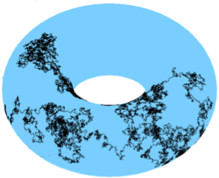
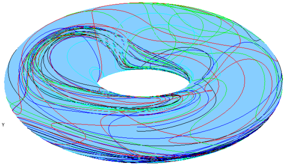
Rough differential equations





A. What this is all about

Make sense of the deterministic controlled ordinary differential equation

$$dx_t = \sum_{i=1}^{\ell} V_i(x_t)dh_t^i,$$

driven by a control h of low regularity, say α-Hölder with $0 < \alpha < 1$, and get a solution x that is a continuous function of the control h, unlike e.g. in Itô' stochastic integration theory where x is only a measurable function of the (semimartingale) control.

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A problem of analysis about products – *The product $V(x)dh \mid \alpha \cdot (\alpha - 1)$, is well-defined as a continuous function of $V(x)$ and dh iff $\alpha + (\alpha - 1) > 0$, i.e. $\alpha > \frac{1}{2}$.*

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What can be done for $\alpha \leq \frac{1}{2}$?

► Lyons' no go theorem – *Given $\alpha < \frac{1}{2}$, there exists no* continuous *functional $I : C^\alpha([0,1],\mathbb{R}) \times C^\alpha([0,1],\mathbb{R}) \to \mathbb{R}$, such that if x, y are trigonometric polynomials, then $I(y,h) = \int_0^1 y_t dh_t$.*

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Different approaches – Lyons (98'), Davie (03'), Gubinelli (04'), Friz-Victoir (08'), Bailleul (12'), Lyons & Yang (15').

B. Constructing flows

A **'numerical' scheme** for a time evolution

$$\mu_{ts} : \mathbb{R}^d \mapsto \mathbb{R}^d, \quad (0 \le s \le t \le T < \infty),$$

approximate description of the evolution of a system between times s and t.
Perturbations of the identity map, for s, t close.

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Self-improving: There is an exponent $a > 1$ such that

$$\left\| \mu_{tu} \circ \mu_{us} - \mu_{ts} \right\|_{C^1} \lesssim |t - s|^a, \quad (0 \leq s \leq u \leq t \leq T).$$

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A **flow** $\varphi = \left(\varphi_{ba} : \mathbb{R}^d \mapsto \mathbb{R}^d \right)_{0 \le a \le b \le T}$

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► Theorem – *One can associate to any self-improving numerical scheme a unique flow* φ *such that*

$$\left\| \varphi_{ts} - \mu_{ts} \right\|_{C^0} \lesssim |t-s|^a.$$

Moreover

$$\left\| \varphi_{ts} - \mu_{\pi_{ts}} \right\|_{C^0} \lesssim |\pi_{ts}|^{a-1},$$

for any partition $\pi_{ts} = \left\{ s < s_1 < \cdots < s_n < t \right\}$ *of any interval* $[s,t]$*, with*

$$\mu_{\pi_{ts}} := \bigcirc_{i=0}^{n} \mu_{s_{i+1} s_i}.$$

C. Rough paths

A generalised notion of **control** $h : [0, T] \to \mathbb{R}^\ell$, in a controlled ordinary differential equation

$$dx_t = \sum_{i=1}^{\ell} V_i(x_t)dh_t^i =: V_i(x_t)dh_t^i.$$

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► Key elementary remark – For all $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$, $0 \leq s \leq t \leq T$,

$$f(x_t) = f(x_s) + h_{ts}^i (V_i f)(x_s) + \left(\int_s^t \int_s^{s_1} dh_{s_2}^j dh_{s_1}^k \right) \big(V_j V_k f\big)(x_s) + O(|t-s|^3),$$

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Pick $2 \le p < 3$. A Hölder ***p*-rough path** is a function

$$\left(\mathbf{X}_{ts} = (X_{ts}, \mathbb{X}_{ts})\right)_{0\le s\le t\le T}, \quad X_{ts} = \left(X_{ts}^i\right)_{1\le i\le \ell} \in \mathbb{R}^\ell,\ \mathbb{X}_{ts} = \left(\mathbb{X}_{ts}^{jk}\right)_{1\le j,k\le \ell} \in \mathbb{R}^\ell \otimes \mathbb{R}^\ell$$

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subject to

- size constraints

$$\big|X_{ts}\big| \lesssim |t-s|^{1/p}, \quad \big|\mathbb{X}_{ts}\big| \lesssim |t-s|^{2/p},$$

- algebraic constraints (relations amongst the coefficients), for all $s \leq u \leq t$,

$$\mathbf{X}_{us}\mathbf{X}_{tu} = \mathbf{X}_{ts}.$$

D. Numerical schemes associated to rough differential equations

Given vector fields $V_1, \dots, V_\ell$ on $\mathbb{R}^d$ and a rough path $\mathbf{X} = (X, \mathbb{X})$, one can construct explicitly a self improving numerical scheme $(\mu_{ts})_{0 \le s \le t \le T}$ such that for all $x \in \mathbb{R}^d$, for all $f \in C_b^3(\mathbb{R}^d, \mathbb{R})$,

$$f\big(\mu_{ts}(x)\big) = f(x) + X_{ts}^i (V_i f)(x) + \mathbb{X}_{ts}^{jk} (V_j V_k f)(x) + O_f\big(|t-s|^{3/p}\big).$$

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Compare with the local expansion property of solutions of controlled ordinary differential equations

$$f(x_t) = f(x_s) + h_{ts}^i (V_i f)(x_s) + \left(\int_s^t \int_s^{s_1} dh_{s_2}^j \, dh_{s_1}^k\right)\big(V_j V_k f\big)(x_s) + O(|t-s|^3).$$

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The unique flow associated with the numerical scheme μ by the above Theorem is the **solution flow** to the rough differential equation

$$dx_t = V(x_t) d\mathbf{X}_t.$$

D. Numerical schemes associated to rough differential equations – The core of the matter

- Rewrite the expansion property

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- One can write

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and $V(\Lambda_{ts})$ is a vector field. Define

$$\mu_{ts} := e^{V(\Lambda_{ts})}$$

as the time 1 map of the ordinary differential equation

$$\dot{y}_u = V\big(\Lambda_{ts}\big)(y_u).$$

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so μ defines indeed an self-improving numerical scheme.

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1. From approximate flows to flows

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► Definition – *A* C^1**-approximate flow on** $\mathbb{R}^d$ *is a family* $(\mu_{ts})_{0 \leq s \leq t \leq T}$ *of* C^2 *maps from* $\mathbb{R}^d$ *into itself, depending continuously on* s, t *in the topology of uniform convergence, such that*

$$\left\| \mu_{ts} - \mathrm{Id} \right\|_{C^2} = o_{t-s}(1) \tag{2}$$

and there exists positive constants c_1 *and* $a > 1$*, such that the inequality*

$$\left\| \mu_{tu} \circ \mu_{us} - \mu_{ts} \right\|_{C^1} \leq c_1 |t - s|^a \tag{3}$$

holds for all $0 \leq s \leq u \leq t \leq T$.

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An example – Euler' scheme

$$\mu_{ts}(x) = x + V(x)(t-s),$$

with $V \in C_b^2(\mathbb{R}^d, \mathbb{R}^d)$. ◄

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Given a partition $\pi_{ts} = \{s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t\}$ of an interval $[s,t] \subset [0,T]$, set

$$\mu_{\pi_{ts}} := \mu_{s_n s_{n-1}} \circ \cdots \circ \mu_{s_1 s_0} = \bigcirc_{i=0}^{n-1} \mu_{s_{i+1} s_i}.$$

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and there exists positive constants c_1 *and* $a > 1$*, such that the inequality*

$$\left\|\mu_{tu} \circ \mu_{us} - \mu_{ts}\right\|_{C^1} \leq c_1 |t-s|^a \tag{5}$$

holds for all $0 \leq s \leq u \leq t \leq T$.

► Theorem 1 (Constructing flows) – *A* C^1*-approximate flow defines a unique flow* $\varphi = \left(\varphi_{ts}\right)_{0\leq s\leq t\leq T}$ *on* $\mathbb{R}^d$ *such that the inequality*

$$\left\|\varphi_{ts} - \mu_{ts}\right\|_\infty \leq c\,|t-s|^a \tag{6}$$

holds for a positive constant c*, for all* $0 \leq s \leq t \leq T$ *sufficiently close, say* $t - s \leq \delta$*. This flow satisfies the inequality*

$$\left\|\varphi_{ts} - \mu_{\pi_{ts}}\right\|_\infty \lesssim c_1^2\, T\, |\pi_{ts}|^{a-1}, \tag{7}$$

for any partition π_{ts} *of any interval* $[s,t]$ *of mesh* $|\pi_{ts}| \leq \delta$.

1. From approximate flows to flows – Step 1 of the proof

- **Definition –** *Let $\epsilon \in (0,1)$ be given. A partition*

$$\pi = \Big\{s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t\Big\}$$

*of an interval $[s,t]$ is said to be **ϵ-special** if it is either trivial or*

  - *one can find an $s_i \in \pi$ such that $\epsilon \leq \frac{s_i - s}{t-s} \leq 1 - \epsilon$,*
  - *and for any choice u of such an s_i, the partitions of $[s,u]$ and $[u,t]$ induced by π are both ϵ-special.*

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A partition of any interval into sub-intervals of equal length is $\frac{1}{3}$-special.

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A partition of any interval into sub-intervals of equal length is $\frac{1}{3}$-special.Set

$$m_\epsilon := \sup_{\epsilon \leq \beta \leq 1-\epsilon} \left(\beta^a + (1-\beta)^a\right) < 1,$$

and pick a constant

$$L > \frac{2c_1}{1 - m_\epsilon},$$

where c_1 is the constant that appears in the definition of a C^1-approximate flow, in equation (5).

1. From approximate flows to flows – Step 1 of the proof

- Proposition 2 – *Let* $(\mu_{ts})_{0\leq s\leq t\leq T}$ *be a* C^1*-approximate flow on* $\mathbb{R}^d$. *Given* $\epsilon > 0$, *there exists a positive constant* δ *such that for any* $0 \leq s \leq t \leq T$ *with* $t - s \leq \delta$, *and any* ϵ*-special partition* π_{ts} *of the interval* $[s, t]$, *we have*

$$\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_{C^1} \leq L\,|t - s|^a. \tag{8}$$

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▶ Proposition 2 – *Let* $(\mu_{ts})_{0\leq s\leq t\leq T}$ *be a* C^1*-approximate flow on* $\mathbb{R}^d$*. Given* $\epsilon > 0$*, there exists a positive constant* δ *such that for any* $0 \leq s \leq t \leq T$ *with* $t - s \leq \delta$*, and any* ϵ*-special partition* π_{ts} *of the interval* $[s, t]$*, we have*

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Proof – We first prove

$$\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_{C^0} \leq L\,|t-s|^a. \tag{9}$$

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The proof of estimate (8) is similar and given later. We proceed by induction on the number n of sub-intervals of the partition.

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$(n = 2)$: This is the C^0 version of identity (5) defining C^1-approximate flows.

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$(n \to n+1)$: Fix $0 \leq s < t \leq T$ with $t - s \leq \delta$, and let π_{ts} be an ϵ-special partition of $[s, t]$, splitting the interval $[s, t]$ into $(n+1)$ sub-intervals. Let u be one of the points of the partition such that $\epsilon \leq \frac{t-u}{t-s} \leq 1-\epsilon$, so the two partitions π_{tu} and π_{us} are both ϵ-special, with respective cardinals no greater than n.

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Then

$$\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_\infty \leq \left\|\mu_{\pi_{tu}} \circ \mu_{\pi_{us}} - \mu_{tu} \circ \mu_{\pi_{us}}\right\|_\infty + \left\|\mu_{tu} \circ \mu_{\pi_{us}} - \mu_{ts}\right\|_\infty$$

1. From approximate flows to flows – Step 1 of the proof

► Proposition 2 – *Let $(\mu_{ts})_{0\leq s\leq t\leq T}$ be a C^1-approximate flow on $\mathbb{R}^d$. Given $\epsilon > 0$, there exists a positive constant δ such that for any $0 \leq s \leq t \leq T$ with $t - s \leq \delta$, and any ϵ-special partition π_{ts} of the interval $[s, t]$, we have*

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Proof – We first prove

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Then

$$\begin{aligned}\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_\infty &\leq \left\|\mu_{\pi_{tu}} \circ \mu_{\pi_{us}} - \mu_{tu} \circ \mu_{\pi_{us}}\right\|_\infty + \left\|\mu_{tu} \circ \mu_{\pi_{us}} - \mu_{ts}\right\|_\infty \\ &\leq \left\|\mu_{\pi_{tu}} - \mu_{tu}\right\|_\infty + \left\|\mu_{tu} \circ \mu_{\pi_{us}} - \mu_{tu} \circ \mu_{us}\right\|_\infty + \left\|\mu_{tu} \circ \mu_{us} - \mu_{ts}\right\|_\infty\end{aligned}$$

1. From approximate flows to flows – Step 1 of the proof

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&\leq \left\|\mu_{\pi_{tu}} - \mu_{tu}\right\|_\infty + \left\|\mu_{tu} \circ \mu_{\pi_{us}} - \mu_{tu} \circ \mu_{us}\right\|_\infty + \left\|\mu_{tu} \circ \mu_{us} - \mu_{ts}\right\|_\infty \\
&\leq L|t-u|^a + \big(1 + o_\delta(1)\big)L\,|u - s|^a + c_1\,|t - s|^a
\end{aligned}$$

by the induction hypothesis and (4) – here the fact that the μ_{ba} are C^1-close to the identity, and (5) – the C^0 version of the C^1-approximate flow property.

1. From approximate flows to flows – Step 1 of the proof

$$\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_\infty \leq L|t-u|^a + \big(1 + o_\delta(1)\big)L\,|u-s|^a + c_1|t-s|^a$$

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$$\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_\infty \leq L|t-u|^a + \big(1 + o_\delta(1)\big)L\,|u-s|^a + c_1|t-s|^a$$

Set $u - s := \beta(t-s)$, with $\epsilon \leq \beta \leq 1-\epsilon$. The above inequality rewrites

$$\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_\infty \leq \Big\{\big(1 + o_\delta(1)\big)\big((1-\beta)^a + \beta^a\big)L + c_1\Big\}\,|t-s|^a.$$

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In order to close the induction, we need to choose δ small enough for the condition

$$c_1 + \big(1 + o_\delta(1)\big)\,m_\epsilon L \leq L \tag{10}$$

to hold; this can be done since $m_\epsilon < 1$.

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One needs to control the derivative of $\mu_{\pi_{ts}} - \mu_{ts}$ to prove (8). One uses the full definition of a C^1-approximate flow for that purpose, and not only its C^0 version; see later. ◀

1. From approximate flows to flows – An elementary identity

Existence and uniqueness both rely on the **elementary identity**

$$f_N \circ \cdots \circ f_1 - g_N \circ \cdots \circ g_1$$
$$= \sum_{i=1}^{N} \Big(g_N \circ \cdots \circ g_{N-i+1} \circ f_{N-i} - g_N \circ \cdots \circ g_{N-i+1} \circ g_{N-i} \Big) \circ f_{N-i-1} \circ \cdots \circ f_1, \tag{11}$$

with g_i and f_i any maps from $\mathbb{R}^d$ into itself.

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$$
\begin{aligned}
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&= \sum_{i=1}^{N} \Big(g_N \circ \cdots \circ g_{N-i+1} \circ f_{N-i} \; - \; g_N \circ \cdots \circ g_{N-i+1} \circ g_{N-i} \Big) \circ f_{N-i-1} \circ \cdots \circ f_1,
\end{aligned}
\tag{11}
$$

with g_i and f_i any maps from $\mathbb{R}^d$ into itself. E.g.

$$
\begin{aligned}
&f \circ g \circ h - f' \circ g' \circ h' \\
&= \big(f \circ g \circ h - f \circ g \circ h' \big) + \big(f \circ g \circ h' - f \circ g' \circ h' \big) + \big(f \circ g' \circ h' - f' \circ g' \circ h' \big).
\end{aligned}
\tag{12}
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\begin{aligned}
& f_N \circ \cdots \circ f_1 \;-\; g_N \circ \cdots \circ g_1 \\
&= \sum_{i=1}^{N} \Big(g_N \circ \cdots \circ g_{N-i+1} \circ f_{N-i} \;-\; g_N \circ \cdots \circ g_{N-i+1} \circ g_{N-i} \Big) \circ f_{N-i-1} \circ \cdots \circ f_1,
\end{aligned}
\tag{11}
$$

with g_i and f_i any maps from $\mathbb{R}^d$ into itself. In particular, *if all the maps* $g_N \circ \cdots \circ g_k$ *are Lipschitz continuous*, with a common upper bound L for their Lipschitz constants, then

$$
\big\| f_N \circ \cdots \circ f_1 - g_N \circ \cdots \circ g_1 \big\|_\infty \leq L \sum_{i=1}^{N} \| f_i - g_i \|_\infty. \tag{12}
$$

1. From approximate flows to flows – Step 2 of the proof

Existence. Set $\mathrm{D}_\delta := \big\{0 \leq s \leq t \leq T\,;\; t - s \leq \delta\big\}$ and $\mathbb{D}_\delta = D_\delta \cap \big\{\text{dyadic numbers}\big\}$.

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Existence. Set $\mathrm{D}_\delta := \{0 \leq s \leq t \leq T\,;\, t-s \leq \delta\}$ and $\mathbb{D}_\delta = D_\delta \cap \{\text{dyadic numbers}\}$. Given $s = a2^{-k_0}$ and $t = b2^{-k_0}$ in $\mathbb{D}_\delta$, define for $n \geq k_0$

$$\mu_{ts}^{(n)} := \mu_{s_{N(n)}s_{N(n)-1}} \circ \cdots \circ \mu_{s_1 s_0},$$

where $s_i = s + i2^{-n}$ and $s_{N(n)} = t$.

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where $s_i = s + i2^{-n}$ and $s_{N(n)} = t$. Given $n \geq k_0$, write

$$\mu^{(n+1)}_{ts} = \bigcirc_{i=0}^{N(n)-1} \big(\mu_{s_{i+1}\, s_i+2^{-n-1}} \circ \mu_{s_i+2^{-n-1}\, s_i}\big)$$

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and use the elementary identity (11) with

$$f_i = \mu_{s_{i+1} s_i + 2^{-n-1}} \circ \mu_{s_i + 2^{-n-1} s_i}, \quad g_i = \mu_{s_{i+1} s_i}$$

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and the fact that the compositions of the g-maps

$$\mu_{s_{N(n)} s_{N(n)-1}} \circ \cdots \circ \mu_{s_{N(n)-i+1} s_{N(n)-i}}$$

are Lipschitz continuous with a common Lipschitz constant L

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$$\mu_{s_{N(n)} s_{N(n)-1}} \circ \cdots \circ \mu_{s_{N(n)-i+1} s_{N(n)-i}}$$

are Lipschitz continuous with a common Lipschitz constant L, to get

1. From approximate flows to flows – Step 2 of the proof

$$\left\| \mu_{ts}^{(n+1)} - \mu_{ts}^{(n)} \right\|_\infty \leq L \sum_{i=0}^{N(n)-1} \left\| \mu_{s_{i+1} s_i + 2^{-n-1}} \circ \mu_{s_i + 2^{-n-1} s_i} - \mu_{s_{i+1} s_i} \right\|_\infty \leq c_1 L T \, 2^{-(a-1)n}.$$

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Uniqueness. Let ψ be any *flow* such that

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One has

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Rough paths are placeholders for the family of coefficients

$$H_{ts} :=$$
$$\left(1, \Big(\int_s^t dh^i_{s_1}\Big)_{1 \le i \le \ell}, \Big(\int_s^t \int_s^{s_1} dh^j_{s_2} dh^k_{s_1}\Big)_{1 \le j,k \le \ell}, \ldots, \Big(\int_{s \le s_1 \le \cdots \le s_n \le t} dh^{i_n}_{s_n} \cdots dh^{i_1}_{s_1}\Big)_{1 \le i_n, \ldots, i_1 \le \ell}\right).$$

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that appear in the expansion, when h is not sufficiently regular for making sense of the iterated integrals, e.g. h is only α-Hölder with $\alpha \le 1/2$. Like the function H, they take values in an algebraic structure that gives much insight on them.

2.1 An algebraic prelude

Collections of real valued coefficients $(a^{i_n \dots i_1})_{1 \leq i_1, \dots, i_n \leq \ell}$, are better seen here as elements of the tensor space $(\mathbb{R}^\ell)^{\otimes n}$.

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For $N \in \mathbb{N} \cup \{\infty\}$, set $T_\ell^N := \bigoplus_{r=0}^{N} \big(\mathbb{R}^\ell\big)^{\otimes r}$, with $\big(\mathbb{R}^\ell\big)^{\otimes 0} := \mathbb{R}$.

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for any $w' \in (\mathbb{R}^\ell)'$. Let $(\epsilon_1, \ldots, \epsilon_\ell)$ stand for the canonical basis of $\mathbb{R}^\ell$. The family $\big(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_k}\big)_{1 \le i_1, \ldots, i_k \le \ell}$ defines the canonical basis of $(\mathbb{R}^\ell)^{\otimes k}$. An element $\mathbf{a} \in (\mathbb{R}^\ell)^{\otimes k}$ is identified with the collection of its coordinates $(a^{i_n \cdots i_1})_{1 \le i_1, \ldots, i_n \le \ell}$ in the canonical basis.

For $N \in \mathbb{N} \cup \{\infty\}$, set $T_\ell^N := \bigoplus_{r=0}^{N} \left(\mathbb{R}^\ell\right)^{\otimes r}$, with $\left(\mathbb{R}^\ell\right)^{\otimes 0} := \mathbb{R}$. For $\mathbf{a} = \oplus_{r=0}^{N} a^r$ and $\mathbf{b} = \oplus_{r=0}^{N} b^r$ in T_ℓ^N

$$\mathbf{a} + \mathbf{b} := \oplus_{r=0}^{N} (a^r + b^r),$$

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2.1 An algebraic prelude

Collections of real valued coefficients $(a^{i_n \cdots i_1})_{1 \le i_1, \ldots, i_n \le \ell}$, are better seen here as elements of the tensor space $(\mathbb{R}^\ell)^{\otimes n}$. One can see any element of $\mathbb{R}^\ell$ as a linear map on the dual space $(\mathbb{R}^\ell)'$. Given $u, v \in \mathbb{R}^\ell$, one has

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The space T_ℓ^N is called the (truncated) **tensor algebra over** $\mathbb{R}^\ell$ (if N is finite).

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Think of

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$$\delta_\lambda(\mathbf{a}) = \left(1, \lambda a^1, \dots, \lambda^N a^N\right),$$

for all $\lambda \in \mathbb{R}$ and $\mathbf{a} \in T_\ell^{N,1}$.

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that is homogeneous with respect to the dilation

$$\left\|\delta_\lambda(\mathbf{a})\right\| = |\lambda|\,\|\mathbf{a}\|.$$

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Elements of $T_\ell^{N,1}$ are invertible, with

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2.2 Hölder p-rough paths

Fix s and look at the evolution of

$$H_{ts} =$$
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Chen's relation

$$\mathbf{X}_{ts} = \mathbf{X}_{us}\mathbf{X}_{tu}$$

holds by definition of the increments.

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For $2 \le p < 3$, Chen's relation is equivalent to

$$X_{ts}^1 = X_{tu}^1 + X_{us}^1, \quad X_{ts}^2 = X_{tu}^2 + X_{us}^1 \otimes X_{tu}^1 + X_{us}^2$$

Condition on X^1 means that X_{ts}^1 is the increment of the $\mathbb{R}^\ell$-valued path $\left(X_{r0}^1\right)_{0 \le r \le T}$.
Condition on X^2 analogue of $\int_s^t \int_s^r = \int_s^u \int_s^r + \int_u^t \int_s^u + \int_u^t \int_u^r$.

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for all $m = 1 \ldots [p]$. *We define the norm of* $\boldsymbol{X}$ *to be*

$$\|\boldsymbol{X}\| := \max_{m=1\ldots[p]} \left\| X^m \right\|_{\frac{m}{p}},$$

and a distance $d(\boldsymbol{X}, \boldsymbol{Y}) := \|\boldsymbol{X} - \boldsymbol{Y}\|$ *on the set of Hölder p-rough path. A* **Hölder weak geometric p-rough path on** $[0, T]$ *is a* $G_\ell^{[p]}$*-valued Hölder p-rough path.*

The metric

$$\overline{d}(\mathbf{X}, \mathbf{Y}) := \left| X_0^1 - Y_0^1 \right| + d(\mathbf{X}, \mathbf{Y})$$

turns the set of all Hölder p-rough paths into a (non-separable) complete metric space.

3. Flows driven by rough paths

3.1 Differential operators

Given a collection of vector fields $V_1, \ldots, V_\ell \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ on $\mathbb{R}^d$, set for $z \in \mathbb{R}^\ell$

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$$V(z_1 \otimes \cdots \otimes z_k) := V(z_1) \circ \cdots \circ V(z_k),$$

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and requiring linearity. In those terms, the expansion property of ODE solutions

$$f(x_t) = f(x_s) + \Big(\int_s^t dh^i_{s_1}\Big)(V_i f)(x_s) + \Big(\int_s^t \int_s^{s_1} dh^j_{s_2}\, dh^k_{s_1}\Big)\big(V_j V_k f\big)(x_s) + (\cdots)$$
$$+ \Big(\int_{s \le s_1 \le \cdots \le s_n \le t} dh^{i_n}_{s_n} \dots dh^{i_1}_{s_1}\Big)\big(V_{i_n} \dots V_{i_1} f\big)(x_s) + O\big(|t-s|^{n+1}\big)$$

rewrites

$$f(x_t) = \big(V(H_{ts}) f\big)(x_s) + O\big(|t-s|^{n+1}\big).$$

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so V is a Lie algebra morphism sending T_ℓ^∞-brackets into brackets of differential operators

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As *brackets of vector fields are vector fields*, $V(g_\ell^N)$ is made up vector fields.

3.2 A ‘numerical’ scheme with the local expansion property

Let $V_1, \ldots, V_\ell \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ be smooth vector fields on $\mathbb{R}^d$, with bounded $2[p] + 1$ derivatives ($V_i \in C_b^{[p]+1}$ suffices).

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► Proposition – *There exists a positive constant c, depending only on the V_i, such that the inequality*

$$\left\| f \circ \mu_{ts} - V(\boldsymbol{X}_{ts}) f \right\|_\infty \leq c\Big(1 + \|\boldsymbol{X}\|^{[p]}\Big) \|f\|_{C^{[p]+1}} |t-s|^{\frac{[p]+1}{p}} \qquad (13)$$

holds for any $f \in C_b^{[p]+1}(\mathbb{R}^d)$.

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$$\sum_{k=0}^{[p]} \frac{1}{k!}\Lambda_{ts}^{*k} = \mathbf{X}_{ts} + O\Big(|t-s|^{\frac{[p]+1}{p}}\Big) \in T_\ell^\infty,$$

where $*$ stands for the multiplication in T_ℓ^∞, while $\mathbf{X}_{ts} \in T_\ell^{[p]} \subset T_\ell^\infty$.

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$$f(y_1) = f(x) + \int_0^1 \big\{V(\Lambda_{ts})f\big\}(y_{u_1})\, du_1$$

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$$f(y_1) = f(x) + \big\{V(\Lambda_{ts})f\big\}(x) + \int_0^1 \int_0^{u_1} \big\{V(\Lambda_{ts})\,V\big(\Lambda_{ts}\big)f\big\}(y_{u_2})\,du_2\,du_1$$

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$$f(y_1) = f(x) + \big\{V(\Lambda_{ts})f\big\}(x) + \int_0^1 \int_0^{u_1} \Big\{V\big(\Lambda_{ts}^{*2}\big)f\Big\}(y_{u_2})\, du_2 du_1$$

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$$f(y_1) = f(x) + \big\{V(\Lambda_{ts})f\big\}(x) + \frac{1}{2}\Big\{V\big(\Lambda_{ts}^{*2}\big)f\Big\}(x) + \int_0^1\int_0^{s_1}\int_0^{s_2}\Big\{V(\Lambda_{ts}^{*3})f\Big\}(y_{u_3})\,du_3du_2du_1,$$

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where $*$ stands for the multiplication in T_ℓ^∞, while $\mathbf{X}_{ts} \in T_\ell^{[p]} \subset T_\ell^\infty$.Then by induction

$$f(y_1) = \left\{ V\left(\sum_{k=0}^{[p]} \frac{1}{k!} (\Lambda_{ts})^{*k} \right) f \right\}(x) + \int_{0 \le u_{[p]+1} \le \cdots \le u_1} \left\{ V\Big(\Lambda_{ts}^{*([p]+1)}\Big) f \right\}(y_{u_{[p]+1}})\, du$$

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$$= \Big(V(\mathbf{X}_{ts})f\Big)(x) + O\Big(|t-s|^{\frac{[p]+1}{p}}\Big).$$

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$$\mu_{tu} \circ \mu_{us} = \Big\{V(\mathbf{X}_{tu})\mathrm{Id}\Big\} \circ \mu_{us} + \epsilon^{\mathrm{Id}_{tu}} \circ \mu_{us}$$

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&= V(\mathbf{X}_{ts})\mathrm{Id} + V\Big(\pi\big(\mathbf{X}_{us}\mathbf{X}_{tu}\big)\Big)\mathrm{Id} + \epsilon_{us}^{V(\mathbf{X}_{tu})\mathrm{Id}} + \epsilon^{\mathrm{Id}_{tu}} \circ \mu_{us}
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if there exists a constant $a > 1$ *independent of* $\boldsymbol{X}$ *and two possibly* $\boldsymbol{X}$*-dependent positive constants* δ *and* c *such that one has*

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► Theorem – *The rough differential equation*

$$d\varphi = V(\varphi)d\mathbf{X}_t$$

has a unique solution flow; it takes values in the space of uniformly Lipschitz continuous homeomorphisms of $\mathbb{R}^d$ with uniformly Lipschitz continuous inverses, and depends continuously on $\mathbf{X}$.

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for all $0 \leq s \leq t \leq T$ with $t - s \leq \delta$, and $f \in C_b^{[p]+1}(\mathbb{R}^d)$.

► Theorem – *The rough differential equation*

$$dz = V(z)d\boldsymbol{X}_t, \quad z_0 = x \in \mathbb{R}^d,$$

has a unique solution path. It is a continuous function of $\boldsymbol{X}$ *in the uniform norm topology.*

3.3 Solution paths to rough differential equations

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The continuous dependence of the solution path $z_\bullet$ with respect to $\mathbf{X}$ is transfered from φ to $z_\bullet$.

Further reading

Written version of the lectures on my teaching web page

https://perso.univ-rennes1.fr/ismael.bailleul/files/M2Course.pdf

Further reading

Branched rough paths (towards regularity structures)

- Ramification of rough paths. M. Gubinelli, *J. Diff. Eq.*, **248**(4):693–721, (2010).
- Geometric versus non-geometric rough paths. M. Hairer and D. Kelly, *Ann. Inst. H. Poincaré Probab. Stat.*, **51**(1):207–251, (2015).
- On the definition of a solution to a rough differential equation. I. Bailleul, to appear in *Ann. Fac. Sci. Toulouse*.

Applications to stochastic analysis... so many!

- Tiny sample in Chap. 5 of my lecture notes, and Chap. 9-11 of Friz-Hairer's book.
- Mean field rough differential equations
  - Evolving communities with individual preferences. T. Cass and T. Lyons, *Proc. London Math. Soc.*, **110**(1):83–107, (2015).
  - Solving mean field rough differential equations. I. Bailleul and R. Catellier and F. Delarue, *Elec. J. Probab.*, **25**(21):1–51, (2020).
  - Pathwise McKean-Vlasov Theory with Additive Noise. M. Coghi and J.D. Deuschel and P. Friz and M. Maurelli, arXiv:1812.11773, (2018).

Further reading

Fast-slow systems

- Deterministic homogenization for fast-slow systems with chaotic noise. D. Kelly and I. Melbourne, *J. Funct. Anal.*, **272**(10):4063–4102, (2017).
- Rough flows and homogenization in stochastic turbulence. I. Bailleul and R. Catellier, *J. Diff. Eq*, **263**(8):4894–4928, (2017).
- Homogenization with fractional random fields. J. Gehringer and X.-M. Li, arXiv:1911.12600, (2019).

Signature, analysis of streams and machine learning

- Uniqueness for the signature of a path of bounded variation and the reduced path group. B. Hambly and T. Lyons, *Ann. Math.*, **171**(1):109–167, (2010).
- The Signature of a Rough Path: Uniqueness. H. Boedihardjo and X. Geng and T. Lyons and D. Yang, *Adv. Math.*, **293**:720–737, (2016).
- Reconstruction for the signature of a rough path. X. Geng, *Proc. London Math. Soc.*, **114**(3):495–526, (2017).
- Rough paths, Signatures and the modelling of functions on streams. T. Lyons, https://arxiv.org/abs/1405.4537, (2014).
- Kernels for sequentially ordered data. F. Kiraly and H. Oberhauser, arXiv:1601.08169, (2016).
- Signature moments to characterize laws of stochastic processes. I. Chevyrev and H. Oberhauser, arXiv:1810.10971, (2018).

On rough paths convergence

- Theorem – *Assume* $^{(n)}\boldsymbol{X}$ *is a sequence of Hölder p-rough paths with uniform bounds*

$$\sup_n \left\| {}^{(n)}X \right\| \leq C < \infty, \tag{14}$$

which converge pointwise, in the sense that $^{(n)}\boldsymbol{X}_{ts}$ *converges to some* $\boldsymbol{X}_{ts}$ *for each* $0 \leq s \leq t \leq 1$*. Then the limit object* $\boldsymbol{X}$ *is a Hölder p-rough path, and* $^{(n)}\boldsymbol{X}$ *converges to* $\boldsymbol{X}$ *as a Hölder q-rough path, for any* $p < q < [p] + 1$*.*

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- Proof – • **X** is a Hölder p-rough path: direct consequence of the uniform bounds (14) and pointwise convergence:

$$\left|X^i_{ts}\right| = \lim_n \left|{}^{(n)}X^i_{ts}\right| \leq C|t-s|^{\frac{i}{p}}.$$

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$$\left| X^i_{ts} - {}^{(n)}X^i_{ts} \right| \leq \epsilon_n, \quad \left| X^i_{ts} - {}^{(n)}X^i_{ts} \right| \leq 2C|t-s|^{\frac{i}{p}}.$$

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Using $a \wedge b \leq a^{1-\theta} b^{\theta}$, with $\theta = \frac{p}{q} < 1$, we have

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which entails the convergence result as a Hölder q-rough path.

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and

$$\mathbf{X}_{\bar{t}\bar{s}} = \mathbf{X}_{s\bar{s}}\mathbf{X}_{ts}\mathbf{X}_{\bar{t}t}, \quad {}^{(n)}\mathbf{X}_{\bar{t}\bar{s}} = {}^{(n)}\mathbf{X}_{s\bar{s}}{}^{(n)}\mathbf{X}_{ts}{}^{(n)}\mathbf{X}_{\bar{t}t}$$

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$$d\Big(\mathbf{X}_{ts}, {}^{(n)}\mathbf{X}_{ts}\Big) \leq d\big(\mathbf{X}_{ts}, \mathbf{X}_{\overline{t}\overline{s}}\big) + d\Big(\mathbf{X}_{\overline{t}\overline{s}}, {}^{(n)}\mathbf{X}_{\overline{t}\overline{s}}\Big) + d\Big({}^{(n)}\mathbf{X}_{\overline{t}\overline{s}}, {}^{(n)}\mathbf{X}_{ts}\Big)$$

and

$$\mathbf{X}_{\overline{t}\overline{s}} = \mathbf{X}_{s\overline{s}}\mathbf{X}_{ts}\mathbf{X}_{\overline{t}t}, \quad {}^{(n)}\mathbf{X}_{\overline{t}\overline{s}} = {}^{(n)}\mathbf{X}_{s\overline{s}}{}^{(n)}\mathbf{X}_{ts}{}^{(n)}\mathbf{X}_{\overline{t}t}$$

and using the uniform estimate (14)

On rough paths convergence

- Theorem – *Assume ${}^{(n)}\boldsymbol{X}$ is a sequence of Hölder p-rough paths with uniform bounds*

$$\sup_n \left\| {}^{(n)}X \right\| \leq C < \infty, \tag{14}$$

which converge pointwise, in the sense that ${}^{(n)}\boldsymbol{X}_{ts}$ converges to some $\boldsymbol{X}_{ts}$ for each $0 \leq s \leq t \leq 1$. Then the limit object $\boldsymbol{X}$ is a Hölder p-rough path, and ${}^{(n)}\boldsymbol{X}$ converges to $\boldsymbol{X}$ as a Hölder q-rough path, for any $p < q < [p] + 1$.

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and using the uniform estimate (14), the first and third terms in the above upper bound can be made arbitrarily small by choosing a partition with a small enough mesh, uniformly in s, t and n.

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and using the uniform estimate (14), the first and third terms in the above upper bound can be made arbitrarily small by choosing a partition with a small enough mesh, uniformly in s, t and n. Second term dealt with the pointwise convergence assumption as it involves only finitely many points once the partition π has been chosen as above.

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Controlled paths and rough integral

- Definition – *Pick an* $\mathbb{R}^{\ell}$*-valued Hölder p-rough path* $\boldsymbol{X} = (X, \mathbb{X})$*, for* $2 \leq p < 3$.

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for all $0 \leq s \leq t \leq 1$, for an $L\big(\mathbb{R}^\ell, \mathbb{R}^d\big)$-valued $\frac{1}{p}$-Lipschitz map $Z'_\bullet$, and some $\mathbb{R}^d$-valued $\frac{2}{p}$-Lipschitz map R.

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For linear maps $A, B \in \mathrm{L}(\mathbb{R}^\ell, \mathbb{R}^d)$, and $a, b \in \mathbb{R}^\ell$, set

$$\big(A \otimes B\big)(a \otimes b) := (Aa) \otimes (Bb).$$

Controlled paths and rough integral

- Theorem – *A family* $(\mu_{ts})_{0\leq s\leq t\leq T}$ *of elements of* $\mathbb{R}^d$ *such that*

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for $a>1$ is said to be ***almost additive***. *There exists a unique $\mathbb{R}^d$-valued function φ such that*

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Controlled paths and rough integral

▶ Proposition – *Let* $\boldsymbol{X} = (X, \mathbb{X})$ *be an* $\mathbb{R}^\ell$*-valued Hölder p-rough path, with* $2 \leq p < 3$*. We define an almost-additive map setting*

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Given vector fields $V_1, \dots, V_\ell$ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, define $F(x) \in L(\mathbb{R}^\ell, \mathbb{R}^d)$ setting

$$F(x)(z) := \sum_{1 \leq i \leq \ell} z^i V_i(x).$$

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► Corollary – A path $x_\bullet$ in $\mathbb{R}^d$ is a solution to the rough differential equation

$$dx_t = F(x_t) d\mathbf{X}_t$$

iff it is a path controlled by X, with derivative $F(x_\bullet)$, and

$$x_t = x_0 + \int_0^t \big(F(x), (DF)(F(x))\big)_s \, d\mathbf{X}_s.$$

4. Applications to stochastic analysis

4.1 The Brownian rough path

Let $\left(B_t\right)_{0 \le t \le 1}$ be an $\mathbb{R}^\ell$-valued Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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for any $0 \le s \le u \le t \le 1$. Recall for $\mathbf{a} \in T^{2,1}_\ell$

$$\|\mathbf{a}\| = \left\|1 \oplus a^1 \oplus a^2\right\| = \left|a^1\right| + \sqrt{\left|a^2\right|}, \quad d(\mathbf{a}, \mathbf{b}) = \left\|\mathbf{a}^{-1}\mathbf{b}\right\|.$$

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So $\mathbf{B}^I$ is a Hölder p-rough path iff it is a $1/p$-Hölder continuous $(T^{2,1}_\ell, d)$-valued path.

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$$\mathbb{E}\left[\left\|\mathbf{B}^I_{ts}\right\|^q\right] \lesssim |t-s|^{q/2},$$

for $0 < \frac{1}{2} - \frac{1}{q} < \frac{1}{p}$.

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$$\mathbb{E}\left[\left\|\mathbf{B}^I_{ts}\right\|^q\right] \lesssim |t-s|^{q/2},$$

for $0 < \frac{1}{2} - \frac{1}{q} < \frac{1}{p}$. Equivalent to requiring

$$\left\|B_{ts}\right\|_{L^q} \lesssim |t-s|^{\frac{1}{2}}, \qquad \left\|\mathbb{B}^I_{ts}\right\|_{L^{\frac{q}{2}}} \lesssim |t-s|.$$

4.1 The Brownian rough path

Let $\left(B_t\right)_{0\leq t\leq 1}$ be an $\mathbb{R}^\ell$-valued Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set

$$\mathbb{B}^I_{ts} := \int_s^t \int_s^u dB_r \otimes dB_u = \int_s^t B_{us} \otimes dB_u.$$

This process satisfies Chen's relation

$$\mathbb{B}^I_{ts} = \mathbb{B}^I_{tu} + \mathbb{B}^I_{us} + B_{us} \otimes B_{tu}$$

for any $0 \leq s \leq u \leq t \leq 1$. Recall for $\mathbf{a} \in T^{2,1}_\ell$

$$\|\mathbf{a}\| = \left\|1 \oplus a^1 \oplus a^2\right\| = \left|a^1\right| + \sqrt{\left|a^2\right|}, \quad d(\mathbf{a}, \mathbf{b}) = \left\|\mathbf{a}^{-1}\mathbf{b}\right\|.$$

So $\mathbf{B}^I$ is a Hölder p-rough path iff it is a $1/p$-Hölder continuous $(T^{2,1}_\ell, d)$-valued path. Use Kolmogorov's criterion

$$\mathbb{E}\left[\left\|\mathbf{B}^I_{ts}\right\|^q\right] \lesssim |t-s|^{q/2},$$

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True as a consequence of the scaling properties of Brownian motion.

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The process $\mathbf{B}^I$ is almost surely a Hölder p-rough path; it is called the **Itô Brownian rough path**.

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The process $\mathbf{B}^S := (B, \mathbb{B}^S)$ is almost surely a Hölder p-rough path; it is called the **Itô Brownian rough path**. Unlike $\mathbf{B}^I$, it is a *weak geometric* Hölder p-rough path.

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► Proposition – *The Hölder p-rough path* $\boldsymbol{B}^{(n)} := \left(B^{(n)}, \mathbb{B}^{(n)}\right)$ *converges almost-surely to* $\boldsymbol{B}^S$ *in the Hölder p-rough path topology.*

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► Proof – Use our statement on rough paths convergence. The almost-sure pointwise convergence follows from the martingale convergence theorem applied to the martingales in (15).

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$$\sup_n \left\|\mathbf{B}^{(n)}\right\| < \infty \tag{16}$$

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so the uniform estimate (16) follows from Doob's maximal inequality.

4.2 Rough and stochastic integrals

▶ Proposition – *Let* $(F_s)_{0\leq s\leq 1}$ *be an* $L(\mathbb{R}^\ell, \mathbb{R}^d)$*-valued path controlled by B, adapted to the Brownian filtration, with derivative process* $(F'_s)_{0\leq s\leq 1}$ *also adapted to that filtration.*

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so pass to the limit $M \to \infty$. ◄

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- Corollary – *Under the above assumptions one has almost surely*

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Denote by $\mathrm{Sym}(A)$ the symmetric part of a matrix A and recall that

$$\frac{1}{2}(t_{i+1} - t_i)\,\mathrm{Id} = \mathrm{Sym}\big(\mathbb{B}^S_{t_{i+1}t_i}\big) - \mathrm{Sym}\big(\mathbb{B}^I_{t_{i+1}t_i}\big) = \frac{1}{2} B^{\otimes 2}_{t_{i+1}t_i} - \mathrm{Sym}\big(\mathbb{B}^I_{t_{i+1}t_i}\big).$$

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One sees as above that $\sum_i F'_{t_i}\mathrm{Sym}\Big(\mathbb{B}^I_{t_{i+1}t_i}\Big)$ converges to 0 in L^2.

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with a well-defined additional term

$$(\star) := \lim_{|\pi|\searrow 0} \sum_i F'_{t_i} \frac{1}{2}(t_{i+1} - t_i)\, \mathrm{Id}.$$

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$$(\star) \overset{a.s.}{=} \lim_{|\pi|\searrow 0} \frac{1}{2} \sum_i F'_{t_i} B^{\otimes 2}_{t_{i+1} t_i}.$$

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We recognize a quantity which converges in probability to the bracket $\langle F, B \rangle$.

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► Proof – It suffices to notice that solving the rough differential equation

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is equivalent to solving equation (18). ◄

Thank you all for attending the lectures!