Relativistic processes: where we are and what is ahead of us

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Spacetime as a whole: infinitesimal structure

- **1905 - Special relativity. Einstein's picture of spacetime.**

  Postulate. There exists a class of distinguished systems of coordinates of spacetime as a whole ($\mathbb{R}^4$ and not only $\mathbb{R}^3$!) in which the mathematical laws of physics take the same form.

They are called *inertial frames*. Let (x, y, z, t) and (x', y', z', t') be coordinates in two inertial frames.

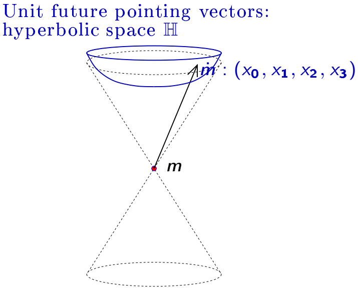
- Free fall = straight line trajectories.
- The speed of light is independent of the inertial frames (Michaelson-Morley experiment).

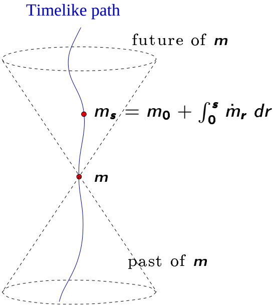
- $A : (x, y, z, t) \mapsto (x', y', z', t')$ is linear.
- $dx^2 + dy^2 + dz^2 - c^2dt^2 = 0$ iff $(dx')^2 + (dy')^2 + (dz')^2 - c^2(dt')^2 = 0$.

Up to a multiplicative constant, $A \in SO(1, 3)$.

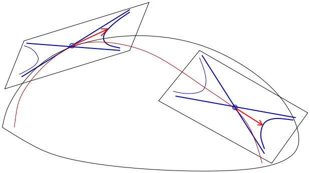
Spacetime as a whole

- **1908 – Minkowski spacetime** $\mathbb{R}^{1,3}$: $g(\dot{m}, \dot{m}) = x_0^2 - x_1^2 - x_2^2 - x_3^2$





- **1915 – General relativity.** To incorporate *gravity* in the picture: **spacetime = Lorentzian manifold**.



Is Lorentzian geometry different from Riemannian geometry?

- **Local objects.** Levi-Civita connection, curvature, geodesics (affine parameter)... are defined as in Riemannian geometry.
- Drastically differs in **global** aspects.
  - **Not a metric space.** Geodesic completeness does not prevent paths of bounded acceleration from exploding!
  - **Causality.** Future and past notions associated with the cone structure.

    Pathologies: there can exist *closed* timelike paths; any compact spacetime has a closed timelike path!

    Led to a very important ladder of *causality conditions*.

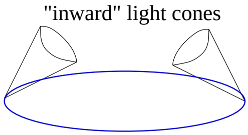
Matter and geometry

- **Einstein's equations.** $Ric - \frac{R}{2}g = T$,

with Ric Ricci tensor, R scalar curvature and T the energy-momentum tensor, given by physics.

- **Big bang, big crunch – Incompleteness issues.** (Hawking, Penrose...) There exists an incomplete lightlike geodesic under the following (physically reasonable!) conditions.
  - *Global/causality condition.* $(\mathbb{M}, g)$ is globally hyperbolic.
  - *Local energy condition.* We have $Ric(v, v) \geqslant 0$, for any lightlike vector field v.
  - *Existence of a trapped surface.* A spacelike 2-dimensional surface starts collapsing.

There are known conditions ensuring the appearance of trapped surfaces when Einstein's equations are viewed as an evolution equation.



A trapped surface

Electromagnetism and other fields

- **Elementary particles and bundles.** Most field theories are expressed in terms of pdes relating the metric on a spacetime and a connection on a (finite or infinite dimensional) bundle above it, e.g. Yang-Mills theory. The geometry of these bundles carry physical information.

- **Conformal invariance.** Maxwell equations for electromagnetism, Yang-Mills field equations... are *conformally invariant*: they take the same form if the metric g is replaced by $\Omega^2 g$, with $\Omega : \mathbb{M} \to \mathbb{R}_+^*$.

- **Boundary matters.** Many physical informations on a field can be seen at infinity (i.e. defined by asymptotic properties/quantities).
  - **Conformal boundary:** take Ω accordingly to bring back ∞ at a finite "location". (Done by embedding conformally $(\mathbb{M}, g)$ as a *compact* subset of a larger spacetime.) Notion used to define a *black hole*.
  - **Causal boundary:** defined from the causal structure only.

Relativistic random processes: the model case

- In Minkowski spacetime $\mathbb{R}^{1,3}$, are there intrinsic Markov processes with *timelike trajectories*? (Intrinsic = distribution invariant by the isometries.)
  - If $X_s = (m_s, \dot{m}_s) \in \mathbb{R}^{1,3} \times \mathbb{H}$ is continuous then $\dot{m}_s$ is a Brownian motion on $\mathbb{H}$. (*Dudley, 1966*)

- **What features of the geometry of $\mathbb{R}^{1,3}$ can this diffusion explore?**

The causal boundary of $\mathbb{R}^{1,3}$ is non-trivial: a pinched cylinder $(\mathbb{R} \cup \{\pm\infty\}) \times \mathbb{S}^2 / \sim$.

  - The *Poisson boundary* of Dudley's hypoelliptic diffusion can be identified with the *causal boundary* of Minkowski spacetime. (*Bailleul, 2007*)

Relativistic processes on any spacetime

- **Generalizations.** (*Franchi-Le Jan, 2007/2011; Bailleul, 2010*)
Random perturbations of the geodesic flow conveniently defined on the frame bundle $\mathbb{OM}$ over $\mathbb{M}$. Given $\Theta : \mathbb{OM} \to \mathbb{R}_+$ (e.g. squared scalar curvature), they have hypoelliptic generator

$$L = H_0 + \frac{1}{2}\sum_{i=1}^{3} V_i\big(\Theta V_i\big).$$

- **What features of the geometry can these diffusions explore?**

Open problems/conjectures.

- Position of the **transience/recurrence** property of Dudley's generalized diffusion in the causality ladder.
- Under reasonable causality conditions, the **Poisson boundary** of the diffusion can be identified with the *causal boundary* of spacetime. (Verified for Minkowski and two other families of spacetimes – *Angst, 2010; Tardif, 2011*.)

Matter and geometry: the incompleteness issue

- *Geodesic* and *stochastic* completeness are not equivalent.
- Some known **non-explosion** criteria (*Bailleul-Franchi, 2011*):
  - explicit and usable for a largely used class of spacetime models (globally hyperbolic),
  - general spacetimes: involve *volume growth rate* of some *sub-Riemannian balls* in $\mathbb{OM}$.
- **Explosion**
  - Some known rough explosion criteria, using *Lyapounov function* methods. (*Bailleul, 2011*)
  - Work in progress on the incompleteness of the *random flow*.

Matter and geometry: typical gas particle

- **Diffusion approx.** of the general *relativistic Boltzmann equation* leads to relativistic diffusions.

  One particle distribution function $h(\Phi_0, \cdot)$: for *any* spacelike hypersurface, with associated hitting time H, the density of $\Phi_H \, \mathbf{1}_{H<\infty}$ is explicit in terms of $h(\Phi_0, \cdot)$.

  $$L^* h(\Phi_0, \cdot) = 0$$

  (*Bailleul, 2010*)

  Conjectures.

  - The function $h(\cdot, \cdot)$ is the Green function of the diffusion.
  - The Poisson/Martin boundary of the diffusion is obtained from renormalized $h(\Phi_0, \cdot)$ with $\Phi_0 \to \infty$.

- **Open problem.** Associate random processes to *non-diffusive approx.* to the general relativistic Boltzmann equation.

Structures on a spacetime: open problems

- **Conformal invariance.**
  - Can one construct interesting *random lightlike paths*?
  - Construct a *conformally invariant* random (Markovian) dynamics. Does it converge towards a point of the causal boundary?

- **Coloured particles.** In relation with different field theories, can one construct interesting random dynamics in some bundles over spacetime? (e.g. on $\mathrm{SU}(n)$, spin or Clifford bundles.)