# Regularity structures and paracontrolled calculus

Joint works with M. Hoshino

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1. Pointwise description devices

2. Fourier-type description devices

3. From paracontrolled systems to models and modelled distributions

# The multiplication problem in singular PDEs

**Singular** PDEs = **multiplication problem**, e.g.

$$(\partial_t - \Delta)u = u\,\zeta, \quad \text{in 2-dimensional torus,}$$
$$(\partial_t - \partial_x^2)u = \xi + (\partial_x u)^2, \quad \text{in 1-dimensional torus,}$$

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Leads to **regularity structures, models and modelled distributions**, and **paracontrolled calculus and paracontrolled systems**.

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Regularity structures (**RS**) and paracontrolled calculus (**PC**) have their roots in rough paths theory for ODEs driven by irregular controls

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Our **aim** in a singular PDE setting

**microscopic** description $\Longleftrightarrow$ **macroscopic** description

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Consistency of repeated re-expansion around different points and requirement that the $g(\tau)$ form a sufficiently rich family to describe an algebra of functions, directly lead to the definition of a concrete regularity structure $\mathscr{T}$ and a model $(g, \Pi)$ on it.

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  - There is a map $\Pi : T \to \mathcal{S}'(\mathbb{T}^d)$, such that

    $$\Pi_x \tau = (\Pi \otimes g_x^{-1})\Delta\tau$$

    has $C^{|\tau|}$-regularity at $x$ (only).

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$$\mathcal{D}^\gamma(T,g) := \Big\{ \mathbf{f} := \big(f^\tau(x)\big)_{\tau\in\mathcal{B},\, x\in\mathbb{T}^d} \,;\, \Big|\mathbf{f}(z) - \widehat{g_{zy}}\big(\mathbf{f}(y)\big), \tau\rangle\Big| \lesssim |z-y|^{\gamma-|\tau|}, \\ \forall \tau \in T, \forall y,z \in \mathbb{T}^d \Big\} : \textbf{ modelled distributions}$$

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**Reconstruction theorem (Hairer)** – *Given a model* $(g,\Pi)$ *on a regularity structure* $\mathscr{T}$, *there exists a linear continuous operator*

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*such that*

$$\Big| \Big\langle \mathbf{Rf} - \sum_\tau f^\tau(x)\Pi_x\tau, \varphi_x^\lambda \Big\rangle \Big| \lesssim \lambda^{|\tau|};$$

this map $\mathbf{R}$ is unique if $\gamma > 0$. It is called the **reconstruction map**.

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The **paraproduct** terms $P_a b, P_b a$ are always well-defined, not the case of the **resonant** term $\Pi(a,b)$. In $P_a b$ one modulates the high frequencies of $b$ by low frequencies of $a$; one can say that $P_a b$ looks like $b$ in a Fourier sense. Write

$$\Delta\sigma = \sum_{\mu \leq \sigma} \mu \otimes (\sigma/\mu) \in T \otimes T^+, \quad \Delta^+\tau = \sum_{\nu \leq^+ \tau} \nu \otimes (\tau/^+\nu) \in T^+ \otimes T^+.$$

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The **paraproduct** terms $P_a b, P_b a$ are always well-defined, not the case of the **resonant** term $\Pi(a,b)$. In $P_a b$ one modulates the high frequencies of $b$ by low frequencies of $a$; one can say that $P_a b$ looks like $b$ in a Fourier sense. Write

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# 2. Fourier-type description devices: micro to macro

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**Theorem (B.-Hoshino 2018)** – *Fix a regularity structure* $\mathscr{T}$ *and a model* $\mathsf{M} = (\mathsf{g}, \Pi)$ *on* $\mathscr{T}$. *One can construct 'reference functions/distributions'* $\{[\tau]^{\mathsf{g}} \in C^{|\tau|}(\mathbb{T}^d)\}_{\tau \in \mathcal{B}^+}$ *and* $\{[\sigma]^{\mathsf{M}} \in C^{|\sigma|}(\mathbb{T}^d)\}_{\sigma \in \mathcal{B}}$ *such that*

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*We talk of* **para-remainders** $[\tau]^{\mathsf{g}}, [\sigma]^{\mathsf{M}}$*; they depend continuously on the model M.*

# 2. Fourier-type description devices: micro to macro

**Theorem (B.-Hoshino 2018)** – *Fix a regularity exponent* $\gamma \in \mathbb{R}$, *and a model* $\mathsf{M} = (\mathrm{g}, \Pi)$ *on the regularity structure* $\mathscr{T}$.

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**Theorem (B.-Hoshino 2018)** – *Fix a regularity exponent $\gamma \in \mathbb{R}$, and a model* $\mathsf{M} = (\mathsf{g}, \Pi)$ *on the regularity structure* $\mathscr{T}$*. One can associate to any modelled distribution*

$$\mathbf{f} = \sum_{\sigma \in \mathcal{B}; |\sigma| < \gamma} f^{\sigma} \sigma \in \mathcal{D}^{\gamma}(T, g),$$

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*for an* $[f^\sigma]^{\mathsf{g}} \in \mathcal{C}^{\gamma-|\sigma|}(\mathbb{T}^d)$. *Moreover, the* **para-remainder map**

$$\mathbf{f} \mapsto \Big([\mathbf{f}]^{\mathsf{M}}, \big([f^\sigma]^{\mathsf{g}}\big)_{\sigma\in\mathcal{B}}\Big)$$

*from* $\mathcal{D}^\gamma(T, g)$ *to* $\mathcal{C}^\gamma(\mathbb{T}^d) \times \prod_{\tau\in\mathcal{B}} \mathcal{C}^{\gamma-|\tau|}(\mathbb{T}^d)$, *is continuous.*

# 3. From paracontrolled systems to models and modelled distributions

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**Proposition** – *If $g$ is given, then for any family $\big([\sigma] \in C^{|\sigma|}(\mathbb{T}^d)\big)_{\sigma\in\mathcal{B}_\bullet, |\sigma|<0}$ there exists a unique model $(g, \Pi)$ on $\mathscr{T}$ such that*

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One assumes fairly weak assumptions on $\mathscr{T}$, satisfied by all reasonable regularity structures, like the regularity structures used for the study of singular PDEs. Assume in particular $\mathcal{B}^+$ freely generated by $\mathcal{B}^+_\bullet$ and monomials.

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  **(1 – Generating set)** *There exists a finite subset $\mathcal{G}^+_\bullet$ of $\mathcal{B}^+_\bullet$ such that*

$$\mathcal{B}^+_\bullet = \bigsqcup_{\tau \in \mathcal{G}^+_\bullet} \Big\{ \tau / X^k \; ; \, k \in \mathbb{N}^d, \; |\tau| - |k| > 0 \Big\}.$$

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**Theorem (B. Hoshino 2019)** – *Under weak assumptions, for any family $\big([\tau] \in C^{|\tau|}(\mathbb{T}^d)\big)_{\tau \in \mathcal{G}^+_\bullet}$ there exists a unique $g$ map on $(T^+, \Delta^+)$ such that*

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**Theorem (B. Hoshino 2019)** – *For any reasonable regularity structure $\mathscr{T}$, one has a bi-Lipschitz parametrization of the space of models by*

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**Theorem (B. Hoshino 2018)** – *For the regularity structures used for singular PDEs, one has a bi-Lipschitz parametrization of the space of admissible models by*

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(Generalizes greatly a result by Tapia and Zambotti (2018) on the parametrization of the set of branched rough paths – they used completely different methods.)

# 3. From paracontrolled systems to models: applications

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*via the paracontrolled representation*

$$f^\sigma = \sum_{\sigma < \mu; |\mu| < \gamma} \mathsf{P}_{f^\mu} [\mu/\sigma]^{\mathrm{g}} + [f^\sigma]^{\mathrm{g}}, \qquad (1)$$

*for $\mathbf{f} = \sum_{\sigma \in \mathcal{B}} f^\sigma \sigma \in \mathcal{D}^\gamma(T, g)$ – recall $\mathcal{B}_\bullet = \mathcal{B} \backslash \text{polynomials}$.*

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Previous assumptions are on the regularity structure. Here is an assumption on a basis of $T$.

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(Proving this statement happens to be equivalent to an extension problem for the map $g$.)

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The use of paracontrolled systems like (1) are the starting point of the paracontrolled approach to singular PDEs.

Thank you for your attention!