Lifetime of relativistic diffusions

Ismaël Bailleul,
(Statistical Laboratory, Cambridge)

Layout of the talk

- Relativistic diffusions
- Lifetime of relativistic diffusions
  - Non-explosion criteria
  - Explosion criteria
- Time function on a Lorentzian manifold
• $m_s = m_0 + \int_0^s \dot{m}_r \, dr$:

$X_s = (m_s, \dot{m}_s) \in \mathbb{R}^{1,3} \times \mathbb{H}$

• Law of $\{X_s\}_{s \geq 0}$ frame-independent, i.e. intrinsic.

**Theorem (Dudley, 66’)**

*If $X$ is continuous then $\dot{m}_s$ is a Brownian motion on $\mathbb{H}$.***
Relativistic diffusions
Life time of relativistic diffusions
Time function on a Lorentzian manifold

Brownian motion on $S^2$. Infinitesimal Euclidean rotations:
\[
E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad e \in O_3 : V_i(e) = e E_i: \text{ invariant vector fields on } O(3).
\]

Brownian motion on $\mathbb{H}$. Idem : use infinitesimal hyperbolic rotations:
\[
E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.}
\]

\[
g_i \parallel \text{parallelly transported along the path } \{B_s\}_{s \in [t, t+\Delta t]}
\]

\[
\circ d e_s = V_1(e_s) \circ dw^1_s + V_2(e_s) \circ dw^2_s
\]
**Brownian motion on** $\mathbb{S}^2$. Infinitesimal Euclidean rotations:

$$
E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad e \in O(3) : V_i(e) = eE_i: \text{ invariant vector fields on } O(3).
$$

**Brownian motion on** $\mathbb{H}$. Infinitesimal hyperbolic rotations:

$$
E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad e \in SO(1, 3) : V_i(e) = eE_i: \text{ invariant vector fields on } SO(1, 3).
$$

**Dudley's diffusion on** $\mathbb{R}^{1,3} \times SO(1, 3)$. 

$$(m_s, e_s) = (m_s, (e_0(s), e_1(s), e_2(s), e_3(s))) \text{ solves the sde}$$

$$
dm_s = e_0(s) \, ds,$$

$$
de_s = V_i(e_s) \circ dw_s^i,$$

has generator

$$
G = H_0 + \frac{1}{2} \sum_{i=1}^{3} V_i^2.
$$
• \((\mathbb{M}, g)\): Lorentzian manifold (oriented, time-oriented) of dimension \(1 + 3\).
• \(T^1\mathbb{M}\): future unit bundle; \((m, \dot{m}) \in T^1\mathbb{M}\).
• \(\mathbb{OM}\): orthonormal frame bundle, \(\Phi = (m, (e_0, \cdots, e_3)) \in \mathbb{OM}\), \(e_0\) timelike and future.
• \(H_0\): generates the geodesic flow, \((V_i)_{1 \leq i \leq 3}\) canonical vertical vector fields.
• \(\Theta\)-diffusion: \(\Theta : T^1\mathbb{M} \rightarrow \mathbb{R}_+\) (function of the curvature, of the stress-energy tensor...)

\[
d\Phi_s = H_0(\Phi_s) \, ds + \frac{1}{4} (V_i \Theta) V_i + \sqrt{\Theta(\Phi_s)} V_i \circ dw_s^i
\]

\[
G = H_0 + \frac{1}{2} \sum_{i=1}^{3} V_i (\Theta V_i)
\]
• Relativistic diffusions

• Lifetime of relativistic diffusions

• Time function on a Lorentzian manifold
Motivations from physics. Undesirable phenomena: exploding curvature invariants, inextendible incomplete geodesics...

**Theorem (Hawking-Penrose)**

*The following three conditions cannot hold at the same time on a Lorentzian manifold*

- *(Causality condition)* The chronology condition holds.
- *(Energy condition)* Any complete causal geodesic has a pair of conjugated points.
- *(Initial/boundary condition)* The space has a “trapped set”.
Non-explosion

Lyapounov functions • If there exists a function $f$ and a positive constant $C$ such that $\mathcal{G}f \leq Cf$, and $f$ diverges to $+\infty$ along any timelike path living any compact set, then the relativistic diffusion has as an infinite lifetime.

• $U$ vector field on $\mathcal{M}$: $f(\Phi) = g(U, e_0)$, for $\Phi = (m, (e_0, \ldots, e_d)) \in \mathcal{O}\mathcal{M}$

• Generalised warped product: $\mathbb{R} \times S$, $ds^2 = a(m)^2 dt^2 - h_{ij}(m) dx^i dx^j$

**Theorem (B.-F. '10)**

Take $\Theta(\Phi) = \Theta(m)$. Then the $\Theta$-diffusion has as an infinite lifetime if the function $(m, \dot{m}) \in T^1\mathcal{M} \to \nabla_m(\log a)$ is bounded below $\iff \nabla a$ is everywhere non-spacelike and future.
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A “volume-growth” condition

Riemannian case. \( X \) a symmetric conservative diffusion, with generator \( L \), \( X_0 \sim \) invariant measure

\[
f(X_s) = f(X_0) + M_s + \int_0^s Lf(X_r) \, dr
\]

\[
f(X_s) = f(X_{T-(T-s)}) = f(X_T) + \tilde{M}_s + \int_0^s Lf(X_{T-(T-r)}) \, dr
\]

so \( df(X_s) = \frac{1}{2}(dM_s + \tilde{M}_s) \): Gaussian control of the increments of \( f(X) \) if \( \langle M \rangle, \langle \tilde{M} \rangle \) controlled.

\( X \): reflected Brownian motion on the boundary of a (large) ball.

**Theorem** (Grigor’yan ’86 / Hsu-Qin ’10)

On a complete Riemannian manifold, Brownian motion is conservative if

\[
\int_1^\infty \frac{r}{\ln |B(r)|} \, dr = \infty.
\]
Lorentzian case. **Difficulties:** no balls on $\mathcal{M}$; what could play the role of the conservative, symmetric diffusion?

- A Riemannian metric on $\mathcal{OM}$? Parallelisable manifold: set $H_0, \ldots, H_d, (V_{ij})_{0 \leq i < j \leq d}$ to be an orthonormal basis everywhere.

**Proposition (B.-F. ’10)**

$\Theta(\Phi) = \Theta(m)$, bounded. The $\Theta$-diffusion is conservative if $\mathcal{OM}$ is complete.

- A weaker geometric control. Controlled dynamics: $\dot{\Psi}_s = H_0 u_0^s + V_i u_i^s$

Reference point $\Phi_{\text{ref}}$. Define $T(\Phi) =$ minimal traveling time from $\Phi_{\text{ref}}$ to $\Phi$, with controls $|u^i|_\infty \leq 1$.

$$|H_0 T| + \sum_{i=1}^{d} |V_i T| \leq 1.$$
Theorem (B.-F. ’10)

The $\Theta$-diffusion is conservative if

$$\int_0^{\infty} \frac{r \, dr}{\Theta_r \log(\Theta_r \text{Vol}(B_r))} = \infty.$$ 

If $M \sim \mathbb{R}^{1,3}$, $\Theta$ bounded, and $g, g^{-1}$ and their first derivative are bounded, then non-explosion.
Write $\zeta$ for the lifetime of the diffusion.

**Proposition**

If $\mathbb{P}_\Phi(\zeta < \infty) > 0$, then $\mathbb{P}_\Phi(\zeta < \epsilon) > 0$, for all $\epsilon > 0$.

A simple explosion criterion. On some manifold $\mathcal{M}$.

**Lemma**

Suppose there exists two smooth functions $f \leq h$ and two constants $0 \leq c' < c$ such that

$$\mathcal{G}f \geq cf \quad \text{and} \quad \mathcal{G}h \leq c' h.$$  

Let $x_0 \in \mathcal{M}$ be such that $f(x_0) > 0$. Then the diffusion with generator $\mathcal{G}$ started from $x_0$ explodes with positive probability.
In our case:

\[ G = H_0 + \frac{1}{2} \sum_{i=1..3} V_i^2. \]

Given \( \Phi = (m, e) = (m, (e_0, \cdots, e_3)) \in \mathcal{OM} \), set

\[ U(\Phi) = \frac{3}{2} \int_{T^1_m\mathcal{M}} G(e_0, y)Ric_m(y, y) \, dy, \]

where \( G \) is the Green function of Laplacian on each fiber of \( T^1_{\mathcal{M}} \). One can apply the preceding explosion lemma to \( f = Ric_{|T^1_{\mathcal{M}}} \) and \( h := Ric_{|T^1_{\mathcal{M}}} + U \), under some conditions. Write \( R \) for the scalar curvature.
Let \((M, g)\) be a Lorentzian manifold satisfying the following conditions.

(1') **Static energy condition.** \(Ric \geq 0\) non-constant, \(R \leq 0\).

(2') **Regularity condition.** \(\exists 0 < \alpha < 1, \ 0 \leq c' < c \text{ and } c < 2 < \frac{c'}{\alpha}\) such that \(c - c' < 2(1 - \alpha)\) and

\[
\frac{1 - \alpha}{\alpha} \left| \text{Ric}_{T^1 M} \right| \leq U.
\]

(3') **Dynamic energy condition.** (i) \(H_0 h \leq (c' - 2\alpha) h\),

(ii) \(H_0 \text{Ric} \geq (c - 2) \text{Ric}\).

Let \(\Phi_0 \in \mathcal{O}_M\) be such that \(\text{Ric}(\Phi_0) > 0\). Then the relativistic diffusion started from \(\Phi_0\) explodes with positive probability.
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Globally hyperbolic spacetimes. A function $T$ on a Lorentzian manifold is said to be a (global) time function if (1) $\nabla T$ is everywhere timelike, (2) each level hypersurface $\{T = t\}$ is a (connected) spacelike submanifold, (3) each integral curve of $\nabla T$ meets each hypersurface $\{T = t\}$ at precisely one point.

**Theorem (Geroch '70)**

A Lorentzian manifold has a global time function iff it is globally hyperbolic.

On $\mathcal{O}M$. A weaker definition: A (smooth) function $T : \mathcal{O}M \to \mathbb{R}$ is a time function if it increases along any (lifting to $\mathcal{O}M$ of a) timelike geodesic.

**Theorem (B. '10)**

In any strongly causal Lorentzian manifold, the orthonormal frame bundle $\mathcal{O}M$ has a time function.
Some questions. (With an eye towards Penrose, Hawking incompleteness theorems...)

- Find a pathwise version of the global explosion criterion used.
- What happens if the diffusion enters a region satisfying the conditions of a geometric incompleteness theorem?
- Can a relativistic diffusion miss a naked singularity? (Like in the fast rotating Kerr black-holes.)