic setting, with our paraproduct Π in the role of the comparison operator.

Theorem 1 (Higher order Taylor expansion). *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a C^4 function with bounded fourth derivative, and let u be a real-valued α -Hölder function on the parabolic space \mathcal{M} , with $0 < \alpha < 1$. Then*

$$\begin{aligned} f(u) &= \Pi_{f'(u)}(u) + \frac{1}{2} \left\{ \Pi_{f^{(2)}(u)}(u^2) - 2\Pi_{f^{(2)}(u)u}(u) \right\} \\ &\quad + \frac{1}{3!} \left\{ \Pi_{f^{(3)}(u)}(u^3) - 3\Pi_{f^{(3)}(u)u}(u^2) + 3\Pi_{f^{(3)}(u)u^2}(u) \right\} + f(u)^\sharp \end{aligned} \quad (2.3)$$

for some remainder $f(u)^\sharp$ of parabolic Hölder regularity 4α . Moreover the remainder term $f(u)^\sharp$ is a locally Lipschitz function of u , in the sense that

$$\|f(u)^\sharp - f(v)^\sharp\|_{C^{4\alpha}} \lesssim (1 + \|u\|_{C^\alpha} + \|v\|_{C^\alpha})^4 \|u - v\|_{C^\alpha}.$$

We give here a proof of this statement in the case where u is a time-independent function on the d -dimension torus and we can use the elementary paraproduct Π^0 instead of Π . The full proof of theorem 1 is given in Appendix B, Theorem 22; we hope this way of proceeding will make the reasoning clear and technical-free.

Proof – Let us prove the second order formula in the special case where $u : \mathbf{T}^d \rightarrow \mathbf{R}$, and we use the elementary paraproduct Π^0 in place of Π . The claim amounts in the case to proving that

$$(\star) := f(u) - \Pi_{f'(u)}^0(u) - \frac{1}{2} \left\{ \Pi_{f^{(2)}(u)}^0(u^2) - 2 \Pi_{f^{(2)}(u)u}^0(u) \right\}$$

is a 3α -Hölder function on the torus. As in the proof of Bony's parilinearisation result, write (\star) under the form

$$\sum \Delta_i(f(u)) - S_{i-1}(f'(u))\Delta_i(u) - \left\{ \frac{1}{2} S_{i-1}(f^{(2)}(u))\Delta_i(u^2) + S_{i-1}(f^{(2)}(u)u)\Delta_i(u) \right\} =: \sum \varepsilon_i.$$

Denote by $D_u^{(k)}f$, the k^{th} -derivative $f^{(k)}(u)$ of f at u . For each $i \geq 1$, we have

$$\begin{aligned} \varepsilon_i(x) = & \int K_i(x, y) K_{\leq i-1}(x, z) \\ & \left\{ \int_0^1 (D_{u(z)+t(u(y)-u(z))}^{(2)} f)(u(y) - u(z))^2 t dt \right. \\ & \left. - \frac{1}{2} (D_{u(z)}^{(2)} f) u^2(y) + (D_{u(z)}^{(2)} f) u(z)u(y) \right\} dz dy, \end{aligned}$$

which we can rewrite as

$$\begin{aligned} \varepsilon_i(x) = & \int K_i(x, y) K_{\leq i-1}(x, z) \\ & \int_0^1 \int_0^1 (D_{u(z)+st(u(y)-u(z))}^{(3)} f)(u(y) - u(z))^3 ds t dt dz dy, \end{aligned}$$

using once again the fact that the kernels $K_i(x, \cdot)$ have null mean. One reads on this expression for ε_i that it is of order $2^{-3i\alpha}$, uniformly in x . See Appendix B for a full proof of the statement, in the parabolic setting.

▷

Observe that the expansion (2.3) is exact, $f(u)^\sharp = 0$, for a polynomial function f of degree at most 3. The above Taylor formula for $f(u)$ is conveniently rewritten under the form

$$f(u) = \Pi_{f'(u)-uf^{(2)}(u)+\frac{1}{2}u^2f^{(3)}(u)}(u) + \frac{1}{2} \Pi_{f^{(2)}(u)-uf^{(3)}(u)}(u^2) + \frac{1}{6} \Pi_{f^{(3)}(u)}(u^3) + f(u)^\sharp.$$

As a reminder for future use, we note here that the general Taylor expansion formula writes

$$f(u) = \sum_{n=1}^k \sum_{j=1}^{n-1} (-1)^j \binom{n}{j} \Pi_{u^j f^{(n)}(u)}(u^{n-j}) + f(u)^\sharp,$$

for a function f of class C^{k+1} with bounded $(k+1)^{\text{th}}$ derivative, and a remainder $f(u)^\sharp$ of parabolic Hölder regularity $(k+1)\alpha$.

3 Toolkit for paracontrolled calculus

We prove in this section a number of continuity results for some operators built from the parabolic paraproduct and resonant operators associated with L . These continuity results will play a crucial role in the analysis of the right hand side $f(u, \zeta)$ of a generic singular PDE such as equation (1.1); the two examples treated in sections 4 and 5 will make that point clear. Together with the Taylor formula of section 2, the results of this section are our main contribution. It is not necessary, for the purpose of solving singular PDEs, to get into the details of the proofs of the different results given here; we invite the reader to have a look at the results only and then go directly to sections 4 and 5 to see them on stage.

We adopt in this section the same pedagogical point of view as in section 2, giving the reader the general statements of our theorems, in the above parabolic setting over a compact manifold that requires the use of the parabolic paraproduct and resonant operators of Appendix A, and only providing here the proofs of their spatial counterparts on the torus, where only time-independent functions are in play and one can use the elementary paraproduct Π^0 in the analysis. A further simplification in the proofs is done here, and detailed below; proofs of the full statements are given in Appendix C. We hope this way of proceeding will convince the reader that the basic ideas involved here are elementary.

3.1 Commutator, corrector and their iterates

Recall from [2] and Appendix C.2 that the modified paraproduct $\tilde{\Pi}$ is defined by the formula

$$\tilde{\Pi}_f(g) = \mathcal{R}\left(\Pi_f(\mathcal{L}g)\right),$$

where \mathcal{L} stands for the parabolic differential operator $(\partial_\tau + L)$ on the parabolic space \mathcal{M} . See section 4.1 of [2] for a study of the continuity properties of $\tilde{\Pi}$. We provide in this section a number of continuity results for some operators involving the paraproduct and resonant operators, together with the modified paraproduct $\tilde{\Pi}$. *We state our results in their general form, in the parabolic setting of section 1.2, and give proofs in the time-independent, space setting of the torus, of versions of each statement where we use Π^0 instead of $\tilde{\Pi}$.* This should make it easier for the reader to go to the core of the machinery without fighting with some possibly overwhelming technicalities; full proofs are given in Appendix C.

We define on the space L^∞ of bounded measurable functions on the parabolic space \mathcal{M} the **commutator** as the operator

$$D(f, g; h) := \Pi\left(\tilde{\Pi}_f(g), h\right) - \Pi_f\left(\Pi(g, h)\right),$$

and the **corrector** as the operator

$$C(f, g; h) := \Pi\left(\tilde{\Pi}_f(g), h\right) - f \Pi(g, h).$$

The first part of the next theorem is the workhorse of the first order paracontrolled calculus, such as devised in [13] by Gubinelli-Imkeller-Perkowski. Note how unfortunate they were in naming the operator C a "commutator"; which is definitely not the case, unlike the operator D – up to the tilde on one of the Π operators in the definition of D . Recall we denote by C^α the spacial Hölder spaces on the torus and by \mathcal{C}^α the parabolic Hölder spaces over the compact manifold M .

Theorem 2. (i) *If α, β and γ are positive, then the commutator D is continuous from $C^\alpha \times C^\beta \times C^\gamma$ to $C^{\alpha+\beta+\gamma}$.*

(ii) *Assume α is positive, $\beta + \gamma$ is negative and $\alpha + \beta + \gamma$ is positive. Then, the corrector C extends continuously as a function from $C^\alpha \times C^\beta \times C^\gamma$ to $C^{\alpha+\beta+\gamma}$.*

Proof – As said above, we prove here these continuity results for simplified versions of the operators D and C . So, assume we are working in the time-independent setting of the d -dimensional torus, with the operators

$$D^0(f, g; h) := \Pi^0\left(\Pi_f^0(g), h\right) - \Pi_f^0\left(\Pi^0(g, h)\right),$$

and

$$C^0(f, g; h) := \Pi^0\left(\Pi_f^0(g), h\right) - f\Pi^0(g, h).$$

We start by proving the claim about the continuity of the corrector C^0 , as a function from $C^\alpha \times C^\beta \times C^\gamma$ to $C^{\alpha+\beta+\gamma}$, under the above sign assumptions on α, β, γ .

(ii) The resonant part is given by

$$\Pi^0(a, b) \simeq \sum \Delta_i(a) \Delta_i(b). \quad (3.1)$$

Write

$$C(f, g; h) = \sum \Delta_i\left(\Pi_f^0(g)\right) \Delta_i h - f \Delta_i(g) \Delta_i(h),$$

and set

$$\varepsilon'_i := \Delta_i\left(\Pi_f^0(g)\right) - f \Delta_i(g),$$

such that

$$C^0(f, g; h) = \sum_i \varepsilon'_i \Delta_i(h).$$

The fact that ε'_i has L^∞ -norm of order $2^{-i(\alpha+\beta)}$ can be guessed on the expression

$$\begin{aligned} \varepsilon'_i(x) &= \int K_i(x, y) \left\{ (\Pi_f^0 g)(y) - f(x)g(y) \right\} dy \\ &= \int K_i(x, y) \left\{ \Pi_{f-f(x)}^0 \mathbf{1}(g) \right\}(y) dy. \end{aligned}$$

As y is concentrated near x , at scale 2^{-i} , and we are looking at the i^{th} Paley-Littlewood block of $\Pi_{f-f(\cdot)} g$, we expect

$$|\varepsilon'_i(x)| \lesssim 2^{-i\beta} \left\| \Pi_{f-f(x)}^0 g \right\|_{C^\beta} \lesssim 2^{-i\beta} \|f - f(x)\|_{L^\infty} \|g\|_{C^\beta},$$

with a term $\|f - f(x)\|_{L^\infty}$ involving only the neighbourhood of x of size 2^{-i} , that is with

$$\|f - f(x)\|_{L^\infty} \lesssim 2^{-i\alpha} \|f\|_{C^\alpha},$$

since f is α -Hölder. Such an estimate would imply the continuity of the corrector C as a function from $C^\alpha \times C^\beta \times C^\gamma$ to $C^{\alpha+\beta+\gamma}$ if $\alpha + \beta + \gamma$, since h is γ -Hölder. This heuristic argument, however, does not make it clear why we need $\beta + \gamma$ to be negative to get the result.

A mathematically correct version of the above sketch of proof is done by estimating the L^∞ -norm of the dyadic blocks of ε'_i . For $j \geq i + 2$ then

$$\Delta_j \varepsilon'_i = -\Delta_j(f \Delta_i(g)) \simeq -\Delta_j(f) \Delta_i(g)$$

hence

$$\|\Delta_j \varepsilon'_i\|_{L^\infty} \lesssim 2^{-j\alpha} 2^{-i\beta} \|f\|_{C^\alpha} \|g\|_{C^\beta}.$$

For $j \leq i - 2$ then

$$\Delta_j \varepsilon'_i = -\Delta_j(f \Delta_i(g)) \simeq -\Delta_j(\Delta_i(f) \Delta_i(g))$$

hence

$$\|\Delta_j \varepsilon'_i\|_{L^\infty} \lesssim 2^{-i(\alpha+\beta)} \|f\|_{C^\alpha} \|g\|_{C^\beta}.$$

We adopt the classical notation $S_{j-1}f$ for the partial sum $\sum_{\ell \leq j-1} f_\ell$ of the Paley-Littlewood decomposition, so for $|i - j| \leq 2$ we have

$$\Delta_j \varepsilon'_i \simeq \Delta_j \left(\Delta_j(g) S_{j-1}(f) - S_{j+2}(f) \Delta_i(g) \right),$$

hence

$$\|\Delta_j \varepsilon'_i\|_{L^\infty} \lesssim 2^{-i(\alpha+\beta)} \|f\|_{C^\alpha} \|g\|_{C^\beta}.$$

As a consequence, we always have the following estimate

$$\|\Delta_j \varepsilon'_i\|_{L^\infty} \lesssim 2^{-i\beta} 2^{-\max(j,i)\alpha} \|f\|_{C^\alpha} \|g\|_{C^\beta}. \quad (3.2)$$

We can then estimate $C^0(f, g; h)$ in some Hölder space. For a non-negative integer k , we have

$$\begin{aligned} \Delta_k \left(C^0(f, g; h) \right) &= \sum_i \Delta_k \left(\varepsilon'_i \Delta_i(h) \right) \\ &\simeq \sum_{i \leq k-2} \Delta_k(\varepsilon'_i) \Delta_i(h) + \sum_{k \leq i-2} \Delta_k \left(\Delta_i(\varepsilon'_i) \Delta_i(h) \right) \\ &\quad + \sum_{|k-i| \leq 2} \Delta_k \left(S_i(\varepsilon'_i) \Delta_i(h) \right) \end{aligned}$$

which is then controlled, using estimate (3.2), by

$$\begin{aligned} &\left\| \Delta_k \left(C^0(f, g; h) \right) \right\|_{L^\infty} \\ &\lesssim \left(\sum_{i \leq k-2} 2^{-i\gamma} 2^{-k\alpha} 2^{-i\beta} + \sum_{k \leq i-2} 2^{-i(\alpha+\beta+\gamma)} + \sum_{|k-i| \leq 2} 2^{-i(\alpha+\beta+\gamma)} \right) \|f\|_{C^\alpha} \|g\|_{C^\beta} \\ &\lesssim 2^{-k(\alpha+\beta+\gamma)} \|f\|_{C^\alpha} \|g\|_{C^\beta}, \end{aligned}$$

where we used the two conditions $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$ along the way. The fact that the latter estimate holds uniformly in k concludes the proof of the $(\alpha + \beta + \gamma)$ -Hölder regularity of the corrector.

(i) We refer the reader to Proposition 25, in Appendix C.1, for a full proof of the regularity statement for the commutator D . Simply mention that in the special case of D^0 , the regularity estimate comes from the following identity

$$\Delta_k(D^0(f, g; h)) = \sum_{\ell \geq k-2} \Delta_k \left(\Delta_\ell(g) S_\ell(f) \Delta_\ell(h) \right) - S_k(f) \Delta_k \left(\Delta_\ell(g) \Delta_\ell(h) \right). \quad (3.3)$$

▷

We emphasize the importance of the above heuristic proof of point **(i)** by introducing a notation. Given a function-valued operator A on some function space, we denote by $\mathcal{C}f$, or $\mathcal{C}_x f$, the function

$$(\mathcal{C}f)(\cdot) := f(\cdot) - f(x),$$

recentered around its value at the 'running' variable x , so that

$$A(\mathcal{C}f)(x) = A(f - f(x))(x).$$

(Strictly speaking, the operator \mathcal{C} is an operator on the space of operators A .) The choice of letter \mathcal{C} for this operator is for 'centering', and we call \mathcal{C} the **outer centering operator**. In those terms, we have

$$\mathbf{C}(f, g; h) = \Pi\left(\tilde{\Pi}_{\mathcal{C}f}(g), h\right), \quad (3.4)$$

and

$$\Pi\left(\Pi_{\mathcal{C}\Pi_{\mathcal{C}b}(c)}(g), h\right)(x) = \Pi\left(\Pi_{\Pi_{b-b(x)}(c)-(\Pi_{b-b(x)}(c))(x)}(g), h\right)(x),$$

for instance. The main property of this operator is the following. For a function $f \in C^\alpha(\mathbf{T}^d)$ with α positive, we have first

$$\begin{aligned} S_k(\mathcal{C}f)(x) &= S_k(f - f(x))(x) = S_k(f)(x) - f(x) \\ &= \sum_{\ell \geq k+1} \Delta_\ell(f)(x). \end{aligned}$$

Since f is supposed to have a positive regularity the dyadic blocks $\Delta_\ell(f)$ have an exponentially decreasing L^∞ size as a function of ℓ , so one has approximately

$$S_k(\mathcal{C}f)(x) \simeq \Delta_k(f)(x). \quad (3.5)$$

A very similar property holds in the parabolic setting, which is used in the proofs of the continuity results of this section given in Appendix C.

The study of singular PDEs happens to require some finer analysis of the operators \mathbf{D} and \mathbf{C} that take the form of some continuity estimates for some 'iterated' versions of them. More precisely, it is possible to decompose further \mathbf{D} and \mathbf{C} in case one of their first two arguments are given in the form of a (modified) paraproduct or an iterated (modified) paraproduct. We introduce here for that purpose a notation. Given a tuple of functions $(a, b, c; g)$, set

$$\tilde{\Pi}_{a,b}^\downarrow(c) := \tilde{\Pi}_{\tilde{\Pi}_a(b)}^\downarrow(c)$$

and

$$\tilde{\Pi}_{a,b,c}^\downarrow(g) := \tilde{\Pi}_{\tilde{\Pi}_{a,b}^\downarrow(c)}^\downarrow(g),$$

and give similar definitions of $\Pi_{a,b}^\downarrow(c)$ and $\Pi_{a,b,c}^\downarrow(g)$ using only Π operators. Depending on whether or not such a paraproduct appears in the low frequency, in place of f , or high frequency, in place of g , in the formulas for the commutator \mathbf{D} or the corrector \mathbf{C} , we shall talk about **lower** or **upper iterated** operators.

Proposition 3. *Given some positive regularity exponents $\alpha, \beta, \gamma, \delta$, the formulas*

$$\mathbf{D}(a, b; g, h) := \mathbf{D}\left(\tilde{\Pi}_a b, g; h\right) - \Pi_a \mathbf{D}(b, g; h), \quad (\text{lower iterated commutator})$$

$$\mathbf{D}(f; a, b; h) := \mathbf{D}\left(f, \tilde{\Pi}_a b; h\right) - \Pi_a \mathbf{D}(f, b; h), \quad (\text{upper iterated commutator})$$

define continuous operators from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma \times \mathcal{C}^\delta$ to $\mathcal{C}^{\alpha+\beta+\gamma+\delta}$.

Proof – As in the proof of Theorem 2, we analyse in this proof what happens in the time-independent setting of the d -dimensional torus, in the case where we also use Π^0 instead of $\tilde{\Pi}$. So we set

$$\mathbf{D}^0(a, b; g, h) := \mathbf{D}^0\left(\Pi_a^0 b, g; h\right) - \Pi_a^0 \mathbf{D}^0(b, g; h)$$

and have a look at its continuity properties on the spacial Hölder spaces. Using formula (3.3), it follows that we roughly have

$$\begin{aligned} \Delta_k(\mathbf{D}^0(a, b; g, h)) &\simeq \Delta_k(\mathbf{D}^0(\Pi_a b, g; h)) - S_{k-2}(a) \Delta_k(\mathbf{D}^0(b, g; h)) \\ &\simeq \sum_{\ell \geq k-2} \Delta_k \left[\Delta_\ell(g) \Delta_\ell(h) \left(S_\ell \Pi_a(b) - S_k \Pi_a(b) - S_k(a) (S_\ell(b) - S_k(b)) \right) \right]. \end{aligned}$$

The quantity inside the brackets is equal to

$$\begin{aligned} S_\ell \Pi_a(b) - S_k \Pi_a(b) - S_k(a) (S_\ell(b) - S_k(b)) &= \sum_{j=k+1}^{\ell} \Delta_j \Pi_a(b) - S_k(a) \Delta_j(b) \\ &\simeq \sum_{j=k+1}^{\ell} S_j(a) \Delta_j(b) - S_k(a) \Delta_j(b) \\ &\simeq \sum_{j=k+1}^{\ell} (S_j(a) - S_k(a)) \Delta_j(b), \end{aligned}$$

which is then easily bounded in L^∞ by

$$\sum_{j=k+1}^{\ell} 2^{-k\alpha} \|a\|_{C^\alpha} 2^{-j\beta} \|b\|_{C^\beta} \lesssim 2^{-k(\alpha+\beta)} \|a\|_{C^\alpha} \|b\|_{C^\beta}.$$

This estimate allows us to conclude that

$$\Delta_k(\mathbf{D}^0(a, b; g, h)) \lesssim 2^{-k(\alpha+\beta+\gamma+\delta)},$$

uniformly in k , which proves the continuity result for the 4-linear operator \mathbf{D}^0 . A very similar proof gives the continuity of the simplified version of the upper iterated commutator; we leave the details to the reader. \triangleright

Theorem 4. *Let $(a, b, c; g)$ in $C^\alpha \times C^\beta \times C^\gamma \times C^{\nu_1}$, with positive regularity exponents, be given, together with $h \in C^{\nu_2}$, with possibly non-positive regularity exponent. Assume*

$$\alpha + \beta + \gamma + \nu_1 + \nu_2 \in (0, 1).$$

Then the lower iterated corrector

$$\Pi \left(\tilde{\Pi}_{a,b,c}^\downarrow(g), h \right) - \left\{ \tilde{\Pi}_{a,b}^\downarrow(c) \Pi(g, h) + \tilde{\Pi}_a(b) \Pi \left(\tilde{\Pi}_{\mathcal{E}c}(g), h \right) + a \Pi \left(\tilde{\Pi}_{\mathcal{E}\tilde{\Pi}_{\mathcal{E}b}(c)}(g), h \right) \right\} \quad (3.6)$$

defines a continuous map from $C^\alpha \times C^\beta \times C^\gamma \times C^{\nu_1} \times C^{\nu_2}$ to $C^{\alpha+\beta+\gamma+\nu_1+\nu_2}$.

Proof – To get a clear idea of the mechanics at play, we prove here a simpler statement and refer the reader to Appendix C.2 for the full proof. We work in the time-independent setting of the flat torus and prove that the formula

$$\Pi^0 \left(\Pi_{a,b,c}^{0,\downarrow}(g), h \right) - \left\{ \Pi_{a,b}^{0,\downarrow}(c) \Pi^0(g, h) + \Pi_a^0(b) \Pi^0 \left(\Pi_{\mathcal{E}c}^0(g), h \right) + a \Pi^0 \left(\Pi_{\mathcal{E}\Pi_{\mathcal{E}b}(c)}^0(g), h \right) \right\}$$

defines a continuous map from $C^\alpha \times C^\beta \times C^\gamma \times C^{\nu_1} \times C^{\nu_2}$ to $C^{\alpha+\beta+\gamma+\nu_1+\nu_2}$, under the above conditions on the regularity exponents. To see how the second term in the expansion arises, use formula (3.4) for the corrector and write

$$\begin{aligned} \left\{ \Pi^0 \left(\Pi_{a,b,c}^{0,\downarrow}(g), h \right) - \Pi_{a,b}^{0,\downarrow}(c) \Pi^0(g, h) \right\}(x) &= \mathbf{C}^0 \left(\Pi_{a,b}^{0,\downarrow}(c), g; h \right)(x) \\ &= \Pi^0 \left(\Pi_{\mathcal{E}\Pi_{a,b}^{0,\downarrow}(c)}^0(g), h \right)(x). \end{aligned}$$

Note that since

$$\Pi_a^0(b) = \left(\Pi_a^0(b) \right)(x) + \mathcal{C}\Pi_a^0(b),$$

we have the identity

$$\mathcal{C}\Pi_{a,b}^{0,\downarrow}(c) = \left(\Pi_a^0(b) \right)(x) \mathcal{C}c + \mathcal{C}\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c).$$

It follows that

$$\Pi^0\left(\Pi_{a,b,c}^{0,\downarrow}(g), h\right) = \Pi_{a,b}^{0,\downarrow}(c) \Pi^0(g, h) + \Pi_a^0(b) \Pi^0\left(\Pi_{\mathcal{C}c}^0(g), h\right) + \Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right).$$

Writing $a = a(x) + \mathcal{C}a$, in the above expression for the remainder yields that the lower iterated corrector

$$\begin{aligned} \Pi^0\left(\Pi_{a,b,c}^{0,\downarrow}(g), h\right) - \left\{ \Pi_{a,b}^{0,\downarrow}(c) \Pi^0(g, h) + \Pi_a^0(b) \Pi^0\left(\Pi_{\mathcal{C}c}^0(g), h\right) + a \Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right) \right\} \\ = \Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right) \end{aligned}$$

defines a $(\alpha + \beta + \gamma + \nu_1 + \nu_2)$ -Hölder function if $\alpha + \beta + \gamma + \nu_1 + \nu_2$ is positive.

Indeed, for every x we have

$$\begin{aligned} \Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right)(x) &\simeq \sum_k \Delta_k \left[\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g) \right](x) \Delta_k[h](x) \\ &\simeq \sum_k S_k \left[\mathcal{C}\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c) \right](x) \Delta_k[g](x) \Delta_k[h](x) \\ &\simeq \sum_k \Delta_k \left[\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c) \right](x) \Delta_k[g](x) \Delta_k[h](x), \end{aligned}$$

where we used (3.5). Iterating the reasoning, we get

$$\Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right)(x) \simeq \sum_k \Delta_k[a](x) \Delta_k[b](x) \Delta_k[c](x) \Delta_k[g](x) \Delta_k[h](x) \quad (3.7)$$

and so since $\alpha + \beta + \gamma + \nu_1 + \nu_2$ is non-negative, we conclude that

$$\begin{aligned} \left| \Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right)(x) \right| &\simeq \sum_k 2^{-k(\alpha+\beta+\gamma+\nu_1+\nu_2)} \|a\|_{C^\alpha} \|b\|_{C^\beta} \|c\|_{C^\gamma} \|g\|_{C_1^\nu} \|h\|_{C_2^\nu} \\ &\lesssim \|a\|_{C^\alpha} \|b\|_{C^\beta} \|c\|_{C^\gamma} \|g\|_{C_1^\nu} \|h\|_{C_2^\nu}, \end{aligned}$$

uniformly in x , which yields that the main quantity defines a bounded function.

Using (3.7), we can also obtain its Hölder character. For $x \neq y$, and writing m for $\|a\|_{C^\alpha} \|b\|_{C^\beta} \|c\|_{C^\gamma} \|g\|_{C_1^\nu} \|h\|_{C_2^\nu}$, we have

$$\begin{aligned} &\left| \Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right)(x) - \Pi^0\left(\Pi_{\mathcal{C}\Pi_a^0(b)}^0(c)(g), h\right)(y) \right| \\ &\lesssim \sum_k \left| \Delta_k[a](x) \Delta_k[b](x) \Delta_k[c](x) \Delta_k[g](x) \Delta_k[h](x) \right. \\ &\quad \left. - \Delta_k[a](y) \Delta_k[b](y) \Delta_k[c](y) \Delta_k[g](y) \Delta_k[h](y) \right| \\ &\lesssim m \left(\sum_{1 \leq 2^k |x-y|} 2^{-k(\alpha+\beta+\gamma+\nu_1+\nu_2)} + \sum_{1 \geq 2^k |x-y|} |x-y| 2^{k-k(\alpha+\beta+\gamma+\nu_1+\nu_2)} \right) \\ &\lesssim m |x-y|^{\alpha+\beta+\gamma+\nu_1+\nu_2}; \end{aligned}$$

in the second sum, over $1 \geq 2^k|x-y|$, we have used the finite increment theorem together with the fact that differentiating one operator Δ_k is equivalent to multiplying it by 2^k , together with the condition $(\alpha + \beta + \gamma + \nu_1 + \nu_2) \in (0, 1)$. \triangleright

The above proof contains the fact that if the regularity exponent ν_1 is allowed to be negative, and $\alpha + \beta + \nu_1 + \nu_2$ is positive, then the 4-lower iterated corrector

$$\begin{aligned} \mathbf{C}(a, b; g, h) &:= \Pi\left(\tilde{\Pi}_{a,b}^\downarrow(g), h\right) - \left\{ \tilde{\Pi}_a(b) \Pi(g, h) + a \Pi\left(\tilde{\Pi}_{\mathcal{C}b}(g), h\right) \right\} \\ &= \mathbf{C}\left(\tilde{\Pi}_a b; g, h\right) - a \mathbf{C}(b, g; h) \end{aligned} \quad (3.8)$$

defines a continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^{\nu_1} \times \mathcal{C}^{\nu_2}$ to $\mathcal{C}^{\alpha+\beta+\nu_1+\nu_2}$.

The 4 and 5-linear **upper iterated correctors** are defined by the formulas

$$\mathbf{C}(f; a, b; h) := \mathbf{C}\left(f, \tilde{\Pi}_a(b); h\right) - a \mathbf{C}(f, b; h).$$

and

$$\mathbf{C}(f; a, (b, c); h) := \mathbf{C}\left(f; a, \tilde{\Pi}_b(c); h\right) - b \mathbf{C}(f; a, c; h).$$

Theorem 5. *The following continuity results for the 4 and 5-linear upper iterated correctors holds.*

(i) *If $\alpha, \beta \in (0, 1)$, the exponents $(\alpha + \nu_1 + \nu_2)$ and $(\beta + \nu_1 + \nu_2)$ are negative and*

$$\alpha + \beta + \nu_1 + \nu_2 > 0,$$

then the 4-linear upper iterated corrector \mathbf{C} defines a continuous linear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^{\nu_1} \times \mathcal{C}^{\nu_2}$ to $\mathcal{C}^{\alpha+\beta+\nu_1+\nu_2}$.

(ii) *If $\alpha, \beta, \gamma \in (0, 1)$, the exponents $(\alpha + \nu_1 + \nu_2)$, $(\beta + \nu_1 + \nu_2)$ and $(\gamma + \nu_1 + \nu_2)$ are negative, and*

$$\alpha + \beta + \gamma + \nu_1 + \nu_2 > 0,$$

then the 5-linear upper iterated corrector \mathbf{C} defines a continuous linear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma \times \mathcal{C}^{\nu_1} \times \mathcal{C}^{\nu_2}$ to $\mathcal{C}^{\alpha+\beta+\gamma+\nu_1+\nu_2}$.

Proof – We only sketch the proof of the continuity result of the 4-linear operator in the model case of the time-independent setting of the flat torus, and rely on formula (3.1) for the diagonal operator $\Pi(\cdot, \cdot)$ for the purpose; see Proposition 27 in Appendix C.2 for a fully detailed proof in the parabolic setting. In the present setting, the quantity $\mathbf{C}^0(f; a, b; g)$ is then given by a sum of the form

$$\mathbf{C}^0(f; a, b; h) = \sum_i \varepsilon'_i \Delta_i h,$$

with

$$\varepsilon'_i = \left\{ \Delta_i(\Pi_f(\Pi_a(b))) - a \Delta_i(\Pi_f(b)) \right\} + f \left\{ a \Delta_i b - \Delta_i(\Pi_a(b)) \right\}$$

We read on the expression

$$\begin{aligned} \varepsilon'_i(x) &= \int K_i(x, y) \left\{ \Pi_f(\Pi_a(b))(y) - a(x) \Pi_f(b)(y) + (fa)(x)b(y) - f(x) \Pi_a(b)(y) \right\} dy \\ &= \int K_i(x, y) \Pi_{f-f(x)\mathbf{1}}\left(\Pi_{a-a(x)\mathbf{1}}(b)\right)(y) dy, \end{aligned}$$

that

$$\varepsilon'_i = \Delta_i\left(\Pi_{\mathcal{C}f}\left(\Pi_{\mathcal{C}a}(b)\right)\right)$$

has L^∞ -norm of order $2^{-i(\nu_1+\alpha+\beta)}$, as a consequence of (3.5). The proof is then not fully completed, since the block $\varepsilon_i' \Delta_i h$ is not perfectly localized in frequency at scale 2^i , so an extra decomposition is necessary. We do not give the details here and refer the reader to the proof of Proposition 27 in Appendix C. \triangleright

3.2 Iterated paraproducts

In addition to the above continuity results for the commutator/corrector and their iterates, we shall also need 'expansion'/continuity results for some iterated paraproducts. This requires the introduction of a notation for a particular difference operator on functions. We give here its definition in the model setting of the time-independent flat torus and refer the reader to Appendix C.2 for the description of how things work in the parabolic setting.

The value at $x \in \mathbf{T}^d$ of some paraproduct $\Pi_u v$ is a sum over the integers i of terms of the form

$$\left(\Pi_u^{0,(i)} v\right)(x) := \iint K_i(x, y) K_{\leq i-1}(x, z) u(z) v(y) dz dy.$$

We thus have for instance, for $f \in L^\infty, g \in \mathcal{C}^\nu$ and $a \in \mathcal{C}^\alpha$ with $\alpha \in (0, 1)$,

$$\begin{aligned} \left(\Pi_f^{0,(i)} \left(\Pi_a^0(g)\right) - \Pi_{fa}^{0,(i)}(g)\right)(x) &= \iint K_i(x, y) K_{\leq i-1}(x, z) f(z) \left(\Pi_{a-a(z)}^0(g)\right)(y) dz dy \\ &=: \iint K_i(x, y) K_{\leq i-1}(x, z) f(z) \Pi_{\mathcal{D}a}^0(g)(y) dz dy, \end{aligned}$$

where we have defined the **inner difference operator** \mathcal{D} ($= \mathcal{D}_z$) by the formula

$$\iint (\mathcal{D}f)(y) g(z) dz dy := \iint (f(y) - f(z)) g(z) dz dy;$$

we may also call this difference operator the *low frequency difference operator* to emphasize the fact that it acts, in the paraproduct formula, on the function that has the low frequencies. In those terms, and given the definition of the difference operator \mathcal{D} given in section A.3 in the parabolic setting, we have

$$\Pi_f^0 \left(\Pi_a^0(g)\right) - \Pi_{fa}^0(g) = \Pi_f^0 \left(\Pi_{\mathcal{D}a}^0(g)\right)$$

and, more generally,

$$\Pi_f \left(\tilde{\Pi}_a(g)\right) - \Pi_{fa}(g) = \Pi_f \left(\tilde{\Pi}_{\mathcal{D}a}(g)\right). \quad (3.9)$$

Compare this expression with the formal multiple integral, where we use the same letters to make it more striking,

$$\int f(z) d \left(\int^z a dg \right) = \int f a dg + \int f(z) d \left(\int^z (a - a(z)) dg \right).$$

Using the fact that $K_i(x, \cdot)$ has null mean (2.2), we can rewrite the preceding quantity as

$$\left(\Pi_f^{0,(i)} \left(\Pi_a^0(g)\right) - \Pi_{fa}^{0,(i)}(g)\right)(x) = \iint K_i(x, y) K_{\leq i-1}(x, z) f(z) (\mathcal{D} \Pi_{\mathcal{D}a}^0(g))(y) dz dy;$$

from which we read off the fact that

$$\mathbb{R}^0(f, a; g) := \Pi_f^0 \left(\Pi_a^0(g)\right) - \Pi_{fa}^0(g) = \Pi_f^0 \left(\Pi_{\mathcal{D}a}^0(g)\right)$$

and, more generally,

$$\mathbf{R}(f, a; g) := \Pi_f\left(\tilde{\Pi}_a(g)\right) - \Pi_{fa}(g) = \Pi_f\left(\tilde{\Pi}_{\mathcal{D}a}(g)\right)$$

are $(\alpha + \nu)$ -Hölder; so the linear map \mathbf{R} is bounded from $L^\infty \times \mathcal{C}^\alpha \times \mathcal{C}^\nu$ to $\mathcal{C}^{\alpha+\nu}$, as soon as $\alpha \in (0, 1)$ – a detailed proof is given in the parabolic setting in Appendix C.2, Proposition 23. This result can be refined if a is given under the form of a paraproduct or a modified paraproduct.

Theorem 6. *Let $f \in L^\infty$ and $g \in \mathcal{C}^\nu$ be given.*

(a) *Let also $a \in \mathcal{C}^\alpha$ and $b \in \mathcal{C}^\beta$ be given with $\alpha, \beta \in (0, 1)$. Then*

$$\begin{aligned} \mathbf{R}(f; a, b; g) &:= \Pi_f\left(\tilde{\Pi}_{\tilde{\Pi}_a(b)}(g)\right) - \Pi_{f\tilde{\Pi}_a(b)}(g) - \Pi_{fa}\left(\tilde{\Pi}_{\mathcal{D}b}(g)\right) \\ &:= \mathbf{R}\left(f, \tilde{\Pi}_a b; g\right) - \mathbf{R}(fa, b; g) \end{aligned}$$

is an element of $\mathcal{C}^{\alpha+\beta+\nu}$.

(b) *If $a \in \mathcal{C}^\alpha, b \in \mathcal{C}^\beta$ and $c \in \mathcal{C}^\gamma$ are given with $\alpha, \beta, \gamma \in (0, 1)$, then*

$$\mathbf{R}\left(f; (a, b), c; g\right) := \mathbf{R}\left(f; \tilde{\Pi}_a b, c; g\right) - \mathbf{R}(fa; b, c; g)$$

is an element of $\mathcal{C}^{\alpha+\beta+\gamma+\nu}$.

We invite the reader to right the analogues of $\Pi_f\left(\tilde{\Pi}_{\tilde{\Pi}_a(b)}(g)\right)$ and $\mathbf{R}\left(f; \tilde{\Pi}_a b, c; g\right)$ in terms of iterated integrals to build his own intuition about the above statement. The range $(0, 1)$ for the exponent α (β and γ) is dictated by the operator \mathcal{D} , which makes appear a first order increment and so can only encode regularity at order at most 1.

Proof – We prove the corresponding statement in the model time-independent setting of the flat torus. Starting from equation (3.2) with $\Pi_a(b)$ instead of a , we see that

$$\Pi_f^0\left(\Pi_{\Pi_a^0(b)}^0(g)\right) - \Pi_{f\Pi_a^0(b)}^0(g) - \Pi_{fa}^0\left(\Pi_{\mathcal{D}b}^0(g)\right) = \Pi_f^0\left(\Pi_{\mathcal{D}\Pi_a^0(b)}^0(g)\right) - \Pi_{fa}^0\left(\Pi_{\mathcal{D}b}^0(g)\right)$$

is a sum over i of double integrals

$$\begin{aligned} &\iint K_i(x, y) K_{\leq i-1}(x, z) f(z) (\mathcal{D}\Pi_{\mathcal{D}(\Pi_a^0(b)-a(z)b)}^0(g))(y) dz dy \\ &= \iint K_i(x, y) K_{\leq i-1}(x, z) f(z) (\mathcal{D}\Pi_{\mathcal{D}\Pi_a^0(b)}^0(g))(y) dz dy \end{aligned}$$

on which we read off that their L^∞ norm is of order $2^{-i(\alpha+\beta+\nu)}$. The proof is then finished, since this last quantity corresponds to the dyadic blocks $\Delta_i[\mathbf{R}(f; a, b; g)]$. \triangleright

A careful examination of the proof reveals that the following finer result holds. If $f \in \mathcal{C}^{\nu_1}$ with $\nu_1 \in (0, 1)$ then item (a) of the previous theorem can be improved to the following expansion

$$\mathbf{R}(f; a, b; g) - \Pi_f(\mathbf{R}(1; a, b; g)) \in \mathcal{C}^{\alpha+\beta+\nu+\nu_1}. \quad (3.10)$$

Beware that the notation $\Pi_{fa}\left(\tilde{\Pi}_{\mathcal{D}b}(g)\right)$ may be a bit misleading, as the function $\tilde{\Pi}_{\mathcal{D}b}(g)$ that appears in this formula is a function of two variables, one of which being the (parabolic equivalent of the) z that is integrated in the integral formula in $dzdy$ defining the i^{th} -term of the paraproduct sum.

Theorem 7. *Let $f \in L^\infty$ and $g \in \mathcal{C}^\nu$ be given. Let also $a \in \mathcal{C}^\alpha$ and $b \in \mathcal{C}^\beta$ be given with $\alpha, \beta \in (0, 1)$. Then*

$$l(f, a, b; g) := \Pi_f\left(\tilde{\Pi}_a(\tilde{\Pi}_b g)\right) - \left\{ \Pi_{fabg} + \Pi_{fa}(\tilde{\Pi}_{\mathcal{D}b}g) + \Pi_f\left(\tilde{\Pi}_{b\mathcal{D}(a)}g\right) \right\}$$

is an element of $\mathcal{C}^{\alpha+\beta+\nu}$.

Here again, we invite the reader to right the analogue of $\Pi_f\left(\tilde{\Pi}_a(\tilde{\Pi}_b g)\right)$ in terms of iterated integrals to built his own intuition about the above statement. Observe that $l(f, a, 1; g) = 0$ as a consequence of the defining relation (3.9).

Proof – Let us prove the statement in the model setting of the time-independent flat torus, with Π operators used in place of $\tilde{\Pi}$. In that case, a dyadic bloc $\Delta_k[l(f, a, b; g)]$ is given by

$$\begin{aligned} \Delta_k[l(f, a, b; g)](x) &= \Delta_k[g](x) \left\{ S_k[b](x) S_k[a](x) S_k[f](x) - S_k[abf](x) \right. \\ &\quad \left. - S_k[fa](x) S_k[\mathcal{D}b](x) - S_k[f](x) S_k[b\mathcal{D}(a)](x) \right\}. \end{aligned}$$

Using the normalization $S_k(1) = 1$, we obtain

$$\Delta_k[l(f, a, b; g)](x) = \Delta_k[g](x) I(x)$$

with

$$\begin{aligned} I(x) &:= \iiint K_{\leq k-1}(x, z_1) K_{\leq k-1}(x, z_2) K_{\leq k-1}(x, z_3) \left\{ b(z_1) a(z_2) f(z_3) - a(z_3) b(z_3) f(z_3) \right. \\ &\quad \left. - a(z_3) f(z_3) (b(z_1) - b(z_3)) - f(z_3) (a(z_2) - a(z_3)) b(z_2) \right\} dz_1 dz_2 dz_3 \\ &= \iiint K_{\leq k-1}(x, z_1) K_{\leq k-1}(x, z_2) K_{\leq k-1}(x, z_3) f(z_3) \\ &\quad \left\{ b(z_1) a(z_2) - a(z_3) b(z_1) - a(z_2) b(z_2) + a(z_3) b(z_2) \right\} dz_1 dz_2 dz_3. \end{aligned}$$

Since a and b have a positive regularity, we deduce that

$$\begin{aligned} \left| b(z_1) a(z_2) - a(z_3) b(z_1) - a(z_2) b(z_2) + a(z_3) b(z_2) \right| &= |a(z_2) - a(z_3)| |b(z_1) - b(z_3)| \\ &\lesssim \max(|z_2 - z_3|, |z_1 - z_3|)^{\alpha+\beta} \|a\|_{\mathcal{C}^\alpha} \|b\|_{\mathcal{C}^\beta} \end{aligned}$$

and so

$$\begin{aligned} \left\| \Delta_k[l(f, a, b; g)] \right\|_{L^\infty} &\lesssim \|\Delta_k g\|_{L^\infty} \|f\|_{L^\infty} 2^{-k(\alpha+\beta)} \|a\|_{\mathcal{C}^\alpha} \|b\|_{\mathcal{C}^\beta} \\ &\lesssim 2^{-k(\alpha+\beta+\nu)} \|f\|_{L^\infty} \|g\|_{\mathcal{C}^\nu} \|a\|_{\mathcal{C}^\alpha} \|b\|_{\mathcal{C}^\beta}, \end{aligned}$$

which concludes the proof. \triangleright

Our last ingredient is a continuity result for the commutator of two para-products, and their iterates. The result stated below in Theorem 8 is fully proved in Appendix C.2. Given bounded functions u, a, b, c, g, f , we define the **modified commutator on para-products** and its iterates by the formulae

$$\mathsf{T}_u(g, f) := \Pi_u\left(\tilde{\Pi}_g(f)\right) - \Pi_g\left(\Pi_u(f)\right),$$

and

$$\mathsf{T}_u(a, b, f) := \mathsf{T}_u\left(\tilde{\Pi}_a(b), f\right) - \Pi_a\left(\mathsf{T}_u(b, f)\right)$$

and

$$\mathbb{T}_u(a, b, c, f) := \mathbb{T}_u(\tilde{\Pi}_a(b), c, f) - \Pi_a(\mathbb{T}_u(b, c, f)).$$

The continuity properties of these operators are given in the following statement.

- Theorem 8.** (a) *Let α, β, γ be Hölder regularity exponents with $\alpha \in \mathbb{R}$, $\beta \in (0, 1)$ and $\gamma \in (-\infty, 0)$ and set $\delta := \alpha + \beta + \gamma$. Then the commutator defines a trilinear continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to \mathcal{C}^δ .*
- (b) *Let $\alpha, \beta, \gamma, \nu$ be Hölder regularity exponents with $\alpha \in \mathbb{R}$, $\beta, \gamma \in (0, 1)$ and $\nu \in (-\infty, 0)$ and set $\delta := \alpha + \beta + \gamma + \nu$. Then the commutator defines a trilinear continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma \times \mathcal{C}^\nu$ to \mathcal{C}^δ . A similar result holds for the 5-linear operator.*

Let us mention here that all the continuity results of section also hold true when we replace $\tilde{\Pi}$ by Π ; the corresponding operators will also be denoted by the same letter as the setting will make the situation clear.

Together with the results on the pair of paraproducts $(\Pi, \tilde{\Pi})$ proved in [2], the Taylor expansion formula of section 2 and the above continuity results provide the technical basis needed to run the paracontrolled analysis of a generic equation of type (1.1), along the lines described in section 1.1. Rather than providing the reader with a general statement identifying a class of equations that can be solved within our setting, we concentrate on what seems to us to be two typical and interesting examples, the study of the 2^+ and 3-dimensional generalised parabolic Anderson model equation (gPAM), and the study of the generalized KPZ equation. Both examples are out of reach of the Gubinelli-Imkeller-Perkowski first order paracontrolled calculus. We find it reasonable to proceed this way in so far as a systematic approach of singular stochastic PDEs requires the development of a systematic approach to renormalisation problems which is still under study in the present setting, and which is only almost achieved within the setting of regularity structures at the time of writing.

4 Nonlinear singular PDEs: a case study (gPAM)

Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a function of class C^3 , with bounded third derivative. We aim here to make sense of, and solve uniquely, the equation

$$(\partial_t + L)u = f(u)\zeta \tag{4.1}$$

in a high order paracontrolled setting, for a spatial 'noise' ζ that is $(\alpha - 2)$ -space-Hölder. For $\alpha \geq \frac{2}{3}$, the first order original formulation of paracontrolled calculus is sufficient for solving equation (4.1); see Gubinelli-Imkeller-Perkowski' seminal work [13] or [1]. We deal with the range of exponents $\frac{1}{2} < \alpha \leq \frac{2}{3}$ in sections 4.1, 4.2 and 4.3, and deal with the range $\frac{2}{5} < \alpha \leq \frac{1}{2}$ in section 4.5 – the latter range of exponents corresponds to the irregularity of space white noise in dimension 3, or space-time white noise in dimension 1. Note that for $\frac{1}{2} < \alpha \leq \frac{2}{3}$ we have $0 < 4\alpha - 2 \leq \alpha$.

We set up the equation in a paracontrolled setting where the spacial distribution ζ is enhanced into a time-space rough distribution $\tilde{\zeta} = (\zeta, \dots)$. The components of this extended 'noise' will appear along the computations done below to give sense to the equation. Write $\mathcal{R} = (\partial_t + L)^{-1}$ for the resolution operator, and set

$$Z_1 := \mathcal{R}(\tilde{\zeta}).$$

Recall that \mathcal{R} sends any space $L_T^\infty C^{\beta-2}$ into C^β , for any β in the interval $(0, 2)$ – see for instance proposition 10 in [2], and notice that $L_T^\infty C^{\beta-2} \subset C^{\beta-2}$ in that case.

We take as a **solution space for equation (4.1)** the set of functions u satisfying the following **second order paracontrolled ansatz**

$$\begin{aligned} u &= \tilde{\Pi}_{u_1}(Z_1) + \tilde{\Pi}_{u_2}(Z_2) + u^\sharp \\ u_1 &= \tilde{\Pi}_{u_{11}}(Z_1) + u_1^\sharp, \end{aligned} \quad (4.2)$$

with $Z_1 = \mathcal{R}(\zeta)$, with 'derivatives' u_1, u_2, u_{11} in C^α , and remainders u^\sharp and u_1^\sharp in $C^{3\alpha}$ and $C^{2\alpha}$ respectively. The functions Z_2 , possibly equal to a tuple (Z_2^1, Z_2^2, \dots) , are constructed from the enhanced noise $\hat{\zeta}$, and are 2α -Hölder continuous. The notation u_2 may stand for a tuple (u_2^1, u_2^2, \dots) , if Z_2 does, in which case the expression $\tilde{\Pi}_{u_2}(Z_2)$ involves an implicit sum.

Our first task is to make sense of the product $f(u)\zeta$ for functions u with the above second order paracontrolled structure; this is where we use the continuity results proved in sections 2 and 3.1. We want for that purpose to give a description of $f(u)\zeta$ under the form

$$f(u)\zeta = \Pi_{f(u)}(\zeta) + \Pi_{v_2}(Y_2) + \Pi_{v_3}(Y_3) + (\sharp), \quad (4.3)$$

up to some remainder term (\sharp) in $C^{4\alpha-2}$, and for some distributions $Y_2 = (Y_2^1, Y_2^2, \dots)$ in $L_T^\infty C^{2\alpha-2}$, $Y_3 = (Y_3^1, \dots)$ in $L_T^\infty C^{3\alpha-2}$, built from the rough distribution $\hat{\zeta}$, and some functions v_2, v_3 of positive regularity, constructed from u, u_1, u_2, u_{11} . It will follow from the defining intertwining relation

$$\mathcal{R}(\Pi_{a_1} a_2) = \tilde{\Pi}_{a_1}(\mathcal{R}a_2)$$

relating Π and $\tilde{\Pi}$, together with the Schauder and the continuity estimates for $\tilde{\Pi}$ proved in [2] that it will make sense to consider terms of the form $\Pi_{v_3}(Y_3)$ and (\sharp) as remainders in the computations to follow. Recall that the model functions Z_i will be defined as $Z_1 = \mathcal{R}(\zeta)$ and $Z_i = \mathcal{R}(Y_i)$ for $i \geq 2$. Writing

$$u = \mathcal{R}(f(u)\zeta) + e^{-\tau L}u_0,$$

that is

$$\tilde{\Pi}_{u_1}(Z_1) + \tilde{\Pi}_{u_2}(Z_2) + u^\sharp = \tilde{\Pi}_{f(u)}(Z_1) + \tilde{\Pi}_{v_2}(Z_2) + \left(\tilde{\Pi}_{v_3}(Z_3) + \mathcal{R}(\sharp) + e^{-\tau L}u_0 \right),$$

will allow us to set up a fixed point problem for (u, u_1, u_2, u_{11}) , and solve it by Banach contraction principle on a small time interval.

4.1 Enhanced distribution

The archetype of equation (4.1) is given by the controlled ordinary differential equation

$$dx_t = V(x_t)dh_t, \quad (4.4)$$

where h is a non-differentiable \mathbf{R}^ℓ -valued control and V an $L(\mathbf{R}^\ell, \mathbf{R}^d)$ -valued one form on \mathbf{R}^d , say. Think of a Brownian path for the control h . One of the deepest insights of T. Lyons in his theory of rough paths [22] was to understand that one needs to change the notion of control to make sense of such an equation, and that this enhanced control takes values in a very specific universal algebraic structure. In simple terms, the enhanced control consists of h and the collection of a number of objects playing the role of the non-existing iterated integrals $\int_{s \leq s_1 \leq \dots \leq s_k \leq t} dh_{s_1} \otimes \dots \otimes dh_{s_1}$ – such

iterated integrals cannot be defined as continuous functions of their integrands, here (h, \dots, h) , if h is not sufficiently regular; see proposition 1.29 in [23]. Once given these extra data, one can make sense of, and solve uniquely, the controlled ordinary differential equation (4.4) under some appropriate regularity conditions on the one form V , and the solution path happens to be a continuous function of the enhanced control, in some appropriate topology. The enhancement of the control cannot be made on a purely analytic basis and requires some extra input, typically the use of probabilistic methods when the control h is random.

Hairer's theory of regularity structures provides a conceptually close framework for the study of a class of singular partial differential equations containing equation (4.1) as a particular case. To make sense of equation (4.1), one needs to enhance the distribution ζ with the a priori datum of a number of other distributions. Contrary to the case of the controlled ordinary differential equation (4.4), this enhanced 'control' takes values in an equation-dependent algebraic structure. The solving process is also different, as the equation is first recast in some abstract space of jets of solutions, where it can be solved under appropriate conditions. This corresponds to looking for a solution in a specific space of distributions where one can actually make sense of all the terms in the equation, especially some a priori undefined products. A fundamental tool, the reconstruction operator, allows then to associate to this abstract solution a classical distribution. The equation-dependent algebraic structure in which the enhanced distribution lives also allows to give sense to this solution distribution as a limit of solutions to some family of classically well-posed equations in which the distribution ζ has been smoothened. The latter point is related to renormalisation matters.

The setting which we develop in the present work shares some common features with Lyons' theory of rough paths and Hairer's theory of regularity structures.

- One needs a notion of enhanced distribution to make sense of the equation.
- This enhancement cannot be made on a purely analytic basis, and requires the use of probabilistic tools when ζ is random.
- Our solutions are described by some kind of Taylor expansion; this is the paracontrolled ansatz (1.2), here (4.2), which defines at the same time the restricted space of functions/distributions where one looks for a solution to the problem.

However, this 'local' description of a possible solution is of a different analytical nature from Hairer's notion of modelled distribution; it is in particular a classically well-defined distribution/function that is defined everywhere in time-space. There is no need as a consequence to rephrase the problem in any abstract space of jets, and the paracontrolled analysis of equation (4.1), or any other singular PDE, is made 'downstairs' with classical objects. Let $\frac{1}{2} < \alpha \leq \frac{2}{3}$, and a finite time interval $[0, T]$, be given.

Definition. *We define the space of enhanced distributions for equation (4.1) as the space*

$$C^{\alpha-2} \times \left(L_T^\infty C^{2\alpha-2} \right)^2 \times \left(L_T^\infty C^{3\alpha-2} \right)^8,$$

and denote by $\hat{\zeta}$ a generic element of that space.

As said above, the elements of this enhanced distribution represent some quantities that are needed to make sense of all the terms of equation (4.1), and that either

one cannot define on a purely analytic basis when ζ is not regular enough or that need to be assumed to be slightly more regular than what analysis gives for free from their expressions . With a smooth ζ , and

$$Z_1 = \mathcal{R}(\zeta),$$

set $\widehat{\zeta} = \left(\zeta, \zeta_1^{(2)}, \zeta_2^{(2)}, (\zeta_i^{(3)})_{i=1..8} \right)$, with

$$\begin{aligned} \zeta_1^{(2)} &:= \Pi(Z_1, \zeta), & \zeta_2^{(2)} &:= \Pi_\zeta Z_1 \\ Y_2 &:= \zeta_1^{(2)} + \zeta_2^{(2)}, & Z_2 &:= \mathcal{R}(Y_2), \end{aligned}$$

and

$$\begin{aligned} \zeta_1^{(3)} &:= \Pi(Z_2, \zeta), & \zeta_2^{(3)} &:= \mathcal{C}(Z_1, Z_1, \zeta), \\ \zeta_3^{(3)} &:= \Pi(\Pi(Z_1, Z_1), \zeta), & \zeta_4^{(3)} &:= \Pi(Z_1, \Pi(Z_1, \zeta)) \\ \zeta_5^{(3)} &:= \mathcal{T}_\zeta(Z_1, Z_1), & \zeta_6^{(3)} &:= \Pi_\zeta(\Pi_{\mathcal{D}Z_1} Z_1), \\ \zeta_7^{(3)} &:= \Pi_\zeta Z_2, & \zeta_8^{(3)} &:= \Pi_\zeta \Pi(Z_1, Z_1). \end{aligned} \tag{4.5}$$

Observe that the last terms $\zeta_i^{(3)}$ (for $i = 4, \dots, 8$) are well-defined and have an analytic sense in $C^{3\alpha-2}$; we need however to assume them well-defined in $L_T^\infty C^{3\alpha-2}$.

One now shows that one can make good sense of the product $f(u)\zeta$, and that it has an expression of the form (4.3), provided one replaces the occurrence of the above quantities in its expansion when ζ is smooth by the above a priori given $\zeta_i^{(j)}$'s, when ζ is only an element of $C^{\alpha-2}$. Note that one adds inside the enhanced distributions those quantities that one needs to make sense of the products

$$Z_1\zeta, Z_1^2\zeta, \mathcal{R}(Z_1\zeta)\zeta,$$

in accordance with what one expects from the theory of regularity structures. The fact that each ill-posed product above is decomposed into three terms in the para-product picture explains why our space of enhanced distributions contains so many elements; there is nothing annoying in that fact. (Note here that, as far as renormalisation matters are concerned, we expect that robust tools that are currently being developed for the study of renormalisation within the theory of regularity structures, by Hairer and his co-authors, to be usable in our paracontrolled setting as well, up to some ad hoc modification.)

4.2 Analysis of the product $f(u)\zeta$.

We start from the paraproduct decomposition, which gives

$$f(u)\zeta = \Pi_{f(u)}\zeta + \Pi_\zeta(f(u)) + \Pi(f(u), \zeta);$$

the first term on the right hand side suits us. We shall use along the way the notation

$$a(u) := f'(u) - uf^{(2)}(u)$$

for this expression of u that appears in the Taylor expansion formula for $f(u)$ in Theorem 1,

$$\begin{aligned} f(u) &= \Pi_{f'(u)-f^{(2)}(u)u}u + \frac{1}{2}\Pi_{f^{(2)}(u)}(u^2) + (3\alpha) \\ &= \Pi_{a(u)}u + \Pi_{f^{(2)}(u)}(\Pi_u u) + \frac{1}{2}\Pi_{f^{(2)}(u)}(\Pi(u, u)) + (3\alpha). \end{aligned} \tag{4.6}$$

Here and below, a term (β) stands for some element in \mathcal{C}^β that depends in a locally Lipschitz way on $u \in \mathcal{C}^\alpha$ – with polynomial dependence on u for the Lipschitz constant. Let first use this Taylor expansion for $f(u)$ to rewrite $f(u)\zeta$ under the form

$$\begin{aligned} & \Pi_{f(u)}\zeta + \left\{ \Pi_\zeta(\Pi_{a(u)}u) + \Pi_\zeta(\Pi_{f^{(2)}(u)}(\Pi_u u)) + \frac{1}{2} \Pi_\zeta(\Pi_{f^{(2)}(u)}(\Pi(u, u))) + \Pi_\zeta(3\alpha) \right\} \\ & + \left\{ \Pi(\Pi_{a(u)}u, \zeta) + \Pi(\Pi_{f^{(2)}(u)}(\Pi_u u), \zeta) + \frac{1}{2} \Pi(\Pi_{f^{(2)}(u)}(\Pi(u, u)), \zeta) + \Pi((3\alpha), \zeta) \right\} \end{aligned}$$

The following intermediate analysis of this expression will be useful in section 4.4 to analyse the dynamical consequences of renormalisation.

Lemma 9. *Let ζ be a continuous function, and let u , or rather $\hat{u} = (u_1^\#, u_2^\#; u_{11}, u_2)$, be a function satisfying the second order paracontrolled ansatz (4.2). Then one can write the product $f(u)\zeta$ under the form*

$$\begin{aligned} f(u)\zeta &= \Pi_{f(u)}\zeta + \Pi_\zeta f(u) + f'(u)u_1 \Pi(Z_1, \zeta) + \left(f'(u)u_{11} + f^{(2)}(u)u_1^2 \right) \mathbf{C}(Z_1, Z_1; \zeta) \\ & \quad + f'(u)u_2 \Pi(Z_2, \zeta) + \frac{1}{2} f^{(2)}(u)u_1^2 \Pi(\Pi(Z_1, Z_1), \zeta) + (\#) \\ & =: \Pi_{f(u)}\zeta + \Pi_\zeta f(u) + \mathbf{F}(\hat{u})\hat{\zeta} + (\#), \end{aligned} \tag{4.7}$$

for some remainder $(\#)$ in $\mathcal{C}^{4\alpha-2}$, that is a continuous function of $u \in \mathcal{C}^\alpha$ and ζ , seen as an element of $L_T^\infty \mathcal{C}^{\alpha-2}$ – the remainder $(\#)$ is in particular of positive Hölder regularity since $\alpha > \frac{1}{2}$.

Proof – We provide more details than necessary as this is the first time that we see the corrector and its iterates in action. We use the term $(\#)$ as in the statement, with different expressions at every occurrence. Let us focus on studying the resonant part $\Pi(f(u), \zeta)$ and use identity (4.6) and the correctors \mathbf{C} to get

$$\begin{aligned} \Pi(\Pi_{a(u)}u, \zeta) &= a(u)\Pi(u, \zeta) + \mathbf{C}(a(u), u; \zeta) \\ &= a(u) \left\{ u_1 \Pi(Z_1, \zeta) + \mathbf{C}(u_1, Z_1; \zeta) + u_2 \Pi(Z_2, \zeta) + \mathbf{C}(u_2, Z_2; \zeta) + \Pi(u^\#, \zeta) \right\} \\ & \quad + \mathbf{C}(a(u), u; \zeta). \end{aligned}$$

We analyze successively the different terms. First $u^\# \in \mathcal{C}^{3\alpha}$ so $\Pi(u^\#, \zeta) \in \mathcal{C}^{4\alpha-2}$, since $4\alpha - 2 > 0$, and this term goes into the remainder $(\#)$. Then, from the ansatz for u , we have

$$\begin{aligned} \mathbf{C}(u_1, Z_1; \zeta) &= \mathbf{C}(\tilde{\Pi}_{u_{11}} Z_1, Z_1; \zeta) + \mathbf{C}((2\alpha), Z_1; \zeta) \\ &= u_{11} \mathbf{C}(Z_1, Z_1; \zeta) + \mathbf{C}(u_{11}, Z_1; Z_1, \zeta) + (4\alpha - 2) \\ &= u_{11} \mathbf{C}(Z_1, Z_1; \zeta) + (4\alpha - 2), \end{aligned}$$

where we used Theorems 2 and 4 on the boundedness of \mathbf{C} and its iterates, equation (3.8). So it comes

$$\begin{aligned} \Pi(\Pi_{a(u)}u, \zeta) &= a(u)u_1 \Pi(Z_1, \zeta) + a(u)u_{11} \mathbf{C}(Z_1, Z_1; \zeta) + a(u)u_2 \Pi(Z_2, \zeta) \\ & \quad + \mathbf{C}(a(u), u; \zeta) + (\#). \end{aligned}$$

For the last commutator in the right hand side of the above equation, we use the ansatz for u to get first

$$\begin{aligned}\mathbf{C}(a(u), u; \zeta) &= \mathbf{C}\left(a(u), \tilde{\Pi}_{u_1} Z_1; \zeta\right) + \mathbf{C}(a(u), (2\alpha); \zeta) \\ &= u_1 \mathbf{C}(a(u), Z_1; \zeta) + \mathbf{C}(a(u); u_1, Z_1; \zeta) + (4\alpha - 2) \\ &= u_1 \mathbf{C}(a(u), Z_1; \zeta) + (4\alpha - 2);\end{aligned}$$

we used the boundedness of the upper iterated commutator, Theorem 5. We can now also parilinearize $a(u)$, with Theorem 1, and by (3.8), it comes

$$\begin{aligned}\mathbf{C}(a(u), u; \zeta) &= u_1 \mathbf{C}(\Pi_{a'(u)}(u), Z_1; \zeta) + (4\alpha - 2) \\ &= u_1 a'(u) \mathbf{C}(u, Z_1; \zeta) + (4\alpha - 2) \\ &= u_1^2 a'(u) \mathbf{C}(Z_1, Z_1; \zeta) + (4\alpha - 2)\end{aligned}$$

At the end, putting these estimates together yields

$$\begin{aligned}\Pi\left(\Pi_{a(u)} u, \zeta\right) &= a(u) u_1 \Pi(Z_1, \zeta) + a(u) u_{11} \mathbf{C}(Z_1, Z_1; \zeta) + a(u) u_2 \Pi(Z_2, \zeta) \\ &\quad + a'(u) u_1^2 \mathbf{C}(Z_1, Z_1; \zeta) + (\#)\end{aligned}$$

and similarly

$$\begin{aligned}\Pi\left(\Pi_{f^{(2)}(u)}(\Pi_u u), \zeta\right) &= f^{(2)}(u) \Pi(\Pi_u u, \zeta) + \mathbf{C}\left(f^{(2)}(u), \Pi_u u; \zeta\right) \\ &= f^{(2)}(u) \left\{ u \Pi(u, \zeta) + \mathbf{C}(u, u; \zeta) \right\} + f^{(3)}(u) u_1^2 u \mathbf{C}(Z_1, Z_1; \zeta) + (\#) \\ &= f^{(2)}(u) \left\{ u u_1 \Pi(Z_1, \zeta) + u u_{11} \mathbf{C}(Z_1, Z_1; \zeta) + u u_2 \Pi(Z_2, \zeta) + (\#) \right. \\ &\quad \left. + u_1^2 \mathbf{C}(Z_1, Z_1; \zeta) + (\#) \right\} + f^{(3)}(u) u_1^2 u \mathbf{C}(Z_1, Z_1; \zeta) + (\#)\end{aligned}$$

and

$$\frac{1}{2} \Pi\left(\Pi_{f^{(2)}(u)} \Pi(u, u), \zeta\right) = \frac{1}{2} f^{(2)}(u) u_1^2 \Pi(\Pi(Z_1, Z_1), \zeta) + (\#).$$

These three identities together give the statement of the lemma. \triangleright

Note that the only term that does not make obvious sense analytically in the decomposition (4.7), given the regularity of the different components of the enhanced distribution $\hat{\zeta}$, is the term $f'(u) u_1 \Pi(Z_1, \zeta)$. To analyse it, note that

$$\begin{aligned}f'(u) u_1 &= \Pi_{f'(u)} u_1 + \Pi_{u_1}(f'(u)) + (2\alpha) \\ &= \Pi_{f'(u)}\left(\tilde{\Pi}_{u_{11}} Z_1\right) + \Pi_{u_1}\left(\Pi_{f^{(2)}(u)}(\Pi_{u_1} Z_1)\right) + (2\alpha) \\ &= \Pi_{f'(u) u_{11} + f^{(2)}(u) u_1^2} Z_1 + (2\alpha),\end{aligned}$$

Hence, one has

$$\begin{aligned}f'(u) u_1 \Pi(Z_1, \zeta) &= \Pi_{f'(u) u_1} \Pi(Z_1, \zeta) + \Pi_{\Pi(Z_1, \zeta)}(f'(u) u_1) + \Pi(f'(u) u_1, \Pi(Z_1, \zeta)) \\ &= \Pi_{f'(u) u_1} \Pi(Z_1, \zeta) + \Pi_{\Pi(Z_1, \zeta)}(f'(u) u_1) \\ &\quad + \left(f'(u) u_{11} + f^{(2)}(u) u_1^2\right) \Pi(Z_1, \Pi(Z_1, \zeta)) + (4\alpha - 2),\end{aligned}$$

from which it appears as a well-defined element of $\mathcal{C}^{2\alpha-2}$.

Proposition 10. *One can decompose the product $f(u)\zeta$ in canonical form*

$$f(u)\zeta = \Pi_{f(u)}\zeta + \Pi_{f'(u)u_1} \left(\Pi(Z_1, \zeta) + \Pi_\zeta Z_1 \right) + \Pi_{v_3} Y_3 + (4\alpha - 2),$$

where the distributions $Y_3 = (Y_3^1, \dots)$ belong to $L_T^\infty C^{3\alpha-2}$, and $v_3 \in C^\alpha$, for some remainder term $(4\alpha - 2)$ in $C^{4\alpha-2}$, whose norm depends polynomially on \hat{u} and $\hat{\zeta}$.

Proof – Given the result of lemma 9 and the fact that

$$f'(u)u_1 = \Pi_{f'(u)u_{11}+f'(u)u_1^2} Z_1 + (2\alpha),$$

we already know that

$$\begin{aligned} f(u)\zeta &= \Pi_{f(u)}\zeta + \Pi_\zeta f(u) + \left\{ \Pi_{f'(u)u_1} \Pi(Z_1, \zeta) \right. \\ &\quad \left. + \Pi_{f'(u)u_{11}+f^{(2)}(u)u_1^2} \left(\Pi_{\Pi(Z_1, \zeta)} Z_1 + \Pi(Z_1, \Pi(Z_1, \zeta)) \right) + (4\alpha - 2) \right\} \\ &\quad + \left(f'(u)u_{11} + f^{(2)}(u)u_1^2 \right) \mathcal{C}(Z_1, Z_1; \zeta) \\ &\quad + f'(u)u_2 \Pi(Z_2, \zeta) + \frac{1}{2} f^{(2)}(u)u_1^2 \Pi(\Pi(Z_1, Z_1), \zeta) + (\#) \\ &= \Pi_{f(u)}\zeta + \Pi_\zeta f(u) + \Pi_{f'(u)u_1} \Pi(Z_1, \zeta) \\ &\quad + \Pi_{f'(u)u_{11}+f^{(2)}(u)u_1^2} \left(\Pi_{\Pi(Z_1, \zeta)} Z_1 + \Pi(Z_1, \Pi(Z_1, \zeta)) + \mathcal{C}(Z_1, Z_1; \zeta) \right) \\ &\quad + \Pi_{f'(u)u_2} \Pi(Z_2, \zeta) + \frac{1}{2} \Pi_{f^{(2)}(u)u_1^2} \left(\Pi(\Pi(Z_1, Z_1), \zeta) \right) + (4\alpha - 2). \end{aligned}$$

It suffices then to decompose the paraproduct $\Pi_\zeta f(u)$ in canonical form to prove the statement of the proposition. Building on the second order Taylor formula (4.6), this is done first by putting each of the terms $\Pi_{a(u)}u$, $\Pi_{f^{(2)}(u)}(\Pi_u u)$ and $\Pi_{f^{(2)}(u)}\Pi(u, u)$ in canonical form, and then commuting the paraproducts with the operator Π_ζ , using the continuity results on the operator \mathbb{T} given in Theorem 8. One has first

$$\begin{aligned} \Pi_{a(u)}u &= \Pi_{a(u)} \left(\tilde{\Pi}_{u_1} Z_1 \right) + \Pi_{a(u)} \left(\tilde{\Pi}_{u_2} Z_2 \right) + (3\alpha) \\ &= \Pi_{a(u)} \left(\tilde{\Pi}_{u_1} Z_1 \right) + \Pi_{a(u)u_2} Z_2 + (3\alpha), \end{aligned}$$

Using Theorem 6 on the continuity of the iterates of \mathbb{R} , we have

$$\begin{aligned} \Pi_{a(u)} \left(\tilde{\Pi}_{u_1} Z_1 \right) &= \Pi_{a(u)u_1} Z_1 + \mathbb{R} \left(a(u), u_1; Z_1 \right) \\ &= \Pi_{a(u)u_1} Z_1 + \mathbb{R} \left(a(u), \tilde{\Pi}_{u_{11}} Z_1; Z_1 \right) + (3\alpha) \\ &= \Pi_{a(u)u_1} Z_1 + \mathbb{R} \left(a(u)u_{11}, Z_1; Z_1 \right) + (3\alpha) \\ &= \Pi_{a(u)u_1} Z_1 + \Pi_{a(u)u_{11}} (\Pi_{\mathcal{D}Z_1} Z_1) + (3\alpha), \end{aligned}$$

using again identity (3.9) at the last line. We thus have

$$\Pi_{a(u)}u = \Pi_{a(u)u_1} Z_1 + \Pi_{a(u)u_{11}} (\Pi_{\mathcal{D}Z_1} Z_1) + \Pi_{a(u)u_2} Z_2 + (3\alpha)$$

at that point. A similar reasoning gives

$$\begin{aligned} \Pi_{f^{(2)}(u)}(\Pi_u u) &= \Pi_{f^{(2)}(u)} \left(\Pi_{uu_1} Z_1 + \Pi_{uu_{11}} (\Pi_{\mathcal{D}Z_1} Z_1) + \Pi_{uu_2} Z_2 + (3\alpha) \right) \\ &= \Pi_{f^{(2)}(u)uu_1} Z_1 + \Pi_{f^{(2)}(u)(u_1^2+2uu_{11})} (\Pi_{\mathcal{D}Z_1} Z_1) + \Pi_{f^{(2)}(u)uu_2} Z_2 + (3\alpha) \end{aligned}$$

and

$$\begin{aligned}\Pi_{f^{(2)}(u)}(\Pi(u, u)) &= \Pi_{f^{(2)}(u)}\left(\Pi_{u_1^2}\Pi(Z_1, Z_1) + (3\alpha)\right) \\ &= \Pi_{f^{(2)}(u)u_1^2}\Pi(Z_1, Z_1) + (3\alpha).\end{aligned}$$

So one can rewrite the Taylor formula for $f(u)$ (equation 4.6) under the form

$$\begin{aligned}f(u) &= \Pi_{f'(u)u_1}Z_1 + \Pi_{(f'(u)+uf^{(2)}(u))u_{11}+f^{(2)}(u)u_1^2}(\Pi_{\mathcal{D}Z_1}Z_1) \\ &\quad + \Pi_{f'(u)u_2}Z_2 + \frac{1}{2}\Pi_{f^{(2)}(u)u_1^2}\Pi(Z_1, Z_1) + (3\alpha).\end{aligned}$$

Using the continuity result on the operator \mathbb{T} , one then gets the decomposition

$$\begin{aligned}\Pi_{\zeta}f(u) &= \Pi_{\zeta}\left\{\Pi_{f'(u)u_1}Z_1 + \Pi_{(f'(u)+uf^{(2)}(u))u_{11}+f^{(2)}(u)u_1^2}(\Pi_{\mathcal{D}Z_1}Z_1)\right. \\ &\quad \left.+ \Pi_{f'(u)u_2}Z_2 + \frac{1}{2}\Pi_{f^{(2)}(u)u_1^2}\Pi(Z_1, Z_1)\right\} + (4\alpha - 2) \\ &= \Pi_{\zeta}\left(\Pi_{f'(u)u_1}Z_1\right) + \Pi_{(f'(u)+uf^{(2)}(u))u_{11}+f^{(2)}(u)u_1^2}\left(\Pi_{\zeta}(\Pi_{\mathcal{D}Z_1}Z_1)\right) \\ &\quad + \Pi_{f'(u)u_2}(\Pi_{\zeta}Z_2) + \frac{1}{2}\Pi_{f^{(2)}(u)u_1^2}\left(\Pi_{\zeta}\Pi(Z_1, Z_1)\right) + (4\alpha - 2) \\ &= \Pi_{\zeta}(\Pi_{f'(u)u_1}Z_1) + \Pi_{v_3}Y_3' + (4\alpha - 2),\end{aligned}$$

for some distributions $Y_3' \in L_T^{\infty}C^{3\alpha-2}$. It remains to explain the decomposition of the first term in the right hand side of the above identity. We use again the commutator \mathbb{T} and its iterates to write

$$\begin{aligned}\Pi_{\zeta}\left(\Pi_{f'(u)u_1}Z_1\right) &= \Pi_{f'(u)u_1}(\Pi_{\zeta}Z_1) + \mathbb{T}_{\zeta}(f'(u)u_1, Z_1) \\ &= \Pi_{f'(u)u_1}(\Pi_{\zeta}Z_1) + \mathbb{T}_{\zeta}(\Pi_{u_1}\Pi_{f^{(2)}(u)}\Pi_{u_1}Z_1, Z_1) \\ &\quad + \mathbb{T}_{\zeta}(\Pi_{f'(u)}\Pi_{u_{11}}Z_1, Z_1) + (4\alpha - 2) \\ &= \Pi_{f'(u)u_1}(\Pi_{\zeta}Z_1) + \Pi_{u_1^2f^{(2)}(u)+u_11f'(u)}\mathbb{T}_{\zeta}(Z_1, Z_1) + (4\alpha - 2).\end{aligned}$$

So at the end, we conclude to

$$\Pi_{\zeta}f(u) = \Pi_{f'(u)u_1}(\Pi_{\zeta}Z_1) + \Pi_{v_3}Y_3' + (4\alpha - 2),$$

for some distributions $Y_3' \in L_T^{\infty}C^{3\alpha-2}$. A careful reading of this proof gives the assertion about the dependence of the norm of the remainder as a function of the norms of \hat{u} and $\hat{\zeta}$.

▷

As a sanity check, we invite the reader to look at the linear case where $f(u) = u$. A number of terms in the analysis disappear or simplify, and one can work with a smaller space of enhanced distributions.

4.3 Solving the equation

Assume that the enhanced distribution $\hat{\zeta}$ is given, together with an initial condition $u_0 \in C^{3\alpha}$. The study of equation (4.1) from the paracontrolled calculus point of view is a three step process.

- (a)** *Set yourself an ansatz for the solution space $\mathcal{S}(\hat{\zeta})$, in the form of a Banach space of paracontrolled functions/distributions.*

- (b) Recast the equation as a fixed point problem for a map Φ from the solution space $\mathcal{S}(\widehat{\zeta})$ to itself.
- (c) Prove that Φ is a contraction of $\mathcal{S}(\widehat{\zeta})$ for a small enough choice of time horizon T .

Fix a finite time horizon T and recall the notation \mathcal{C}_w^α for the weighted spaces introduced in Appendix A.2, for a weight depending on a non-negative parameter κ ; all these spaces are equal as a set, with equivalent norms, for κ in a bounded set. All of the above continuity results hold in these spaces, with implicit constants independent of κ in a bounded set, as the weight is non-decreasing and all the approximation operators have temporal support in $[0, \infty)$. This elementary fact will allow us to gain in some estimate a crucial multiplicative factor depending on κ that will eventually provide the contraction property for Φ .

Given $\frac{1}{2} < \beta < \alpha \leq \frac{2}{3}$, with $3\alpha + \beta > 2$, we choose to work with the functions satisfying the second order paracontrolled ansatz

$$\begin{aligned} u &= \widetilde{\Pi}_{u_1}(Z_1) + \widetilde{\Pi}_{u_2}(Z_2) + u^\sharp \\ u_1 &= \widetilde{\Pi}_{u_{11}}(Z_1) + u_1^\sharp, \end{aligned} \tag{4.8}$$

with remainders $u^\sharp \in \mathcal{C}_w^{2\alpha+\beta}$ and $u_1^\sharp \in \mathcal{C}_w^{\alpha+\beta}$, and u_2, u_{11} in \mathcal{C}_w^β . Note that we use the operator $\widetilde{\Pi}$ introduced in [2] rather than the usual paraproduct operator Π ; the advantage of this choice will appear clearly in the proof of theorem 11 given below. Here the parameter β has to be thought as very close to α and will play the same role as α . The main trick is to use another parameter β , slightly lower than α , in order to prove the contraction property of the map Φ . We write

$$\widehat{u} := (u; u_1, u_2; u_{11})$$

and set

$$\widehat{u}_0 := (u_0; f(u_0), f'(u_0)f(u_0); f'(u_0)f(u_0)),$$

and turn the solution space

$$\mathcal{S}(\widehat{\zeta}) := \{\widehat{u}_{\tau=0} = \widehat{u}_0\}$$

into a Banach space by defining its norm as

$$\|\widehat{u}\| := \|u_2\|_{\mathcal{C}_w^\beta} + \|u_{11}\|_{\mathcal{C}_w^\beta} + \|u_1^\sharp\|_{\mathcal{C}_w^{\alpha+\beta}} + \|u^\sharp\|_{\mathcal{C}_w^{2\alpha+\beta}}.$$

The analysis of the product $f(u)\zeta$ done in section 4.2 corresponds to working with $\beta = \alpha$. Everything works verbatim under the assumption that $3\alpha + \beta > 2$, by replacing $(2\alpha - 2)$, $(3\alpha - 2)$ and $(4\alpha - 2)$ by $(\alpha + \beta - 2)$, $(2\alpha + \beta - 2)$ and $(3\alpha + \beta - 2)$, respectively; the product $f(u)\zeta$ is in particular well-defined for functions u , or rather \widehat{u} , satisfying the second order paracontrolled ansatz (4.8). We adopt the notations of equation (4.3) and write

$$f(u)\zeta = \Pi_{f(u)}(\zeta) + \Pi_{f'(u)u_1}(Y_2) + \Pi_{v_3}(Y_3) + (\sharp).$$

A better notation for $f(u)\zeta$ would be $\widehat{f}(\widehat{u})\widehat{\zeta}$, emphasizing the dependence on \widehat{u} and $\widehat{\zeta}$ of this notion of product between $f(u)$ and ζ – we stick to the former notation however. We define the map Φ by setting

$$\Phi(\widehat{u}) = (v; f(u), f'(u)u_1; f'(u)u_1),$$

where v is the solution to the equation

$$(\partial_t + L)v = f(u)\zeta,$$

with initial condition $v_{\tau=0} = u_0$. Notice that the definition of the space $\mathcal{S}(\widehat{\zeta})$ and the map Φ implicitly depend on the finite time interval $[0, T]$ on which we are working. We define a solution of the equation

$$(\partial_t + L)u = f(u)\zeta,$$

as a fixed point of the map Φ .

Theorem 11. *Let a function $f \in C_b^3(\mathbb{R})$, an enhanced distribution $\widehat{\zeta}$, and an initial condition $u_0 \in C^{3\alpha}$ be given. For any interval of time $[0, T]$, the map Φ has a unique fixed point \widehat{u} in $\mathcal{S}(\widehat{\zeta})$.*

Proof – The proof is an elementary application of Banach fixed point theorem. Let us explain it in details.

Let us fix a time interval $[0, T]$ and agree that all the implicit constants below are allowed to depend on T . Recall that we denote by P the free evolution given by the semigroup

$$P(u_0) := (\tau, x) \mapsto e^{-\tau L}(u_0)(x).$$

Given $\widehat{u} \in \mathcal{S}(\widehat{\zeta})$, the solution v of the well-posed parabolic equation

$$(\partial_t + L)v = f(u)\zeta, \quad v_{\tau=0} = u_0$$

is given by

$$v = \mathcal{R}(f(u)\zeta) + P(u_0)$$

Since we assume the initial data u_0 to be in space Hölder space $C^{3\alpha}$, then $P(u_0)$ belongs to the parabolic Hölder space $\mathcal{C}_w^{3\alpha}$. So to prove that

$$\Phi(\widehat{u}) = \left(v; f(u), f'(u)u_1; f'(u)u_1 \right),$$

belongs to $\mathcal{S}(\widehat{\zeta})$, it suffices to see that the map

$$\Psi(\widehat{u}) := \left(\mathcal{R}(f(u)\zeta); f(u), f'(u)u_1; f'(u)u_1 \right)$$

sends $\mathcal{S}(\widehat{\zeta})$ into itself. This is precisely what is given by Proposition 10, the regularity properties of $\widetilde{\Pi}$ and Schauder estimates, Theorem 18, which altogether show that $\Psi(\widehat{u})$ is in $\mathcal{S}(\widehat{\zeta})$, and

$$\|v_1^\sharp\|_{\mathcal{C}_w^{\alpha+\beta}} + \|v^\sharp\|_{\mathcal{C}_w^{2\alpha+\beta}} \lesssim \kappa^{-(\alpha-\beta)/2} C(\|\widehat{u}\|)$$

where C is a positive constant that depends polynomially on $\|\widehat{u}\|$. At the same time, the paracontrolled structure of $\mathcal{R}(f(u)\zeta)$, and Schauder estimates, also give

$$\|\mathcal{R}(f(u)\zeta)\|_{\mathcal{C}_w^\beta} \lesssim \kappa^{-\varepsilon} C(\|\widehat{u}\|),$$

giving a control of $\mathcal{R}(f(u)\zeta)$ by a small factor $\kappa^{-\varepsilon}$. Unfortunately, there is no reason so that the three paracontrolled derivatives of $\mathcal{R}(f(u)\zeta)$ enjoy that property, although they are given in terms of \widehat{u} . We iterate the map Φ to get around this problem. Indeed, by iterating four times the map Φ we observe that $\Phi(\widehat{u})$ is also a paracontrolled function of the space $\mathcal{S}(\widehat{\zeta})$ whose derivatives are given in the iterative process by the heat resolution \mathcal{R} of some functions; as such

one can use Schauder estimates to estimate them in the corresponding Holder space with a small factor of order $\kappa^{-\varepsilon}$. We deduce from that fact that

$$\Phi^{\circ 4}(\hat{u}) = \hat{w}$$

with $w = P(u_0) + \tilde{w}$ and

$$\|\hat{w}\| \lesssim \kappa^{-(\alpha-\beta)/2} C(\|\hat{u}\|).$$

So $\Phi^{\circ 4}$ is indeed a small perturbation of the constant map $\hat{u} \rightarrow P(u_0)$. Then it is standard that if one chooses κ big enough for $\kappa^{-(\alpha-\beta)/2}$ to be small enough, the map $\Phi^{\circ 4}$ will send a large enough ball of the space $\mathcal{S}(\hat{\zeta})$ into itself.

It remains us to see that $\Phi^{\circ 4}$ is a contraction. Indeed, we have

$$\Phi^{\circ 4}(\hat{u}^1) - \Phi^{\circ 4}(\hat{u}^2) = \hat{w}^1 - \hat{w}^2$$

where \hat{w}^1 and \hat{w}^2 , and their derivatives, are paracontrolled distributions obtained by iterating four times the map $\mathcal{R}(f(\bullet)\zeta)$, applied to \hat{u}^1 and \hat{u}^2 , respectively. This map is locally Lipschitz from the continuity results of section 3, and taking advantage of the game between α and β , it follows from Schauder estimates that

$$\|\hat{w}^1 - \hat{w}^2\| \lesssim \kappa^{-(\alpha-\beta)/2} C(\|\hat{u}^1\|, \|\hat{u}^2\|) \|\hat{u}^1 - \hat{u}^2\|,$$

where C is some polynomial function of two variables. So we conclude that $\Phi^{\circ 4}$ is a contraction of any large enough ball of $\mathcal{S}(\hat{\zeta})$, for a large enough choice of constant κ .

▷

Remarks.

- *A local in time well-posedness result can be proved following the same reasoning, assuming only that the nonlinearity f is of class C^3 , with a bounded third derivative.*
- *We assume here that the initial condition is in $C^{3\alpha}$. We use that fact to put the term $P(u_0)$ in the remainder. One can improve upon this constraint on u_0 and only require that $u_0 \in C^\alpha$, at the price of working with weighted Hölder spaces with an explosive weight. This is well explained in Lemma A.7 and A.9 of [13].*
- *So far, the theory of regularity structures has not been developed in a manifold setting. A forthcoming work of Dahlqvist-Diehl-Driver shows how this can be done in the simplest case where the noise is not too rough, corresponding in our setting to a regularity exponent $\alpha > \frac{2}{3}$. A first order description of the objects is sufficient in that setting, as was the case in our previous work [1], whose content covers partly their results. It is very likely that one can improve upon the Dahlqvist-Diehl-Driver approach to regularity structures on a manifold by working on the second order frame bundle in order to study the (gPAM) equation in the range of regularity exponents $\frac{1}{2} < \alpha \leq \frac{2}{3}$ for the noise – this is how the story of stochastic differential equations on manifolds can be told from Schwartz-Meyer’s point of view. This potential extension of the work of Dahlqvist-Diehl-Driver is what is covered by the results of the present section, in our paracontrolled setting. On the other hand, it is not clear to us what geometric setting will be needed to get the equivalent of the results we obtain in section 4.5, where the exponents α is in the range $\frac{2}{5} < \alpha \leq \frac{1}{2}$.*

One gets as a direct consequence of the fact that the solution u to equation (4.1) has the form

$$u = \Pi_{f(u)}Z_1 + (2\alpha),$$

the following corollary; it is the analogue of a result of Hairer and Pardoux [20, Corollary 1.11] – their result is a direct consequence of the content of section 4.5. Recall ρ stands for the parabolic distance on \mathcal{M} ; it was introduced in section 1.2.

Corollary 12. *Let f be C_b^5 . For $0 < t < T$, there exists a positive constant C such that one has the estimate*

$$|u(e') - u(e) - f(u(e))(Z_1(e') - Z_1(e))| \leq C\rho(e', e),$$

uniformly in $e' = (\sigma, y)$ and $e = (\tau, x)$ with $|\tau - \sigma| \leq \frac{T}{2}$.

Proof – The proof is a direct application of the representation of the solution u as a paracontrolled distribution

$$u = \Pi_{f(u)}Z_1 + (2\alpha)$$

together with Proposition 21. ▷

A similar result holds in the rougher case where $\frac{2}{5} < \alpha \leq \frac{1}{2}$, studied in section 4.5, with the exponent 1 for $\rho(e', e)$ in the right hand side of the estimate of corollary 12 replaced by an exponent 2α , in accordance with the above mentioned result of Hairer and Pardoux.

4.4 Symmetry group

The study of equation (4.1) is particularly motivated when ζ is assumed to be the realization $\zeta(\omega)$ of a random field ζ , defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, typically a Gaussian spatial noise of Hölder regularity $(\alpha - 2)$, with α in the range $(\frac{1}{2}, \frac{2}{3}]$. One needs to assume that we are able to construct on that probability space a random enhanced distribution $\hat{\zeta}$ to use the above deterministic machinery for each realization $\hat{\zeta}(\omega)$ of $\hat{\zeta}$, and construct in this way a random solution $u(\omega)$ to equation (4.1) – the measurability of $u(\omega)$ as a function of ω comes from the fact that $u(\omega)$ is a continuous function of $\hat{\zeta}(\omega)$. Although it is always possible to enhance $\zeta(\omega)$ in an arbitrary (measurable) way (with respect to ω), it makes sense to

- (a) ask for some more or less canonical way of doing the enhancement,
- (b) relate the solution to the singular equation (4.1), such as built and understood in section 4.3, to some family of solutions to some classically well-posed partial differential equations.

The most natural and naive way of defining the random variable $\hat{\zeta}$ is to smoothen ζ into ζ^ε by any deterministic classical mean, such as convolution with a smooth kernel, define its associated enhancement $\hat{\zeta}^\varepsilon$, via formula (4.5), and pass to the limit. Unfortunately, this family of random variables cannot converge in any sensible sense as ε goes to 0, and it is the object of renormalisation to provide a robust approach to this problem, by taking deterministic special linear combinations of these otherwise diverging quantities to make them converge. See the forthcoming works of Bruned-Hairer-Zambotti and Chandra-Hairer for a systematic study of these questions within the setting of regularity structures; note that the renormalisation of

the term $\Pi(Z_1, \zeta)$ was already done in [1]. This renormalisation story has direct consequences on point (b).

The analysis of equation (4.1) done in section 4.3 shows that the solution \hat{u} to equation (4.1) is a continuous function of $\hat{\zeta}$; write

$$\hat{u} = \left(u; f(u), f'(u)f(u); f'(u)f(u) \right) =: \mathfrak{J}(\hat{\zeta}).$$

Better, one can write

$$(\partial_t + L)u = \Pi_{f(u)}\zeta + \Pi_\zeta(f(u)) + F(\hat{u})\hat{\zeta} + (4\alpha - 2),$$

for some continuous map F of \hat{u} and $\hat{\zeta}$, that is linear with respect to $\hat{\zeta}$, and some remainder $(4\alpha - 2)$ that is a continuous function of \hat{u} and ζ *only*. The first two paraproduct terms on the right hand side also have the latter property. Precisely, one knows from lemma 9 that

$$\begin{aligned} F(\hat{u})\hat{\zeta} &= f'(u)f(u)\Pi(Z_1, \zeta) + f'(u)^2f(u)\Pi(Z_2, \zeta) + \frac{1}{2}f^{(2)}(u)f(u)^2\Pi(\Pi(Z_1, Z_1), \zeta) \\ &\quad + \left(f'(u)^2f(u) + uf^{(2)}(u)f(u)^2 \right) C(Z_1, Z_1, \zeta) \\ &=: g_2(u)\hat{\zeta}^{(2)} + \sum_{i=1}^3 g_i(u)\hat{\zeta}_i^{(3)}. \end{aligned} \tag{4.9}$$

The renormalisation procedure provides in the present case a deterministic, possibly constant, element $C^\varepsilon = (0, C_2^\varepsilon, C_3^\varepsilon)$ in the space of enhanced distributions such that the family $(\hat{\zeta}^\varepsilon - C^\varepsilon)$ converges in that space, in probability say, as ε goes to 0. Set

$$\hat{u}^\varepsilon := (u^\varepsilon, \dots) := \mathfrak{J}(\hat{\zeta}^\varepsilon - C^\varepsilon);$$

so this family converges in probability to $\hat{u} = \mathfrak{J}(\hat{\zeta})$, by the continuity of the solution map \mathfrak{J} . One reads on equation (4.9) the effect of adding C^ε into the dynamics. The function u^ε is a solution to the well-posed equation

$$(\partial_t + L)u^\varepsilon = f(u^\varepsilon)\zeta^\varepsilon + C_2^\varepsilon g_2(u^\varepsilon) + \sum_{i=1}^3 C_{3,i}^\varepsilon g_3^{(i)}(u^\varepsilon),$$

and it converges in \mathcal{C}^α , in probability, to the first component u of the solution \hat{u} to equation (4.1).

4.5 Rougher noise ζ .

The above methods are robust enough to deal with the generalized parabolic Anderson model equation

$$(\partial_t + L)u = f(u)\zeta$$

when the spatial noise ζ has the regularity $(\alpha - 2)$ of a 3-dimensional space white noise, that is ζ is $(\alpha - 2)$ -Hölder regular, for some $\alpha < \frac{1}{2}$, with $\alpha > \frac{2}{5}$ say. We describe in this section the essentials of the analysis of the product term $f(u)\zeta$ that one can do to study the equation; the fixed point problem is tackled with the very same tools as those used in section 4.3.

Fix some regularity exponents $\frac{2}{5} < \beta \leq \alpha \leq \frac{1}{2}$, and assume we are given some reference functions

$$Z_1 = \mathcal{R}(\zeta), \quad Z_2 = \mathcal{R}(Y_2), \quad Z_3 = \mathcal{R}(Y_3)$$

with $Y_i \in L_T^\infty C^{i\alpha-2}$ to be determined latter from consistency conditions; from Schauder estimates, these regularity assumptions on the Y_i ensure that Z_i is $(i\alpha)$ -parabolic Hölder continuous. We take as a solution space for equation (4.1) the set of functions satisfying the following **third order paracontrolled ansatz**

$$\begin{aligned} u &= \tilde{\Pi}_{u_1} Z_1 + \tilde{\Pi}_{u_2} Z_2 + \tilde{\Pi}_{u_3} Z_3 + u^\sharp \\ u_1 &= \tilde{\Pi}_{u_{11}} Z_1 + \tilde{\Pi}_{u_{12}} Z_2 + u_1^\sharp \\ u_2 &= \tilde{\Pi}_{u_{21}} Z_1 + u_2^\sharp \\ u_{11} &= \tilde{\Pi}_{u_{111}} Z_1 + u_{11}^\sharp \end{aligned} \tag{4.10}$$

with $u_3, u_{12}, u_{21}, u_{111}$ in \mathcal{C}^β and the remainders $u_{11}^\sharp, u_2^\sharp$ in $\mathcal{C}^{\alpha+\beta}$, with u_1^\sharp in $\mathcal{C}^{2\alpha+\beta}$ and u^\sharp in $\mathcal{C}^{3\alpha+\beta}$. Note here again that we use the $\tilde{\Pi}$ operator introduced in [2] rather than the usual paraproduct operator Π . The set of all such tuples

$$\hat{u} := \left(u ; u_1, u_2, u_3 ; u_{11}, u_{12}, u_{21} ; u_{111} \right)$$

satisfying identity (4.10) is turned into a Banach space setting

$$\|\hat{u}\| := \|u_3\|_{\mathcal{C}^\beta} + \|u_{12}\|_{\mathcal{C}^\beta} + \|u_{21}\|_{\mathcal{C}^\beta} + \|u_{111}\|_{\mathcal{C}^\beta} + \|u_{11}^\sharp\|_{\mathcal{C}^{\alpha+\beta}} + \|u_2^\sharp\|_{\mathcal{C}^{\alpha+\beta}} + \|u_1^\sharp\|_{\mathcal{C}^{2\alpha+\beta}} + \|u^\sharp\|_{\mathcal{C}^{3\alpha+\beta}}.$$

One says that u is in **(dressed) canonical form** (4.10) to mean that we are given \hat{u} as here. The **naked canonical form** consists of a similar decomposition for u , but with the Π operator used in place of $\tilde{\Pi}$; we use the expression *canonical form* for *dressed canonical form*. One gets a clear picture of the product $f(u)\zeta$, or rather $\hat{f}(\hat{u})\hat{\zeta}$, by

- (a) showing that, for \hat{u} in *dressed* canonical form, one can write $f(u)$ in *naked* canonical form,
- (b) for $\hat{v} = (v ; v_1, v_2, v_3 ; \dots)$ in *dressed or naked* canonical form, the product $v\zeta$, or rather $\hat{v}\hat{\zeta}$, is well-defined and

$$v\zeta = \Pi_v \zeta + \Pi_{v_1} Y_2 + \Pi_{v_2} Y_3 + \Pi_{v_3} Y_4 + (4\alpha + \beta - 2),$$

for some $Y_4 \in L_T^\infty C^{3\alpha+\beta-2}$, and $v_i \in \mathcal{C}^\beta$.

Consistency conditions imply some relations between the Z_i . These two steps also dictate the choice of Y_i and single out the different components of the space of enhanced distributions, as those expressions in Z_1, ζ that do not make sense on a purely analytic basis. One uses the full strength of the Taylor formula stated in theorem 1 to deal with point (a). Given identity (2.3) and the fact that

$$\begin{aligned} u^2 &= 2\Pi_u u + \Pi(u, u), \\ u^3 &= 2\Pi_u(\Pi_u u) + \Pi_{u^2} u + \Pi_u(\Pi(u, u)) + 2\Pi(u, \Pi_u u) + \Pi(u, \Pi(u, u)), \end{aligned}$$

we see that point (a) holds if the following condition holds.

- (a') For u and v in *dressed* canonical form and g satisfying the second order paracontrolled ansatz (4.2), then $\Pi_g u$ and $\Pi(u, v)$ can be written in *naked* canonical form.

Proposition 13. *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a function of class \mathcal{C}^4 , with bounded fourth derivative. For a function u in dressed canonical form $f(u)$ can be decomposed in naked canonical form.*

Proof – We prove point (a') and start with $\Pi_g u$ – recall we are working up to elements in $(3\alpha + \beta)$. We have

$$\Pi_g\left(\tilde{\Pi}_{u_1} Z_1\right) = \Pi_{gu_1} Z_1 + \Pi_g\left(\tilde{\Pi}_{\mathcal{D}u_1} Z_1\right),$$

with $u_1 = \tilde{\Pi}_{u_{11}} Z_1 + \tilde{\Pi}_{u_{12}} Z_2 + (2\alpha + \beta)$. One has

$$\begin{aligned} \Pi_g\left(\tilde{\Pi}_{\mathcal{D}\tilde{\Pi}_{u_{11}} Z_1} Z_1\right) &= \Pi_{gu_{11}}\left(\Pi_{\mathcal{D}Z_1} Z_1\right) + \mathbf{R}(g; u_{11}, Z_1; Z_1) \\ &= \Pi_{gu_{11}}\left(\Pi_{\mathcal{D}Z_1} Z_1\right) + \mathbf{R}(gu_{111}, ; Z_1, Z_1; Z_1) + (3\alpha + \beta) \\ &= \Pi_{gu_{11}}\left(\Pi_{\mathcal{D}Z_1} Z_1\right) + \Pi_{gu_{111}}\left(\mathbf{R}(1; Z_1, Z_1; Z_1)\right) + (3\alpha + \beta), \end{aligned}$$

after (3.10); we also have

$$\Pi_g\left(\tilde{\Pi}_{\mathcal{D}\tilde{\Pi}_{u_{12}} Z_2} Z_1\right) = \Pi_{gu_{12}}\left(\Pi_{\mathcal{D}Z_2} Z_1\right) + (3\alpha + \beta).$$

This gives us as a decomposition for $\Pi_g(\tilde{\Pi}_{u_1} Z_1)$ the sum

$$\begin{aligned} \Pi_g\left(\tilde{\Pi}_{u_1} Z_1\right) &= \Pi_{gu_1} Z_1 + \Pi_{gu_{11}}\left(\Pi_{\mathcal{D}Z_1} Z_1\right) + \Pi_{gu_{111}}\left(\mathbf{R}(1; Z_1, Z_1; Z_1)\right) \\ &\quad + \Pi_{gu_{12}}\left(\Pi_{\mathcal{D}Z_2} Z_1\right) + (3\alpha + \beta). \end{aligned}$$

The same computations shows that

$$\Pi_g\left(\tilde{\Pi}_{u_2} Z_2\right) = \Pi_{gu_2} Z_2 + \Pi_{gu_{21}}\left(\Pi_{\mathcal{D}Z_1} Z_2\right) + (3\alpha + \beta)$$

and

$$\Pi_g\left(\tilde{\Pi}_{u_3} Z_3\right) = \Pi_{gu_3} Z_3 + (3\alpha + \beta),$$

which shows that indeed the operator Π_g transforms a function u in dressed canonical form into an object in naked canonical form, under the assumption that g satisfies the second order paracontrolled ansatz (4.2) – the latter assumption is needed to ensure that the different derivatives of $\Pi_g u$ satisfy the structure equation imposed to u_1, u_2, u_{111} in (4.10).

To analyse the term $\Pi(u, v)$, look first at

$$\begin{aligned} \Pi\left(\tilde{\Pi}_{u_1} Z_1, \tilde{\Pi}_{v_1} Z_1\right) &= \Pi_{u_1}\left(\Pi(Z_1, \Pi_{v_1} Z_1)\right) + \mathbf{D}(u_1, Z_1, \Pi_{v_1} Z_1) \\ &= \Pi_{u_1}\left(\Pi_{v_1}\Pi(Z_1, Z_1) + \mathbf{D}(v_1, Z_1, Z_1)\right) + \Pi_{v_1}\mathbf{D}(u_1, Z_1, Z_1) + (4\alpha) \\ &= \Pi_{u_1}\left(\Pi_{v_1}\Pi(Z_1, Z_1)\right) + \Pi_{u_1}\left(\Pi_{v_{11}}\mathbf{D}(Z_1, Z_1, Z_1)\right) + (4\alpha) \\ &\quad + \Pi_{v_1}\left(\Pi_{u_{11}}\mathbf{D}(Z_1, Z_1, Z_1)\right) + (4\alpha) \\ &= \Pi_{u_1}\left(\Pi_{v_1}\Pi(Z_1, Z_1)\right) + \Pi_{u_1 v_{11} + v_1 u_{11}}\left(\mathbf{D}(Z_1, Z_1, Z_1)\right) + (4\alpha), \end{aligned}$$

and note that the term $\Pi_{u_1}\left(\Pi_{v_1}\Pi(Z_1, Z_1)\right)$ can be analysed as the term $\Pi_g u$ above. For $\Pi(\Pi_{u_1} Z_1, \Pi_{v_2} Z_2)$ or $\Pi(\Pi_{u_2} Z_2, \Pi_{v_1} Z_1)$, write simply

$$\Pi(\Pi_{u_1} Z_1, \Pi_{v_2} Z_2) = \Pi_{u_1 v_2} \Pi(Z_1, Z_2),$$

and

$$\Pi(\Pi_{u_2} Z_2, \Pi_{v_1} Z_1) = \Pi_{u_2 v_1} \Pi(Z_1, Z_2).$$

In the end, one sees that all the terms of the Taylor expansion formula for $f(u)$ can be decomposed in naked canonical form.

▷

Recall that each Z_i may have several components $(Z_i^k)_k$, in which case the notation $\Pi_\bullet Z_i$ stands for an implicit sum

$$\Pi_\bullet Z_i = \sum_k \Pi_{\bullet_k} Z_i^k.$$

The above proof shows that for *consistency purposes* the reference operators $\Pi_\bullet Z_2$ need to have *at least* the following components

$$\Pi_\bullet(\Pi_{\mathcal{O}Z_1} Z_1), \quad \Pi_\bullet(\Pi(Z_1, Z_1)),$$

and the operators $\Pi_\bullet Z_3$ the following components

$$\begin{aligned} & \Pi_\bullet(\Pi_{\mathcal{O}Z_2} Z_1), \quad \Pi_\bullet(\Pi_{\mathcal{O}Z_1} Z_2), \quad \Pi_\bullet(\Pi_{\mathcal{O}Z_1} \Pi(Z_1, Z_1)) \\ & \Pi_\bullet(\mathbb{R}(1, Z_1, Z_1, Z_1)), \quad \Pi_\bullet(\mathbb{D}(Z_1, Z_1, Z_1)), \quad \Pi_\bullet(\Pi(Z_1, Z_2)) \\ & \Pi_\bullet(\Pi_{\mathcal{O}Z_1} \Pi(Z_1, Z_1)). \end{aligned}$$

Other components of the operators $\Pi_\bullet Z_2$ and $\Pi_\bullet Z_3$ will pop out from the proof of the next statement.

Proposition 14. *For $\hat{v} = (v; v_1, v_2, v_3; \dots)$ in dressed or naked canonical form, the product $v\zeta$ is well-defined and*

$$v\zeta = \Pi_v \zeta + \Pi_{v_1} Y_2 + \Pi_{v_2} Y_3 + \Pi_{v_3} Y_4 + (4\alpha + \beta - 2), \quad (4.11)$$

for some $Y_4 \in L_T^\infty C^{3\alpha+\beta-2}$ and $v_i \in \mathcal{C}^\beta$.

Proof – We do the proof for \hat{v} in dressed canonical form; cosmetic changes are needed to deal with the other case. Given that

$$v\zeta = \Pi_v \zeta + \Pi_\zeta v + \Pi(v, \zeta),$$

it should be clear to the reader that the main work is to show that $\Pi_\zeta(\tilde{\Pi}_{v_1} Z_1)$ and $\Pi(\tilde{\Pi}_{v_1} Z_1, \zeta)$ can be written under the form (4.11) – which also justifies that the latter a priori undefined term makes sense. We give the details for the analysis of these two terms and trust the reader for completing the analysis of the other, easier, terms in the expansion of $v\zeta$. We use the continuity results proved in sections 3.1 and 3.2 along the way without explicit mention.

• Let start with the term $\Pi_\zeta(\tilde{\Pi}_{v_1} Z_1)$, of parabolic regularity $(2\alpha - 2)$. One has

$$\begin{aligned} \Pi_\zeta(\tilde{\Pi}_{v_1} Z_1) &= \Pi_{v_1}(\Pi_\zeta Z_1) + \mathbb{T}_\zeta(v_1, Z_1) \\ &= \Pi_{v_1}(\Pi_\zeta Z_1) + \mathbb{T}_\zeta(\tilde{\Pi}_{v_{11}} Z_1, Z_1) + \mathbb{T}_\zeta(\tilde{\Pi}_{v_{12}} Z_2, Z_1) + (4\alpha + \beta - 2) \\ &= \Pi_{v_1}(\Pi_\zeta Z_1) + \Pi_{v_{11}}(\mathbb{T}_\zeta(Z_1, Z_1)) + \mathbb{T}_\zeta(v_{11}, Z_1; Z_1) \\ & \quad + \Pi_{v_{12}}(\mathbb{T}_\zeta(Z_2, Z_1)) + (4\alpha + \beta - 2) \\ &= \Pi_{v_1}(\Pi_\zeta Z_1) + \Pi_{v_{11}}(\mathbb{T}_\zeta(Z_1, Z_1)) + \Pi_{v_{111}}(\mathbb{T}_\zeta(Z_1, Z_1; Z_1)) \\ & \quad + \Pi_{v_{12}}(\mathbb{T}_\zeta(Z_2, Z_1)) + (4\alpha + \beta - 2). \end{aligned}$$

• We start from the identity

$$\Pi(\tilde{\Pi}_{u_1} Z_1, \zeta) = u_1 \Pi(Z_1, \zeta) + \mathbb{C}(u_1, Z_1, \zeta)$$

to analyse the term $\Pi\left(\tilde{\Pi}_{u_1} Z_1, \zeta\right)$, and look at each term on the right hand side separately. First, we have

$$u_1 \Pi(Z_1, \zeta) = \Pi_{u_1}\left(\Pi(Z_1, \zeta)\right) + \Pi_{\Pi(Z_1, \zeta)} u_1 + \Pi\left(u_1, \Pi(Z_1, \zeta)\right)$$

with

$$\begin{aligned} \Pi\left(u_1, \Pi(Z_1, \zeta)\right) &= u_{11} \Pi\left(Z_1, \Pi(Z_1, \zeta)\right) + \mathbf{C}\left(u_{11}, Z_1; \Pi(Z_1, \zeta)\right) \\ &\quad + u_{12} \Pi\left(Z_2, \Pi(Z_1, \zeta)\right) + (4\alpha + \beta - 2) \\ &= \Pi_{u_{11}}\left(\Pi(Z_1, \Pi(Z_1, \zeta))\right) + \Pi_{\Pi(Z_1, \Pi(Z_1, \zeta))}(u_{11}) + \Pi\left(u_{11}, \Pi(Z_1, \Pi(Z_1, \zeta))\right) \\ &\quad + u_{111} \mathbf{C}\left(Z_1, Z_1; \Pi(Z_1, \zeta)\right) + \Pi_{u_{12}}\left(\Pi(Z_2, \Pi(Z_1, \zeta))\right) + (4\alpha + \beta - 2) \\ &= \Pi_{u_{11}}\left(\Pi(Z_1, \Pi(Z_1, \zeta))\right) + \Pi_{u_{111}}\left\{\Pi_{\Pi(Z_1, \Pi(Z_1, \zeta))} Z_1\right. \\ &\quad \left. + \Pi(Z_1, \Pi(Z_1, \Pi(Z_1, \zeta))) + \mathbf{C}\left(Z_1, Z_1, \Pi(Z_1, \zeta)\right)\right\} \\ &\quad + \Pi_{u_{12}}\left(\Pi(Z_2, \Pi(Z_1, \zeta))\right) + (4\alpha + \beta - 2) \end{aligned}$$

and

$$\begin{aligned} \Pi_{\Pi(Z_1, \zeta)} u_1 &= \Pi_{\Pi(Z_1, \zeta)}(\Pi_{u_{11}} Z_1) + \Pi_{\Pi(Z_1, \zeta)}(\Pi_{u_{12}} Z_2) + (4\alpha + \beta - 2) \\ &= \Pi_{u_{11}}(\Pi_{\Pi(Z_1, \zeta)} Z_1) + \mathbf{T}_{\Pi(Z_1, \zeta)}(u_{11}, Z_1) + \Pi_{u_{12}}(\Pi_{\Pi(Z_1, \zeta)} Z_2) + (4\alpha + \beta - 2) \\ &= \Pi_{u_{11}}(\Pi_{\Pi(Z_1, \zeta)} Z_1) + \Pi_{u_{111}}\left(\mathbf{T}_{\Pi(Z_1, \zeta)}(Z_1, Z_1)\right) + \Pi_{u_{12}}(\Pi_{\Pi(Z_1, \zeta)} Z_2) \\ &\quad + (4\alpha + \beta - 2). \end{aligned}$$

Second, the term

$$\mathbf{C}(u_1, Z_1, \zeta) = u_{11} \mathbf{C}(Z_1, Z_1, \zeta) + \mathbf{C}(u_{11}, Z_1; Z_1, \zeta)$$

has the same structure as the term $\Pi(u_1, \Pi(Z_1, \zeta))$ analysed above; one can repeat the same computations. We are then left with checking that the distributions Y_i that appear in this decomposition of $v\zeta$ are indeed in $L_T^\infty C^{i\alpha}$; the assumptions on the enhanced distribution $\hat{\zeta}$ are made on purpose.

- It is straightforward to adapt the above computations to the analysis of the terms $\Pi\left(\tilde{\Pi}_{u_2} Z_2, \zeta\right)$ and $\Pi\left(\tilde{\Pi}_{u_3} Z_3, \zeta\right)$, by tracking the indices and running the computations up to remainders of regularity $(4\alpha + \beta - 2)$.

▷

One gets from propositions 13 and 14 that for \hat{u} in canonical form (4.10) one can write the product $f(u)\zeta$ under the form

$$f(u)\zeta = \Pi_{f(u)}\zeta + \Pi_{v_2} Y_2 + \Pi_{v_3} Y_3 + \Pi_{v_4} Y_4 + (4\alpha + \beta - 2),$$

with Y_2 depending only on ζ and $Z_1 = \mathcal{R}(\zeta)$, with Y_3 depending on ζ, Z_1 and $Z_2 = \mathcal{R}(Y_2)$, and so on. The consistency relation

$$\mathcal{R}(f(u)\zeta) = \tilde{\Pi}_{f(u)} Z_1 + \tilde{\Pi}_{v_2} Z_2 + \tilde{\Pi}_{v_3} Z_3 + (3\alpha + \beta),$$

determines then uniquely the choice of Z_1, Z_2 and Z_3 , or rather the operators $\Pi_\bullet Z_i$. The different components of $\hat{\zeta}$ also pop out of the above computations, as those expressions in Z_1, ζ that do not make sense on a purely analytic basis. From the study of the term $\Pi\left(\tilde{\Pi}_{u_1} Z_1, \zeta\right)$, we have just singled out

$$\Pi_\zeta Z_1 \quad \text{and} \quad \Pi(Z_1, \zeta)$$

to be assumed in $L_T^\infty C^{2\alpha-2}$,

$$\Pi(Z_1, \zeta), \quad \Pi(Z_1, \Pi(Z_1, \zeta)), \quad \mathsf{T}_\zeta(Z_1, Z_1) \quad \text{and} \quad \Pi_{\Pi(\zeta, Z_1)} Z_1$$

in $L_T^\infty C^{3\alpha-2}$ and

$$\begin{aligned} & \mathsf{T}_\zeta(Z_1, Z_1, Z_1), \quad \mathsf{T}_\zeta(Z_2, Z_1), \quad \mathsf{C}(Z_1, Z_1, \zeta), \quad \Pi(Z_2, \Pi(Z_1, \zeta)), \\ & \Pi\left(Z_1, \Pi(Z_1, \Pi(Z_1, \zeta))\right), \quad \mathsf{C}(Z_1, Z_1, \Pi(Z_1, \zeta)), \quad \mathsf{C}(Z_1, Z_1; Z_1, \zeta), \quad \mathsf{C}(Z_2, Z_1, \zeta), \\ & \Pi_{\Pi(Z_1, \Pi(Z_1, \zeta))} Z_1, \quad \Pi_{\Pi(Z_1, \zeta)} Z_2, \quad \mathsf{T}_{\Pi(Z_1, \zeta)}(Z_1, Z_1), \end{aligned}$$

in $L_T^\infty C^{4\alpha-2}$. The study of the terms corresponding to $\tilde{\Pi}_{u_2} Z_2$ and $\tilde{\Pi}_{u_3} Z_3$, only add the expressions

$$\begin{aligned} & \Pi_\zeta(Z_2), \quad \Pi(Z_2, \zeta), \quad \Pi(Z_1, \Pi(Z_2, \zeta)), \quad \mathsf{C}(Z_1, Z_2, \zeta), \\ & \Pi_{\Pi(\zeta, Z_2)} Z_1, \quad \Pi_\zeta(Z_3), \quad \Pi(Z_3, \zeta) \end{aligned}$$

to the above list; this is the list of the components of the enhanced distribution $\widehat{\zeta}$. One sees that they correspond to the terms needed to make sense of the products

$$Z_1 \zeta; \quad Z_1^2 \zeta, \quad Z_2 \zeta; \quad Z_1 Z_2 \zeta, \quad Z_1^3 \zeta, \quad Z_3 \zeta,$$

in accordance with the overall picture provided by the theory of regularity structures – see Hairer and Pardoux work [20] for a study of equation (4.1) from the regularity structure point of view, amongst other things. Here again, recall that to each product in the theory of regularity structures are associated three terms in our paracontrolled setting, so the reader should not be afraid to see so many terms in our enhancement $\widehat{\zeta}$ of the noise ζ .

One can proceed, from that point on, to the analysis of equation (1.4) by the fixed point method of section 4.3 by following almost verbatim the details given there. The analysis of the symmetry group of this equation in the present low regularity regime is done in exactly the same way as in section 4.4, and requires from the reader to write the explicit formula for the function $F(\widehat{u})\widehat{\zeta}$ by collecting its different pieces from the above computations; we leave her/him the task of doing that.

5 Generalized KPZ equation

We provide in this section sufficiently many details on the study of the generalized KPZ equation

$$(\partial_t + L)u = f(u)\zeta + g(u)(\partial u)^2, \tag{5.1}$$

for the reader to fill in the gaps herself/himself. The noise ζ is here a one dimensional space-time noise on $[0, T] \times \mathbb{S}^1$, almost surely of parabolic regularity $(\alpha - 2)$, and the symbol ∂ stands for the derivative with respect to the space variable. Such a kind of equation appears in the study of the random motion of a string on a manifold [19], where $\alpha < \frac{1}{2}$ in that case; its study in the setting of regularity structures is the object of Bruned-Hairer-Zambotti's forthcoming work [3]. The renormalisation of the 50ish terms that appear in the models for this equation motivated the development of systematic renormalisation procedures. This is the content of Bruned-Haire-Zambotti's and Chandra-Hairer's forthcoming works [3, 5].

Theorem 15. *For $\frac{1}{2} < \alpha$, one can formulate the generalized KPZ equation (5.1) as a well-posed differential equation within the setting of paracontrolled calculus.*

We show here how some elementary, and relatively short, computations allow for the analysis of this equation within the paracontrolled calculus setting developed here, in the mild case where $\frac{1}{2} < \alpha \leq \frac{2}{3}$, and the second order paracontrolled calculus suffices for the analysis. Similar computations can be done in the space-time white noise case $\frac{2}{5} < \alpha < \frac{1}{2}$, to the price of some heavier, unappealing, computations. We do not touch upon the renormalisation problem, which is a different subject.

We set the scene in the second order paracontrolled setting of section 4.3, for some generalized KPZ enhancement $\widehat{\zeta}$ of ζ to be identified from the analysis of equation (5.1). The term $(\partial u)^2$ is of parabolic regularity $(2\alpha - 2)$, more regular than the term $f(u)\zeta$, of regularity $\alpha - 2$. The main task in the analysis of the generalized KPZ equation (5.1) is to put the term $g(u)(\partial u)^2$ in the form

$$g(u)(\partial u)^2 = \Pi_{v_2}\mathfrak{Z}_2 + \Pi_{v_3}\mathfrak{Z}_3 + (4\alpha - 2) \quad (5.2)$$

for some reference distributions \mathfrak{Z}_i in $L_T^\infty C^{i\alpha-2}$, $i \in \{2, 3\}$, depending only on an enhancement $\widehat{\zeta}$ of ζ , and some functions v_2, v_3 in some Hölder space – typically \mathcal{C}^β , for some $\beta < \alpha$, as in section 4.3. The analysis proceeds in two elementary steps. To lighten notations, we do the computations here in the case where the regularity exponent β equals α ; only cosmetic changes are needed in the case where $\beta < \alpha$ is close enough to α .

Proof of theorem 15 – We provide a **sketch** of proof, living the details to the reader; we proceed in two steps.

Step 1 – $(\partial u)^2$. Given u with the second order paracontrolled structure (4.2), one has

$$\partial u = \widetilde{\Pi}_{u_1}(\partial Z_1) + \left(\widetilde{\Pi}_{\partial u_1}(Z_1) + \widetilde{\Pi}_{u_1}(\partial Z_2) \right) + (3\alpha - 1),$$

so the only ill-defined terms in the product $(\partial u)^2$ are the three terms

$$\left\{ \widetilde{\Pi}_{u_1}(\partial Z_1) \right\}^2, \quad \left\{ \widetilde{\Pi}_{u_1}(\partial Z_1) \right\} \left\{ \widetilde{\Pi}_{\partial u_1}(Z_1) \right\}, \quad \left\{ \widetilde{\Pi}_{u_1}(\partial Z_1) \right\} \left\{ \widetilde{\Pi}_{u_2}(\partial Z_2) \right\}.$$

We analyse in detail the worst term $\left\{ \widetilde{\Pi}_{u_1}(\partial Z_1) \right\}^2$, of regularity $(2\alpha - 2)$; the two other, more regular, terms are easier to study. All the computations below use the continuity results proved in section 3. We have

$$\begin{aligned} \left\{ \widetilde{\Pi}_{u_1}(\partial Z_1) \right\}^2 &= 2\Pi_{\widetilde{\Pi}_{u_1}(\partial Z_1)} \left(\widetilde{\Pi}_{u_1}(\partial Z_1) \right) + \Pi \left(\widetilde{\Pi}_{u_1}(\partial Z_1), \widetilde{\Pi}_{u_1}(\partial Z_1) \right) \\ &= 2\Pi_{u_1} \left(\Pi_{\widetilde{\Pi}_{u_1}(\partial Z_1)}(\partial Z_1) \right) + 2\mathbb{T}_{\widetilde{\Pi}_{u_1}(\partial Z_1)}(u_1, \partial Z_1) \\ &\quad + u_1 \Pi \left(\partial Z_1, \widetilde{\Pi}_{u_1}(\partial Z_1) \right) + \mathbb{C} \left(u_1, \partial Z_1, \widetilde{\Pi}_{u_1}(\partial Z_1) \right) \\ &= 2\Pi_{u_1^2} \left(\widetilde{\Pi}_{\partial Z_1}(\partial Z_1) \right) + 2\mathbb{R}(u_1; u_1, \partial Z_1; \partial Z_1) \\ &\quad + 2\mathbb{T}_{\widetilde{\Pi}_{u_1}(\partial Z_1)} \left(\widetilde{\Pi}_{u_{11}} Z_1 + (2\alpha); \partial Z_1 \right) \\ &\quad + u_1^2 \Pi(\partial Z_1, \partial Z_1) + 2u_1 \mathbb{C}(u_1, \partial Z_1, \partial Z_1) + (4\alpha - 2) \\ &= 2\Pi_{u_1^2} \left(\widetilde{\Pi}_{\partial Z_1}(\partial Z_1) \right) + 2\Pi_{u_{11}^2} \left(\mathbb{R}(Z_1; Z_1, \partial Z_1; \partial Z_1) \right) + (4\alpha - 2) \\ &\quad + \Pi_{u_1 u_{11}} \left(\mathbb{T}_{\partial Z_1}(Z_1; \partial Z_1) \right) + (4\alpha - 2) \\ &\quad + u_1^2 \Pi(\partial Z_1, \partial Z_1) + 2\Pi_{u_1 u_{11}} \mathbb{C}(Z_1, \partial Z_1, \partial Z_1), \end{aligned}$$

with

$$\begin{aligned} u_1^2 \Pi(\partial Z_1, \partial Z_1) &= \Pi_{u_1^2}(\Pi(\partial Z_1, \partial Z_1)) + \Pi_{\Pi(\partial Z_1, \partial Z_1)}(u_1^2) + \Pi(u_1^2, \Pi(\partial Z_1, \partial Z_1)) \\ &= \Pi_{u_1^2}(\Pi(\partial Z_1, \partial Z_1)) \\ &\quad + 2\Pi_{u_1 u_{11}}\left(\Pi_{\Pi(\partial Z_1, \partial Z_1)} Z_1 + \Pi(Z_1, \Pi(\partial Z_1, \partial Z_1))\right) + (4\alpha - 2). \end{aligned}$$

This computation shows what terms need to be considered as part of the enhanced distribution and that $\left\{\tilde{\Pi}_{u_1}(\partial Z_1)\right\}^2$ can indeed be written under the form

$$\left\{\tilde{\Pi}_{u_1}(\partial Z_1)\right\}^2 = \Pi_{\bullet_2} \mathfrak{Y}_2 + \Pi_{\bullet_3} \mathfrak{Y}_3 + (4\alpha - 2). \quad (5.3)$$

The very same kind of computations shows that we have in the end

$$\begin{aligned} (\partial u)^2 &= \Pi_{u_1^2}\left(2\Pi_{\partial Z_1} \partial Z_1 + \Pi(\partial Z_1, \partial Z_1)\right) + \Pi_{\bullet_3} \mathfrak{Y}_3 + (4\alpha - 2) \\ &=: \Pi_{u_1^2} \mathfrak{Y}_2 + \Pi_{\bullet_3} \mathfrak{Y}_3 + (4\alpha - 2), \end{aligned}$$

for some \mathfrak{Y}_i in $L_T^\infty C^{3\alpha-2}$ – and a definition of \mathfrak{Y}_3 different from its definition in equation (5.3).

Step 2 – $g(u)(\partial u)^2$. We finally have the decomposition

$$\begin{aligned} g(u)(\partial u)^2 &= \Pi_{g(u)}\left(\Pi_{u_1^2} \mathfrak{Y}_2\right) + \Pi_{\Pi_{u_1^2} \mathfrak{Y}_2}(g(u)) + \Pi\left(g(u), \Pi_{\bullet_2} \mathfrak{Y}_2\right) + \Pi_{g(u)\bullet_3} \mathfrak{Y}_3 \\ &\quad + (4\alpha - 2) \\ &= \Pi_{g(u)u_1^2} \mathfrak{Y}_2 + \Pi_{g(u)}\left(\Pi_{\mathcal{D}(u_1^2)} \mathfrak{Y}_2\right) + \Pi_{g'(u)u_1^3}\left(\Pi_{\mathfrak{Y}_2} Z_1 + \Pi(Z_1, \mathfrak{Y}_2)\right) \\ &\quad + \Pi_{g(u)\bullet_3} \mathfrak{Y}_3 + (4\alpha - 2) \\ &= \Pi_{g(u)u_1^2} \mathfrak{Y}_2 + \Pi_{2g(u)u_1 u_{11}}\left(\Pi_{\mathcal{D}Z_1} \mathfrak{Y}_2\right) + \Pi_{g'(u)u_1^3}\left(\Pi_{\mathfrak{Y}_2} Z_1 + \Pi(Z_1, \mathfrak{Y}_2)\right) \\ &\quad + \Pi_{g(u)\bullet_3} \mathfrak{Y}_3 + (4\alpha - 2), \end{aligned}$$

in the required form (5.2). ▷

It is easy, although a bit tedious, to give from that point on an explicit description of the space of enhanced distributions for equation (5.1), and prove its well-posed character in the present second order paracontrolled setting. It is of fundamental interest that the solution map for the equation is a continuous solution of the enhanced distribution and the sufficiently regular initial condition.

- It is elementary to describe the **symmetry group** of the generalized KPZ equation, in the present mild setting where $\alpha > \frac{1}{2}$. As in section 4.4, one can indeed write the right hand side $f(u)\zeta + g(u)(\partial u)^2$ of the generalized KPZ equation under the form

$$f(u)\zeta + g(u)(\partial u)^2 = \mathsf{H}(\hat{u}, \zeta) + \mathsf{K}(\hat{u})\hat{\zeta},$$

for some continuous functions H , of \hat{u} and $\zeta \in L_T^\infty C^{\alpha-2}$, and a continuous function K of \hat{u} and $\hat{\zeta}$ that is linear with respect to $\hat{\zeta}$. Such a decomposition for the product $f(u)\zeta$ was given in section 4.4, and an elementary computations shows that one has

$$\begin{aligned} (\partial u)^2 &= (\checkmark) + u_1^2 \Pi(\partial Z_1, \partial Z_1) + 2(u_1 + \partial u_1)u_{11} \mathsf{C}(Z_1, \partial Z_1, \partial Z_1) + 2u_1 \partial u_1 \Pi(\partial Z_1, Z_1) \\ &\quad + 2u_1 u_{11} \mathsf{C}(Z_1, Z_1, \partial Z_1) + 2u_1^2 \Pi(\partial Z_1, \partial Z_2) + (4\alpha - 2), \end{aligned}$$

with

$$\begin{aligned} (\checkmark) &:= 2\Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right) + \Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{\partial u_1}(Z_1)\right) + \Pi_{\tilde{\Pi}_{\partial u_1}(Z_1)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right) \\ &\quad + \Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{u_2}(\partial Z_2)\right) + \Pi_{\tilde{\Pi}_{u_2}(\partial Z_2)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right) \\ &= 2\Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right) + (3\alpha - 2), \end{aligned}$$

with $(3\alpha - 2)$ a continuous function of \hat{u} and ζ . Note that the term $\mathbf{C}(Z_1, Z_1, \partial Z_1)$ in the formula for $(\partial u)^2$ has positive Hölder regularity, so it will be part of $\mathbf{H}(\hat{u}, \zeta)$, after multiplication by $g(u)$. Now, for the term (\checkmark) , we have

$$\begin{aligned} g(u) \Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right) &= \Pi_{g(u)}\left(\Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right)\right) + \Pi_{\Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}(\tilde{\Pi}_{u_1}(\partial Z_1))}(g(u)) \\ &\quad + g'(u)u_1\Pi\left(Z_1, \Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right)\right) + (4\alpha - 2); \end{aligned}$$

the first two paraproducts are continuous functions of \hat{u} and ζ , and, since $2\alpha - 1$ is positive,

$$\begin{aligned} g'(u)u_1\Pi\left(Z_1, \Pi_{\tilde{\Pi}_{u_1}(\partial Z_1)}\left(\tilde{\Pi}_{u_1}(\partial Z_1)\right)\right) &= \tilde{\Pi}_{u_1}(\partial Z_1)g'(u)u_1\Pi\left(Z_1, \tilde{\Pi}_{u_1}(\partial Z_1)\right) + g'(u)u_1\mathbf{C}\left(\tilde{\Pi}_{u_1}(\partial Z_1), Z_1, \tilde{\Pi}_{u_1}(\partial Z_1)\right) \\ &= \tilde{\Pi}_{u_1}(\partial Z_1)g'(u)u_1\Pi\left(Z_1, \tilde{\Pi}_{u_1}(\partial Z_1)\right) + g'(u)u_1^3\mathbf{C}(\partial Z_1, Z_1, \partial Z_1) + (4\alpha - 2) \\ &= \left(u_1\partial Z_1 + (2\alpha - 1)\right)g'(u)u_1\left(u_1\Pi(Z_1, \partial Z_1) + (3\alpha - 1)\right) \\ &\quad + g'(u)u_1^3\mathbf{C}(\partial Z_1, Z_1, \partial Z_1) \\ &= g'(u)u_1^3\left((\partial Z_1)\Pi(Z_1, \partial Z_1) + \mathbf{C}(\partial Z_1, Z_1, \partial Z_1)\right) + (4\alpha - 2) \end{aligned}$$

for some continuous function (\dots) of \hat{u} and ζ . In the end, we have

$$\begin{aligned} \mathbf{K}(\hat{u})\hat{\zeta} &= \mathbf{F}(\hat{u})\hat{\zeta} + g(u)\left\{u_1^2\Pi(\partial Z_1, \partial Z_1) + 2(u_1 + \partial u_1)u_{11}\mathbf{C}(Z_1, \partial Z_1, \partial Z_1)\right. \\ &\quad \left.+ 2u_1\partial u_1\Pi(\partial Z_1, Z_1) + 2u_1^2\Pi(\partial Z_1, \partial Z_2)\right\} \\ &\quad + 2g'(u)u_1^3\left((\partial Z_1)\Pi(Z_1, \partial Z_1) + \mathbf{C}(\partial Z_1, Z_1, \partial Z_1)\right), \end{aligned}$$

with the function \mathbf{F} that appears in the decomposition of $f(u)\zeta$ given in lemma 9. Note that the additional terms that appear in this formula for \mathbf{K} , compared to the formula for \mathbf{F} , are precisely those terms that are needed to make sense of the products

$$(\partial Z_1)^2, Z_1(\partial Z_1)^2, (\partial Z_1)(\partial Z_2), Z_1\partial Z_1, Z_1^2\partial Z_1,$$

once again in accordance with the theory of regularity structures.

List the elements of $\hat{\zeta}$ in non-decreasing order of regularity. Buiding on the continuity of the solution map for the generalized KPZ equation, one readily sees the effect on the dynamics of a renormalisation procedure of the form

$$\mathcal{M} : \hat{\zeta} \mapsto T\hat{\zeta} - \Xi,$$

for some upper triangular constant matrix T , with a unit diagonal, and some possibly space-time dependent renormalisation functions/constants Ξ . If ζ^ε stands for a regularized noise, with associated canonical enhancement $\hat{\zeta}^\varepsilon$, and if $\mathcal{M}^\varepsilon\hat{\zeta}^\varepsilon$ converges

in probability to some limit element in the space of enhanced distributions for the generalized KPZ equation (5.1), then the solution to the well-posed equation

$$(\partial_t + L)u^\varepsilon = f(u^\varepsilon)\zeta^\varepsilon + g(u^\varepsilon)(\partial u^\varepsilon)^2 + K(u^\varepsilon)(\mathcal{M}^\varepsilon - \text{Id})\widehat{\zeta}^\varepsilon$$

converges in probability to the first component of the solution to the generalized KPZ equation constructed in the present third order paracontrolled setting. (The different components of \widehat{u}^ε are all explicit functions of u^ε , which is why we abuse slightly notations above and write $K(u^\varepsilon)$ instead of $K(\widehat{u}^\varepsilon)$.)

A Details on the parabolic setting

For the reader's convenience, we recall in this Appendix a number of notions/facts introduced and studied in detail in our previous work [2], with the hope that this will make the reading of the present work self-contained. We refer the reader to [2] for the proofs of the different statements given here. We describe in section A.1 a class of operators with some cancellation property. Parabolic Hölder spaces are described in section A.2, together with the fundamental Schauder estimates in this scale of spaces. We introduce the pair $(\Pi, \widetilde{\Pi})$ of paraproducts in section A.3. The statements given here are explicitly used in the proofs of the continuity results of section 3, given in Appendix C.

We use the notations introduced in section 1.2 and assume the operator L satisfies the assumption stated there. Recall in particular that we denote by e a generic element of the parabolic space \mathcal{M} .

A.1 Approximation operators

The use of paraproducts and other kind of singular operators involve the fundamental notion of approximation operators, of which we discuss some aspects in this section.

The following parabolic Gaussian-like kernels $(\mathcal{G}_t)_{0 < t \leq 1}$ will be used as reference kernels. For $0 < t \leq 1$ and $\sigma \leq \tau$, set

$$\mathcal{G}_t((\tau, x), (\sigma, y)) := \nu\left(B_{\mathcal{M}}((\tau, x), \sqrt{t})\right)^{-1} \left(1 + c \frac{\rho((\tau, x), (\sigma, y))^2}{t}\right)^{-\ell_1}$$

and set $\mathcal{G}_t \equiv 0$ if $\tau \leq \sigma$. We do not emphasize the dependence of \mathcal{G} on the positive constant c in the above definition, and we shall allow ourselves to abuse notations and write \mathcal{G}_t for two functions corresponding to two different values of that constant. So we have for instance, for $s, t \in (0, 1)$, the estimate

$$\int_{\mathcal{M}} \mathcal{G}_t((\tau, x), (\sigma, y)) \mathcal{G}_s((\sigma, y), (\lambda, z)) \nu(d\sigma dy) \lesssim \mathcal{G}_{t+s}((\tau, x), (\lambda, z)). \quad (\text{A.1})$$

Presently, note that a large enough choice of constant ℓ_1 ensures that we have

$$\sup_{t \in (0, 1]} \sup_{(\tau, x) \in \mathcal{M}} \int_{\mathcal{M}} \mathcal{G}_t((\tau, x), (\sigma, y)) \nu(d\sigma dy) < \infty,$$

so any linear operator on \mathcal{M} , with a kernel pointwisely bounded by some \mathcal{G}_t is bounded in $L^p(\nu)$ for every $p \in [1, \infty]$.

Definition. We shall denote throughout by \mathbf{G} the set of families $(\mathcal{P}_t)_{0 < t \leq 1}$ of linear operators on \mathcal{M} with kernels pointwisely bounded by

$$\left| K_{\mathcal{P}_t}(e, e') \right| \lesssim \mathcal{G}_t(e, e').$$

Given a real-valued integrable function ϕ on \mathbf{R} , set

$$\phi_t(\cdot) := \frac{1}{t} \phi\left(\frac{\cdot}{t}\right);$$

the family $(\phi_t)_{0 < t \leq 1}$ is uniformly bounded in $L^1(\mathbf{R})$. We also define the ‘‘convolution’’ operator ϕ^\star associated with ϕ via the formula

$$\phi^\star(f)(\tau) := \int_0^\infty \phi(\tau - \sigma) f(\sigma) d\sigma.$$

Note that if ϕ has support in \mathbf{R}_+ , then the operator ϕ^\star has a kernel supported on the same set $\{(\sigma, \tau) ; \sigma \leq \tau\}$ as our Gaussian-like kernel. Moreover, we let the reader check that if ϕ_1, ϕ_2 are two L^1 -functions with ϕ_2 supported on $[0, \infty)$ then

$$(\phi_1 * \phi_2)^\star = \phi_1^\star \circ \phi_2^\star,$$

where $\phi_1 * \phi_2$ stand for the usual convolution of ϕ_1 and ϕ_2 .

Given an integer $b \geq 1$, we define a special family of operators on $L^2(M)$ setting

$$Q_t^{(b)} := \gamma_b^{-1} (tL)^b e^{-tL} \quad \text{and} \quad -t\partial_t P_t^{(b)} = Q_t^{(b)},$$

with $\gamma_b := (b-1)!$; so $P_t^{(b)}$ is an operator of the form $p_b(tL)e^{-tL}$, for some polynomial p_b of degree $b-1$, with value 1 in 0. Under the assumptions on L stated in section 1.2, the operators $P_t^{(b)}$ and $Q_t^{(b)}$ both satisfy the Gaussian regularity estimates

$$\left| K_{t^{\frac{|J|}{2}} V_I R}(x, y) \right| \vee \left| K_{t^{\frac{|J|}{2}} R V_I}(x, y) \right| \lesssim \frac{1}{\mu(B(x, \sqrt{t}))} e^{-c \frac{d(x, y)^2}{t}},$$

with R standing here for $P_t^{(b)}$ or $Q_t^{(b)}$, as well as the pointwise regularity estimates. For $d(x, z) \leq \sqrt{t}$, we have

$$\left| K(x, y) - K(z, y) \right| \lesssim \frac{d(y, z)}{\sqrt{t}} \frac{1}{V(x, \sqrt{t})} e^{-c \frac{d(x, y)^2}{t}},$$

where K is the kernel of either $t^{\frac{|J|}{2}} V_I R$ or $t^{\frac{|J|}{2}} R V_I$.

The parameters b and ℓ_1 are chosen large enough and fixed once and for all – see [2] to see how this choice needs to be done. The reader should just keep in mind that the higher b and ℓ_1 are, the higher order of regularity we can deal with. In our applications, we need all the objects to have a regularity order in the range $(-3, 3)$, so b and ℓ_1 are chosen big enough to allow for this range in all the following continuities result.

Definition. Let an integer $a \in \llbracket 0, 2b \rrbracket$ be given. The following collection of families of operators is called the **standard collection of operators with cancellation of order a** , denoted by StGC^a . It is made up of all the space-time operators

$$\left((t^{\frac{|J|}{2}} V_J)(tL)^{\frac{a-|J|-2k}{2}} P_t^{(c)} \otimes m_t^\star \right)_{0 < t \leq 1}$$

where k is an integer with $2k + |J| \leq a$, and $c \in \llbracket 1, b \rrbracket$, and m is any smooth function supported on $[\frac{1}{2}, 2]$ such that

$$\int \tau^i m(\tau) d\tau = 0, \tag{A.2}$$

for all $0 \leq i \leq k - 1$, with the first b derivatives bounded by 1. These operators are uniformly bounded in $L^p(\mathcal{M})$ for every $p \in [1, \infty]$, as functions of the scaling parameter t . We also set

$$\text{StGC}^{[0,2b]} := \bigcup_{0 \leq a \leq 2b} \text{StGC}^a.$$

The above mentioned cancellation effect is quantified by the property (A.3) stated in Proposition 16 below; note here that it makes sense at an intuitive level to say that $L^{\frac{a-|J|-2k}{2}}$ encodes cancellation in the space-variable of order $a - |J| - 2k$, that V_J encodes a cancellation in space of order $|J|$ and that the moment condition (A.2) encodes a cancellation property in the time-variable of order k for the convolution operator m_t^\star . Since we are in the parabolic scaling, a cancellation of order k in time corresponds to a cancellation of order $2k$ in space, so that $V_J L^{\frac{a-|J|-2k}{2}} P_t^{(c)} \otimes m_t^\star$ has a space-time cancellation property of order a . We give one more definition before stating the cancellation property.

Definition. Given an operator $Q := V_I \phi(L)$, with $|I| \geq 1$, defined by functional calculus from some appropriate function ϕ , we write Q^\bullet for the **formal dual operator**

$$Q^\bullet := \phi(L) V_I.$$

For $I = \emptyset$, and $Q = \phi(L)$, we set $Q^\bullet := Q$. For an operator Q as above we set

$$(Q \otimes m^\star)^\bullet := Q^\bullet \otimes m^\star.$$

Note that the above definition is *not* related to any classical notion of duality and let emphasize that we do *not assume* that L is self-adjoint in $L^2(\mu)$. This notation is only used to indicate that a Q_t operator, resp. a Q_t^\bullet operator, can be composed on the right, resp. on the left, by another operator $\psi(L)$, for a suitable function ψ , due to the functional calculus on L .

Proposition 16. Consider $\mathcal{Q}^1 \in \text{StGC}^{a_1}$ and $\mathcal{Q}^2 \in \text{StGC}^{a_2}$ two standard collections with cancellation, and set $a := \min(a_1, a_2)$. Then for every $s, t \in (0, 1]$, the composition $\mathcal{Q}_s^1 \circ \mathcal{Q}_t^{2\bullet}$ has a kernel pointwisely bounded by

$$\left| K_{\mathcal{Q}_s^1 \circ \mathcal{Q}_t^{2\bullet}}(e, e') \right| \lesssim \left(\frac{ts}{(s+t)^2} \right)^{\frac{a}{2}} \mathcal{G}_{t+s}(e, e'). \quad (\text{A.3})$$

The above mentioned *orthogonality* property of standard operators with cancellation is encoded in the factor $\left(\frac{ts}{(s+t)^2} \right)^{\frac{a}{2}}$ that appears in the above estimate. This factor is small as soon as s or t is small compared to the other.

Definition. Let $0 \leq a \leq 2b$ be an integer. We define the subset GC^a of \mathbf{G} of **families of operators with the cancellation property of order a** as the set of elements \mathcal{Q} of \mathbf{G} with the following cancellation property. For every $0 < s, t \leq 1$ and every standard family $\mathcal{S} \in \text{StGC}^{a'}$, with $a' \in \llbracket a, 2b \rrbracket$, the operator $\mathcal{Q}_t \circ \mathcal{S}_s^\bullet$ has a kernel pointwisely bounded by

$$\left| K_{\mathcal{Q}_t \circ \mathcal{S}_s^\bullet}(e, e') \right| \lesssim \left(\frac{st}{(s+t)^2} \right)^{\frac{a}{2}} \mathcal{G}_{t+s}(e, e'). \quad (\text{A.4})$$

We introduced above the operators $Q_t^{(b)}$ and $P_t^{(b)}$ acting on functions/distributions on M ; we now their parabolic counterpart. Choose arbitrarily a smooth real-valued

function φ on \mathbf{R} , with support in $[\frac{1}{2}, 2]$, unit integral and such that for every integer $k = 1, \dots, b$

$$\int \tau^k \varphi(\tau) d\tau = 0.$$

Set

$$\mathcal{P}_t^{(b)} := P_t^{(b)} \otimes \varphi_t^* \quad \text{and} \quad \mathcal{Q}_t^{(b)} := -t \partial_t \mathcal{P}_t^{(b)}.$$

An easy computation yields that

$$\mathcal{Q}_t^{(b)} = Q_t^{(b)} \otimes \varphi_t^* + P_t^{(b)} \otimes \psi_t$$

where $\psi(\sigma) = \varphi(\sigma) + \sigma \varphi'(\sigma)$. Note that, from its very definition, a parabolic operator $\mathcal{Q}_t^{(b)}$ belongs at least to \mathbf{GC}^2 , for $b \geq 2$. Note also that due to the normalization of φ , then for every $f \in L^p(\mathbf{R})$ supported on $[0, \infty)$ then we have the L^p convergence

$$\varphi_t^*(f) \xrightarrow[t \rightarrow 0]{} f.$$

So, the operators \mathcal{P}_t weakly tend to the identity on $L_0^p(\mathcal{M})$ (the set of functions $f \in L^p(\mathcal{M})$ with time-support included in $[0, \infty)$), $p \in [1, \infty)$, and the set of functions $f \in C^0(\mathcal{M})$ with time-support included in $[0, \infty)$, as t goes to 0; so we have the following **Calderón reproducing formula**. For every continuous function $f \in L^\infty(\mathcal{M})$ with time-support in $[0, \infty)$, then

$$f = \int_0^1 \mathcal{Q}_t^{(b)}(f) \frac{dt}{t} + \mathcal{P}_1^{(b)}(f). \quad (\text{A.5})$$

Noting that the measure $\frac{dt}{t}$ gives unit mass to intervals of the form $[2^{-i-1}, 2^{-i}]$, and considering the operator $\mathcal{Q}_t^{(b)}$ as a kind of multiplier roughly localized at frequencies of size $t^{-\frac{1}{2}}$, Calderón's formula appears as nothing else than a continuous time analogue of the Paley-Littlewood decomposition of f , with $\frac{dt}{t}$ in the role of the counting measure.

A.2 Parabolic Hölder spaces and Schauder estimates

We recall in this section the definitions and basic properties of the space and space-time weighted Hölder spaces, with possibly negative regularity index. We also recall the fundamental regularization properties of the heat operator, quantified by Schauder estimates.

Let us start recalling the following well-known facts about Hölder space on M , and single out a good class of weights on M . Given $0 < \alpha \leq 1$, the classical metric Hölder space H^α is defined as the set of real-valued functions f on M with finite H^α -norm, defined by the formula

$$\|f\|_{H^\alpha} := \|f\|_{L^\infty(M)} + \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty.$$

Definition. For $\alpha \in (-3, 3)$, define $C^\alpha := C^\alpha(M)$ as the closure of the set of bounded and continuous functions for C^α -norm, defined by the formula

$$\|f\|_{C^\alpha} := \|e^{-L} f\|_{L^\infty(M)} + \sup_{0 < t \leq 1} t^{-\frac{\alpha}{2}} \left\| Q_t^{(a)} f \right\|_{L^\infty(M)};$$

this norm does not depend on the integer $a > \frac{|\alpha|}{2}$, and the two spaces H^α and C^α coincide and have equivalent norms when $0 < \alpha < 1$ – see for instance [1].

These notions have parabolic counterparts which we now introduce. Recall we work with the parabolic space $\mathcal{M} = [0, T] \times M$, for a finite time horizon; the introduction of a time weight in the next definition thus has no effect on the space involved, nor on its topology. Its introduction happens however to be a convenient freedom which allows to simplify a number of arguments. Let then a positive parameter κ be given and denote by w the weight

$$w(\tau) := e^{\kappa\tau}. \quad (\text{A.6})$$

For $0 < \alpha \leq 1$, the metric parabolic Hölder space $\mathcal{H}^\alpha = \mathcal{H}^\alpha(\mathcal{M})$ is defined as the set of all functions on \mathcal{M} with finite \mathcal{H}^α -norm, defined by the formula

$$\|f\|_{\mathcal{H}^\alpha} := \|w^{-1}f\|_{L^\infty(\mathcal{M})} + \sup_{0 < \rho((\tau,x),(\sigma,y)) \leq 1; \tau \geq \sigma} \frac{|f(\tau,x) - f(\sigma,y)|}{w^{-1}(\tau) \rho((\tau,x),(\sigma,y))^\alpha}.$$

As in the above space setting one can recast this definition in a more functional setting, using the parabolic standard operators. A set of distributions was introduced in [2], whose precise definition is irrelevant here.

Definition. For $\alpha \in (-3, 3)$, we define the parabolic Hölder space $\mathcal{C}^\alpha := \mathcal{C}^\alpha(\mathcal{M})$ as the closure, in the set of distributions, of the set of bounded and continuous functions on \mathcal{M} for the \mathcal{C}^α -norm, defined by

$$\|f\|_{\mathcal{C}^\alpha} := \sup_{\substack{\mathcal{Q} \in \mathcal{S}\mathcal{O}^k \\ 0 \leq k \leq 2b}} \|w^{-1}\mathcal{Q}_1(f)\|_{L^\infty(\mathcal{M})} + \sup_{\substack{\mathcal{Q} \in \mathcal{S}\mathcal{O}^k \\ |\alpha| < k \leq 2b}} \sup_{0 < t \leq 1} t^{-\frac{\alpha}{2}} \|w^{-1}\mathcal{Q}_t(f)\|_{L^\infty(\mathcal{M})}.$$

We write \mathcal{C}_w if we want to emphasize the dependence of the norm on w . The following result was proved in [2] building on Calderón's formula (A.5).

Proposition 17. Given $\alpha \in (0, 2)$, set

$$\mathcal{E}^\alpha := \left(C_\tau^{\alpha/2} L_x^\infty \right) \cap \left(L_\tau^\infty C_x^\alpha \right),$$

and endow this space with its natural norm. Then \mathcal{E}^α is continuously embedded into \mathcal{C}^α . Furthermore, if $\alpha \in (0, 1)$, the spaces $\mathcal{E}^\alpha, \mathcal{C}^\alpha$ and \mathcal{H}^α are equal, with equivalent norms.

The weighted version $\left(L_\tau^\infty C_x^\alpha \right)_w$ of $L_\tau^\infty C_x^\alpha$ is the same space, equipped with the norm

$$\|f\|_{\left(L_\tau^\infty C_x^\alpha \right)_w} := \sup_{0 \leq \tau \leq T} e^{-\kappa\tau} \|f(\tau, \cdot)\|_{\mathcal{C}^\alpha}.$$

We use in the body of the work the following regularization properties of the heat operator associated with L – it is proved under this form in section 3.4 of [2]. This property is used crucially in the fixed point argument in the resolution process of singular PDEs in our paracontrolled setting.

Theorem 18 (Schauder estimates). For any choice of parameters β and $\varepsilon > 0$, such that $-2 + 2\varepsilon < \beta < 0$, we have

$$\|\mathcal{R}(v)\|_{\mathcal{C}_w^{\beta+2-2\alpha-2\varepsilon}} \lesssim_T \kappa^{-\varepsilon} \|v\|_{\left(L_T^\infty C_x^\beta \right)_w}.$$

Before turning to the definition of an intertwined pair of parabolic paraproducts we close this section with two other useful continuity properties involving the Hölder spaces \mathcal{C}_w^σ – recall the manifold M is compact.

Proposition 19. *Given $\alpha \in (0, 1)$, a space-time weight ω , some integer $a \geq 0$ and a standard family $\mathcal{P} \in \text{StGC}^a$, there exists a constant c depending only on the weight ω , such that*

$$\omega(\tau)^{-1} \left| (\mathcal{P}_t f)(e) - (\mathcal{P}_s f)(e') \right| \lesssim (s + t + \rho(e, e')^2)^{\frac{\alpha}{2}} \|f\|_{\mathcal{C}_\omega^\alpha},$$

uniformly in $s, t \in (0, 1]$ and $e = (\tau, x)$ and $e' = (\sigma, y) \in \mathcal{M}$, with $\tau \geq \sigma$.

A.3 Parabolic paraproducts

We give a quick presentation in this subsection of the pair of intertwined paraproducts introduced in [2], following the semigroup approach developed first in [1]. The starting point for the introduction of the operator Π is Calderón's reproducing formula (A.5). Using iteratively the Leibniz rule for the differentiation operators V_i or ∂_τ , we have the following decomposition

$$fg = \sum_{\mathcal{I}_b} a_{k,\ell}^{I,J} \int_0^1 \left(\mathcal{A}_{k,\ell}^{I,J}(f, g) + \mathcal{A}_{k,\ell}^{I,J}(g, f) \right) \frac{dt}{t} + \sum_{\mathcal{I}_b} b_{k,\ell}^{I,J} \int_0^1 \mathcal{B}_{k,\ell}^{I,J}(f, g) \frac{dt}{t},$$

where

- \mathcal{I}_b is the set of all tuples (I, J, k, ℓ) with the tuples I, J and the integers k, ℓ satisfying the constraint

$$\frac{|I| + |J|}{2} + k + \ell = \frac{b}{2};$$

- $a_{k,\ell}^{I,J}, b_{k,\ell}^{I,J}$ are bounded sequences of numerical coefficients;
- for $(I, J, k, \ell) \in \mathcal{I}_b$, $\mathcal{A}_{k,\ell}^{I,J}(f, g)$ has the form

$$\mathcal{A}_{k,\ell}^{I,J}(f, g) := \mathcal{P}_t^{(b)} \left(t^{\frac{|I|}{2} + k} V_I \partial_\tau^k \right) \left(\mathcal{S}_t^{(b/2)} f \cdot (t^{\frac{|J|}{2} + \ell} V_J \partial_\tau^\ell) \mathcal{P}_t^{(b)} g \right)$$

with $\mathcal{S}^{(b/2)} \in \text{GC}^{b/2}$;

- for $(I, J, k, \ell) \in \mathcal{I}_b$, $\mathcal{B}_{k,\ell}^{I,J}(f, g)$ has the form

$$\mathcal{B}_{k,\ell}^{I,J}(f, g) := \mathcal{S}_t^{(b/2)} \left(\left\{ (t^{\frac{|I|}{2} + k} V_I \partial_\tau^k) \mathcal{P}_t^{(b)} f \right\} \cdot \left\{ (t^{\frac{|J|}{2} + \ell} V_J \partial_\tau^\ell) \mathcal{P}_t^{(b)} g \right\} \right)$$

with $\mathcal{S}^{(b/2)} \in \text{GC}^{b/2}$.

Definition. *Given f in $\bigcup_{s \in (0,1)} \mathcal{C}^s$ and $g \in L^\infty(\mathcal{M})$, we define the **paraproduct** $\Pi_g^{(b)} f$ by the formula*

$$\Pi_g^{(b)} f := \int_0^1 \left\{ \sum_{\mathcal{I}_b; \frac{|I|}{2} + k > \frac{b}{4}} a_{k,\ell}^{I,J} \mathcal{A}_{k,\ell}^{I,J}(f, g) + \sum_{\mathcal{I}_b; \frac{|I|}{2} + k > \frac{b}{4}} b_{k,\ell}^{I,J} \mathcal{B}_{k,\ell}^{I,J}(f, g) \right\} \frac{dt}{t},$$

and the **resonant term** $\Pi^{(b)}(f, g)$ by the formula

$\Pi^{(b)}(f, g) :=$

$$\int_0^1 \left\{ \sum_{\mathcal{I}_b; \frac{|I|}{2} + k \leq \frac{b}{4}} a_{k,\ell}^{I,J} \left(\mathcal{A}_{k,\ell}^{I,J}(f, g) + \mathcal{A}_{k,\ell}^{I,J}(g, f) \right) + \sum_{\mathcal{I}_b; \frac{|I|}{2} + k = \frac{|J|}{2} + \ell = \frac{b}{4}} b_{k,\ell}^{I,J} \mathcal{B}_{k,\ell}^{I,J}(f, g) \right\} \frac{dt}{t}.$$

With these notations, Calderón's formula becomes

$$fg = \Pi_g^{(b)}(f) + \Pi_f^{(b)}(g) + \Pi^{(b)}(f, g) + \Delta_{-1}(f, g)$$

with the "low-frequency part"

$$\Delta_{-1}(f, g) := \mathcal{P}_1^{(b)} \left(\mathcal{P}_1^{(b)} f \cdot \mathcal{P}_1^{(b)} g \right).$$

If b is chosen large enough, then all the operators involved in the paraproduct and resonant terms have a kernel pointwisely bounded by a kernel \mathcal{G}_t at the right scaling. Moreover,

- (a) the paraproduct term $\Pi_g^{(b)}(f)$ is a finite linear combination of operators of the form

$$\int_0^1 \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2 f \cdot \mathcal{P}_t^1 g \right) \frac{dt}{t}$$

with $\mathcal{Q}^1, \mathcal{Q}^2 \in \text{StGC}^{\frac{b}{4}}$, and $\mathcal{P}^1 \in \text{StGC}$,

- (b) the resonant term $\Pi^{(b)}(f, g)$ is a finite linear combination of operators of the form

$$\int_0^1 \mathcal{P}_t^1 \left(\mathcal{Q}_t^1 f \cdot \mathcal{Q}_t^2 g \right) \frac{dt}{t}$$

with $\mathcal{Q}^1, \mathcal{Q}^2 \in \text{StGC}^{\frac{b}{4}}$ and $\mathcal{P}^1 \in \text{StGC}$.

We invite the reader to see what happens of all this when working with in the flat torus with its associated Laplacian. Note also that $\Pi_f^{(b)}(\mathbf{1}) = \Pi^{(b)}(f, \mathbf{1}) = 0$, and that we have the identity

$$\Pi_{\mathbf{1}}^{(b)}(f) = f - \mathcal{P}_{\mathbf{1}}^{(b)} \mathcal{P}_{\mathbf{1}}^{(b)} f,$$

as a consequence of our choice of the renormalizing constant. Therefore the paraproduct with the constant function $\mathbf{1}$ is equal to the identity operator, up to the strongly regularizing operator $\mathcal{P}_{\mathbf{1}}^{(b)} \mathcal{P}_{\mathbf{1}}^{(b)}$. The regularity properties of the paraproduct and resonant operators can be described as follows; it behaves as it classical, Fourier-based, counterpart (2.1).

Proposition 20. (a) *For every real-valued regularity exponent α, β , and every positive regularity exponent γ , we have*

$$\|\Delta_{-1}(f, g)\|_{\mathcal{C}^\gamma} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}$$

for every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$.

- (b) *For every $\alpha \in (-3, 3)$ and $f \in \mathcal{C}^\alpha$, we have*

$$\left\| \Pi_g^{(b)}(f) \right\|_{\mathcal{C}^\alpha} \lesssim \|g\|_\infty \|f\|_{\mathcal{C}^\alpha}$$

for every $g \in L^\infty$, and

$$\left\| \Pi_g^{(b)}(f) \right\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|g\|_{\mathcal{C}^\beta} \|f\|_{\mathcal{C}^\alpha}$$

for every $g \in \mathcal{C}^\beta$ with $\beta < 0$ and $\alpha + \beta \in (-3, 3)$.

- (c) *For every $\alpha, \beta \in (-\infty, 3)$ with $\alpha + \beta > 0$, we have the continuity estimate*

$$\left\| \Pi^{(b)}(f, g) \right\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}$$

for every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$.

Definition. We define a **modified paraproduct** $\tilde{\Pi}^{(b)}$ setting

$$\tilde{\Pi}_g^{(b)}(f) := \mathcal{R}\left(\Pi_g^{(b)}(\mathcal{L}f)\right).$$

The next proposition shows that if one chooses the parameters ℓ_1 that appears in the reference kernels \mathcal{G}_t , and the exponent b in the definition of the paraproduct large enough, then the modified paraproduct $\tilde{\Pi}_g^{(b)}(\cdot)$ has the same algebraic/analytic properties as $\Pi_g^{(b)}(\cdot)$.

Proposition. • For a large enough choice of constants ℓ_1 and b , the modified paraproduct $\tilde{\Pi}_g f$ is a finite linear combination of operators of the form

$$\int_0^1 \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2 f \cdot \mathcal{P}_t^1 g \right) \frac{dt}{t}$$

with $\mathcal{Q}^1 \in \text{GC}^{\frac{b}{8}-2}$, $\mathcal{Q}^2 \in \text{StGC}^{\frac{b}{4}}$ and $\mathcal{P}^1 \in \text{StGC}$.

- For every $\alpha \in (-3, 3)$ and $\varepsilon \in (0, 1)$ with $\alpha - \varepsilon \in (-3, 3)$ and $f \in \mathcal{C}^\alpha$, we have

$$\left\| \tilde{\Pi}_g^{(b)}(f) \right\|_{\mathcal{C}_w^{\alpha-\varepsilon}} \lesssim \kappa^{-\varepsilon} \|w^{-1}g\|_\infty \|f\|_{\mathcal{C}^\alpha},$$

for every $g \in L^\infty$.

Note that the norm $\|f\|_{\mathcal{C}^\alpha}$ above has no weight. Note here the normalization identity

$$\tilde{\Pi}_1^{(b)} f = f - \mathcal{R}\mathcal{P}_1^{(b)}\mathcal{P}_1^{(b)}(\mathcal{L}f)$$

for every distribution in $f \in \mathcal{S}'_0$; it reduces to

$$\tilde{\Pi}_1^{(b)} f = f - \mathcal{P}_1^{(b)}\mathcal{P}_1^{(b)}(f)$$

if $f|_{\tau=0} = 0$.

Following the definition of the **inner difference operator** \mathcal{D} given in subsection 3.2, we extend it to a parabolic version by defining $\mathcal{D} (= \mathcal{D}_{e'})$ by the formula

$$\iint_{\mathcal{M}^2} (\mathcal{D}f)(e')g(e')\nu(de)\nu(de') := \iint_{\mathcal{M}^2} (f(e') - f(e))g(e)\nu(de)\nu(de');$$

with this notation, the crucial motivating relation

$$\Pi_j\left(\tilde{\Pi}_a(g)\right) - \Pi_{fa}(g) = \Pi_f\left(\tilde{\Pi}_{\mathcal{D}a}(g)\right)$$

holds indeed.

Last, we prove an elementary property of the modified paraproduct that provides some pointwise information on the solutions to singular PDEs constructed via paracontrolled calculus.

Proposition 21. Let α be a positive regularity exponent, and let $u, v, Z \in \mathcal{C}^\alpha$ be given, with $Z(0, \cdot) = 0$. Assume that

$$u - \tilde{\Pi}_v Z \in \mathcal{C}^{2\alpha},$$

and define $\beta := \min(2\alpha, 1)$. If $\alpha \neq \frac{1}{2}$, we have

$$|u(e) - u(e') - v(e)(Z(e) - Z(e'))| \lesssim \rho(e, e')^\beta,$$

uniformly in $e, e' \in \mathcal{M}$ with $\rho(e, e') \leq 1$. If $\alpha = \frac{1}{2}$, we have a logarithmic loss

$$|u(e) - u(e') - v(e)(Z(e) - Z(e'))| \lesssim \rho(e, e') \log \left(1 + \rho(e, e')^{-1} \right).$$

Proof – Due to the assumption, one has

$$|u(e) - u(e') - v(e)(Z(e) - Z(e'))| \lesssim \rho(e, e')^\beta + (\star)$$

with

$$(\star) := \left| \left(\tilde{\Pi}_v Z \right)(e) - \left(\tilde{\Pi}_v Z \right)(e') - v(e)(Z(e) - Z(e')) \right|.$$

Using Calderón reproducing formula, or the normalization which yields

$$\tilde{\Pi}_1 Z = Z$$

since $Z(0, \cdot) = 0$, we see that (\star) is equal to

$$\left| \int_0^1 \mathcal{Q}_t^\bullet [\mathcal{Q}_t Z \mathcal{P}_t v](e) - \mathcal{Q}_t^\bullet [\mathcal{Q}_t Z \mathcal{P}_t v](e') - v(e) \mathcal{Q}_t^\bullet [\mathcal{Q}_t Z](e) + v(e) \mathcal{Q}_t^\bullet [\mathcal{Q}_t Z](e') \frac{dt}{t} \right|,$$

so

$$(\star) \lesssim \int_0^1 \left| \int (K_{\mathcal{Q}_t^\bullet}(e, a) - K_{\mathcal{Q}_t^\bullet}(e', a)) \mathcal{Q}_t Z(a) (\mathcal{P}_t v(a) - v(e)) \nu(da) \right| \frac{dt}{t}.$$

Using the regularity estimates on v and on the kernel of the approximation operators, one sees that

$$\begin{aligned} (\star) &\lesssim \|v\|_{\mathcal{C}^\alpha} \int_0^1 \int \min \left\{ 1, \frac{\rho(e, e')}{\sqrt{t}} \right\} \mathcal{G}_t(e, a) |\mathcal{Q}_t Z(a)| (t + \rho(a, e)^2)^{\beta/4} \nu(da) \frac{dt}{t} \\ &\lesssim \|v\|_{\mathcal{C}^\alpha} \|Z\|_{\mathcal{C}^\alpha} \int_0^{\rho^2} t^{(2\alpha+\beta)/4} \frac{dt}{t} + \|v\|_{\mathcal{C}^\alpha} \|Z\|_{\mathcal{C}^\alpha} \int_{\rho^2}^1 \int \frac{\rho(e, e')}{\sqrt{t}} t^{(2\alpha+\beta)/4} \frac{dt}{t} \\ &\lesssim \|v\|_{\mathcal{C}^\alpha} \|Z\|_{\mathcal{C}^\alpha} \rho(e, e')^\beta, \end{aligned}$$

which concludes the proof. \triangleright

The next proposition gets its flavour from the remark that a function defined up to some remainder by a paraproduct may have different derivatives. Consider for example real-valued functions on the interval $(0, 1)$, and take $Z = t$. A smooth function u of time, seen as an element of \mathcal{C}^α , with $0 < \alpha < 1$, satisfies both

$$u = \Pi_0 Z + (2\alpha)$$

and

$$u = \Pi_1 Z + (2\alpha) = Z + (2\alpha),$$

since Z itself can go inside the remainder (2α) . In other terms, the derivative of a paracontrolled function is not generically determined by the function itself. This happens, however, if the reference function Z is sufficiently 'wiggly'. Let a positive index β be given. Following Friz and Shekar in their study of controlled paths [12], we say that a parabolic function Z is **β -truly rough** at space-time point e if

$$\limsup_{e' \rightarrow e} \frac{|Z(e') - Z(e)|}{d(e', e)^\beta} = \infty.$$

It is said to be β -truly rough if it is β -truly rough at a dense set of points in \mathcal{M} . The following result stating that the derivative of a paracontrolled function is determined by the function itself if the reference function is truly rough comes as a direct consequence of proposition 21.

Corollary. *Let $\alpha < \beta \leq 2\alpha$ be positive exponents. Let $Z \in \mathcal{C}^\alpha$ be a β -truly rough function such that $Z(0, \cdot) = 0$, and let also u, v be elements of \mathcal{C}^α such that*

$$u - \tilde{\Pi}_v Z \in \mathcal{C}^{2\alpha}.$$

Then $v = 0$, if $u = 0$.

It is elementary to proceed as in [12] and check that if ζ stands for a d -dimension space white noise in M , for $d = 2$ or 3 , then $\mathcal{R}(\zeta)$ is almost surely $(4 - d)^-$ -truly rough. A sufficient condition for a function for being truly rough is provided by Hairer-Pillai's notion of θ -rough function [21]; see for instance section 6.4 of Friz-Hairer's lecture notes [11]. It may be interesting to note that Norris lemma holds in that case, giving a control of the L^∞ -norm of v in terms of the modulus of continuity of u and the 2α -norm of $(u - \tilde{\Pi}_v Z)$. The proof that Brownian motion is Hölder rough given in section 6.5 of [11] shows that $Z = \mathcal{R}(\zeta)$ is Hölder rough if ζ stands for space white noise in the flat torus, with L its associated Laplace operator. We shall show elsewhere that this result also holds true in our closed manifold setting, as expected.

B Taylor expansion formula

We give in this section a detailed and rigorous proof of Theorem 1. The parameter b is fixed, and we note Π for $\Pi^{(b)}$.

Theorem 22 (Higher order Taylor expansion). *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a C^4 function, and let u be a real-valued and C^α function on \mathcal{M} , with $\alpha \in (0, 1)$. Then*

$$\begin{aligned} f(u) &= \Pi_{f'(u)}(u) + \frac{1}{2} \left\{ \Pi_{f^{(2)}(u)}(u^2) - 2\Pi_{f^{(2)}(u)u}(u) \right\} \\ &+ \frac{1}{3!} \left\{ \Pi_{f^{(3)}(u)}(u^3) - 3\Pi_{f^{(3)}(u)u}(u^2) + 3\Pi_{f^{(3)}(u)u^2}(u) \right\} + f(u)^\sharp \end{aligned} \quad (\text{B.1})$$

for some remainder $f(u)^\sharp \in \mathcal{C}^{4\alpha}$. If moreover f is of class C^5 , then the remainder term $f(u)^\sharp$ is locally-Lipschitz with respect to u , in the sense that

$$\|f(u)^\sharp - f(v)^\sharp\|_{\mathcal{C}^{4\alpha}} \lesssim (1 + \|u\|_{\mathcal{C}^\alpha} + \|v\|_{\mathcal{C}^\alpha})^4 \|u - v\|_{\mathcal{C}^\alpha}.$$

Proof – Let us give a detailed proof of the third order expansion, that claims that

$$(\star) := f(u) - \Pi_{f'(u)}(u) - \frac{1}{2} \left\{ \Pi_{f^{(2)}(u)}(u^2) - 2\Pi_{f^{(2)}(u)u}(u) \right\}$$

is a 3α -Hölder function. We invite the reader to follow what comes next in the light of the proof given in section 2 in the time-independent, flat, model setting of the torus.

As, by definition, the paraproduct operator $\Pi_g(\cdot)$ is a finite sum of different terms, each of them of the form

$$\mathcal{A}_g^1(\cdot) := \int_0^1 \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2(\cdot) \mathcal{P}_t^1(g) \right) \frac{dt}{t},$$

with $\mathcal{Q}^1, \mathcal{Q}^2$ at least to StGC^3 , it is sufficient to prove that the following function

$$\begin{aligned} (\star) &:= f(u) - \int_0^1 \left[\mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2(u) \mathcal{P}_t^1(f'(u)) \right) + \frac{1}{2} \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2(u^2) \mathcal{P}_t^1(f^{(2)}(u)) \right) \right. \\ &\quad \left. - \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2(u) \mathcal{P}_t^1(f^{(2)}(u)u) \right) \right] \frac{dt}{t} \end{aligned}$$

is an element of $\mathcal{C}^{3\alpha}$. Using Calderón's reproducing formula together with the normalization of the paraproduct, we have

$$f(u) \simeq \int_0^1 \mathcal{Q}_t^{1\bullet} \mathcal{Q}_t^2(f(u) \mathcal{P}_t^1(1)) \frac{dt}{t}$$

up to a remainder quantity corresponding to the low frequency part that is as smooth as we want. So one can write (\star) under the form

$$(\star) = \int_0^1 \mathcal{Q}_t^{1\bullet}(\varepsilon_t) \frac{dt}{t}, \quad (\text{B.2})$$

with

$$\begin{aligned} \varepsilon_t := & \mathcal{Q}_t^2(f(u)) \mathcal{P}_t^1(1) - \mathcal{Q}_t^2(u) \mathcal{P}_t^1(f'(u)) \\ & - \frac{1}{2} \mathcal{Q}_t^2(u^2) \mathcal{P}_t^1(f^{(2)}(u)) + \mathcal{Q}_t^2(u) \mathcal{P}_t^1(f^{(2)}(u)u). \end{aligned}$$

Due to the orthogonality/cancellation property of the operators $\mathcal{Q}_t^{1\bullet}$, it suffices for us to get an L^∞ control of ε_t . Using the kernel representation of the different operators, we have for every $e \in \mathcal{M}$

$$\begin{aligned} \varepsilon_t(e) = & \iint_{\mathcal{M}^2} K_{\mathcal{Q}_t^2}(e, e') K_{\mathcal{P}_t^1}(e, e'') \left\{ f(u(e')) - u(e') f'(u(e'')) \right. \\ & \left. - \frac{1}{2} u^2(e') f^{(2)}(u(e'')) + u(e') f^{(2)}(u(e'')) u(e'') \right\} \nu(de') \nu(de'') \end{aligned}$$

Note also that we have from the usual Taylor formula for f

$$\begin{aligned} & f(u(e')) - u(e') f'(u(e'')) - \frac{1}{2} u^2(e') f^{(2)}(u(e'')) + u(e') f^{(2)}(u(e'')) u(e'') \\ & = \iiint_{[0,1]^3} f^{(3)}\left(u(e'') + \alpha\beta\gamma(u(e') - u(e''))\right) \beta\gamma(u(e') - u(e''))^3 d\alpha d\beta d\gamma \\ & \quad + f(u(e'')) + u(e'') f'(u(e'')) + \frac{1}{2} u^2(e'') f^{(2)}(u(e'')). \end{aligned}$$

When we integrate against $K_{\mathcal{Q}_t^2}(e, e') K_{\mathcal{P}_t^1}(e, e'')$ a quantity depending only in e'' has no contribution, since the latter kernel satisfies a cancellation property along the e' -variable; so we have exactly

$$\begin{aligned} \varepsilon_t(e) = & \iint_{\mathcal{M}^2} K_{\mathcal{Q}_t^2}(e, e') K_{\mathcal{P}_t^1}(e, e'') \\ & \left(\iiint_{[0,1]^3} f^{(3)}\left(u(e'') + \alpha\beta\gamma(u(e') - u(e''))\right) \beta\gamma(u(e') - u(e''))^3 d\alpha d\beta d\gamma \right) \nu(de') \nu(de''). \end{aligned}$$

Since $K_{\mathcal{Q}_t^2}$ and $K_{\mathcal{P}_t^1}$ are both pointwisely dominated by the Gaussian kernel \mathcal{G}_t , and using the fact that $f^{(3)}$ is bounded on the range of u , we obtain the uniform control

$$\begin{aligned} |\varepsilon_t(e)| & \lesssim \iint_{\mathcal{M}^2} \mathcal{G}_t(e, e') \mathcal{G}_t(e, e'') (u(e') - u(e''))^3 \nu(de') \nu(de'') \\ & \lesssim \|u\|_{\mathcal{C}^\alpha}^3 t^{3\alpha/2}, \end{aligned}$$

from which the fact that (\star) belongs to $\mathcal{C}^{3\alpha}$ follows from (B.2). We used for that purpose the identity

$$u(e') - u(e'') = (u(e') - u(e)) + (u(e) - u(e'')),$$

together with Proposition 17 on the characterization of parabolic regularity in terms of increments, to see that

$$|u(e') - u(e'')| \lesssim (d(e', e) + d(e'', e))^\alpha \|f\|_{\mathcal{C}^\alpha}.$$

The fourth order expansion of the statement is proved by a very similar reasoning left to the reader. ▷

C Continuity results

Recall the definitions of the corrector

$$\mathcal{C}(f, g; h) := \Pi \left(\tilde{\Pi}_f(g), h \right) - f \Pi(g, u),$$

and the (modified) commutators

$$\begin{aligned} \mathcal{D}(f, g; h) &= \Pi \left(\tilde{\Pi}_f(g), h \right) - \Pi_f \left(\Pi(g, h) \right), \\ \mathcal{T}_u(g, f) &:= \Pi_u \left(\tilde{\Pi}_g(f) \right) - \Pi_g \left(\Pi_u(f) \right), \end{aligned}$$

and their iterates, introduced in section 3; they are initially defined on the space of smooth functions. We prove in this last Appendix the continuity results on these operators stated in section 3.

C.1 Boundedness of commutators/correctors

We start by looking at the case of the operator \mathcal{T} .

Proposition 23. *• Let α, β, γ be Hölder regularity exponents with $\alpha \in (-3, 3)$, $\beta \in (0, 1)$ and $\gamma \in (-\infty, 0)$. Then if*

$$\alpha + \beta < 3, \quad \text{and} \quad \delta := \alpha + \beta + \gamma \in (-3, 3),$$

we have

$$\|\mathcal{T}_u(g, f)\|_{\mathcal{C}^\delta} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|u\|_{\mathcal{C}^\gamma}, \quad (\text{C.1})$$

for every $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$ and $u \in \mathcal{C}^\gamma$; so the modified commutator on para-products extends naturally into a trilinear continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{C}_\omega^\delta$.

- *If $\gamma = 0$ then the product ug has a sense for $u \in L^\infty(\mathcal{M})$ and $g \in \mathcal{C}^\beta$, and we have*

$$\|\mathcal{R}(u, g; f)\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|u\|_{L^\infty}. \quad (\text{C.2})$$

Proof – Recall that the operators $\Pi_g^{(b)}(\cdot)$, respectively $\tilde{\Pi}_g^{(b)}(\cdot)$, are given by a finite sum of operators of the form

$$\mathcal{A}_g^1(\cdot) := \int_0^1 \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2(\cdot) \mathcal{P}_t^1(g) \right) \frac{dt}{t},$$

respectively

$$\tilde{\mathcal{A}}_g^1(\cdot) := \int_0^1 \tilde{\mathcal{Q}}_t^{1\bullet} \left(\tilde{\mathcal{Q}}_t^2(\cdot) \mathcal{P}_t^1(g) \right) \frac{dt}{t},$$

where $\mathcal{Q}^1, \mathcal{Q}^2, \tilde{\mathcal{Q}}^2$ belong at least to StGC^3 and $\tilde{\mathcal{Q}}^1$ is an element of GC^3 . We describe similarly the operator $\Pi_u^{(b)}(\cdot)$ as a finite sum of operators of the form

$$\mathcal{A}_u^2(\cdot) := \int_0^1 \mathcal{Q}_t^{3\bullet} \left(\mathcal{Q}_t^4(\cdot) \mathcal{P}_t^2(u) \right) \frac{dt}{t}.$$

Thus, we need to study a generic modified commutator

$$\mathcal{A}_u^2 \left(\tilde{\mathcal{A}}_g^1(f) \right) - \mathcal{A}_g^1 \left(\mathcal{A}_u^2(f) \right),$$

and introduce for that purpose the intermediate quantity

$$\mathcal{E}(f, g, u) := \int_0^1 \mathcal{Q}_s^{3\bullet} \left(\mathcal{Q}_s^4(f) \cdot \mathcal{P}_s^1(g) \cdot \mathcal{P}_s^2(u) \right) \frac{ds}{s}.$$

Note here that due to the normalization $\Pi_{\mathbf{1}} \simeq \text{Id}$, up to some strongly regularizing operator, there is no loss of generality in assuming that

$$\int_0^1 \tilde{\mathcal{Q}}_t^{1\bullet} \tilde{\mathcal{Q}}_t^2 \frac{dt}{t} = \int_0^1 \mathcal{Q}_t^{1\bullet} \mathcal{Q}_t^2 \frac{dt}{t} = \int_0^1 \mathcal{Q}_t^{3\bullet} \mathcal{Q}_t^4 \frac{dt}{t} = \text{Id}. \quad (\text{C.3})$$

Step 1. Study of $\mathcal{A}_u^2 \left(\tilde{\mathcal{A}}_g^1(f) \right) - \mathcal{E}(f, g, u)$. We shall use a family \mathcal{Q} in StGC^a , for some $a > |\delta|$, to control the Hölder norm of that quantity. By definition, and using the normalization (C.3), the quantity $\mathcal{Q}_r \left(\mathcal{A}_u^2 \left(\tilde{\mathcal{A}}_g^1(f) \right) - \mathcal{E}(f, g, u) \right)$ is, for every $r \in (0, 1)$, equal to

$$\begin{aligned} & \int_0^1 \int_0^1 \mathcal{Q}_r \mathcal{Q}_s^{3\bullet} \left\{ \mathcal{Q}_s^4 \tilde{\mathcal{Q}}_t^{1\bullet} \left(\tilde{\mathcal{Q}}_t^2(f) \mathcal{P}_t^1(g) \right) \cdot \mathcal{P}_s^2(u) \right\} \frac{ds dt}{st} - \int_0^1 \mathcal{Q}_r \mathcal{Q}_s^{3\bullet} \left(\mathcal{Q}_s^4(f) \cdot \mathcal{P}_s^1(g) \cdot \mathcal{P}_s^2(u) \right) \frac{ds}{s} \\ &= \int_0^1 \int_0^1 \mathcal{Q}_r \mathcal{Q}_s^{3\bullet} \left\{ \mathcal{Q}_s^4 \tilde{\mathcal{Q}}_t^{1\bullet} \left(\tilde{\mathcal{Q}}_t^2(f) (\mathcal{P}_t^1(g) - \mathcal{P}_s^1(g)) \right) \cdot \mathcal{P}_s^2(u) \right\} \frac{ds dt}{st}, \end{aligned}$$

where in the last line the variable of $\mathcal{P}_s^1(g)$ is that of $\mathcal{Q}_s^{3\bullet}$, and so it is frozen through the action of $\tilde{\mathcal{Q}}_s^4 \mathcal{Q}_t^{1\bullet}$. Then using that $g \in \mathcal{C}^\beta$ with $\beta \in (0, 1)$, we know by Proposition 19 that we have, for $\tau \geq \sigma$,

$$\left| (\mathcal{P}_s^1 g)(x, \tau) - (\mathcal{P}_t^1 g)(y, \sigma) \right| \lesssim \left(s + t + \rho((x, \tau), (y, \sigma))^2 \right)^{\frac{\beta}{2}} \|g\|_{\mathcal{C}^\beta}.$$

Note that it follows from equation (A.1) that the kernel of $\mathcal{Q}_s^4 \tilde{\mathcal{Q}}_t^{1\bullet}$ is pointwisely bounded by \mathcal{G}_{t+s} , and allowing different constants in the definition of \mathcal{G} , we have

$$\mathcal{G}_{t+s}((x, \tau), (y, \sigma)) (s + t + d(x, y)^2)^{\frac{\beta}{2}} \lesssim (s + t)^{\frac{\beta}{2}} \mathcal{G}_{t+s}((x, \tau), (y, \sigma)). \quad (\text{C.4})$$

So using the cancellation property of the operators \mathcal{Q} , resp. \mathcal{Q}^i and $\tilde{\mathcal{Q}}^i$, at an order no less than a , resp. 3, we deduce that

$$\begin{aligned} & \left\| \mathcal{Q}_r \left(\mathcal{A}_u^2 \left(\tilde{\mathcal{A}}_g^1(f) \right) - \mathcal{E}(f, g, u) \right) \right\|_\infty \\ & \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|u\|_{\mathcal{C}^\gamma} \int_0^1 \int_0^1 \left(\frac{sr}{(s+r)^2} \right)^{\frac{a}{2}} \left(\frac{st}{(s+t)^2} \right)^{\frac{3}{2}} t^{\frac{a}{2}} (s+t)^{\frac{\beta}{2}} s^{\frac{\gamma}{2}} \frac{ds dt}{st}, \end{aligned}$$

where we used that γ is negative to control $\mathcal{P}_s^2(u)$. The integral over $t \in (0, 1)$ can be computed since $\alpha > -3$ and $\alpha + \beta < 3$, and we have

$$\begin{aligned} & \left\| \mathcal{Q}_r \left(\mathcal{A}_u^2 \left(\tilde{\mathcal{A}}_g^1(f) \right) - \mathcal{E}(f, g, u) \right) \right\|_\infty \\ & \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|u\|_{\mathcal{C}^\gamma} \int_0^1 \int_0^1 \left(\frac{sr}{(s+r)^2} \right)^{\frac{\alpha}{2}} s^{\frac{\delta}{2}} \frac{ds}{s} \\ & \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|u\|_{\mathcal{C}^\gamma} r^{\frac{\delta}{2}}, \end{aligned}$$

uniformly in $r \in (0, 1)$ because $|a| > \delta$. That concludes the estimate for the high frequency part. We repeat the same reasoning for the low-frequency part by replacing \mathcal{Q}_r with \mathcal{Q}_1 and conclude that

$$\left\| \mathcal{A}_u^2 \left(\tilde{\mathcal{A}}_g^1(f) \right) - \mathcal{E}(f, g, u) \right\|_{\mathcal{C}^\delta} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|u\|_{\mathcal{C}^\gamma}.$$

Step 2. Study of $\mathcal{A}_g^1(\mathcal{A}_u^2(f)) - \mathcal{E}(f, g, u)$. This term is almost the same as that of Step 1 and can be treated in exactly the same way. Note that $\mathcal{Q}_r(\mathcal{A}_g^1(\mathcal{A}_u^2(f)) - \mathcal{E}(f, g, u))$ is equal, for every $r \in (0, 1)$, to

$$\begin{aligned} & \int_0^1 \int_0^1 \mathcal{Q}_r \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2 \mathcal{Q}_s^{3\bullet} \left[\mathcal{Q}_s^4(f) \mathcal{P}_s^2(u) \right] \cdot \mathcal{P}_t^1(g) \right) \frac{ds dt}{st} - \int_0^1 \mathcal{Q}_r \mathcal{Q}_s^{3\bullet} \left(\mathcal{Q}_s^4(f) \cdot \mathcal{P}_s^1(g) \cdot \mathcal{P}_s^2(u) \right) \frac{ds}{s} \\ & = \int_0^1 \int_0^1 \mathcal{Q}_r \mathcal{Q}_t^{1\bullet} \left\{ \mathcal{Q}_t^2 \mathcal{Q}_s^{3\bullet} \left(\mathcal{Q}_s^4(f) (\mathcal{P}_t^1(g) - \mathcal{P}_s^1(g)) \cdot \mathcal{P}_s^2(u) \right) \right\} \frac{ds dt}{st}, \end{aligned}$$

where in the last line the variable of $\mathcal{P}_t^1(g)$ is that of $\mathcal{Q}_t^{1\bullet}$, so it is frozen through the action of $\mathcal{Q}_s^{3\bullet}$. The same proof as in Step 1 can be repeated, which gives the first statement of the theorem.

Step 3. Proof of the second statement. For the second statement, Step 1 still holds. So it only remains to compare $\mathcal{E}(f, g, u)$ with $\mathcal{A}_{ug}^2(f)$. This amounts to compare $\mathcal{P}_t^2(ug)$ with $\mathcal{P}_t^1(g)\mathcal{P}_t^2(u)$. Using the regularity of $g \in \mathcal{C}^\beta$ and the uniform boundedness of $u \in L^\infty$, we get

$$\left\| \mathcal{P}_t^2(ug) - \mathcal{P}_t^1(g)\mathcal{P}_t^2(u) \right\|_{L^\infty} \lesssim t^{\beta/2}$$

which allows us to conclude. ▷

Remark 24. *The above proof actually shows the following property of the operator*

$$\bar{\mathbb{T}}_{u,f} := g \mapsto \mathbb{T}_u(g, f)$$

where $f \in \mathcal{C}^\alpha$ and $u \in \mathcal{C}^\nu$ are fixed. For all families $\mathcal{Q}^1, \mathcal{Q}^2 \in \text{GC}^a$ for some $a > 0$, the linear operator $\mathcal{Q}_t^1 \bar{\mathbb{T}}_{u,f} \mathcal{Q}_s^{2\bullet}$ has a kernel pointwisely bounded by

$$(t+s)^{\frac{\beta+\nu}{2}} \left(\frac{st}{(s+t)^2} \right)^{\frac{\alpha}{2}} \mathcal{G}_{t+s}(e, e') \|f\|_{\mathcal{C}^\alpha} \|u\|_{\mathcal{C}^\nu}.$$

Proposition 25. • Let α, β, γ be Hölder regularity exponents with $\alpha \in (0, 1), \beta \in (-3, 3)$ and $\gamma \in (-\infty, 3]$. Set

$$\delta := (\alpha + \beta) \wedge 3 + \gamma.$$

If

$$0 < \alpha + \beta + \gamma < 1 \quad \text{and} \quad \beta + \gamma < 0$$

then the corrector \mathbf{C} extends continuously into a trilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to \mathcal{C}^δ .

- If α, β, γ are positive then the commutator \mathbf{D} is a continuous trilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to \mathcal{C}^δ .

Proof – The result on \mathbf{C} was already proved in [1, Proposition 3.6] in a more general setting. We only focus here on proving the boundedness of \mathbf{D} . As already done above, we represent the operator $\Pi_f^{(b)}(\cdot)$ under the form

$$\mathcal{A}_f(\cdot) := \int_0^1 \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2(\cdot) \mathcal{P}_t^1(f) \right) \frac{dt}{t},$$

and the resonant term $\Pi^{(b)}(g, h)$ as

$$\mathcal{B}(g, h) := \int_0^1 \mathcal{P}_t^{2\bullet} \left(\mathcal{Q}_t^3(g) \mathcal{Q}_t^4(h) \right) \frac{dt}{t}.$$

Thus, we need to study a generic modified commutator

$$\begin{aligned} (\star) &:= \mathcal{B}(\mathcal{A}_f(g), h) - \mathcal{A}_f(\mathcal{B}(g, h)) \\ &= \int_0^1 \int_0^1 \mathcal{P}_t^{2\bullet} \left(\mathcal{Q}_t^3 \mathcal{Q}_s^{1\bullet} \left(\mathcal{Q}_s^2(g) \mathcal{P}_t^1(f) \right) \mathcal{Q}_t^4(h) \right) \frac{ds}{s} \frac{dt}{t} \\ &\quad - \int_0^1 \int_0^1 \mathcal{Q}_s^{1\bullet} \left(\mathcal{Q}_s^2 \mathcal{P}_t^{2\bullet} \left(\mathcal{Q}_t^3(g) \mathcal{Q}_t^4(h) \right) \mathcal{P}_s^1(f) \right) \frac{ds}{s} \frac{dt}{t}, \end{aligned}$$

and introduce for that purpose the intermediate quantity

$$\mathcal{E}(f, g, h) := \int_0^1 \mathcal{P}_t^{2\bullet} \left(\mathcal{P}_t^1(f) \mathcal{Q}_t^3(g) \mathcal{Q}_t^4(h) \right) \frac{dt}{t}.$$

Then we compare the two quantities with $\mathcal{E}(f, g, h)$, such as done previously. Each of these two comparisons makes appear an exact commutation on the function f , due to our choice of normalization for our paraproducts. Using the \mathcal{C}^α regularity on f together with the cancellation property of the \mathcal{Q} operators, we get

$$\begin{aligned} \|\mathcal{Q}_r(\star)\|_{L^\infty} &\lesssim \int_0^1 \int_0^1 \left(\frac{r}{r+t} \right)^3 \left(\frac{st}{(s+t)^2} \right)^3 s^{\beta/2} t^{\gamma/2} (s+t)^{\alpha/2} \frac{dt}{t} \frac{ds}{s} \\ &\quad + \int_0^1 \int_0^1 \left(\frac{rs}{(r+s)^2} \right)^3 \left(\frac{s}{s+t} \right)^3 t^{\beta/2} t^{\gamma/2} (s+t)^{\alpha/2} \frac{dt}{t} \frac{ds}{s} \\ &\lesssim \int_0^1 \left(\frac{r}{r+t} \right)^3 t^{(\alpha+\beta+\gamma)/2} \frac{dt}{t} + \int_0^1 \left(\frac{r}{r+t} \right)^3 t^{\beta/2} t^{\gamma/2} (r+t)^{\alpha/2} \frac{dt}{t} \\ &\lesssim r^{\delta/2}, \end{aligned}$$

which shows that (\star) belongs to \mathcal{C}^δ .

▷

C.2 Boundedness of iterated commutators/correctors

We now turn to the study of the continuity properties of the iterated versions of commutators/correctors, and start with the (modified) iterated commutator on paraproducts.

Proposition 26. • Let $\alpha, \beta, \gamma, \nu$ be Hölder regularity exponents with $\alpha \in (-3, 3)$, $\beta, \gamma \in (0, 1)$ and $\nu \in (-\infty, 0)$. Then if

$$\alpha + \beta + \gamma < 3, \quad \text{and} \quad \delta := \alpha + \beta + \gamma + \nu \in (-3, 3),$$

we have

$$\|\mathbb{T}_u(h, g; f)\|_{\mathcal{C}^\delta} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma} \|u\|_{\mathcal{C}^\nu}, \quad (\text{C.5})$$

for every $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$, $h \in \mathcal{C}^\gamma$ and $u \in \mathcal{C}^\nu$; so the commutator defines a trilinear continuous map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma \times \mathcal{C}^\nu$ to $\mathcal{C}_\omega^\delta$.

- A similar result holds for the 5-linear iterate of \mathbb{T} .

Proof – Fix some functions $u \in \mathcal{C}^\nu$ and $f \in \mathcal{C}^\alpha$; we have

$$\mathbb{T}_u(h, g; f) := \mathbb{T}_u(\tilde{\Pi}_h g, f) - \Pi_h(\mathbb{T}_u(g, f)).$$

With the same notations as in the proof of Proposition 23, for which we have relations (C.3), we write

$$\begin{aligned} \Pi_h[\mathbb{T}_u(g, f)] &= \int_0^1 \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2[\mathbb{T}_u(g, f)] \cdot \mathcal{P}_t^1 h \right) \frac{dt}{t} \\ &= \int_0^1 \int_0^1 \mathcal{Q}_t^{1\bullet} \left(\mathcal{Q}_t^2[\mathbb{T}_u(\tilde{\mathcal{Q}}_s^{1\bullet} \tilde{\mathcal{Q}}_s^2 g, f)] \cdot \mathcal{P}_t^1 h \right) \frac{ds dt}{s t}. \end{aligned}$$

Expanding $\mathbb{T}_u(\tilde{\Pi}_h g, f)$ correspondingly, we get

$$\mathbb{T}_u(h, g; f) = \int_0^1 \int_0^1 \mathcal{Q}_t^{1\bullet} \left\{ \mathcal{Q}_t^2[\mathbb{T}_u(\tilde{\mathcal{Q}}_s^{1\bullet} \tilde{\mathcal{Q}}_s^2 g, f)] \cdot (\mathcal{P}_t^1 h - \mathcal{P}_s^1 h) \right\} \frac{ds dt}{s t}, \quad (\text{C.6})$$

where the variable of $\mathcal{P}_t^1 h$ is that of $\mathcal{Q}_t^{1\bullet}$. Since h belongs to \mathcal{C}^γ , with $\gamma \in (0, 1)$, we know from Proposition 19 that

$$\left| (\mathcal{P}_t^1 h)(e) - (\mathcal{P}_s^1 h)(e') \right| \lesssim (t + s + \rho(e, e')^2)^{\frac{\gamma}{2}} \|h\|_{\mathcal{C}^\gamma},$$

for all $e, e' \in \mathcal{M}$. As above, fix a collection \mathcal{Q} of $s\text{fStGC}^a$, for some $a > 3$, to control Hölder norms. We need to estimate

$$\left\| \mathcal{Q}_r \mathbb{T}_u(h, g; f) \right\|_{L^\infty(\mathcal{M})}.$$

Using decomposition (C.6), we have

$$\left\| \mathcal{Q}_r \mathbb{T}_u(h, g; f) \right\|_{L^\infty(\mathcal{M})} \lesssim \int_0^1 \int_0^1 \left(\frac{rt}{(r+t)^2} \right)^{\frac{a}{2}} I_{s,t} \frac{ds dt}{s t}, \quad (\text{C.7})$$

where

$$I_{s,t} := \sup_{e \in \mathcal{M}} \mathcal{Q}_t^2 \left[\mathbb{T}_u \left(\tilde{\mathcal{Q}}_s^{1\bullet} \tilde{\mathcal{Q}}_s^2 g, f \right) \cdot (\mathcal{P}_t^1 h(e) - \mathcal{P}_s^1 h) \right](e).$$

Due to Remark 24, we have a pointwise estimate of the kernel of the operator $\mathcal{Q}_t^2 \mathbb{T}_u(\mathcal{Q}_s^{1\bullet}(\cdot), f)$, so with the pointwise regularity estimate on h and (C.4), we deduce that

$$\begin{aligned} I_{s,t} &\lesssim (s+t)^{\frac{\alpha+\gamma+\nu}{2}} \|\mathcal{Q}_s^2 g\|_{L^\infty} \|f\|_{\mathcal{C}^\alpha} \|h\|_{\mathcal{C}^\gamma} \|u\|_{\mathcal{C}^\nu} \\ &\lesssim (s+t)^{\frac{\delta}{2}} \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma} \|u\|_{\mathcal{C}^\nu}. \end{aligned}$$

It follows from that estimate and the fact that $|\sigma| < a$, that

$$\left\| \mathcal{Q}_r \mathbb{T}_u(h, g; f) \right\|_{L^\infty(\mathcal{M})} \lesssim r^{\frac{\delta}{2}} \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma} \|u\|_{\mathcal{C}^\nu},$$

uniformly in $r \in (0, 1)$. A similar analysis of the low frequency of $\mathbb{T}_u(h, g; f)$ can be done and completes the proof of the Hölder estimate. \triangleright

Proposition 27. Let $\alpha, \beta \in (0, 1)$, $\nu_1 \in (-3, 3)$ and $\nu_2 \in (-\infty, 3]$. Assume that $\alpha + \beta + \nu_1 < 3$ with

$$\delta := \alpha + \beta + \nu_1 + \nu_2 \in (0, 1), \quad \alpha + \nu_1 + \nu_2 < 0 \quad \text{and} \quad \beta + \nu_1 + \nu_2 < 0.$$

Then the iterated corrector \mathbb{C} is a continuous trilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^{\nu_1} \times \mathcal{C}^{\nu_2}$ to \mathcal{C}^δ .

Proof – Fix some functions $f \in \mathcal{C}^\alpha$ and $h \in \mathcal{C}^{\nu_2}$ and define the operator

$$\bar{\mathbb{C}} : \phi \mapsto \mathbb{C}(f, \phi; h),$$

so that

$$\mathbb{C}(f; a, b; h) = \bar{\mathbb{C}}\left(\tilde{\Pi}_a(b)\right) - a \bar{\mathbb{C}}(b).$$

Using the same notation as previously, and omitting for convenience the indices on the different collections \mathcal{Q} and \mathcal{P} , we write

$$\begin{aligned} \bar{\mathbb{C}}\left(\tilde{\Pi}_a(b)\right) &= \int_0^1 \bar{\mathbb{C}} \tilde{\mathcal{Q}}_s^\bullet \left(\tilde{\mathcal{Q}}_s b \cdot \mathcal{P}_s a \right) \frac{ds}{s}, \\ a \bar{\mathbb{C}}(b) &= a \bar{\mathbb{C}}\left(\tilde{\Pi}_1(b)\right) = a \int_0^1 \bar{\mathbb{C}} \tilde{\mathcal{Q}}_s^\bullet \left(\tilde{\mathcal{Q}}_s b \cdot \mathcal{P}_s \mathbf{1} \right) \frac{ds}{s}. \end{aligned}$$

Note that due to the conservation property of the heat semigroup associated with L , the quantity $\mathcal{P}_s \mathbf{1}$ is either constant equal to 1 or to 0, depending on whether \mathcal{P}_s encodes some cancellation or not. Thus, given $e = (x, \tau) \in \mathcal{M}$, and setting

$$F_{s,e} := \tilde{\mathcal{Q}}_s b \cdot (\mathcal{P}_s a - \mathcal{P}_s \mathbf{1}) \cdot b(e),$$

we have

$$\mathbb{C}(f; a, b, ; h)(e) = \bar{\mathbb{C}}\left(\tilde{\Pi}_a(b)\right)(e) - a(e) \bar{\mathbb{C}}(b)(e) = \int_0^1 \bar{\mathbb{C}}\left(\tilde{\mathcal{Q}}_s^\bullet F_{s,e}\right)(e) \frac{ds}{s}.$$

As before, we can use that $a \in \mathcal{C}^\beta$, with $\beta \in (0, 1)$. We have for $e, e' \in \mathcal{M}$ and $s > 0$

$$|a(e) - a(e')| \lesssim \rho(e, e')^\beta \|a\|_{\mathcal{C}^\beta},$$

and therefore, using the ‘‘Gaussian bounds’’ for \mathcal{P}_s ,

$$|(\mathcal{P}_s a)(e') - (\mathcal{P}_s \mathbf{1})(e') a(e)| \lesssim (s + \rho(e, e')^2)^{\frac{\beta}{2}} \|a\|_{\mathcal{C}^\beta}.$$

As done in the proof of Proposition 25 – see also [1, Proposition 3.6], we introduce an intermediate quantity of the form

$$S(f, b, h) := \int_0^1 \mathcal{P}_t \left(\mathcal{Q}_t b \cdot \mathcal{Q}_t h \cdot \mathcal{P}_t f \right) \frac{dt}{t},$$

and write

$$\begin{aligned} \bar{\mathbb{C}}\left(\tilde{\mathcal{Q}}_s^\bullet F_{s,e}\right)(e) &= \Pi\left(\tilde{\Pi}_f\left(\tilde{\mathcal{Q}}_s^\bullet F_{s,e}\right), h\right)(e) - S\left(f, \left(\tilde{\mathcal{Q}}_s^\bullet F_{s,e}\right), h\right)(e) \\ &\quad + S\left(f, \tilde{\mathcal{Q}}_s^\bullet F_{s,e}, h\right)(e) - f(e) \cdot \Pi\left(\tilde{\mathcal{Q}}_s^\bullet F_{s,e}, h\right)(e) \\ &=: I_1(s) + I_2(s). \end{aligned} \tag{C.8}$$

- We start with the estimate for I_2 . One can then write with generic notations for the resonant term Π

$$\left(S(f, b, h) - f \cdot \Pi(b, h)\right)(e) = \int_0^1 \mathcal{P}_t \left(\mathcal{Q}_t b \cdot \mathcal{Q}_t h \cdot (\mathcal{P}_t f - f(e)) \right)(e) \frac{dt}{t},$$

and it is known that the integrand is pointwisely bounded by $t^{\frac{\alpha+\nu_1+\nu_2}{2}}$. Since this argument only uses pointwise estimates, we can replace b by $\mathcal{Q}_s^\bullet F_{s,e}$. Therefore, by writing

$$\int_0^1 I_2(s) \frac{ds}{s} = \int_0^1 \int_0^1 \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e} \cdot \mathcal{Q}_t h \cdot (\mathcal{P}_t f - f(e)) \right)(e) \frac{dt ds}{t s}$$

and using

$$\|\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet \phi\|_{L^\infty(\mathcal{M})} \lesssim \left(\frac{st}{(s+t)^2} \right)^{3/2} \|\phi\|_{L^\infty(\mathcal{M})}, \quad (\text{C.9})$$

with $\phi = F_{s,e}$, we obtain

$$\begin{aligned} & \left\| \int_0^1 I_2(s) \frac{ds}{s} \right\|_{L^\infty(\mathcal{M})} \\ & \leq \int_0^1 \int_0^1 \left\| e \mapsto \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e} \cdot \mathcal{Q}_t h \cdot (f(e) - \mathcal{P}_t f) \right)(e) \right\|_{L^\infty} \frac{dt ds}{t s} \\ & \lesssim \|b\|_{\mathcal{C}^{\nu_1}} \|a\|_{\mathcal{C}^\beta} \|f\|_{\mathcal{C}^\alpha} \|h\|_{\mathcal{C}^{\nu_2}} \\ & \quad \times \int_0^1 \int_0^1 \left(\frac{st}{(s+t)^2} \right)^{\frac{3}{2}} \mathcal{G}_{t+s}(e, e') \left(s + \rho(e, e')^2 \right)^{\frac{\beta}{2}} s^{\nu_1/2} t^{\frac{\alpha+\nu_2}{2}} \frac{ds dt}{s t} \\ & \lesssim \|f\|_{\mathcal{C}^\alpha} \|a\|_{\mathcal{C}^\beta} \|b\|_{\mathcal{C}^{\nu_1}} \|h\|_{\mathcal{C}^{\nu_2}} \int_0^1 \int_0^1 \left(\frac{st}{(s+t)^2} \right)^{\frac{3}{2}} s^{\nu_1/2} (s+t)^{\beta/2} t^{\frac{\alpha+\nu_2}{2}} \frac{ds dt}{s t} \\ & \lesssim \|f\|_{\mathcal{C}^\alpha} \|a\|_{\mathcal{C}^\beta} \|b\|_{\mathcal{C}^{\nu_1}} \|h\|_{\mathcal{C}^{\nu_2}}, \end{aligned}$$

since $\alpha + \beta + \nu_1 + \nu_2 > 0$.

- Let us now estimate the regularity of $I_2(s)$. Let $e, e' \in \mathcal{M}$ with $\rho(e, e') \leq 1$. We split the integral in t into two parts, corresponding to $t < \rho(e, e')^2$ or $t > \rho(e, e')^2$. In the first case, note that

$$\int_0^{\rho(e, e')^2} t^{(\alpha+\beta+\nu_1+\nu_2)/2} \frac{dt}{t} \lesssim \rho(e, e')^{\alpha+\beta+\nu_1+\nu_2},$$

so that by repeating the arguments above, we get the desired estimate. In the case $t > \rho^2$ with $\rho := \rho(e, e')$, write for $s \in (0, 1)$

$$\begin{aligned}
& \int_{\rho^2}^1 \left\{ \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e} \cdot \mathcal{Q}_t h \cdot (f(e) - \mathcal{P}_t f) \right) (e) \right. \\
& \quad \left. - \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e'} \cdot \mathcal{Q}_t h \cdot (f(e') - \mathcal{P}_t f) \right) (e') \right\} \frac{dt}{t} \\
&= \int_{\rho^2}^1 \left\{ \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e} \cdot \mathcal{Q}_t h \cdot (f(e) - \mathcal{P}_t f) \right) (e) \right. \\
& \quad \left. - \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e} \cdot \mathcal{Q}_t h \cdot (f(e) - \mathcal{P}_t f) \right) (e') \right\} \frac{dt}{t} \\
& \quad + (a(e) - a(e')) \int_{\rho^2}^1 \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet \tilde{\mathcal{Q}}_s b \cdot \mathcal{Q}_t h \cdot (f(e') - \mathcal{P}_t f) \right) (e') \frac{dt}{t} \\
& \quad - (f(e) - f(e')) \int_{\rho^2}^1 \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e} \cdot \mathcal{Q}_t h \right) (e') \frac{dt}{t}. \tag{C.10}
\end{aligned}$$

For the second and third term, we can assume $s \simeq t$ by (C.9). One obtains

$$\begin{aligned}
& |a(e) - a(e')| \int_{\rho^2}^1 \left| \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet \tilde{\mathcal{Q}}_s b \cdot \mathcal{Q}_t h \cdot (f(e') - \mathcal{P}_t f) \right) (e') \right| \frac{dt}{t} \\
& \lesssim \|f\|_{C^\alpha} \|a\|_{C^\beta} \|b\|_{C^{\nu_1}} \|h\|_{C^{\nu_2}} \rho^\beta \int_{\rho^2}^1 t^{\frac{\alpha+\nu_1+\nu_2}{2}} \frac{dt}{t} \\
& \lesssim \|f\|_{C^\alpha} \|a\|_{C^\beta} \|b\|_{C^{\nu_1}} \|h\|_{C^{\nu_2}} \rho^{\alpha+\beta+\nu_1+\nu_2},
\end{aligned}$$

since $\alpha + \nu_1 + \nu_2$ is negative, and

$$\begin{aligned}
& |f(e) - f(e')| \int_{\rho^2}^1 \left| \mathcal{P}_t \left(\mathcal{Q}_t \tilde{\mathcal{Q}}_s^\bullet F_{s,e} \cdot \mathcal{Q}_t h \right) (e') \right| \frac{dt}{t} \\
& \lesssim \|f\|_{C^\alpha} \|a\|_{C^\beta} \|b\|_{C^{\nu_1}} \|h\|_{C^{\nu_2}} \rho^\alpha \int_{\rho^2}^1 t^{\frac{\beta+\nu_1+\nu_2}{2}} \frac{dt}{t} \\
& \lesssim \|f\|_{C^\alpha} \|a\|_{C^\beta} \|b\|_{C^{\nu_1}} \|h\|_{C^{\nu_2}} \rho^{\alpha+\beta+\nu_1+\nu_2},
\end{aligned}$$

since $\beta + \nu_1 + \nu_2$ is also negative. For the first term in (C.10), we now repeat the arguments of the proof of Proposition 25, which rely on the Lipschitz regularity of the heat kernel as well as the fact that $\alpha + \beta + \nu_1 + \nu_2 \in (0, 1)$. Summarising the above, we have shown that for $e, e' \in \mathcal{M}$ with $\rho(e, e') \leq 1$

$$\begin{aligned}
& \left| \int_0^1 \left(I_2(s)(e) - I_2(s)(e') \right) \frac{ds}{s} \right| \\
& \lesssim \rho(e, e')^{\alpha+\beta+\nu_1+\nu_2} \|f\|_{C^\alpha} \|a\|_{C^\beta} \|b\|_{C^{\nu_1}} \|h\|_{C^{\nu_2}}.
\end{aligned}$$

Let us now come to $I_1(s)$ as defined in (C.8). We write with $\phi := \tilde{\mathcal{Q}}_s^\bullet F_{s,e}$

$$\left| \Pi(\tilde{\Pi}_f(\phi), h) - S(f, b, h) \right| \leq \int_0^1 \left| \mathcal{P}_t(A_t(\phi, f) \cdot \mathcal{Q}_t h) \right| \frac{dt}{t}$$

with

$$A_t(\phi, f) := \mathcal{Q}_t \left(\int_0^1 \mathcal{P}_t \tilde{\mathcal{Q}}_r^\bullet (\tilde{\mathcal{Q}}_r \phi \cdot \mathcal{P}_r f) \frac{dr}{r} - \mathcal{P}_t f \mathcal{P}_t \phi \right).$$

Following the proof of Proposition 25, and using (C.9), one obtains

$$\begin{aligned} & \left\| A_t(\tilde{\mathcal{Q}}_s^\bullet F_{s,e}, u) \right\|_{L^\infty(\mathcal{M})} \\ & \lesssim \int_0^1 \left(\frac{rt}{(r+t)^2} \right)^{\frac{3}{2}} \left(\frac{sr}{(s+r)^2} \right)^{\frac{3}{2}} s^{\frac{nu_1}{2}} (r+t)^{\frac{\alpha+\beta}{2}} \frac{dr}{r} \|f\|_{\mathcal{C}^\alpha} \|a\|_{\mathcal{C}^\beta} \|b\|_{\mathcal{C}^{\nu_1}}, \end{aligned}$$

hence

$$\begin{aligned} & \left\| \int_0^1 I_1(s) \frac{ds}{s} \right\|_{L^\infty(\mathcal{M})} \lesssim \|f\|_{\mathcal{C}^\alpha} \|a\|_{\mathcal{C}^\beta} \|b\|_{\mathcal{C}^{\nu_1}} \|h\|_{\mathcal{C}^{\nu_2}} \\ & \quad \times \int_0^1 \int_0^1 \int_0^1 \left(\frac{rt}{(r+t)^2} \right)^{\frac{3}{2}} \left(\frac{sr}{(s+r)^2} \right)^{\frac{3}{2}} s^{\frac{\nu_1}{2}} (r+t)^{\frac{\alpha+\beta}{2}} t^{\frac{\nu_2}{2}} \frac{dr}{r} \frac{ds}{s} \frac{dt}{t}, \end{aligned}$$

and the triple integral is finite since $\alpha + \beta + \nu_1 + \nu_2$ is positive.

- For the regularity estimate of $I_1(s)$, consider

$$\int_0^1 \left\{ \mathcal{P}_t \left(A_t(\tilde{\mathcal{Q}}_s^\bullet F_{s,e}, f) \cdot \mathcal{Q}_t h \right) (e) - \mathcal{P}_t \left(A_t(\tilde{\mathcal{Q}}_s^\bullet F_{s,e'}, f) \cdot \mathcal{Q}_t h \right) (e') \right\} \frac{dt}{t}.$$

The estimate of this expression is similar, though simpler, compared to the one of $I_2(s)$, as here e is frozen only in one spot. As before, one deals with this terms using the heat kernel regularity of \mathcal{P}_t and the regularity estimate for a .

▷

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