Mean field rough differential equations

Joint work with R. Catellier (Nice) and F. Delarue (Nice)

1. Motivation

- A particle system of N individuals, subject to symmetric influence of all individuals

$$dX_t^i = b\big(t, X_t^i, \mu_t^N\big)dt + F\big(t, X_t^i, \mu_t^N\big)\, dB_t^i,$$

with $\mu_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$, the **empirical measure** of the time t system. (Think e.g. of $b(x,\mu)$ or $F(x,\mu) = \int_{\mathbb{R}^d} g(x-y)d\mu(y)$, for μ probability measure.) **Exchangeable system**.

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► Theorem – **Propagation of chaos**. *For any fixed integer $k \geq 1$,*

$$\mathcal{L}\Big((X_t^1)_{0\leq t\leq T}, \cdots, (X_t^k)_{0\leq t\leq T}\Big) \underset{N\to\infty}{\longrightarrow} \mathcal{L}\Big((X_t)_{0\leq t\leq T}\Big)^{\otimes k}$$

where $(X_t)_{0\leq t\leq T}$ *solves* **McKean-Vlasov/(mean field) SDE**

$$dX_t = b\big(t, X_t, \mathcal{L}(X_t)\big)dt + F\big(t, X_t, \mathcal{L}(X_t)\big)\, dB_t.$$

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What if B is not Brownian motion? If F is random non-adapted? $\rightsquigarrow$ **Rough paths world**

2.1 Lift of an irregular trajectory

- Realization $(W_t(\omega))_{0\le t\le T}$ of $\mathbb{R}^\ell$-valued rough random input with same regularity as a Brownian path

$$\big|W_t(\omega) - W_s(\omega)\big| \le C(\omega)|t-s|^\alpha, \quad \alpha \in (1/3, 1/2]$$

  - $\alpha > 1/2$: no need of rough paths, but doesn't cover typical Brownian trajectory
  - $\alpha \le 1/3$: Rough paths theory applies in a more elaborate form

- **Goal**: define $\int Y_t(\omega)\, dW_t(\omega)$ for some path $(Y_t(\omega))_{0\le t\le T}$
  - Not doable for any $\big(Y_t(\omega)\big)_{0\le t\le T}$
  - First question: does it work for $Y_t(\omega) = W^j(\omega)$?

- **Iterated integral of** W – willing to define

$$\mathbb{W}^{i,j}_{st}(\omega) := \int_s^t \big(W^i_r(\omega) - W^i_s(\omega)\big)\, dW^j_r(\omega).$$

Think of Wiener case: several ways to define the stochastic integral. **No canonical choice of iterated integral if** $\alpha \le 1/2$.

- If $(W_t(\omega))_{0\le t\le T}$ smooth curve

$$\mathbb{W}^{i,j}_{st}(\omega) = \int_s^t \big(W^i_r(\omega) - W^i_s(\omega)\big)\dot{W}^j_r(\omega)dr$$

one checks **Chen's relation** for $r \le s \le t$

$$\mathbb{W}^{i,j}_{rt}(\omega) = \mathbb{W}^{i,j}_{rs}(\omega) + \mathbb{W}^{i,j}_{st}(\omega) + \big(W^i_s(\omega) - W^i_r(\omega)\big)\big(W^j_t(\omega) - W^j_s(\omega)\big)$$

- We require from a lift $\boldsymbol{W}(\omega) := \big((W_t(\omega))_{0\le t\le T}, (\mathbb{W}_{st}(\omega))_{0\le s\le t\le T}\big)$
  - **algebraic Chen relation**
  - **analytic regularity** property

$$\big|\mathbb{W}_{st}(\omega)\big| \le C(\omega)\,|t-s|^{2\alpha}$$

If W is $1d$, a natural candidate is $\mathbb{W}_{st}(\omega) = \frac{1}{2}\big(W_t(\omega) - W_s(\omega)\big)^2$.

If $\dim \ge 2$, "cross integrals" may not exist from analytical arguments: **use probabilistic constructions**.

$\rightsquigarrow$ Stratonovich/Itô integral of two independent Brownian motions

$\rightsquigarrow$ Friz-Victoir integral of two independent Gaussian processes

2.2 Rough integral and RDEs

- Back to $\int_s^t Y_u dW_u(\omega)$... **Controlled path** $Y(\omega)$

$$Y_t(\omega) - Y_s(\omega) = \delta_x Y_s(\omega)\left(W_t(\omega) - W_s(\omega)\right) + R^Y_{st}(\omega)$$

with $\delta_x Y_s(\omega)$: "**derivative**", in $L(\mathbb{R}^m, \mathbb{R}^d)$, and $R^Y_{st}(\omega)$: "**remainder**", in $\mathbb{R}^d$, and

$$\left|\delta_x Y_t(\omega) - \delta_x Y_s(\omega)\right| \leq C^Y(\omega)\,|t-s|^\alpha, \quad \left|R^Y_{s,t}(\omega)\right| \leq C^Y(\omega)\,|t-s|^{2\alpha}$$

- **Rough integral** $(\alpha > 1/3)$

$$\int_s^t Y_r(\omega)\, d\mathbf{W}_r(\omega) \approx \sum_{i=0}^{N-1} Y_{t_i}(\omega)\left(W_{t_{i+1}} - W_{t_i}\right)(\omega) + \sum_{i=0}^{N-1} \delta_x Y_{t_i}(\omega)\,\mathbb{W}_{t_i t_{i+1}}(\omega)$$

and

$$\left|\int_s^t Y_r(\omega)\, d\mathbf{W}_r(\omega) - \left(Y_s(\omega)\left(W_t - W_s\right)(\omega) + \delta_x Y_s(\omega)\,\mathbb{W}_{st}(\omega)\right)\right| \leq \kappa(\omega)\,|t-s|^{3\alpha}$$

Solve

$$dX_t(\omega) = F(X_t(\omega))d\mathbf{W}_t(\omega)$$

- **Stability of controlled paths by** F, for $F \in \mathcal{C}_b^2$,
  - take controlled path $(X_t(\omega))_{0\leq t\leq T}$ and expand $\big(F(X_t(\omega))\big)_{0\leq t\leq T}$

$$\begin{aligned} F\big(X_t(\omega)\big) &= F\big(X_s(\omega)\big) + F'\big(X_s(\omega)\big)\,(X_t - X_s)(\omega) + \cdots \\ &= F\big(X_s(\omega)\big) + F'\big(X_s(\omega)\big)\delta_x X_s(\omega)\,(W_t - W_s)(\omega) + R_{st}^{F(X)}(\omega) \end{aligned}$$

  - this makes it possible to define $\int F(X_r(\omega))d\mathbf{W}_r(\omega)$ if X is controlled
- **Fixed point** for the **rough differential equation**, for $F \in \mathcal{C}_b^3$,
  - **input** $=$ controlled path $\big(X_t(\omega), \delta_x X_t(\omega)\big)_{0\leq t\leq T}$,
  - **output** $=$ controlled path $\Big(X_0(\omega) + \int_0^t F\big(X_r(\omega)\big)d\mathbf{W}_r(\omega), F\big(X_t(\omega)\big)\Big)_{0\leq t\leq T}$.

► Theorem **(Lyons, Gubinelli)** – *The intergal map* Γ *:* **input** $\rightsquigarrow$ **output**, *is a contraction in small time* – $\mathcal{C}^\alpha$ norm on $\big(\delta_x X_t(\omega)\big)_{0\leq t\leq T}$ and $\mathcal{C}^{2\alpha}$ on $\big(R_{s,t}^X(\omega)\big)_{0\leq s\leq t\leq T}$.

3.1 Mean field RDEs: the main problem

Define

$$\int_s^t F\Big(X_r(\omega), \mathcal{L}(X_r)\Big)\, d\mathbf{W}_r(\omega)$$

Doesn't suffice to have a rough lift $\mathbf{W}(\omega)$ for any ω, for dependence on r in $\mathcal{L}(X_r)$ is not good enough to make sense of the integral.

∘ Replace $\mathcal{L}(X_r)$ by $\mathcal{L}(W_r)$ and choose $(W_t)_{0\leq t\leq T}$ as a centered Gaussian process

$$W_2\Big(\mathcal{L}(W_t), \mathcal{L}(W_s)\Big) = \left|\sqrt{\mathbb{V}(W_t)} - \sqrt{\mathbb{V}(W_s)}\right|.$$

One can cook an example such that

$$W_2\Big(\mathcal{L}(W_t), \mathcal{L}(W_s)\Big) = (t-s)^\alpha, \quad t > s \geq 0$$

for infinitely many pairs (s, t), so, for all $p \leq 2$,

$$\sup_{0=t_0<\cdots<t_N=T} \sum_{i=0\cdots N-1} W_2\Big(\mathcal{L}(W_{t_{i+1}}), \mathcal{L}(W_{t_i})\Big)^p = \infty,$$

one cannot define $\int_s^t F\Big(\mathcal{L}(W_r)\Big)\, d\mathbf{W}_r(\omega)$ as a Young integral. One can however define this integral using our approach.

3.1 Mean field RDEs: the main problem

Define

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Doesn't suffice to have a rough lift $\mathbf{W}(\omega)$ for any ω, for dependence on r in $\mathcal{L}(X_r)$ is not good enough to make sense of the integral.

∘ *So far* [Cass-Lyons '14, Bailleul '15, Deuschel et al. '17], *no mean field dependence in diffusivity*

$$dX_t(\omega) = b\big(X_t(\omega), \mathcal{L}(X_t)\big)\, dt + F\big(X_t(\omega)\big)\, dW_t(\omega).$$

∘ **Mean field structure interacts with rough set-up**. Requires to expand $F\big(X_r(\omega), \mathcal{L}(X_r)\big)$ in the measure argument; use P.L. **Lions' approach to differential calculus on Wasserstein space**.

3.2 Derivative on Wasserstein space

Given $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, lift $\mathcal{U}$ into

$$\widehat{\mathcal{U}} : L^2(\Omega, \mathbb{P}) \ni X \mapsto \mathcal{U}\big(\mathcal{L}(X)\big)$$

Say $\mathcal{U}$ is differentiable if $\widehat{\mathcal{U}}$ is Fréchet differentiable.

- Derivative of $\mathcal{U}$ – Fréchet differential of $\widehat{\mathcal{U}}$

$$D\widehat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \mu = \mathcal{L}(X)$$

∘ If $\mathcal{U}(\mu) = \int_{\mathbb{R}^d} h(x) d\mu(x)$, with ∇h at most of linear growth, then

$$\partial_\mu \mathcal{U}(\mu)(v) = h'(v)$$

∘

$$\partial_{x_i}\left[\mathcal{U}\left(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right)\right] = \frac{1}{N}\partial_\mu \mathcal{U}\left(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right)(x_i), \quad x_1, \ldots, x_N \in \mathbb{R}$$

∘ If X and X' are two random variables

$$\begin{aligned}\mathcal{U}\big(\mathcal{L}(X')\big) - \mathcal{U}\big(\mathcal{L}(X)\big) &= \mathbb{E}\Big[\partial_\mu \mathcal{U}\big(\mathcal{L}(X)\big)\big(X(\cdot)\big)\big(X' - X\big)(\cdot)\Big] + \cdots \\ &= \int \partial_\mu \mathcal{U}\big(\mathcal{L}(X)\big)(x)\,\big(x' - x\big)\,\mathcal{L}(X, X')(dxdx') + \cdots\end{aligned}$$

3.3 Expanding a function of $X_t(\omega)$ and $\mathcal{L}(X_t)$

Take a *measurable collection* of controlled trajectories as before

$$\Big(X_t(\omega) - X_s(\omega) = \delta_x X_s(\omega)\big(W_t(\omega) - W_s(\omega)\big) + R_{s,t}(\omega)\Big)_{\omega\in\Omega}.$$

Take a function $F(x,\mu)$ with $\partial_x F(x,\mu)$ and $\partial_\mu F(x,\mu)(v)$ bounded and Lipschitz in (x,μ,v). Expand $F_t := F\big(X_t(\omega), \mathcal{L}(X_t)\big)$:

$$\begin{aligned}
F_t - F_s &= \partial_x F\big(X_s(\omega), \mathcal{L}(X_s)\big)\partial_x X_s(\omega)\big(W_t(\omega) - W_s(\omega)\big) \\
&\quad + \mathbb{E}\Big[\partial_\mu F\big(X_s(\omega), \mathcal{L}(X_s)\big)(X_s(\cdot))\partial_x X_s(\cdot)\,(W_s - W_t)(\cdot)\Big] + R^F_{s,t}(\omega) \\
&=: (\delta_x F)_s(\omega)\big(W_t(\omega) - W_s(\omega)\big) + \mathbb{E}\Big[(\delta_\mu F)_s(\omega,\cdot)\,\big(W_t - W_s\big)(\cdot)\Big] \\
&\quad + R^F_{s,t}(\omega),
\end{aligned}$$

where

$$\begin{aligned}
(\delta_x F)_s(\omega) &:= \partial_x F\big(X_s(\omega), \mathcal{L}(X_s)\big)\, F\big(X_s(\omega), \mathcal{L}(X_s)\big), \\
(\delta_\mu F)_s(\omega,\omega') &:= \partial_\mu F\big(X_s(\omega), \mathcal{L}(X_s)\big)\big(X_s(\omega')\big)\, F\big(X_s(\omega), \mathcal{L}(X_s)\big).
\end{aligned}$$

The path $F_t = F\big(X_t(\omega), \mathcal{L}(X_t)\big)$ is not a trajectory controlled by $W_\cdot(\omega)$.

3.4 Extended rough set-up

Above, for fixed ω we need two increments

- $W_t(\omega) - W_s(\omega)$, in $\mathbb{R}^\ell$,
- $W_t(\cdot) - W_s(\cdot) = \big(W_t(\omega') - W_s(\omega')\big)_{\omega' \in \Omega}$, in $L^2(\Omega; \mathbb{R}^\ell)$.

Not sufficient to have $\mathbb{W}(\omega)$ iterated integral of $W(\omega)$, also need

$$\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \omega') = \int_s^t \big(W_r - W_s\big)(\omega') dW_r(\omega), \quad (\omega, \omega') \in \Omega^2$$

where $(\omega, \omega') \mapsto W(\omega')$ independent copy of $(\omega, \omega') \mapsto W(\omega)$ on $\Omega^2 \rightsquigarrow \mathbb{W}^{\perp\!\!\!\perp}$ is the iterated integral of two independent copies of the noise

- Requires a convenient form of Chen identity for $\mathbb{W}^{\perp\!\!\!\perp}$.
- Requires a convenient form of regularity

$$\mathbb{E}'\Big[\big|\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \cdot)\big|^2\Big]^{1/2} \leq C(\omega)\,|t - s|^{2\alpha}.$$

(We actually require higher q-moments, with $q \geq 8$.)

3.5 Rough integral in the extended set-up

Take a measurable collection of controlled trajectory as before

$$\Big(X_t(\omega) - X_s(\omega) = \delta_x X_s(\omega)\big(W_t(\omega) - W_s(\omega)\big) + R^X_{s,t}(\omega)\Big)_{\omega\in\Omega}$$

- no derivative the direction of μ: $\big[\partial_\mu X(\omega)\big]_s = 0$
- require integrability of the Hölder norms of $\big(\delta_x X_t(\omega)\big)_{0\le t\le T}$ and $\big(R^X_{s,t}(\omega)\big)_{0\le s\le t\le T}$

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Expand

$$\begin{aligned} & F\big(X_t(\omega), \mathcal{L}(X_t)\big) - F\big(X_s(\omega), \mathcal{L}(X_s)\big) \\ & = [\delta_x F]_s(\omega)\big(W_t(\omega) - W_s(\omega)\big) + \mathbb{E}\Big[[\delta_\mu F]_s(\omega,\cdot)\big(W_t - W_s\big)(\cdot)\Big] + R^F_{s,t}(\omega) \end{aligned}$$

With $F_r := F\big(X_r(\omega), \mathcal{L}(X_r)\big)$,

$$\begin{aligned} \int_s^t F_r(\omega) d\boldsymbol{W}_r(\omega) \approx & \sum_{i=0}^{N-1} F_{t_i}(\omega)\big(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\big) \\ & + \sum_{i=0}^{N-1} \delta_x F_{t_i}(\omega)\mathbb{W}_{t_i,t_{i+1}}(\omega) + \sum_{i=0}^{N-1} \mathbb{E}\Big[\delta_\mu F_{t_i}(\omega,\cdot)\mathbb{W}^{\perp\!\!\!\perp}_{t_i,t_{i+1}}(\omega,\cdot)\Big] \end{aligned}$$

∘ No derivative in $\mu \rightsquigarrow$ keep stable the form of $X(\omega)$

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Expand

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With $F_r := F\big(X_r(\omega), \mathcal{L}(X_r)\big)$,

$$\begin{aligned} \int_s^t F_r(\omega) d\boldsymbol{W}_r(\omega) \approx &\sum_{i=0}^{N-1} F_{t_i}(\omega)\big(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\big) \\ &+ \sum_{i=0}^{N-1} \delta_x F_{t_i}(\omega)\mathbb{W}_{t_i,t_{i+1}}(\omega) + \sum_{i=0}^{N-1} \mathbb{E}\Big[\delta_\mu F_{t_i}(\omega,\cdot)\mathbb{W}^{\perp\!\!\!\perp}_{t_i,t_{i+1}}(\omega,\cdot)\Big] \end{aligned}$$

◦ If W is a BM and adapted $F \rightsquigarrow$ recover standard Itô integral

4.1 Fixed point procedure for mean field RDEs

Similar procedure as in the non-mean field case for the integral map Γ

- **input** $\rightsquigarrow$ collection $\Big((X_t(\omega), \delta_x X_t(\omega))_{0\leq t\leq T}\Big)_{\omega\in\Omega}$
- **output** $\rightsquigarrow$ collection of controlled paths

$$\bigg(\Big(X_0(\omega) + \int_0^t F\big(X_r(\omega), \mathcal{L}(X_r\big) d\mathbf{W}_r(\omega), F\big(X_t(\omega), \mathcal{L}(X_t)\big)\Big)_{0\leq t\leq T}\bigg)_{\omega\in\Omega}$$

In the usual random RDE case, one shows that Γ is a **contraction** on a small enough **random interval** $[0, T(\omega)]$. **No more possible** to do that because of mean field dependency in the dynamics: We can at best obtain $\mathcal{L}\big(X_t(\cdot)\mathbf{1}_{t<T(\cdot)}\big)$, rather than $\mathcal{L}\big(X_r(\cdot-\big)$. **One needs to prove contraction on a deterministic time interval** $[0, T]$. A variant of Gronwall lemma? Not easy in a rough paths setting...

4.2 A global in time stability estimate

- **Trick**: find a **random norm** $\|\cdot\|_\omega$ on controlled paths, a constant ρ and a r.v. $\mathcal{A}(\omega)$ s.t.

$$\left\|\Gamma(\boldsymbol{X})(\omega) - \Gamma(\boldsymbol{X}')(\omega)\right\|_\omega \leq \rho\, \mathcal{A}(\omega) \left(\int_\Omega \left\|\boldsymbol{X}(\omega') - \boldsymbol{X}'(\omega')\right\|_\omega^p \mathbb{P}(d\omega') \right)^{1/p}$$

for $\big(X_0(\omega), \partial_x X_0(\omega)\big) = \big(X_0'(\omega), \partial_x X_0'(\omega)\big)$, with

$$\rho < 1, \quad \text{and} \quad \int_\Omega \mathcal{A}^p(\omega) d\mathbb{P}(\omega) \to 1, \quad \text{as } T \text{ tends to } 0.$$

If so, taking the $\mathbb{L}^p$ norm in the left hand side shows that Γ is a contraction.

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∘ Choices of $\|\cdot\|_\omega$ and $\mathcal{A}(\omega)$:

$\vartheta(s,t,\omega)$ a variation-type norm involving $W(\omega), W(\cdot), \mathbb{W}(\omega), \mathbb{W}^\perp(\omega,\cdot)$ and $\mathbb{W}^\perp(\cdot,\cdot)$

$N(\omega)$, accumulated local variation: lowest number of points $(t_i)_{1\leq i\leq N(\omega)}$ with $\vartheta(t_i, t_{i+1}, \omega) = \epsilon < 1$, fixed, and $t_{N(\omega)} \geq T$

$\rightsquigarrow \rho = \rho_\epsilon \searrow 0$ as $\epsilon \searrow 0$,

$\rightsquigarrow \mathcal{A}(\omega) = C(1 + \vartheta(0,T,\omega))^{N(\omega)}$,

$$\left\|\boldsymbol{X}(\omega)\right\|_\omega := \left|(X_0, \delta_x X_0)(\omega)\right| + \sup_{[s,t]\subset[0,T]}\left(\frac{\left|\delta_x X_t(\omega) - \partial_x X_s(\omega)\right|}{\vartheta(s,t,\omega)^\alpha} + \frac{\left|R_{s,t}(\omega)\right|}{\vartheta(s,t,\omega)^{2\alpha}}\right)$$

4.3 A well-posedness statement

- **Regularity and boundedness assumptions on** $F, \partial_x F, \partial_\mu F$ and second order derivatives. Example $F(x,\mu) = g\big(x, \int f(x,y)\mu(dy)\big)$

- **Tail assumptions on the noise**: **tails** of $\vartheta(0,T,\cdot)$ in $\exp(-r^{\eta_1})$ and **tails** of $N(\cdot)$ in $\exp(-r^{1+\eta_2})$. Example fBM with Hurst between $1/3$ and $1/2$.

► Theorem – *For initial condition in L^2, **existence and uniqueness** of a solution to the mean field equation*

$$dX_t = F\big(X_t, \mathcal{L}(X_t)\big)\, dW_t;$$

*moreover, $\mathcal{L}(X)$ **depends continuously on** $\mathcal{L}(\mathbf{W})$.*

Continuity leads to **propagation of chaos**. One even has **sharp convergence rate** for the empirical measure of the particle system to its limit.

Open directions

- Allow diffusivity with **linear growth**, as in the Curie-Weiss model, where $F(x,\mu) = \nabla U(x) + \int (x - y)\mu(dy)$.
- **Malliavin calculus** and existence of densities for time marginals of solutions driven by appropriate random rough paths (Gaussian processes, random Fourier series, Markov processes associated with Dirichlet forms...).
- After propagation of chaos, other **limit theorems**, e.g. central limit theorem, large and moderate deviations.
- Particle systems with **common noise**, e.g. motion in a random velocity field, as in classical stochastic flow theory, that introduces strong correlations at small distances and decorrelation at large distances.

Thank you for you attention!