Paracontrolled calculus for quasilinear singular PDEs

I. BAILLEUL[1] and A. MOUZARD

Abstract. We develop further in this work the high order paracontrolled calculus setting to deal with the analytic part of the study of quasilinear singular PDEs. Continuity results for a number of operators are proved for that purpose. Unlike the regularity structures approach of the subject by Gerencser & Hairer and Otto, Sauer, Smith & Weber, or Furlan & Gubinelli' study of the two dimensional quasilinear parabolic Anderson model equation, we do not use parametrised families of models or paraproducts to set the scene. We use instead infinite dimensional paracontrolled structures that we introduce here.

1 – Introduction

This work is dedicated to the study of the quasilinear singular partial differential equation (PDE)

$$\partial_t u - d(u)Au = f(u)\zeta, \tag{1.1}$$

where ζ stands for a spacetime noise of parabolic Hölder regularity $\alpha - 2$, with $2/5 < \alpha < 1/2$, with a real-valued unknown u defined on a 3-dimensional closed Riemannian smooth manifold M and A a smooth elliptic operator on M. The function $d(\cdot)$ – for diffusivity, is supposed to be smooth enough and to take its values in a compact subset of $(0, +\infty)$. We assume here for simplicity that the initial condition u_0 in equation (1.1) is regular enough to treat the free propagation of the initial condition as a remainder term and avoid the technical use of weighted norms. The problems in the resolution of such equations are twofold. First, the low regularity of the noise ζ allows only a low regularity of the potential solutions u, which is not sufficient to make sense of a number of the terms $d(u)Lu$ and $f(u)\zeta$. This type of multiplication problems is commom to a whole class of equations that has received a lot of attention over the past years with the concomitant introduction of regularity structures [24] by Hairer and of paracontrolled calculus [20] by Gubinelli, Imkeller and Perkowski. This class of equations is now referred to as *singular stochastic PDEs* and general methods for solving (subcritical) semilinear stochastic PDEs have been devised, following both approaches. For quasilinear equation a serious additionnal difficulty arises since the nonlinearity in the leading order term is itself ill-defined. In the present work we extend the tools of paracontrolled calculus to deal with the analytic part of the study of such equations. The reader acquainted with the results of Bailleul and Bernicot's work [3] on the high order paracontrolled calculus will see that our method for the study of equation (1.1), and the tools introduced along the way, give a direct access to the analysis of the quasilinear generalised (KPZ) equation

$$\partial_t u - d(u)\partial_x^2 u = f(u)\zeta + g(u)|\partial_x u|^2,$$

or any other quasilinear version of parabolic semilinear equations, or systems of equations, that can be studied within the setting of the high order paracontrolled calculus. Like the works [30, 32], the present work is purely analytical and does not consider the important

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problem of renormalization. This amounts here to assuming that a sequence of multilinear functions of the noise are given a priori as elements of their natural spaces with natural bounds on their norms. In particular, we dot explain how to build these random variables as limits of random variables built from a regularized noise and how to relate the notion of solution that we capture here to the solutions to a family of renormalized equations driven by the regularized noises.

Paracontrolled calculus was introduced in Gubinelli, Imkeller and Perkowski' seminal work [20] as a first order 'expansion machinery' for the study of a number of singular stochastic PDEs. Despite the first order limitation the paracontrolled approach to the study of singular PDEs has been very successful, as testify, amongst others, the works of Gubinelli and Perkowski [22, 23] on the KPZ and stochastic Burgers equations, Catellier-Chouk, Mourrat-Weber and Gubinelli & co-authors works [13, 26, 27, 9, 19, 25] on the Φ^4 scalar equation from quantum field theory, the works [1, 15] of Chouk and co-authors on the spectral theory for the two-dimensional Laplacian with white noise potential, and the very recents works on hyperbolic singular PDEs [21, 29]. The scope of the first order paracontrolled calculus was much extended in [2, 3, 4], and the high order paracontrolled calculus offers now a convenient setting for the study of a whole class of singular parabolic PDEs, in diverse geometric settings. The high order paracontrolled calculus was for instance used in [28] to prove Weyl law for the counting function of the Anderson operator on two-dimensionnal closed Riemannian manifolds.

The study of quasilinear singular PDEs was launched by the works [31] of Otto and Weber, [16] of Furlan and Gubinelli, and [6] of Bailleul, Debussche and Hofmanová, that all appeared within a few months. Interestingly, each of these works used different methods to tackle the same equation: The 2-dimensional quasilinear parabolic Anderson model equation. Otto and Weber introduced a rough paths flavoured variant of regularity structures while Furlan and Gubinelli introduced a variant of the first order paracontrolled calculus using paracomposition operators instead of paraproducts; both methods rely on a parametric Ansatz. Bailleul, Debussche and Hofmanová showed that the original first order paracontrolled calculus is sufficient to prove well-posedness of the equation on a small time interval, for an equation involving a spatial noise. Gerencsér and Hairer then showed in [18] that the study of a whole class of quasilinear singular parabolic PDEs can be done within the setting of regularity structures, in the regime $2/5 < \alpha \leqslant 1/2$ for the regularity exponent α, giving results way beyond the scope of what was proved in [31, 16, 6] and Otto, Sauer, Smith and Weber's followup work [30]. The only caveat to their remarkable results is the fact that their formulation of the quasilinear equation does not allow for a clean treatment of the renormalization problem yet. See however Gerencsér's recent work [17] for a first result in this direction.

By adding a few results to the toolkit of the high order paracontrolled calculus [4] we are able to prove a local in time well-posedness result for equation (1.1), with the same line of attack as in [6]. The latter used the classical space paraproduct on the 2-dimensional torus. By working with the spacetime paraproducts from [3] and the high order paracontrolled calculus from [4] we are able here to work with spacetime noises on a manifold. Note that the method works mutatis mutandis for the study of the quasilinear generalised (KPZ) equation or the quasilinear version of the geometric stochastic heat equation. We reformulate the quasilinear equation (1.1) under a semilinear-like form

$$(\partial_t + L)u = f(u)\zeta + \varepsilon(u,\cdot)Lu + \sum_{i=0}^{\ell} a_i(u,\cdot)V_i u,$$

for a smooth second order differential operator L, smooth vector fields V_i and functions $a_i(\cdot), \varepsilon(\cdot)$ of u and the space argument. As the name suggests the function $\varepsilon(\cdot)$ will be small. At first sight this type equation seems to be critical, in the sense that one does not get any regularizing/contraction effect from the fixed point formulation of the equation due to the second order differential operator L in the right hand side. However, each iteration of the fixed point will come with a factor $\varepsilon(u,\cdot)$ close to 0 at time 0. Working with the a priori knowledge that u is of positive Hölder regularity this term will indeed be small for a small time horizon and this will allow us to get around the 'criticality' problem. We will be able to define a paracontrolled structure and formulate the equation as a fixed point for a contracting map defined on this structure. The well-posedness result obtained from that formulation of equation (1.1) will be our main result, stated in Theorem 10. The other approaches developed so far for the study of quasilinear singular PDEs all require an infinite dimensional ingredient that involves a parametrized family of symbols [18] or operators [31, 16]. This is the quasilinear effect. It takes adifferent form here, where we use paracontrolled structures involving *series* rather that a finite number of terms as in the setting of semilinear singular PDEs. The introduction of this structure is our main technical insight; it pops out naturally as explained in Section 2.1. In retrospect, it looks fortunate that the authors of [6] were able to use a usual, finite, paracontrolled structure to study the 2-dimensional quasilinear (PAM) equation driven by a space white noise, rather than an infinite dimensional paracontrolled structure. This is only due to the fact that the noise in the equation was time-independent and the dimension of the manifold equal to 2. Infinite dimensional structures are needed if one works with a time-dependent noise, even in dimension 2, or if the dimension is equal to 3, as is the case here. This will be explained after the proof of our main result, Theorem 10.

We set the scene of paracontrolled calculus in Section 2, in the form that we need here. Section 3 is dedicated to the proof of the well-posedness result in small time for equation (1.1), stated in Theorem 10. We give in Appendix A a bird's eye view on the results from [4] on the high order paracontrolled calculus that we use here. The proofs of a number of new continuity results for operators needed for the study of quasilinear equations are collected in Appendix B and Appendix C. A minimum of familiarity with the tools of paracontrolled calculus is needed to get the best of what is presented here.

Notations. We gather here a number of notations used below.

– *We denote by M a 3-dimensional closed Riemannian manifold and set $\mathcal{M} := [0,T]\times M$, for a finite positive time horizon T. Given $\alpha\in\mathbb{R}$, we denote by C^α the space of α-Hölder functions on M, defined as the Besov space $B^\alpha_{\infty\infty}$, and write $\mathcal{C}^\alpha$ for the parabolic Hölder spaces. We refer the reader to Appendix A for more information about these spaces.*

– *It will be useful sometimes to denote by (β) an element of the parabolic Hölder space $\mathcal{C}^\beta$ with exponent $\beta\in\mathbb{R}$, whose only noticeable feature is its regularity.*

– *We will denote by $[\![0,b]\!]$ the set of integers of the interval $[0,b]$, for any real number b.*

2 – Paracontrolled calculus

We introduce in this section the tools from (the high order) paracontrolled calculus that we need to build a setting for the study of the quasilinear equation (1.1). The tools developed so far in [2, 3, 4] are not sufficient for our needs but only a little more is needed; it is given in Section 2.2. Paracontrolled calculus uses as a basic tool paraproduct and resonant operators. We recall the reader of the essential features of these bilinear operators before describing in a nutshell the paracontrolled approach to the study of singular semilinear parabolic PDEs. Our starting point for the analysis of (1.1) is a rewriting of this equation as a semilinear-like equation, equation (2.7) below. The special features of quasilinear equations appears clearly on this rewriting. Section 2.1 introduces the notion of paracontrolled system needed for the study of quasilinear equations and Section 2.2 completes our toolkit from [4] to control the operators that pop out only in the quasilinear setting. The study of equation (1.1) with these tools is the subject of Section 3.

As said in the introduction, in order to treat the free propagation of the initial condition as a remainder term in the paracontrolled analysis we will assume in this work that the initial condition u_0 has Hölder regularity 4α. We would otherwise need to work with spaces whose norms involve time weights. We refrain from doing so to keep concentrated on the quasilinear feature of the equation and emphasize the simplicity of our approach.

○ *Paraproduct and resonant operators.* Singular stochastic PDEs are characterized by the fact that they involve ill-defined products. Whatever notion of regularity is chosen (Hölder, Besov, Triebel-Lizorkin...) it happens indeed that the product of two distributions of regularity r_1 and r_2 makes sense as a continuous function of its arguments if and only if $r_1 + r_2 > 0$. Recall we work with a spacetime noise ζ of parabolic regularity $\alpha - 2$, for $\alpha < 1/2$. One has multiplication problems in equation (1.1) in the terms $d(u)\Delta u$ and $f(u)\zeta$, as one cannot expect from any solution theory that it gives u a regularity better than α while $\alpha + (\alpha - 2) < 0$. The starting point of the paracontrolled approach to the study of singular stochastic PDEs is the use of the paraproduct and resonant operators to disentangle this product problem. We describe them here in the simple setting of a finite dimensional torus to give the reader an elementary idea of what these operators are and some of their properties.

On the torus the Paley-Littlewood decomposition allows to represent any distribution f as a sum of smooth functions

$$f = \sum_{i \geqslant -1} \Delta_i(f),$$

with the Fourier transform of $\Delta_i(f)$ being essentially supported on the set of frequencies of order 2^i. This decomposition can be used formally to split a product as

$$\begin{aligned} fg = \sum_{i,j \geqslant -1} \Delta_i(f)\Delta_j(g) &= \sum_{i<j-1} \Delta_i(f)\Delta_j(g) + \sum_{|i-j|\leqslant 1} \Delta_i(f)\Delta_j(g) + \sum_{i-1>j} \Delta_i(f)\Delta_j(g) \\ &=: P_f g + \Pi(f,g) + P_g f. \end{aligned} \tag{2.1}$$

The bilinear operator P defined here is called the paraproduct operator and the bilinear operator Π is called the resonant operator. They were first introduced by Bony in his seminal work [10]. The interest of this decomposition is that the paraproduct operator is well-defined whatever distributions f and g are given as its arguments. It even sends continuously $C^\alpha \times C^\beta$ into $C^{\beta+0\wedge\alpha}$. On the other hand the resonant operator Π is well-defined and continuous on $C^\alpha \times C^\beta$ only if $\alpha + \beta > 0$, in which case it takes values in $C^{\alpha+\beta}$ – in accordance with the above mentioned rule on the well-posed character of the product operation. (We refer the reader to [5] for the basics on Littlewood-Paley decomposition and paraproduct and resonant operators in a Euclidean space.)

The definition of the actual paraproduct and resonant operators that we will use in the sequel is more involved as it provides parabolic operators and is tailor made to the equation we will study. The above Fourier-transform based Littlewood-Paley decomposition is in particular replaced by the Calderón decomposition associated with the semigroup generated by the second order differential operator L in (2.7). The reader will find in Appendix A a description of what is involved here. (The work [28] contains a self-contained introduction to these operators in a geometric elliptic setting.) There is in any case no need to masterize the details of the construction of these operators to use them efficiently.

○ *Solving semilinear singular PDEs using paracontrolled calculus.* Following [3] one can associate to any sufficiently regular second order differential operator L in Hörmander form its parabolic operator

$$\mathscr{L} := \partial_t + L$$

with inverse map $\mathscr{L}^{-1} : v \mapsto u$, giving the solution of the equation $\mathscr{L}u = v$ with zero intial condition, and a paraproduct P and its companion paraproduct $\widetilde{\mathsf{P}}$, intertwined to P by the relation

$$\mathscr{L}^{-1} \circ \mathsf{P} = \widetilde{\mathsf{P}} \circ \mathscr{L}^{-1}. \tag{2.2}$$

A resonant operator Π is also constructed from L. One can describe as follows the paracontrolled approach to the study of a generic *semilinear* singular parabolic PDE of the form

$$\mathscr{L}u = f(u, \partial u, \zeta),$$

with a function f that is affine in its ζ-argument. Denote by $\mathcal{F}$ the resolution operator of the free heat equation

$$\mathcal{F}u_0 := (\tau, x) \mapsto \big(e^{-\tau L}u_0\big)(x).$$

1. **Paracontrolled ansatz.** *The irregularity of the noise ζ dictates the choice of a solution space made up of functions/distributions of the form*

$$u = \sum_{i=1}^{k_0} \widetilde{\mathsf{P}}_{u_i} Z_i + u^\sharp, \tag{2.3}$$

for reference functions/distributions Z_i of regularity $i\alpha$ that depend formally only on ζ, to be determined later. The order of the expansion is chosen in such a way that $(k_0+1)\alpha + (\alpha-2) > 0$. The 'derivatives' u_i of u also need to satisfy similar structure equations to a lower order; their 'derivatives' as well, and so on. Denote by $\widehat{u}^\sharp$ the datum of all the remainders in these expansions; together with the Z_i's they determine entirely this triangular system.

2. **Right hand side.** *Rewrite the right hand side $f(u, \partial u, \zeta)$ of the equation in the canonical form*

$$f\big(u, \partial u, \zeta\big) = \sum_{j=1}^{k_0} \mathsf{P}_{v_j} Y_j + (\flat) \tag{2.4}$$

where $(\flat)$ is a nice remainder and the distributions Y_j depend only on ζ and the Z_i. Both the v_j and $(\flat)$ depend explicitly on u and all its derivatives, that is on $\widehat{u}^\sharp$.

3. **Fixed point.** *The fixed point relation*

$$\begin{aligned} u &= \mathcal{F}u_0 + \mathscr{L}^{-1}\big(f(u, \partial u, \zeta)\big) = \mathcal{F}u_0 + \sum_{j=1}^{k_0} \mathscr{L}^{-1}\Big(\mathsf{P}_{v_j} Y_j\Big) + \mathscr{L}^{-1}(\flat) \\ &= \mathcal{F}u_0 + \sum_{j=1}^{k_0} \widetilde{\mathsf{P}}_{v_j} Z_j + \mathscr{L}^{-1}(\flat), \end{aligned}$$

imposes a number of consistency relations on the choice of the $Z_i = \mathscr{L}^{-1}(Y_i)$ that define them uniquely as functions of ζ, and induces a fixed point relation for $\widehat{u}^\sharp$.

• We illustrate this mechanics on the example of the 2-dimensional parabolic Anderson model equation on the torus

$$\mathscr{L}u = u\zeta = \mathsf{P}_u\zeta + \mathsf{P}_\zeta u + \Pi(u, \zeta), \tag{2.5}$$

with constant initial condition $u_0 = c \neq 0$ and with $L = \Delta$. The noise ζ is almost surely in the parabolic Hölder space $\mathcal{C}^{\alpha-2}$, for α any positive real number strictly smaller than 1. (This is specific of dimension 2.) Whereas the paraproduct terms in (2.5) always make sense for arguments in Hölder spaces of positive or negative exponents, the resonant term is well-defined only if the sum of the Hölder regularity exponents of u and ζ add up to a positive real number. With ζ of Hölder regularity $\alpha-2$ and $\alpha<1$, one has $\alpha+(\alpha-2)<0$, and we fall short of fulfilling this positivity constraint. Rather than looking for a solution of the equation in the class $\mathcal{C}^\alpha$ of α-Hölder parabolic function we look for a solution in a restricted class of $\mathcal{C}^\alpha$ functions of the form

$$u = \widetilde{\mathsf{P}}_{u'} Z + u^\sharp, \tag{2.6}$$

for a reference function $Z \in \mathcal{C}^\alpha$, to be determined from the noise only and from the equation, with a remainder $u^\sharp \in \mathcal{C}^{2\alpha}$ of parabolic Hölder regularity 2α. With the notations of (2.3) one has here $k_0 = 1$ and $Z_1 = Z$. Given Z the unknown becomes the pair $(u', u^\sharp)$, with u' in a well-chosen function space. The special paracontrolled form of u allows to make sense of the a priori ill-defined resonant term $\mathsf{\Pi}(u,\zeta)$ under the assumption that $\mathsf{\Pi}(Z,\zeta)$ is given as an element of $\mathcal{C}^{2\alpha-2}$ – this is Gubinelli, Imkeller and Perkowski's fundamental 'commutator lemma', Lemma 2.4 in [20]. We write

$$\mathscr{L}u = \mathsf{P}_u\zeta + (2\alpha-2)(u', u^\sharp),$$

for a function $(2\alpha-2)(u',u^\sharp)$ depending implicitly on ζ, Z and $\mathsf{\Pi}(Z,\zeta)$, as a continuous function of all its arguments. With the notations of equation (2.4) one has $v_1 = u, Y_1 = \zeta$ and $(\flat) = (2\alpha-2)(u',u^\sharp)$. From the defining intertwining relation (2.2), the fixed point formulation of equation (2.5) then reads

$$\widetilde{\mathsf{P}}_{u'} Z + u^\sharp = u = \widetilde{\mathsf{P}}_u(\mathscr{L}^{-1}\zeta) + \mathscr{L}^{-1}\big((2\alpha-2)(u',u^\sharp)\big) + c,$$

– recall we assume for simplicity $u_0 = c \neq 0$ is constant. We now identify the terms on both sides of the equality according to their regularity so as to have a paracontrolled expression stable under the fixed point map. One then has $Z = \mathscr{L}^{-1}(\zeta)$ on the one hand, and

$$u' = u = \widetilde{\mathsf{P}}_{u'} Z + u^\sharp, \quad u^\sharp = \mathscr{L}^{-1}\big((2\alpha-2)(u',u^\sharp)\big) + c,$$

on the other hand. (Note that if we were working in this paragraph in dimension 3 the noise would be $(\alpha-2)$-Hölder regular, with $2/5 < \alpha < 1/2$ and more work would be required to defined the term $(2\alpha-2)(u',u^\sharp)$; the tools of the high order paracontrolled calculus can be used for that purpose.)

• Two different questions are addressed in Step **2**. Making sense of the ill-defined products, characteristic of singular PDEs, and putting the right hand side of the equation in the form (2.4), for an easy formulation of the fixed point in Step **3**. One of the main findings of [4] is that, at the end of the day, each of these two tasks are dealt with repeating essentially only one operation for each. See Section 2.2.4 for an explanation of the mechanics.

○ *A special feature of quasilinear equations.* One can rewrite equation (1.1) as a semilinear-like equation. The high order paracontrolled calculus developed in [4] requires that we work with an operator in Hörmander form involving vector fields with sufficient regularity. This is not a constraint in so far as smooth second order differential operators always have that form up to the addition of a vector field, so

$$A = \sum_{i=1}^{\ell} A_i^2 + A_0,$$

for smooth vector fields A_0, A_i. (What follows works for vector fields of class C^6. As we are not interested here in optimizing the degree of regularity of thedifferent objects involved in the analysis we stick to the smooth setting. As a matter of fact we could even write A as a sum of square of vector fields, without the drift A_0. This refined description of A would make no difference for us here.) With this in mind, let us introduce a smooth function $\overline{u_0}$ close enough to u_0 – this will be quantified later, in the proof of Theorem 10, and a solution-independent operator

$$L := -\sum_{i=1}^{\ell} V_i^2, \qquad V_i := \sqrt{d(\overline{u_0})}\, A_i.$$

We rewrite equation (1.1) under the form of an evolution equation

$$\begin{aligned}\mathscr{L}u := \;(\partial_t + L)u =& f(u)\zeta + \big(d(u) - d(\overline{u_0})\big)Au + \sum_{i=1}^{\ell} A_i\big(d(\overline{u_0})^{1/2}\big)V_i u \\ =& \, f(u)\zeta - d(\overline{u_0})^{-1}\big(d(u) - d(\overline{u_0})\big)Lu \\ &+ \sum_{i=1}^{\ell}\Big(1 - d(\overline{u_0})^{-1}\big(d(u) - d(\overline{u_0})\big)\Big)A_i\big(d(\overline{u_0})^{1/2}\big)V_i u \\ &+ d(u)A_0 u,\end{aligned}$$

involving the solution-independent operator L in Hörmander from. We write this equation as

$$\mathscr{L}u =: f(u)\zeta + \varepsilon(u,\cdot)Lu + \sum_{i=0}^{\ell} a_i(u,\cdot)V_i u. \tag{2.7}$$

As its name suggests the quantity $\varepsilon(\cdot)$ is expected to be small. The nonlinear term

$$\varepsilon(u,\cdot)Lu = d(\overline{u_0})^{-1}\big(d(u) - d(\overline{u_0})\big)Lu$$

in the right hand side still involves a second order term, a feature of quasilinear equations. (The dot sign in $\varepsilon(u,\cdot)$ stands for the dependence on $x \in M$ of ε, via $d(\overline{u_0})$.) This formulation of the quasilinear equation (1.1) in the semilinear-like form (2.7) involves the second order term $\varepsilon(u,\cdot)Lu$, specific to the quasilinear setting. This is why equation (2.7) is *not* a semilinear equation. Writing

$$\varepsilon(u,\cdot)Lu = \mathsf{P}_{\varepsilon(u,\cdot)}Lu + \mathsf{P}_{Lu}\varepsilon(u,\cdot) + \Pi\big(\varepsilon(u,\cdot), Lu\big), \tag{2.8}$$

the operators

$$\mathsf{P}_{La}b, \quad \Pi(La, b)$$

that appear in the last two terms of the right hand side of identity (2.8) turn out to be of the same type as the resonant operator $(a,b) \mapsto \Pi(a,b)$. Their analysis is thus similar to what was done in [4] for the resonant operator via the introduction of the corrector C and its iterates. (Similar things happen in the analysis of the (KPZ) equation with the terms $\mathsf{P}_{\partial u}\partial u$ and $\Pi(\partial u, \partial u)$.) The operator

$$\mathsf{P}_a(Lb)$$

that appears in the first term of the right hand side of (2.8) does not show up in the study of semilinear singular PDEs and requires a specific treatment. The analysis of these terms will be the object of Section 2.2.

2.1 Paracontrolled systems for quasilinear equations

We introduce in Section 2.1.2 the particular paracontrolled structure that we use for the study of quasilinear singular PDEs. Unlike Furlan & Gubinelli's paracomposition approach [16] or their regularity structures counterparts [31, 30, 32, 18] our paracontrolled structure does not have the form of a parametric finite paracontrolled structures. Rather it involves series in the paracontrolled expansion, as opposed to the finite expansion used for the study of semilinear singular PDEs. To motivate this structure we first test on a model quasilinear singular equation the above three step methodology that was designed for the study of semilinear singular PDEs. Its implementation leads naturally to the structure introduced in Section 2.1.2.

2.1.1 – A naive trial on a model case. Recall our discussion of the paracontrolled approach to the parabolic Anderson model equation and the form (2.5) that we gave to the quasilinear equation (1.1). The main feature of the quasilinear setting is the presence of a second order term Lu in the right hand side of the equation. Consider, as a motivation, the model equation

$$\begin{aligned}\mathscr{L}u &= u\zeta + uLu \\ &= \mathsf{P}_u\zeta + \mathsf{P}_u(Lu) + \Big(\mathsf{P}_\zeta u + \Pi(u,\zeta) + \mathsf{P}_{Lu}u + \Pi(u,Lu)\Big),\end{aligned} \tag{2.9}$$

still in the setting where $2/3 < \alpha < 1$ is close to 1. As above, one problem is to make sense of the resonant term $\Pi(u, Lu)$; this can be done assuming that the term $\Pi\big(Z, LZ\big)$ makes sense as an element of the parabolic Hölder space of exponent $2\alpha - 2$. This assumption allows to define the term in parentheses in the right hand side of (2.9) as an element of $\mathcal{C}^{2\alpha-2}$. One can further see that for u of paracontrolled form (2.6) one has

$$\mathsf{P}_u(Lu) \simeq \mathsf{P}_{u'u}(LZ),$$

up to a term in $\mathcal{C}^{2\alpha-2}$. A naive fixed point formulation of equation (2.9) then reads

$$\widetilde{\mathsf{P}}_{u'}Z + u^\sharp = \widetilde{\mathsf{P}}_u\big(\mathscr{L}^{-1}(\zeta)\big) + \widetilde{\mathsf{P}}_{u'u}\big(\mathscr{L}^{-1}(LZ)\big) + (2\alpha)(u', u^\sharp) + c. \tag{2.10}$$

Note that the operator $\mathscr{L}^{-1}L$ sends any $\mathcal{C}^\beta$ into itself, with no regularization property. Since we want to have a paracontrolled expression stable under the fixed point map encoded in identity (2.10) it imposes that Z is actually made up of two components $Z = \big(Z^{(1)}, Z^{(2)}\big)$, with $Z^{(1)} = \mathscr{L}^{-1}(\zeta)$ and $Z^{(2)} = \mathscr{L}^{-1}(LZ^{(1)})$. The function u' should have as a consequence two components as well so equation (2.9) rewrites

$$\sum_{k=1}^{2} \widetilde{\mathsf{P}}_{u'_k} Z^{(k)} + u^\sharp = \widetilde{\mathsf{P}}_u\big(\mathscr{L}^{-1}(\zeta)\big) + \sum_{k=1}^{2} \widetilde{\mathsf{P}}_{u'_k u}\big(\mathscr{L}^{-1}(LZ^{(k)})\big) + (2\alpha)(u', u^\sharp) + c,$$

with terms $\mathscr{L}^{-1}\big(\Pi(Z^{(i)}, LZ^{(j)})\big)$ inside the remainder $(2\alpha)(\cdots)$ given a priori. The first two terms in the right hand side are taken care of by the $Z^{(1)}$ and $Z^{(2)}$ terms in the left hand side. This is not the case of the term $\widetilde{\mathsf{P}}_{u'_2 u}\mathscr{L}^{-1}(LZ^{(2)})$ in the right hand side. Consistency then imposes that we actually add a third component to Z and u', to take care of $\widetilde{\mathsf{P}}_{u'_2 u}\mathscr{L}^{-1}(LZ^{(2)})$. The story then repeats itself and we are led to consider as a priori form for the solution an infinite paracontrolled expansion

$$u = \sum_{k\geqslant 1} \widetilde{\mathsf{P}}_{u'_k} Z^{(k)} + u^\sharp,$$

with $Z^{(k)} = (\mathscr{L}^{-1}L)^{k-1}Z^{(1)}$ for $k > 1$, and $Z^{(1)} = \mathscr{L}^{-1}(\zeta)$. All the $Z^{(k)}$ are elements of $\mathcal{C}^\alpha$ here since the operator $\mathscr{L}^{-1}L$ does not improve nor worsen the regularity. This infinite dimensional paracontrolled structure is a characteristic feature of the paracontrolled approach of quasilinear singular equations. The convergence of the preceding sum needs to be built in the setting, together with the a priori data of the terms $\Pi(Z^{(i)}, LZ^{(j)})$ as elements of $\mathcal{C}^{2\alpha-2}$. Anticipating over the results to follow, the reference functions in the paracontrolled expansion of a solution to equation (1.1) have the same tree-like structure as the reference functions of the corresponding semilinear equation. This comes from their inductive definition. However, each edge in a 'tree' now has a length, corresponding to composing first the operator represented by the edge by the operator $(\mathscr{L}^{-1}L)^k$, for some $k \geqslant 0$. This echoes Gerencsér and Hairer's work [18], where each symbol represents an infinite dimensional space. This is the quasilinear effect. The approach works under the quantitative assumption that each a priori term has a natural norm bounded above by a constant multiple of C^k, for a constant $C > 1$, and k the number of times that the operator $\mathscr{L}^{-1}L$ appears in the formal definition of the term – the total "length" of the tree.

2.1.2 – Paracontrolled systems for quasilinear equations. Motivated by the analysis of Section 2.1.1 we set up in this section the notations needed to describe solution spaces based on infinite paracontrolled systems. Fix $0 < \alpha < 1$. Let an integer $n \geqslant 1$ be given, together with countable families $\mathscr{T}_1, \dots, \mathscr{T}_n$ of real-valued non-null functions on $[0,T] \times M$ with each $[\tau] \in \mathscr{T}_i$ of parabolic Hölder regularity $|\tau| := i\alpha$. We distinguish the function $[\tau]$ from its label τ by using brackets to denote the function. Write

$$\mathscr{T} := \mathscr{T}_1 \cup \cdots \cup \mathscr{T}_n.$$

A generic finite word with letters in $\mathscr{T}$ will be denoted by $w = \tau_1 \dots \tau_k$ and assigned a **homogeneity**

$$|w| := |\tau_1| + \cdots + |\tau_k|.$$

Define

$$\mathscr{W} := \varnothing \cup \Big\{ w = \tau_1 \dots \tau_k \;; k \geqslant 1,\; |w| \leqslant n\alpha \Big\}.$$

This is the set of words with letters in the alphabet $\mathscr{T}$ and homogeneity no greater than $n\alpha$. This set depends on n, which will be fixed in each application. We do no record the dependence of $\mathscr{W}$ on n in the notation. For a word $w = \tau_1 \dots \tau_k \in \mathscr{W}$ and $\tau \in \mathscr{T}$ we denote by $w\tau$ the concatenation of w and τ, so $|w\tau| = |w| + |\tau|$. Set $(\!|\varnothing|\!) := 1$, and for $w = \tau_1 \dots \tau_k \in \mathscr{W}$, set

$$(\!|w|\!) := (\!|\tau_1 \dots \tau_k|\!) := \big\|[\tau_1]\big\|_{\mathcal{C}^{|\tau_1|}} \cdots \big\|[\tau_k]\big\|_{\mathcal{C}^{|\tau_k|}};$$

this is *not* a norm. The following definition of a paracontrolled system coincides with the notion used in the study of semilinear singular PDEs, where $\mathscr{T}$ can be chosen to be a finite set rather than infinite countable set.

Definition 1. *Let $(\beta_w)_{w \in \mathscr{W}}$ be a family of positive real numbers. A* **system paracontrolled by $\mathscr{T}$ at order** *n is a family $\widehat{u} = (u_w)_{w \in \mathscr{W}}$ of parabolic functions such that one has*

$$u_w = \sum_{\tau \in \mathscr{T}; |w\tau| \leqslant n\alpha} \widetilde{\mathsf{P}}_{u_{w\tau}}[\tau] + u_w^\sharp, \tag{2.11}$$

with $u_w^\sharp \in \mathcal{C}^{n\alpha + \beta_w - |w|}$ for all $w \in \mathscr{W}$, and

$$|\!|\!|\widehat{u}|\!|\!| := \sum_{w \in \mathscr{W}} (\!|w|\!)\, \|u_w^\sharp\|_{\mathcal{C}^{n\alpha + \beta_w - |w|}} < \infty. \tag{2.12}$$

The convergence condition (2.12) is always fulfilled in a semilinear setting where one can work with a finite set $\mathscr{T}$, so $\mathscr{W}$ is finite. In the case where $\mathscr{W}$ is infinite the weights $(\!|w|\!)$ in (2.12) are here to guarantee the convergence in a proper space of the sum (2.11). A reasonable choice for the constants β_w would be to take them all equal to α. This is not a convenient choice from the technical point of view and all of them will be chosen in the interval $(2/5, \alpha)$ in a particular way explained in Section 3 before Theorem 10. In particular, they satisfy $\beta_w < \beta_{w'}$ for any $w, w' \in \mathscr{W}$ with w a subword of w'. These regularity exponents play a crucial role in proving that the fixed point formulation of the equation involves a contracting map. We note that all u_w with $|w| < n\alpha$ are $\mathcal{C}^\alpha$, while the u_w with $|w| = n\alpha$, are elements of $\mathcal{C}^{\beta_w}$. Notice that a paracontrolled system is triangular: The bigger $|w|$ the lesser we expand u_w. The study of the quasilinear equation (1.1) will require below the use of systems paracontrolled at order 3. Note also that a paracontrolled system is actually determined by the set $\widehat{u}^\sharp := (u_w^\sharp)_{w \in \mathscr{W}}$ of all remainders in the paracontrolled expansion (2.11). Putting together all the contributions from each $\mathscr{T}_i$ each u_w in a paracontrolled system is in particular required to have an expansion of the form

$$u_w = (\alpha) + (2\alpha) + \ldots + \big(n\alpha + \beta_w - |w|\big)$$

as will be proved in the following proposition.

Proposition 2. *Let $\widehat{u}$ be a system paracontrolled by $\mathscr{T}$ at order n. One has*

$$\sum_{w\in\mathscr{W}} (\!|w|\!)\, \|u_w\|_{\mathcal{C}^{\beta_w}} \lesssim |\!|\!|\widehat{u}|\!|\!|.$$

This implies in particular that one has for all $w \in \mathscr{W}$ the estimate

$$\|u_w\|_{\mathcal{C}^{\beta_w}} \lesssim |\!|\!|\widehat{u}|\!|\!|.$$

Proof – Let $w \in \mathscr{W}$. We proceed by a finite induction. The case where $|w| = n\alpha$ is well controlled. For $|w| < n\alpha$, we have

$$\begin{aligned} \|u_w\|_{\mathcal{C}^{\beta_w}} &\lesssim \sum_{\tau\in\mathscr{T};|w\tau|\leqslant n\alpha} \|u_{w\tau}\|_{\mathcal{C}^{\beta_{w\tau}}} \|\tau\|_{\mathcal{C}^{|\tau|}} + \|u_w^\sharp\|_{\mathcal{C}^{\beta_w}} \\ &\lesssim \sum_{w'\in\mathscr{W};|ww'|\leqslant n\alpha} (\!|w'|\!)\, \|u^\sharp_{ww'}\|_{\mathcal{C}^{\beta_{ww'}}} \end{aligned}$$

and Tthis yields

$$\sum_{w\in\mathscr{W}} (\!|w|\!)\, \|u_w\|_{\mathcal{C}^{\beta_w}} \lesssim \sum_{w''\in\mathscr{W}} (\!|w''|\!)\, \|u^\sharp_{w''}\|_{\mathcal{C}^{\beta_{w''}}} \lesssim |\!|\!|\widehat{u}|\!|\!|.$$

▷

We note here for later use that if $\widehat{u}$ is paracontrolled at order 3 then for any function $h \in C_b^6$ then $h(u)$ is the $\varnothing$-component of a paracontrolled system at order 2. Indeed, identity (36) of [4] tells us first that there is an element $(\sharp) \in \mathcal{C}^{3\alpha}$ such that

$$h(u) = \mathsf{P}_{h'(u)}u + \frac{1}{2}\Big(\mathsf{P}_{h^{(2)}(u)}u^2 - \mathsf{P}_{h^{(2)}(u)u}u\Big) + (\sharp).$$

Denote by (3α) an element of $\mathcal{C}^{3\alpha}$ that may change from line to line. If one sets

$$\mathsf{R}(\mathbf{1}, b, c) := \widetilde{\mathsf{P}}_b c - \mathsf{P}_b c, \qquad \mathsf{R}^\circ(a, b, c) := \mathsf{P}_a\big(\mathsf{P}_b c\big) - \mathsf{P}_{ab}c,$$

and one uses the properties of these operators proved in Proposition 3 of [4], together with the properties of the D operator proved in Proposition 2 therein, one sees that

$$\begin{aligned} h(u) &= \mathsf{P}_{h'(u)}\big(\widetilde{\mathsf{P}}_{u_\tau}[\tau]\big) + \frac{1}{2}\,\mathsf{P}_{h^{(2)}(u)u_{\tau_1}u_{\tau_2}}\big(\Pi([\tau_1],[\tau_2])\big) + (3\alpha) \\ &= \mathsf{P}_{h'(u)u_\tau}[\tau] + \mathsf{P}_{h'(u)}\big(\mathsf{R}(\mathbf{1}, u_\tau, [\tau])\big) + \frac{1}{2}\,\mathsf{P}_{h^{(2)}(u)u_{\tau_1}u_{\tau_2}}\big(\Pi([\tau_1],[\tau_2])\big) + (3\alpha) \\ &= \mathsf{P}_{h'(u)u_\tau}[\tau] + \mathsf{P}_{h'(u)u_{\tau\sigma}}\big(\mathsf{R}(\mathbf{1}, [\sigma], [\tau])\big) + \frac{1}{2}\,\mathsf{P}_{h^{(2)}(u)u_{\tau_1}u_{\tau_2}}\big(\Pi([\tau_1],[\tau_2])\big) + (3\alpha). \end{aligned} \tag{2.13}$$

The implicit sums are restricted to indices $\tau \in \mathscr{W}$ with $|\tau| \leqslant 2\alpha$ in the first term, to $\tau, \sigma \in \mathscr{W}$ such that $|\tau| = |\sigma| = \alpha$ in the second term, and to $\tau_1, \tau_2 \in \mathscr{W}$ such that $|\tau_1| = |\tau_2| = \alpha$ in the third term. We obtain directly from the identity (2.13) the full description of the paracontrolled system at order 2 corresponding to $h(u)$.

We described in the three step scheme of the introduction of Section 2 the paracontrolled approach to solving semilinear singular PDEs. In this scheme the paracontrolled structure of the elements of the solution space is used to take profit from the continuity properties of a number of operators that come in the analysis of the product problem. In a nutshell, while some operator $M(\cdot)$ may not make sense on a $\mathcal{C}^\beta$ space it can make sense on a subspace of $\mathcal{C}^\beta$ whose elements are of paraproduct form $\mathsf{P}_a b$, or sums of such terms, up to a regular enough remainder term, provided the quantities $M(b)$ are given off-line. The product $\varepsilon(u,\cdot)Lu$ in (2.7) is specific to the quasilinear setting and its analysis requires the use of a number of continuity results for some new operators. The next section presents these results. They will be used jointly with the infinite dimensional paracontrolled structure of Definition 1 in Section 3 to give a proof of the locally well-posed character of equation (1.1).

2.2 Additional correctors

We saw above that the analysis of the quasilinear equation (2.7) requires in addition to the study of terms already encountered in a semilinear setting the study of operators of the form

$$\mathsf{P}_a(Lb), \quad \mathsf{P}_{La}b, \quad \mathsf{\Pi}(La,b).$$

This section is dedicated to the study of these quantities and their expansion properties – when the a argument is of paracontrolled form. Our results come under the form of a number of continuity results whose proofs are given in Appendix B; all the proofs are variations on the pattern of proofs of continuity results from [4]. The continuity results from this section are all we need in addition to the results of [4] to study equation (1.1), and more generally a whole class of quasilinear singular PDEs. The reader is welcome to skip the proofs of the different statements below and directly jump to Section 3 to see them in action.

Given that the technical setting of [3, 4] is likely not to be familiar to most readers we give in this section the proofs of some of the statements in the time-independent model setting of the 3-dimensional flat torus. The paraproduct and resonant operators P_ab and $\Pi(a,b)$ are defined classically in terms of Fourier projectors as in (2.1).

2.2.1 – Operators $\mathsf{P}_{La}b$ *and* $\mathsf{\Pi}(La,b)$*.* These two operators are defined by similar formulas as the resonant operator in terms of the parabolic approximation operators $\mathcal{Q}_t$ from [4]. It is thus natural that they satisfy expansion rules similar to the expansion rules satisfied by the resonant operator. Introduce for that purpose the operators

$$\begin{aligned}
\mathsf{C}_L^-\Big((a_1,a_2),b\Big) &:= \mathsf{P}_{L\widetilde{\mathsf{P}}_{a_1}a_2}b - a_1\mathsf{P}_{La_2}b,\\
\mathsf{C}_L^+\Big(a,(b_1,b_2)\Big) &:= \mathsf{P}_{La}\Big(\widetilde{\mathsf{P}}_{b_1}b_2\Big) - b_1\mathsf{P}_{La}b_2,\\
\mathsf{C}_L\Big((a_1,a_2),b\Big) &:= \mathsf{\Pi}\Big(L\widetilde{\mathsf{P}}_{a_1}a_2,b\Big) - a_1\mathsf{\Pi}\Big(La_2,b\Big).
\end{aligned}$$

We choose the notation $-$ in the exponent of C_L^- to emphasize that the paraproduct term is in the low 'frequency' part of the operator, while it is in the high 'frequency' part in C_L^+. The following theorem is proved here in the time-independent model setting of the flat torus; see Appendix B for a proof in the parabolic setting.

Theorem 3. *The following two statements hold true.*

- *Let* $\alpha_1\in(0,1)$ *and* $\alpha_2,\beta\in(-3,3)$ *be such that* $\alpha_1+\alpha_2\in(-3,3)$*. If*
$$\alpha_2+\beta-2<0 \quad \text{and} \quad \alpha_1+\alpha_2+\beta-2>0 \tag{2.14}$$
then the operators C_L^- *and* C_L *extend as continuous operators from* $\mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^{\beta}$ *into* $\mathcal{C}^{\alpha_1+\alpha_2+\beta-2}$.
- *Let* $\beta_1\in(0,1)$ *and* $\alpha,\beta_2\in(-3,3)$ *be such that* $\beta_1+\beta_2\in(-3,3)$*. If*
$$\alpha+\beta_2-2<0 \quad \text{and} \quad \alpha+\beta_1+\beta_2-2>0$$
then the operator C_L^+ *extends as a continuous operator from* $\mathcal{C}^{\alpha}\times\mathcal{C}^{\beta_1}\times\mathcal{C}^{\beta_2}$ *into* $\mathcal{C}^{\alpha+\beta_1+\beta_2-2}$.

It is possible to explain in a non-technical way the mechanics at work in the proof of Theorem 3. Following [4] we define the *outer centering operator* $\mathscr{C}$ as an operator acting on a space of operators on functions by

$$(\mathscr{C}M)(f)(x) := M\big(f-f(x)\big)(x)$$

and set also

$$(\mathscr{C}f)(x) := f(\cdot)-f(x).$$

One can rewrite in terms of $\mathscr{C}$ the corrector

$$\mathsf{C}(f,g,h) = \Pi(\widetilde{\mathsf{P}}_f g,h) - f\Pi(g,h)$$

as

$$\mathsf{C}(f,g,h) = \Pi\big(\widetilde{\mathsf{P}}_{\mathscr{C}f} g,h\big).$$

The iterated corrector

$$\mathsf{C}\big((f_1,f_2),g,h\big) := \mathsf{C}\big(\widetilde{\mathsf{P}}_{f_1} f_2,g,h\big) - f_1\mathsf{C}(f_2,g,h)$$

from [4] writes in those terms

$$\mathsf{C}\big((f_1,f_2),g,h\big) := \Pi\Big(\widetilde{\mathsf{P}}_{\mathscr{C}\mathsf{P}_{\mathscr{C}f_1}f_2} g,h\Big)$$

Similarly, one has

$$\begin{aligned}
\mathsf{C}_L^-\big((a_1,a_2),b\big) &= \mathsf{P}_{L\widetilde{\mathsf{P}}_{\mathscr{C}a_1}a_1} b,\\
\mathsf{C}_L^+\big(a,(b_1,b_2)\big) &= \mathsf{P}_{La}\big(\widetilde{\mathsf{P}}_{\mathscr{C}b_1}b_2\big),\\
\mathsf{C}_L\big((a_1,a_2),b\big) &= \Pi\big(L\widetilde{\mathsf{P}}_{\mathscr{C}a_1}a_1,b\big).
\end{aligned}$$

On a general basis a paraproduct $\widetilde{\mathsf{P}}_f g$ has the same regularity as g if f has positive regularity. So for $f\in\mathcal{C}^a, g\in\mathcal{C}^b, h\in\mathcal{C}^c$ the quantity $\mathsf{C}(f,g,h)$ should for instance only make sense if $b+c>0$. The effect of the outer recentering is that $\widetilde{\mathsf{P}}_{\mathscr{C}f}g$ behaves inside the resonant operator as a function of regularity $a+b$. This dovetails nicely with the fact that the corrector makes sense if $a+b+c>0$. The very same thing happens for the operators $\mathsf{C}_L^-, \mathsf{C}_L^+, \mathsf{C}_L$.

Proof – Write Δ for the usual Laplace operator on the flat d-dimensional torus and denote here by C^β the associated Hölder spaces, for any $\beta\in\mathbb{R}$.

- Set

$$C_\Delta(a_1,a_2,b) := \Pi\big(\Delta P_{a_1}a_2, v\big) - a_1\Pi(\Delta a_2,b).$$

We prove that for α_1,α_2 and β such that inequalities (2.14) hold true, the operator C_Δ is continuous from $C^{\alpha_1}\times C^{\alpha_2}\times C^\beta$ into $C^{\alpha_1+\alpha_2+\beta-2}$. We have

$$C_\Delta(a_1,a_2,b) = \sum_{|i-j|<1} \Delta_i\left(P_{a_1}a_2\right)\Delta_j(b) - a_1\Delta_i(a_2)\Delta_j(b).$$

Setting

$$\varepsilon_i := \Delta_i\big(\Delta P_{a_1}a_2\big) - a_1\Delta_i(\Delta a_2),$$

we have

$$C_\Delta(a,b,c) = \sum_{|i-j|<1} \varepsilon_i\,\Delta_j(b).$$

As in the proof of the estimate for the classic corrector C, one sees that one has

$$\|\Delta_k\varepsilon_i\|_{L^\infty} \lesssim 2^{2i}\,2^{-i\alpha_2}2^{-\max(i,k)\alpha_1}\,\|a_1\|_{C^{\alpha_1}}\|a_2\|_{C^{\alpha_2}};$$

the factor 2^{2i} comes from the Δ operator. Writing

$$\begin{aligned}
\Delta_k\left(C_\Delta(a_1,a_2,b)\right) &= \sum_{|i-j|\leqslant 1} \Delta_k\big(\varepsilon_i\,\Delta_j(b)\big)\\
&= \sum_{\substack{j<k-2\\|i-j|\leqslant 1}} \Delta_k(\varepsilon_i)\Delta_j(b) + \sum_{\substack{k<j-2\\|i-j|\leqslant 1}} \Delta_k\big(\Delta_i(\varepsilon_i)\Delta_j(b)\big) + \sum_{\substack{|k-j|\leqslant 1\\|i-j|\leqslant 1}} \Delta_k\big(S_i(\varepsilon_i)\Delta_j(b)\big),
\end{aligned}$$

we see that

$$\begin{aligned}
\Big\|\Delta_k\left(C_\Delta(a_1,a_2,b)\right)\Big\|_{L^\infty} &\lesssim \sum_{i<k-2} 2^{-i(\alpha_2+\beta-2)}2^{-k\alpha_1} + \sum_{k<i-2} 2^{-i(\alpha_1+\alpha_2+\beta-2)} + \sum_{|i-k|\leqslant 1} 2^{-i(\alpha_1+\alpha_2+\beta-2)}\\
&\lesssim 2^{-k(\alpha_1+\alpha_2+\beta-2)}
\end{aligned}$$

using that $(\alpha_2+\beta-2)<0$ and $(\alpha_1+\alpha_2+\beta-2)>0$. The implicit multiplicative factor is a multiple of $\|a_1\|_{C^{\alpha_1}}\|a_2\|_{C^{\alpha_2}}\|b\|_{C^\beta}$.

- Set now

$$C_\Delta^-(a_1,a_2,b) := P_{\Delta P_{a_1}a_2}b - a_1 P_{\Delta a_2}b = \sum_{i<j-2}\varepsilon_i\,\Delta_j(b).$$

We prove that for α_1,α_2 and β such that inequalities (2.14) hold true, C_Δ^- is continuous from $C^{\alpha_1}\times C^{\alpha_2}\times C^\beta$ into $C^{\alpha_1+\alpha_2+\beta-2}$. This can be seen by writing

$$\Delta_k\left(C_\Delta^-(a_1,a_2,b)\right) = \sum_{\substack{j<k-2\\ i<j-2}}\Delta_k(\varepsilon_i)\Delta_j(b) + \sum_{\substack{k<j-2\\ i<j-2}}\Delta_k\big(\Delta_i(\varepsilon_i)\Delta_j(b)\big) + \sum_{\substack{|k-j|\leqslant 1\\ i<j-2}}\Delta_k\big(S_i(\varepsilon_i)\Delta_j(b)\big),$$

from which one sees that $\left\|\Delta_k\left\{C_\Delta^-(a_1,a_2,b)\right\}\right\|_{L^\infty}$ is bounded above by a multiple of

$$\sum_{j<k-2}2^{-i(\alpha_2+\beta-2)}2^{-k\alpha_1} + \sum_{k<j-2}2^{-i(\alpha_1+\alpha_2+\beta-2)} + \sum_{|j-k|\leqslant 1}2^{-i(\alpha_1+\alpha_2+\beta-2)} \lesssim 2^{-k(\alpha_1+\alpha_2+\beta-2)},$$

for an implicit multiplicative constant proportional to $\|a_1\|_{C^{\alpha_1}}\|a_2\|_{C^{\alpha_2}}\|b\|_{C^\beta}$. ▷

We also have continuity estimates on iterated $\mathsf{C}_L^{+,-}$ correctors, as in [4]. Given the proof of Theorem 3 given in Appendix B it will be clear to the reader that their statements and proofs are identical to what is done in [4] for the iterated correctors – see Section 3.1.3 therein. We leave their statements and proofs to the reader.

The continuity results on the corrector and its iterates, or on the operators $\mathsf{C}_L^{(\pm)}$and their iterates, are used in the analysis of singular PDEs to take profit from the paracontrolled structure of a potential solution to get expansions of the form

$$\begin{aligned} M\big(\widetilde{\mathsf{P}}_{u_\tau}[\tau]\big) &= u_\tau M([\tau]) + M'(u_\tau) \\ &= \mathsf{P}_{u_\tau}M([\tau]) + \mathsf{P}_{M([\tau])} + M''(u_\tau), \end{aligned}$$

for operators M',M'' that have the same expansion properties as M itself. So one can iterate the expansion as long as u_τ and its 'derivatives' have a paracontrolled structure. The paracontrolled structure of a potential solution will however involve remainder terms for which one cannot use expansions of the previous kind as the only information we have on these remainders are their regularity. One defines 'refined correctors' to take profit from their good regularity properties.

The continuity results from Theorem 3 can only take profit only from the Hölder regularity of the arguments a_1 or b_1 of $\mathsf{C}_L^-\big((a_1,a_2),b\big)$ or $\mathsf{C}_L^-\big((a,(b_1,b_2)\big)$ for regularity exponents in $(0,1]$. As in the semilinear case we need to introduce refined correctors to refine the estimates if a_1 or b_1 is α_1 or β_1-Lipscthiz, with α_1 or β_1 of regularity exponent in the interval $(1,2)$. We set for that purpose, for a generic spacetime point e,

$$\mathsf{C}_{L,(1)}^-\Big(a_1,a_2,b\Big)(e) := \mathsf{C}_L^-(a_1,a_2,b)(e) - d\big(\overline{u_0}(e)\big)^{-1}\sum_{i=1}^{\ell}(V_ia_1)(e)\,\Big(\mathsf{P}_{L\widetilde{\mathsf{P}}_{\delta_i(e,\cdot)}a_2}b\Big)(e),$$

$$\mathsf{C}_{L,(1)}^+\Big(a,b_1,b_2\Big)(e) := \mathsf{C}_L^+(a,b_1,b_2)(e) - d\big(\overline{u_0}(e)\big)^{-1}\sum_{i=1}^{\ell}(V_ib_1)(e)\,\Big(\mathsf{P}_{La}\widetilde{\mathsf{P}}_{\delta_i(e,\cdot)}b_2\Big)(e),$$

$$\mathsf{C}_{L,(1)}\Big(a_1,a_2,b\Big)(e) := \mathsf{C}_L(a_1,a_2,b)(e) - d\big(\overline{u_0}(e)\big)^{-1}\sum_{i=1}^{\ell}(V_ia_1)(e)\,\mathsf{\Pi}\Big(L\widetilde{\mathsf{P}}_{\delta_i(e,\cdot)}a_2,b\Big)(e),$$

where the functions δ_i are defined in Appendix B. Keep in mind right now that in the setting of the flat torus one has

$$d\big(\overline{u_0}(e)\big)^{-1}V_i = \partial_i,$$

the partial derivative in the $i^{\rm th}$ space direction, and

$$\delta_i(e,e') = d\big(\overline{u_0}(x)\big)^{1/2}(x_i - x_i'),$$

for spacetime points $e=(t,x)$ and $e'=(t',x')$. The rationale for the introduction of these refined correctors is that they correspond to refined recentering operators for which $\mathscr{C}f$ corresponds to removing from f its first order Taylor expansion at the running point x rather than just removing its value at x.

Theorem 4. *The following two statements hold true.*

- *Let* $\alpha_1 \in (1,2)$ *and* $\alpha_2, \beta \in (-3,3)$ *such that* $\alpha_1+\alpha_2 \in (-3,3)$*. If*
$$\alpha_2+\beta-2<0 \quad \textit{and} \quad \alpha_1+\alpha_2+\beta-2>0$$
then the operators $\mathsf{C}^-_{L,(1)}$ *and* $\mathsf{C}_{L,(1)}$ *extends as continuous operators from* $\mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^{\beta}$ *into* $\mathcal{C}^{\alpha_1+\alpha_2+\beta-2}$.
- *Let* $\beta_1 \in (1,2)$ *and* $\alpha, \beta_2 \in (-3,3)$ *such that* $\beta_1+\beta_2 \in (-3,3)$*. If*
$$\alpha+\beta_2-2<0 \quad \textit{and} \quad \alpha+\beta_1+\beta_2-2>0$$
then the operator $\mathsf{C}^+_{L,(1)}$ *extends as a continuous operator from* $\mathcal{C}^{\alpha}\times\mathcal{C}^{\beta_1}\times\mathcal{C}^{\beta_2}$ *into* $\mathcal{C}^{\alpha+\beta_1+\beta_2-2}$.

2.2.2 – L *operator.* We define the operator

$$\mathsf{L}(a,b) := L(\widetilde{\mathsf{P}}_a b) - \mathsf{P}_a(Lb).$$

Continuity results on this operator will allow us to get from a paracontrolled expansion for u an expansion for Lu of the form

$$Lu = \sum \mathsf{P}_{u'_\tau}(L[\tau]) + (4\alpha-2),$$

for some u'_τ. A paracontrolled expansion for a term of the form $\mathsf{P}_a(Lu)$ can then be obtained.

Theorem 5. *The following statements hold true.*

- *Let* $\alpha\in(0,1)$ *and* $\beta\in(-3,3)$*, be such that* $\alpha+\beta<3$*, and* $\alpha+\beta-2\in(-3,3)$*. Then the operator* L *extends as a continuous operator from* $\mathcal{C}^{\alpha}\times\mathcal{C}^{\beta}$ *into* $\mathcal{C}^{\alpha+\beta-2}$.
- *Let* $\alpha_1,\alpha_2\in(0,1)$ *and* $\beta\in(-3,3)$ *such that* $\alpha_1+\beta<3$ *and* $\alpha_1+\alpha_2+\beta-2\in(-3,3)$*. Then the iterated operator*
$$\mathsf{L}\big((a_1,a_2),b\big) := \mathsf{L}\big(\widetilde{\mathsf{P}}_{a_1}a_2, b\big) - a_1\mathsf{L}(a_2,b) \tag{2.15}$$
extends as a continuous operator from $\mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^{\beta}$ *into* $\mathcal{C}^{\alpha_1+\alpha_2+\beta-2}$.
- *Let* $\alpha_1,\alpha_2,\alpha_3\in(0,1)$ *and* $\beta\in(-3,3)$ *such that* $\alpha_1+\alpha_2+\beta<3$*,* $\alpha_2+\beta<3$ *and* $\alpha_1+\alpha_2+\beta-2\in(-3,3)$*. Then the iterated operator*
$$\mathsf{L}\Big(\big((a_1,a_2),a_3\big),b\Big) := \mathsf{L}\big((\widetilde{\mathsf{P}}_{a_1}a_2,a_3),b\big) - a_1\mathsf{L}\big((a_2,a_3),b\big)$$
extends as a continuous operator from $\mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^{\alpha_3}\times\mathcal{C}^{\beta}$ *into* $\mathcal{C}^{\alpha_1+\alpha_2+\alpha_3+\beta-2}$.

The mechanics behind this statement is easy to understand on the following example set in the torus with the elementary paraproduct operator P from (2.1). One sees indeed on the formula

$$\Delta(P_a b) - P_a(\Delta b) = P_{\Delta a}b + 2P_{\nabla a}(\nabla b) \tag{2.16}$$

that this operator takes its values in $C^{\alpha+\beta-2}$ if $a \in C^\alpha$, $b \in C^\beta$ and $0 \leqslant \alpha < 2$. Dealing with parabolic paraproducts and conjugated operators $\mathsf{P}, \widetilde{\mathsf{P}}$ only adds a layer of technicality and does not change the mechanics. We see on formula (2.16) that the reason why L has the expansion property (2.15) is because the operators $P_{\Delta a} b$ and $P_{\nabla a}(\nabla b)$ have that property – see Section 3.3 in [4] for an explanation.

The continuity result for L and its iterate stated in the first two items of Theorem 5 do not allow to take profit of a possible better regularity exponent for a. It happens however to be necessary for the analysis of singular PDEs. One has to define for that purpose the refined operator

$$\mathsf{L}_{(1)}(a,b) := L(\widetilde{\mathsf{P}}_a b) - \mathsf{P}_a(Lb) - \sum_{i=1}^{\ell} \mathsf{P}^{(i)}_{d(\overline{u}_0)^{-1}V_i a}(Lb)$$

to deal with arguments a in $\mathsf{L}(a,b)$ with regularity exponent greater in the interval $(1,2)$. (We also had to define in Section 3.1.2 of [4] a refined corrector to deal with 'high' regularity arguments in a resonant term.) The operators $\mathsf{P}^{(i)}$ are defined by for any e in the parabolic space $\mathcal{M}$ by

$$\left(\mathsf{P}^{(i)}_a b\right)(e) := \int_{e',e''\in\mathcal{M}} K(e;e',e'')a(e')\left(\widetilde{\mathsf{P}}_{\delta_i(\cdot,e')} b\right)(e'')\,\nu(\mathrm{d}e')\nu(\mathrm{d}e'')$$

with K the kernel of the bilinear operator $(a,b) \mapsto \mathsf{P}_a b$. See Appendix A for the notations and details on the parabolic setting – these details are not so important when it comes to using the continuity results stated here or in [4], as opposed to proving them. The following theorem is proved in Appendix B.

Theorem 6. *Let $\alpha \in (1,2)$ and $\beta \in (-3,3)$, be such that $\alpha+\beta < 3$, and $(\alpha+\beta-2) \in (-3,3)$. Then the operator $\mathsf{L}_{(1)}$ extends as a continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ into $\mathcal{C}^{\alpha+\beta-2}$.*

Theorem 3, Theorem 4 and Theorem 5-6 take care of the specific features of quasilinear equations, compared to their semilinear analogue. Formulation (2.7) also involves the term $a_i(u,\cdot)V_i u$ that can appear in a semilinear setting as well. The last paragraph of this section state the results that we need about it.

2.2.3 – Dealing with the term $a_i(u,\cdot)V_i u$. We have the following continuity results for the operators

$$\begin{aligned}
\mathsf{C}^-_{V_i}\Big(a_1,a_2,b\Big) &:= \mathsf{P}_{V_i\widetilde{\mathsf{P}}_{a_1}a_2} b - a_1\mathsf{P}_{V_i a_2} b,\\
\mathsf{C}^+_{V_i}\Big(a,b_1,b_2\Big) &:= \mathsf{P}_{V_i a}\Big(\widetilde{\mathsf{P}}_{b_1} b_2\Big) - b_1\mathsf{P}_{V_i a} b_2,\\
\mathsf{C}_{V_i}\Big(a_1,a_2,b\Big) &:= \mathsf{\Pi}\Big(V_i\widetilde{\mathsf{P}}_{a_1}a_2, b\Big) - a_1\mathsf{\Pi}\Big(V_i a_2, b\Big);
\end{aligned}$$

see Appendix B for a proof.

Theorem 7. *The following two statements hold true.*

- *Let $\alpha_1 \in (0,1)$ and $\alpha_2, \beta \in (-3,3)$ such that $\alpha_1+\alpha_2 \in (-3,3)$. If*
$$\alpha_2+\beta-1 < 0 \quad \textit{and} \quad \alpha_1+\alpha_2+\beta-1 > 0 \tag{2.17}$$
then the operators $\mathsf{C}^-_{V_i}$ and C_{V_i} have natural extensions as continuous operators from $\mathcal{C}^{\alpha_1}\times\mathcal{C}^{\alpha_2}\times\mathcal{C}^\beta$ into $\mathcal{C}^{\alpha_1+\alpha_2+\beta-1}$.
- *Let $\beta_1 \in (0,1)$ and $\alpha, \beta_2 \in (-3,3)$ such that $\beta_1+\beta_2 \in (-3,3)$. If*
$$\alpha+\beta_2-1 < 0 \quad \textit{and} \quad \alpha+\beta_1+\beta_2-1 > 0$$
then the operator $\mathsf{C}^+_{V_i}$ has a natural extension as a continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^{\beta_1}\times\mathcal{C}^{\beta_2}$ into $\mathcal{C}^{\alpha+\beta_1+\beta_2-1}$.

Proof – We prove here this continuity result for a simplified version of the operator C_{V_i} in the time-independent case of the flat torus, with the constant vector field ∂_1 in the role of V_i; we refer the reader to Appendix B for the proof of Theorem 7 in the general setting. Set

$$C_{\partial_1}(a,b,c) := \Pi\big(\partial_1 P_a b, c\big) - a\Pi(\partial_1 b, c).$$

We prove that for α, β and γ such that inequalities (2.17) hold true, the operator C_{∂_1} is continuous from $C^\alpha \times C^\beta \times C^\gamma$ into $C^{\alpha+\beta+\gamma-2}$. Using that $\Delta_i(\partial_1 f) \simeq O(2^i)\Delta_i(f)$, for a function $O(2^i)$ with uniform norm of order 2^i, we have

$$C_{\partial_1}(a,b,c) \simeq \sum_{|i-j|<1} O(2^i)\Delta_i\,(\mathsf{P}_a b)\,\Delta_j(c) - a\,O(2^i)\Delta_i(b)\Delta_j(c),$$

so

$$C_{\partial_1}(a,b,c) = \sum_{|i-j|<1} O(2^i)\varepsilon_i\Delta_j(c).$$

The same computations as above then yield the estimate

$$\big\|\Delta_k\big(C_V(a,b,c)\big)\big\|_{L^\infty} \lesssim 2^{-k(\alpha+\beta+\gamma-1)}\|a\|_{C^\alpha}\|b\|_{C^\beta}\|c\|_{C^\gamma}.$$

▷

Associate with each vector field V_i the operator

$$\mathsf{V}_i(a,b) := V_i\big(\widetilde{\mathsf{P}}_a b\big) - \mathsf{P}_a(V_i b).$$

Theorem 8. *The following two statements hold true.*

- *Let $\alpha, \beta \in (-3,3)$ such that $\alpha+\beta-1 \in (-3,3)$. Then the operator V_i has a natural extension as a continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha+\beta-1}$.*
- *Let $\alpha_1, \alpha_2 \in (0,1)$ and $\beta \in (-3,3)$ such that $\alpha_1+\beta < 3$ and $\alpha_1+\alpha_2+\beta-1 \in (-3,3)$. Then the iterated operator*

$$\mathsf{V}_i((a_1,a_2),b) := \mathsf{V}_i(\widetilde{\mathsf{P}}_{a_1}a_2, b) - a_1\mathsf{V}_i(a_2,b) \tag{2.18}$$

extends as a continuous operator from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^\beta$ to $\mathcal{C}^{\alpha_1+\alpha_2+\beta-1}$.

Like for the operator L the mechanics behind this statement is easy to understand on the following example set in the torus, where one has the formula

$$\partial_i(P_a b) - P_a(\partial_i b) = P_{\partial_i a} b. \tag{2.19}$$

We see on formula (2.19) that the reason why V_i has the expansion property (2.18) is because the operator $P_{V_i a} b$ have that property.

2.2.4 – A convenient formalism. One can conveniently use a synthetic notation and encode the expansion rules satisfied by the operators

$$\mathsf{\Pi}(a,b),\quad \mathsf{\Pi}(La,b),\quad \mathsf{P}_{La}b,\quad \mathsf{P}_{V_i a}b,\quad \mathsf{\Pi}(V_i a, b),$$

as functions of a, encoded in the statements of the preceding sections. Fix b and consider any of the preceding operators as a function of a,. For $a = \widetilde{\mathsf{P}}_{a_1}a_2$, under proper regularity conditions specified by the above statements one has the expansion property

$$\mathsf{E}\big(\widetilde{\mathsf{P}}_{a_1}a_2\big) = a_1\mathsf{E}(a_2) + \mathsf{E}'(a_1,a_2), \tag{2.20}$$

for another operator E'. The rule of thumb is that the argument a_2 in our computations will always depend only on the noise so it will be convenient to skip it from the notations and only keep track of its regularity. The variable of interest in the expansion will be a_1. The operator E', seen as a function of a_1, will enjoy the same type of expansion property as the operator E. We will use the notation $\mathsf{E}^\beta(a)$ to denote an operator that sends $\mathcal{C}^{|a|}$ into $\mathcal{C}^{|a|+\beta}$ under proper regularity assumptions on its argument. One has for instance

$$\mathsf{\Pi}(\cdot,c) = \mathsf{E}^{|c|}(\cdot),$$

for $c \in \mathcal{C}^{|c|}$. In those terms, for $a_2 \in \mathcal{C}^{|a_2|}$, identity (2.20) rewrites

$$\mathsf{E}^\beta\big(\widetilde{\mathsf{P}}_{a_1} a_2\big) = a_1 \mathsf{E}^\beta(a_2) + \mathsf{E}^{\beta+|a_2|}(a_1).$$

The operators $\mathsf{P}_b a, \mathsf{P}_b La$ and D and S from [4], satisfy a different type of expansion rule. As above, we are interested in the case where b, or other arguments as in D and S, depends only on the noise and a can have a paracontrolled structure. If, for b fixed, the operator sends continuously any space $\mathcal{C}^{|a|}$ into $\mathcal{C}^{|a|+\beta}$ we denote it by $\mathsf{F}^\beta(a)$. One has for instance

$$\mathsf{P}_\zeta a = \mathsf{F}^{\alpha-2}(a).$$

The expansion rule for such operators is

$$\mathsf{F}^\beta\big(\widetilde{\mathsf{P}}_{a_1} a_2\big) = \mathsf{P}_{a_1}\big(\mathsf{F}^\beta(a_2)\big) + \mathsf{F}^{\beta+|a_2|}(a_1),$$

for another operator $\mathsf{F}^{\beta+|a_2|}$ enjoying the same type of expansion rule as F^β, with β replaced by $\beta + |a_2|$.

If one agrees to use the same letter for objects that are possibly different but have the same expansion rule, one has for instance the identity

$$\begin{aligned}\mathsf{E}^\beta\big(\widetilde{\mathsf{P}}_{a_1} a_2\big) &= a_1\, \mathsf{E}^\beta(a_2) + \mathsf{E}^{\beta+|a_2|}(a_1) \\ &= \mathsf{P}_{a_1}\big(\mathsf{E}^\beta(a_2)\big) + \mathsf{P}_{\mathsf{E}^\beta(a_2)} a_1 + \Pi\big(a_1, \mathsf{E}^\beta(a_2)\big) + \mathsf{E}^{\beta+|a_2|}(a_1) \\ &= \mathsf{P}_{a_1}\big(\mathsf{E}^\beta(a_2)\big) + \mathsf{F}^{\beta+|a_2|}(a_1) + \mathsf{E}^{\beta+|a_2|}(a_1).\end{aligned}$$

We can see on this expression that if a_1 itself is given in paracontrolled form then we can re-expand the E and F functions of a_1 above. This is the core of the machinery of the high order paracontrolled calculus, the mechanics that allows to run step **2** in the three step resolution scheme of semilinear singular PDEs described in the introduction of Section 2.

3 – Quasilinear generalised (PAM) equation

We use the generic three step process from Section 2 to solve the quasilinear generalised (PAM) equation (2.7). Recall that in the end we want to have an infinite paracontrolled system stable under the fixed point formulation and to correctly tune the family $(\beta_a)_{a\in\mathscr{W}}$ of regularity exponents in Definition 1 to get a contraction for a small enough horizon time T.

• **Step 1.** We have $2/5 < \alpha < 1/2$, so *we choose to work with a third order paracontrolled expansion*, with a remainder term $u^\sharp \in \mathcal{C}^{4\alpha}$ in the paracontrolled expansion of u whose product with any distribution of Hölder regularity $\alpha - 2$ is well-defined, since $5\alpha - 2 > 0$.

• **Step 2.** *We use the continuity results for correctors, commutators and their iterates proved in* [4] *and in Section 2.2 to put the right hand side of equation* (2.7) *in the canonical form* (2.4)*.*

This is what the next statement does. Recall $d_0(\cdot) = d(\overline{u_0}(\cdot))$ is smooth. Indices w, w', w'' below are in $\mathscr{W}$ while $[\tau] \in \mathscr{T}$. We use below the notation s in $\zeta^{(1)}_{(s)}, \zeta^{(2)}_{(s)}$ for a 'sentence' of the form $w, (w, w')$ or (w, w', w''), for words $w, w,', w'' \in \mathscr{W}$.

Proposition 9. *Assume we are given a system $(u_w)_{w\in\mathscr{W}}$ paracontrolled by a family $\mathscr{T}$ at order* 3*. Then*

$$\begin{aligned} f(u)\zeta + \varepsilon(u,\cdot)Lu + \sum_{i=0}^{\ell} a_i(u,\cdot)V_i u \\ = \mathsf{P}_{f(u)}\zeta + \sum_{|w|\leqslant 2\alpha} \mathsf{P}_{f'(u)u_w}\big(\zeta_w^{(1)}\big) + \sum_{|ww'|\leqslant 2\alpha} \mathsf{P}_{f^{(2)}(u)u_w u_{w'}}\big(\zeta_{(w,w')}^{(1)}\big) \\ + \sum_{\tau\in\mathscr{T}} \mathsf{P}_{\varepsilon(u,\cdot)u_\tau}\big(L[\tau]\big) + \sum_{|w|\leqslant 3\alpha; w\in\mathscr{W}\setminus\mathscr{T}} \mathsf{P}_{\varepsilon(u,\cdot)u_w}\big(\zeta_w^{(2)}\big) \\ + \sum_{|ww'|\leqslant 3\alpha} \mathsf{P}_{d_0^{-1}d'(u)u_w u_{w'}}\big(\zeta_{(w,w')}^{(2)}\big) + \sum_{|ww'w''|\leqslant 3\alpha} \mathsf{P}_{d_0^{-1}d^{(2)}(u)u_w u_{w'}u_{w''}}\big(\zeta_{(w,w',w'')}^{(2)}\big) \\ + \sum_{|\tau|=\alpha; 1\leqslant j\leqslant \ell} \mathsf{P}_{a_j(u,\cdot)u_\tau}\big(\zeta_{j,\tau}\big) + (\sharp), \end{aligned} \tag{3.1}$$

for distributions $\zeta_s^{(1)}, \zeta_s^{(2)}, \zeta_{j,\tau}$ that depend only on ζ and $\mathscr{T}$, with $\zeta_s^{(1)}$ of regularity $|s|+\alpha-2$, with $\zeta_s^{(2)}$ of regularity $|s|-2$ and $\zeta_{j,\tau}$ of regularity $|\tau|-1$. The remainder $(\sharp)$ is an element of $\mathcal{C}^{4\alpha-2}$.

In (3.1), the terms with exponent (1) come from the analysis of the product $f(u)\zeta$ while the terms with exponent (2) come from the analysis of the product $\varepsilon(u,\cdot)Lu$. Proposition 9 is the analog of the paracontrolled expansion of $f(u)\xi$ for the semilinear generalised (PAM) equation for an arbitrary paracontrolled system $\widehat{u}$ – Proposition 4 in [4]. As always in the analytic part of the study of a singular PDE, one needs to assume that the distributions $\zeta_s^{(1)}, \zeta_s^{(2)}, \zeta_{j,\tau}$ are given off-line as elements of their natural spaces. The remainder term $(\sharp)$ also involves off-line data. The point with *stochastic* singular PDEs is that one can construct these data by probabilistic means; this is what renormalization is about. It comes as a by-product of the proof that the remainder $(\sharp)$ in (3.1) is the sum of a term of regularity $4\alpha-2$ involving the off-line data and a term of regularity $5\alpha-2$ that is a continuous function of the paracontrolled system $\widehat{u}$ and all the off-line data.

Proof – Below we invite the reader to check the convergence of all implicit infinite sums of $\mathscr{T}$ using the convergence condition (2.12) in the definition of a paracontrolled system; we do not do that explicitly each time. Recall we denote by (β) an element of the parabolic Hölder space $\mathcal{C}^\beta$ with regularity exponent $\beta\in\mathbb{R}$ whose only noticeable feature is its regularity. Its expression may change from line to line. Recall also from Appendix A the definition of the operator

$$\mathsf{R}^\circ(a,b,c) = \mathsf{P}_a(\mathsf{P}_b c) - \mathsf{P}_{ab}c,$$

its continuity and expansion properties. *To shorten notations, we sometimes use implicit summation on repeated indices.*

- The term $f(u)\zeta$ is the same as in the semilinear (gPAM) equation so its decomposition is given by proposition 17 of [4], that is

$$f(u)\zeta = \mathsf{P}_{f(u)}\zeta + \sum_{|w|\leqslant 2\alpha} \mathsf{P}_{f'(u)u_w}\big(\zeta_w^{(1)}\big) + \sum_{|ww'|\leqslant 2\alpha} \mathsf{P}_{f^{(2)}(u)u_w u_{w'}}\big(\zeta_{ww'}^{(1)}\big) + (4\alpha-2).$$

We now deal with the analysis of the term $\varepsilon(u,\cdot)Lu$ by looking first at $\mathsf{P}_{\varepsilon(u,\cdot)}(Lu)$ and then at $\mathsf{P}_{Lu}\varepsilon(u,\cdot)$ and $\mathsf{\Pi}\big(\varepsilon(u,\cdot),Lu\big)$.

- For the term $\mathsf{P}_{\varepsilon(u,\cdot)}(Lu)$ we have from Theorem 5

$$\begin{aligned}Lu &= L\big(\widetilde{\mathsf P}_{u_\tau}[\tau]\big) + (4\alpha-2)\\ &= \mathsf P_{u_\tau}(L[\tau]) + \mathsf L\big(u_\tau,[\tau]\big) + (4\alpha-2)\\ &= \mathsf P_{u_\tau}([L\tau]) + \mathsf P_{u_{\tau\sigma}}\mathsf L\big([\sigma],[\tau]\big) + \mathsf L\big((u_{\tau\sigma},[\sigma]),[\tau]\big) + (4\alpha-2)\\ &= \mathsf P_{u_\tau}([L\tau]) + \mathsf P_{u_{\tau\sigma}}\mathsf L\big([\sigma],[\tau]\big) + \mathsf P_{u_{\tau\sigma\gamma}}\mathsf L\big(([\gamma],[\sigma]),[\tau]\big) + (4\alpha-2).\end{aligned} \tag{3.2}$$

One takes care of remainder terms in the expansions of the u_τ's with $|\tau| = \alpha$, in the expression $\mathsf L(u_\tau,[\tau])$, using the operator $\mathsf L_{(1)}$. This remainder term $\mathsf L\big(u_\tau,[\tau]\big)$ provides an element of $\mathcal C^{4\alpha-2}$ that goes in the term $(4\alpha-2)$. Write identity (3.2) under the form

$$Lu =: \mathsf P_{u_w}\xi_w^{(2)} + (4\alpha-2),$$

with $\xi_w^{(2)}$ of regularity $|w|-2$. Keeping in mind that the expression $(4\alpha-2)$ may change from line to line, this yields

$$\begin{aligned}\mathsf P_{\varepsilon(u,\cdot)}(Lu) &= \mathsf P_{\varepsilon(u,\cdot)}\big(\mathsf P_{u_w}\xi_w^{(2)}\big) + (4\alpha-2)\\ &= \mathsf P_{\varepsilon(u,\cdot)u_w}\big(\xi_w^{(2)}\big) + \mathsf R^\circ\big(\varepsilon(u,\cdot),u_w,\xi_w^{(2)}\big) + (4\alpha-2)\\ &= \mathsf P_{\varepsilon(u,\cdot)u_w}\big(\xi_w^{(2)}\big) + \mathsf R^\circ\big(\varepsilon(u,\cdot),u_\tau,L[\tau]\big) + (4\alpha-2)\end{aligned}$$

using the definition (3.2) of the $\xi_w^{(2)}$ and the fact that the terms where $|w| > \alpha$ go in the remainder. Using that each u_τ is itself paracontrolled, we get

$$\begin{aligned}\mathsf P_{\varepsilon(u,\cdot)}(Lu) &= \mathsf P_{\varepsilon(u,\cdot)u_w}\big(\xi_w^{(2)}\big) + \mathsf R^\circ\big(\varepsilon(u,\cdot)u_{\tau\sigma},[\sigma],L[\tau]\big) + (4\alpha-2)\\ &= \mathsf P_{\varepsilon(u,\cdot)u_w}\big(\xi_w^{(2)}\big) + \mathsf P_{d_0^{-1}d'(u)u_\gamma u_{\tau\sigma}+\varepsilon(u,\cdot)u_{\tau\sigma\gamma}}\mathsf R^\circ\big([\gamma],[\sigma],L[\tau]\big) + (4\alpha-2)\\ &= \mathsf P_{\varepsilon(u,\cdot)u_w}\big(\zeta_w^{(2)}\big) + \mathsf P_{d_0^{-1}d'(u)u_\gamma u_{\tau\sigma}}\mathsf R^\circ\big([\gamma],[\sigma],L[\tau]\big) + (4\alpha-2).\end{aligned}$$

(Here again remainders in a paracontrolled expansion contribute as remainder terms that go inside the $(4\alpha-2)$ term.) In the last equality, the term

$$\mathsf P_{\varepsilon(u,\cdot)u_{\tau\sigma\gamma}}\Big(\mathsf R^\circ\big([\gamma],[\sigma],L[\tau]\big)\Big)$$

has been added to $\mathsf P_{\varepsilon(u,\cdot)u_w}\big(\xi_w^{(2)}\big)$, with $w=\tau\sigma\gamma$, resulting in changing $\xi_w^{(2)}$ to $\zeta_w^{(2)}$. We rewrite this formula under the form

$$\begin{aligned}\mathsf P_{\varepsilon(u,\cdot)}(Lu) = \sum_{\tau\in\mathscr T}\mathsf P_{\varepsilon(u,\cdot)u_\tau}\big(L[\tau]\big) &+ \sum_{w\in\mathscr W\setminus\mathscr T,|w|\leqslant 3\alpha}\mathsf P_{\varepsilon(u,\cdot)u_w}\big(\zeta_w^{(2)}\big)\\ &+ \mathsf P_{d_0^{-1}d'(u)u_\gamma u_{\tau\sigma}}\mathsf R^\circ\big([\gamma],[\sigma],L[\tau]\big) + (4\alpha-2).\end{aligned}$$

Note that the terms with $|\tau| = \alpha$ are the only terms in the right hand side of equation (2.7) that have the same regularity as the noise ζ.

- We deal with the terms

$$\mathsf P_{Lu}\varepsilon(u,\cdot) = \mathsf P_{Lu}\big(d_0^{-1}d(u)\big) - \mathsf P_{Lu}\mathbf 1 = \mathsf P_{Lu}\big(d_0^{-1}d(u)\big) + (4\alpha-2)$$

and

$$\mathsf\Pi\big(\varepsilon(u,\cdot),Lu\big) = \mathsf\Pi\big(d_0^{-1}d(u),Lu\big) - \mathsf\Pi(\mathbf 1,Lu) = \mathsf\Pi\big(d_0^{-1}d(u),Lu\big) + (4\alpha-2)$$

using the correctors $\mathsf C, \mathsf C_L^-, \mathsf C_L^+$ and $\mathsf C_L$, to take care of paraproducts that appear in the paraproduct plus resonant decomposition of the product $d_0^{-1}d(u)$. The refined versions of the correctors will be used to take care of remainder terms in paracontrolled expansions.

Recall from Section 2.2.4 the $\mathsf E/\mathsf F$-type form of the continuity statements on these operators and let us agree to denote by $\mathsf E^\beta(\cdot,\cdot)$ a bilinear operator that has the $\mathsf E$-type expansion property with respect to each of its two variables. We use the same convention for a trilinear operator $\mathsf{E}^\beta(\cdot,\cdot,\cdot)$. Let us also agree to denote here by E^β, with no argument, an element of $\mathcal{C}^\beta$. Last, recall also that the functions $d_0(\cdot)$ and $d_0^{-1}(\cdot)$ are smooth. Using the E-notation for operators of E-type, such as in the introduction of Section 2, we have

$$\begin{aligned}\mathsf{P}_{Lu}\big(d_0^{-1}d(u)\big) + \Pi\big(d_0^{-1}d(u), Lu\big) &= \mathsf{E}^{-2}\big(d_0^{-1}d(u), u\big)\\ &= d_0^{-1}d'(u)\,\mathsf{E}^{-2}(u,u) + d_0^{-1}d^{(2)}(u)\,\mathsf{E}^{-2}(u,u,u) + (4\alpha-2).\end{aligned}$$

The analysis of the term $\mathsf{E}^{-2}(u,u)$ is conveniently done as follows. (This computation was already done at length in [4].) We first write the term $\mathsf{E}^{-2}(u,u)$ in *multiplicative form*

$$\begin{aligned}\mathsf{E}^{-2}(u,u) &= u_{\tau_1}\mathsf{E}^{-2+|\tau_1|}(u) + \mathsf{E}^{-2+|\tau_1|}(u_{\tau_1},u) + (5\alpha-2)\\
&= \Big\{u_{\tau_1}u_{\tau_2}\mathsf{E}^{-2+|\tau_1|+|\tau_2|} + u_{\tau_1}\mathsf{E}^{-2+|\tau_1|+|\tau_2|}(u_{\tau_2}) + (5\alpha-2)\Big\}\\
&\quad + \Big\{u_{\tau_1\sigma_1}\mathsf{E}^{-2+|\tau_1|+|\sigma_1|}(u) + \mathsf{E}^{-2+|\tau_1|+|\sigma_1|}(u_{\tau_1\sigma_1},u) + (5\alpha-2)\Big\}\\
&\quad + (5\alpha-2)\\
&= \Big\{u_{\tau_1}u_{\tau_2}\mathsf{E}^{-2+|\tau_1|+|\tau_2|} + u_{\tau_1}u_{\tau_2\sigma_2}\mathsf{E}^{-2+|\tau_1|+|\tau_2|+|\sigma_2|}\\
&\qquad + u_{\tau_1}u_{\tau_2\sigma_2\mu_2}\mathsf{E}^{-2+|\tau_1|+|\tau_2|+|\sigma_2|+|\mu_2|} + (5\alpha-2)\Big\}\\
&\quad + \Big\{u_{\tau_1\sigma_1}\mathsf{E}^{-2+|\tau_1|+|\sigma_1|+|\tau_2|} + u_{\tau_1\sigma_1}u_{\tau_2\sigma_2}\mathsf{E}^{-2+|\tau_1|+|\sigma_1|+|\tau_2|+|\sigma_2|} + (5\alpha-2)\\
&\qquad + u_{\tau_1\sigma_1\mu_1}u_{\tau_2}\mathsf{E}^{-2+|\tau_1|+|\sigma_1|+|\mu_1|+|\tau_2|} + (5\alpha-2)\Big\}\\
&\quad + (5\alpha-2).\end{aligned}$$

(All the remainder terms $(5\alpha-2)$ are well-defined.) Each term above that is not a remainder $(5\alpha-2)$ is of the form

$$(\star)\,\mathsf{E}^\beta = \mathsf{P}_{(\star)}\mathsf{E}^\beta + \mathsf{F}^\beta(\star) + \mathsf{E}^\beta(\star),$$

fordifferent values of β, with E^β depending only on the noise and the reference functions $[\tau]$ in the paracontrolled structure and $(\star)$ either of the form u_w or $u_w u_{w'}$, with $w,w'\in\mathscr{W}$. The term $\mathsf{P}_{(\star)}\mathsf{E}^\beta$ has the expected form. We use the paracontrolled structure of u_w and the F-expansion property to deal with $\mathsf{F}^\beta(u_w)$. To deal with $\mathsf{F}^\beta(u_w u_{w'})$, write first

$$\mathsf{F}^\beta(u_w u_{w'}) = \mathsf{F}^\beta\big(\mathsf{P}_{u_w}u_{w'}\big) + \mathsf{F}^\beta\big(\mathsf{P}_{u_{w'}}u_w\big) + \mathsf{F}^\beta\big(\Pi(u_w,u_{w'})\big),$$

and use the F-expansion property for the first two terms. For the resonant term, we use the commutator operator D and its continuity properties, recalled in Appendix A, to expand first the resonant term in the form

$$\Pi(u_w,u_{w'}) = \mathsf{P}_{u_{w\tau}}\Pi([\tau],u_{w'}) + \mathsf{D}\big(u_{w\tau},[\tau],u_{w'}\big),$$

and then expand the paraproduct inside the operators Π and D using the paracontrolled forms of $u_{w'}$ and $u_{w\tau}$. We leave the details to the reader as these computations were already done at length in sections 3 and 4 of [4]. All these operations are only done up to remainders of positive regularity $5\alpha-2$. These computations give in the end an expansion of the form

$$\begin{aligned}&\mathsf{P}_{Lu}\big(d_0^{-1}d(u)\big) + \Pi\big(d_0^{-1}d(u), Lu\big)\\ &\qquad = \mathsf{P}_{d_0^{-1}d'(u)u_wu_{w'}}\big(\zeta^{(2)}_{ww'}\big) + \mathsf{P}_{d_0^{-1}d^{(2)}(u)u_wu_{w'}u_{w''}}\big(\zeta^{(2)}_{ww'w''}\big) + (4\alpha-2),\end{aligned}$$

for functionals $\zeta^{(2)}_{ww'}$ and $\zeta^{(2)}_{ww'w''}$ of ζ and the $[\tau]\in\mathscr{T}$.

- For the terms involving the vector fields $a_i(u,\cdot)V_i u$ we simply note that
$$\mathsf{P}_{V_i u} a_i(u,\cdot) + \Pi\big(a_i(u,\cdot), V_i u\big) = (2\alpha-1) = (4\alpha-2),$$
since $2\alpha - 1 > 4\alpha - 2$, and use Theorem 8 to write
$$\begin{aligned} V_i u &= V_i\big(\widetilde{\mathsf{P}}_{u_\tau}\tau\big) + (4\alpha-2) \\ &= \mathsf{P}_{u_\tau}(V_i\tau) + \mathsf{V}_i(u_\tau,\tau) + (4\alpha-2) \\ &= \mathsf{P}_{u_\tau}(V_i\tau) + (4\alpha-2). \end{aligned}$$

▷

We insist again on the fact that all the implicit sums on repeated indices above converge as a consequence of the bound (2.12) satisfied by paracontrolled systems and from the continuity estimates from Section 2.2.

Remark – *A reader familiar with the setting of regularity structures may wonder what plays here the role of the polynomial component of a modelled distribution and the role of the symbols $X\Xi$ and $X\mathcal{I}(\Xi)\xi$ that already appear in the analysis of the semilinear generalised (PAM) equation. For u itself it is the C^1 part of that function, given by*
$$u^{(1)} := u - \sum_{|\tau|\leqslant 2\alpha} \mathsf{P}_{u_\tau}[\tau] = \sum_{|\tau|=3\alpha} \mathsf{P}_{u_\tau}[\tau] + u^\sharp.$$
This C^1 is in particular a non-local functional of $\widehat{u}^\sharp$ since the paraproduct operator is non-local. This echoes the non-local character of the polynomial part of the lift to a regularity structure of the spacetime convolution operator with the heat kernel. Similar explicit formulas for the C^1 part of functions $g(u)$ of u can be given using the high order paracontrolled expansion formula from Theorem 2 of [4] – see Section 3.5.1 therein. While the contribution of the X component of a modelled distribution is associated with the two symbols $X\Xi$ and $X\mathcal{I}(\Xi)\Xi$ the contribution of $u^{(1)}$ in the paracontrolled analysis appears differently as the contributions of the terms u_τ with $|\tau| = 3\alpha$ and $u^\sharp$. Note of the distributions $\zeta_s^{(1)}$ or $\zeta_s^{(2)}$ correspond in particular to one of the symbols. Rather it is a linear combination of the $\zeta_s^{(1)}$ or $\zeta_s^{(2)}$ that would correspond to each of them.

- **Step 3.** *Consistency of the fixed point relation*
$$u = \mathcal{F}u_0 + \mathscr{L}^{-1}\Big(f(u)\zeta + \varepsilon(u,\cdot)Lu + \sum_{i=0}^{\ell} a_i(u,\cdot)V_i u\Big)$$
imposes the choice of $\mathscr{T}$ and induces a fixed point relation for $\widehat{u}^\sharp$.

◦ *Constructing $\mathscr{T}$.* One identifies from equation (3.1) a number of constraints that $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$ needs to satisfy to write a consistent fixed point formulation of equation (2.7). Denote by $s = (w_1,\dots,w_k)$ a generic sentence with words in $\mathscr{W}$, with $|s| := |w_1| + \cdots + |w_k|$. Consistency imposes that one has
$$\left\{\begin{array}{rll} \mathscr{L}^{-1}(\zeta) & \in \mathscr{T}_1, & \\ \big(\mathscr{L}^{-1}L\big)(\mathscr{T}_i) & \subset \mathscr{T}_i, & \text{for all } 1 \leqslant i \leqslant 3, \\ \mathscr{L}^{-1}\big(\zeta_s^{(1)}\big) & \subset \mathscr{T}_{i+1}, & \text{for all } |s| = i\alpha \leqslant 2\alpha, \\ \mathscr{L}^{-1}\big(\zeta_s^{(2)}\big) & \subset \mathscr{T}_i, & \text{for all } |s| = i\alpha \leqslant 3\alpha, \text{ and } s \notin \mathscr{T}, \\ \mathscr{L}^{-1}\big(\zeta_{j,\mathscr{T}_1}\big) & \subset \mathscr{T}_3. & \end{array}\right. \tag{3.3}$$
Requiring further
$$\Big\{\Pi\big([\tau],[\sigma]\big),\ \widetilde{\mathsf{P}}_{[\tau]}[\sigma] - \mathsf{P}_{[\tau]}[\sigma]\,;\, \tau,\sigma \in \mathscr{T}_1\Big\} \subset \mathscr{T}_2, \tag{3.4}$$
ensures moreover that for u paracontrolled to order 3 by the reference set $\mathscr{T}$ all the functions $f(u), f'(u)u_a, f^{(2)}(u)u_a u_b$, etc. that appear as lower arguments of the paraproducts

in the paracontrolled expansion (3.1) of the right hand side of (2.7) have a second order paracontrolled expansion with respect to that reference set $\mathscr{T}$.

We define $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$, as the smallest set of reference functions satisfying (3.3) *and* (3.4)*.*

Let us emphasize the triangular/iterative nature of (3.3)-(3.4) behind the notations $\zeta_s^{(1)}, \zeta_s^{(2)}$ and $\zeta_{j,\tau}$. The elements of $\mathscr{T}_{i+1}$ are only built from ζ and elements of $\mathscr{T}_i$. This construction recipe for $\mathscr{T}$ gives back the finite set $\mathscr{T}^\circ$ used for the study of the 3-dimensional semilinear generalised (PAM) equation in [4] if one replaces the preceding infinite set $\mathscr{T}_1$ be the one point set $\big\{\mathscr{L}^{-1}(\zeta)\big\}$. In a sense, one can see $\mathscr{T}^\circ$ as the 'skeleton' of $\mathscr{T}$, where each occurence of $\mathscr{L}^{-1}(\zeta)$ in an element of $\mathscr{T}^\circ$ is in $\mathscr{T}$ any of the elements of $\mathscr{T}_1$. Given $[\tau] \in \mathscr{T}$ denote by n_τ the total number of times that the operator $\mathscr{L}^{-1}L$ appears in the formal expression for $[\tau]$.

A proper definition of $\mathscr{T}$ when the noise ζ is space white noise requires the implementation of a renormalization procedure. The works [12, 14] by Bruned, Hairer and Zambotti and Chandra & Hairer provide a systematic approach of that question in a semilinear setting in the context of regularity structures. While [7] gives a strong hint that this result can be translated from the regularity structure world to the paracontrolled world a purely paracontrolled analysis of the renormalization problem is still missing. A number of investigations on the renormalization problem in a quasilinear setting have been done [18, 17]. Given these first results, it is most likely that an ad hoc renormalization process for the quasilinear setting will be given by the same renormalization process as in the semilinear setting, with trees with branches of ad hoc length used instead of their skeleton 'semilinear' trees. As a consequence, one expects estimates of the form

$$\|\tau\|_{\mathcal{C}^{|\tau|}} \leqslant k_{\tau^\circ}\, C^{n_\tau},$$

with τ° the skeleton tree corresponding to τ in the semilinear setting and $C > 1$ a constant depending only on the operator $\mathscr{L}^{-1}L$. As there are only finitely many trees τ° in a subcritical regime one should be able to take a uniform constant k instead of k_{τ°. We capture this discussion under the form of an assumption – which holds true when the noise is smooth.

Assumption (A). *There exists positive constants k and $C > 1$ such that one has*

$$\|\tau\|_{\mathcal{C}^{|\tau|}} \leqslant k\, C^{n_\tau}$$

for all $\tau \in \mathscr{T}$.

◦ *Fixed point formulation of the equation.* With that choice of reference set $\mathscr{T}$ and given a system $\widehat{u}$ paracontrolled by $\mathscr{T}$ at order 3, the function

$$\mathscr{L}^{-1}\left(f(u)\zeta + \varepsilon(u,\cdot)Lu + \sum_{i=1}^{\ell} a_i(u,\cdot)V_i u\right)$$

is the first element of a system paracontrolled by $\mathscr{T}$ that we denote by $\Psi\big(\widehat{u}\big)$. Write

$$\Phi\big(\widehat{u}^\sharp\big) \in \prod_{w\in\mathscr{W}} \mathcal{C}^{3\alpha+\beta_w-|w|}$$

for the associated map that gives the collection of all the remainders in the paracontrolled expansion of the different elements of $\Psi\big(\widehat{u}\big)$. Note that the fixed point identity

$$u = \sum_{\tau\in\mathscr{T}} \widetilde{\mathsf{P}}_{u_\tau}\tau + u^\sharp = \mathscr{L}^{-1}\Big(f(u)\zeta + \varepsilon(u,\cdot)Lu + \sum_{i=1}^{\ell} a_i(u,\cdot)V_i u\Big) + \mathcal{F}u_0$$

identifies then each u_τ in the left hand side to an explicit function $h_\tau(u)$ of u only. One has for instance

$$\varepsilon(u,\cdot)^k f(u), \text{ for } \tau = u_\tau = (\mathscr{L}^{-1}L)^k\big(\mathscr{L}^{-1}(\zeta)\big).$$

More generally formula (2.13) can be used to identify explicit functions h_w such that $u_w = h_w(\widehat{u})$.

We can distinguish three generic methods we could be used to prove that a map like Φ is a contraction provided the time horizon T is small enough. In a setting where we work with time weighted functional spaces, like in regularity structures, one can use a form of Schauder estimate telling in particular that our map takes values in a space of functions with better behaviour near time 0. This gain in the explosion weight give a small constant over a small time interval. We do not choose this strategy as we have chosen not to work with time weighted functional spaces. In a different direction one can try and use scaling arguments as in [20]. This strategy is efficient only when coupled with the scaling property of a random noise provided the spacetime scaling and the noise scaling work appropriately. We do not use that strategy as we stick here to a deterministic setting. We take a different road and work with well-chosen sub-optimal functional spaces for the remainders $\prod_{w\in\mathscr{W}} \mathcal{C}^{3\alpha+\beta_w-|w|}$ in order to take profit from the a priori fact that the solution of the fixed point equation will actually have better regularity. We will use for that purpose the elementary fact that if $0<\beta_1<\beta<1$ and $v\in\mathcal{C}^\beta$ is *null at time* 0 then

$$\|v\|_{\mathcal{C}^{\beta_1}} \lesssim T^{\frac{\beta_2-\beta_1}{2}} \|v\|_{\mathcal{C}^{\beta_2}}. \tag{3.5}$$

This gain of parabolic regularity will provide us with a small factor that will give in the end the contraction property.

A choice of regularity exponents – *We choose the exponents $(\beta_w)_{w\in\mathscr{W}}$ in $(2/5,\alpha)$ in such a way that $\beta_w>\beta_{w'}$ if the word w has more letters that w', and $\beta_w>\beta_{w'}$ if w and w' have the same number of letters and $|w|<|w'|$. Given the above skeleton picture of $\mathscr{T}$, this can be done in such a way that the β_w take only finitely many values.*

Denote by $\widehat{u}^\sharp = (u_w^\sharp)_{w\in\mathscr{W}}$ a generic element of the product space

$$\prod_{w\in\mathscr{W}} \mathcal{C}^{3\alpha+\beta_w-|w|},$$

endowed with the norm

$$|||\widehat{u}^\sharp||| := \sum_{w\in\mathscr{W}} (\!|w|\!)\, \|u_w^\sharp\|_{\mathcal{C}^{3\alpha+\beta_w-|w|}},$$

so if one denotes by $\widehat{u}$ the paracontrolled system associated by (2.11) to the collection $\widehat{u}^\sharp$ of remainders then one has $|||\widehat{u}^\sharp||| = |||\widehat{u}|||$. Given $u_0\in C^{4\alpha}$ set $h_\varnothing(u_0) := u_0$, and define

$$\mathcal{S}(u_0) := \Big\{\widehat{u}^\sharp;\ |||\widehat{u^\sharp}||| < \infty,\ \text{and}\ u^\sharp_{w|_{t=0}} = h_w(u_0),\ \forall\, w\in\mathscr{W}\Big\};$$

this is a closed subspace of $\Big(\prod_{w\in\mathscr{W}} \mathcal{C}^{3\alpha+\beta_w-|w|}, |||\cdot|||\Big)$.

Theorem 10. *The map Φ is a contraction of $\mathcal{S}(u_0)$ provided the positive time horizon T is small enough.*

This statement means that equation (1.1) has a unique local in time solution in the space $\mathcal{S}(u_0)$; it depends continuously on $\mathscr{T}$. The choice of $\overline{u_0}$ is made at the end of point **(i)** of the proof.

Proof – Recall we use exclusively the symbols τ,σ for letters from the alphabet $\mathscr{T}$, while we write w,w',w'' for elements of $\mathscr{W}$ – possibly words with only one lettre.

- *We first prove that Φ is a well-defined map from $\mathcal{S}(u_0)$ into itself.* This means that the condition

$$\sum_{w\in\mathscr{W}} (\!|w|\!)\, \|u_w^\sharp\|_{\mathcal{C}^{3\alpha-|w|+\beta_w}} < \infty$$

is stable by Φ. We decompose this sum according to the value of $|w|$.

– For $|w| = 3\alpha$, one has $v_w = v_w^\sharp \in \mathcal{C}^{\beta_w}$ and the condition reads

$$\sum_{|w|=3\alpha} (\!|w|\!)\, \|v_w\|_{\mathcal{C}^{\beta_w}} < \infty.$$

We read on formula (3.1) the different possibilities for w, of the form $\mathscr{L}^{-1}(\zeta_s^{(1)})$, with $s \in \{w_1, (w_2, w_3)\}_{w_1,w_2,w_3 \in \mathscr{W}}$ and $|w_1| = 2\alpha$ or $|w_2 w_3| = 2\alpha$, etc. If for instance $w = \mathscr{L}^{-1}(\zeta_{w_1}^{(1)})$ with $|w_1| = 2\alpha$ we need to show that

$$\sum_{|w_1|=2\alpha} \big\|g'(u) u_{w_1}\big\|_{\mathcal{C}^{\beta_{w_1}}} \big\|\mathscr{L}^{-1}(\zeta_{w_1}^{(1)})\big\|_{\mathcal{C}^{2\alpha}} < \infty.$$

This can be seen from a direct computation

$$\begin{aligned}
&\sum_{|w_1|=2\alpha} \big\|f'(u) u_{w_1}\big\|_{\mathcal{C}^{\beta_{w_1}}} \big\|\mathscr{L}^{-1}(\zeta_{w_1}^{(1)})\big\|_{\mathcal{C}^{2\alpha}} \\
&\quad = \sum_{|\tau|=2\alpha} \big\|f'(u) u_\tau\big\|_{\mathcal{C}^{\beta_{w_1}}} \big\|\mathscr{L}^{-1}(\zeta_\tau^{(1)})\big\|_{\mathcal{C}^{2\alpha}} + \sum_{|\sigma|=|\gamma|=\alpha} \big\|f'(u) u_{\sigma\gamma}\big\|_{\mathcal{C}^{\beta_{w_1}}} \big\|\mathscr{L}^{-1}(\zeta_{\sigma\gamma}^{(1)})\big\|_{\mathcal{C}^{2\alpha}} \\
&\quad \lesssim \|f'(u)\|_{\mathcal{C}^\alpha} \left(\sum_{|\tau|=2\alpha} (\!|w_1|\!)\, \|u_\tau\|_{\mathcal{C}^{\beta_{w_1}}} + \sum_{|\sigma|=|\gamma|=\alpha} (\!|\sigma\gamma|\!)\, \|u_{\sigma\gamma}\|_{\mathcal{C}^{\beta_{w_1}}} \right) \\
&\quad \lesssim \sum_{|\tau|=2\alpha} (\!|\tau|\!)\, \|u_\tau\|_{\beta_\tau} + \sum_{|\sigma|=|\gamma|=\alpha} (\!|\sigma\gamma|\!)\, \|u_{\sigma\gamma}\|_{\mathcal{C}^{\beta_{\sigma\gamma}}} \\
&\quad \lesssim \|\!|\widehat{u}^\sharp|\!\| < \infty,
\end{aligned}$$

using that $\beta_\sigma > \beta_{w_1}$ since $|\sigma| < |w_1|$ and $\beta_{\sigma\gamma} > \beta_{w_1}$. Another example is given by in the case where w is a word with only one letter τ and we have $\tau = (\mathscr{L}^{-1}L)\sigma$ for $|\sigma| = 3\alpha$ and we need to show that

$$\sum_{|\sigma|=3\alpha} \|\varepsilon(u,\cdot) u_\sigma\|_{\mathcal{C}^{\beta_\sigma}} \|(\mathscr{L}^{-1}L)\sigma\|_{\mathcal{C}^{3\alpha}} < \infty.$$

This indeed holds with

$$\begin{aligned}
\sum_{|\sigma|=3\alpha} \|\varepsilon(u,\cdot) u_\sigma\|_{\mathcal{C}^{\beta_\tau}} \|(\mathscr{L}^{-1}L)\sigma\|_{\mathcal{C}^{3\alpha}} &\lesssim \sum_{|\sigma|=3\alpha} \|u_\sigma\|_{\mathcal{C}^{\beta_\tau}} \|\varepsilon(u,\cdot)\| \|(\mathscr{L}^{-1}L)\sigma\|_{\mathcal{C}^{3\alpha}} \\
&\lesssim \sum_{|\sigma|=3} \|u_\sigma\|_{\mathcal{C}^{\beta_\tau}} (\!|\sigma|\!) \\
&\lesssim \|\!|\widehat{u}^\sharp|\!\| < \infty,
\end{aligned}$$

using that $\beta_\tau = \beta_\sigma$ since $|\tau| = |\sigma|$. Remark that this term corresponds to the quasi-linear character of the equatiton. We let the reader check the other cases.

– For $|w| = 2\alpha$, we need to show

$$\sum_{|w|=2\alpha} (\!|w|\!)\, \|v_w^\sharp\|_{\alpha+\beta_w} < \infty.$$

So we need to compute the remainders $v_w^\sharp$ for all such w; they are given by the formula

$$v_w = \sum_{|\tau|=\alpha} \widetilde{\mathsf{P}}_{v_{w\tau}} \tau + v_w^\sharp.$$

Here again different cases can happen depending on w. If for instance $w = \mathscr{L}^{-1}(\zeta_{w'}^{(1)})$, with $|w'| = \alpha$ so $w' = \sigma \in \mathscr{T}_1$, we have $v_w = f'(u) u_\sigma$ and

$$f'(u) u_\sigma = \mathsf{P}_{f'(u)} u_\sigma + \mathsf{P}_{u_\sigma}\big(f'(u)\big) + \mathsf{\Pi}\big(f'(u), u_\sigma\big)$$

$$\begin{aligned} f'(u)u_\sigma &= \widetilde{\mathsf{P}}_{f'(u)u_{\sigma\gamma}+f^{(2)}(u)u_\sigma u_\gamma}\big([\gamma]\big) \\ &\quad + \Big\{\mathsf{R}\big(f'(u),u_{\sigma\gamma},[\gamma]\big)+\mathsf{R}\big(\mathbf{1},f'(u)u_{\sigma\gamma},\gamma\big)+\mathsf{R}\big(f^{(2)}(u)u_\sigma,u_\gamma,[\gamma]\big) \\ &\qquad + \mathsf{R}(\mathbf{1},f^{(2)}(u)u_\sigma u_\gamma,[\gamma])+\mathsf{P}_{u_\sigma}\big(f'(u)^\sharp\big)+\mathsf{\Pi}\big(f'(u),u_\sigma\big)\Big\} \\ &=: \widetilde{\mathsf{P}}_{v_{w\gamma}}\big([\gamma]\big)+v_w^\sharp \end{aligned}$$

where all term in the remainder $v_w^\sharp$ satisfies the convergence condition. As an example, we have

$$\sum_{|\sigma|=\alpha}\sum_{|\gamma|=\alpha}(\!|\sigma|\!)\,\big\|\mathsf{R}(\mathbf{1},f'(u)u_{\sigma\gamma},\gamma)\big\|_{\alpha+\beta_w}\lesssim\|g'(u)\|_\alpha\sum_{|\sigma|=|\gamma|=\alpha}(\!|\sigma\gamma|\!)\,\|u_{\sigma\gamma}\|_{\beta_{\sigma\gamma}}\lesssim\|\!|\widehat{u}^\sharp|\!\|<\infty.$$

Another example is given by $w=\mathscr{L}^{-1}\big(\zeta^{(2)}_{w'}\big)$ with $|w'|=2\alpha$. If $w'=\tau\in\mathscr{T}_2$, the computation is similar to the one where $|w'|=3\alpha$ thus we consider the case $w'=\tau\sigma$ with $\tau,\sigma\in\mathscr{T}_1$. We have $v_w=\varepsilon(u,\cdot)u_{w'}$ and the similar computation

$$\begin{aligned} \varepsilon(u,\cdot)u_{\tau\sigma} &= \mathsf{P}_{\varepsilon(u,\cdot)}u_{\tau\sigma}+\mathsf{P}_{u_{\tau\sigma}}\big(\varepsilon(u,\cdot)\big)+\mathsf{\Pi}\big(\varepsilon(u,\cdot),u_{\tau\sigma}\big) \\ &= \widetilde{\mathsf{P}}_{\varepsilon(u,\cdot)u_{\tau\sigma\gamma}+d_0^{-1}d'(u)u_{\tau\sigma}u_\gamma}\big([\gamma]\big) \\ &\quad + \Big\{\mathsf{R}\big(\varepsilon(u,\cdot),u_{\tau\sigma\gamma},[\gamma]\big)+\mathsf{R}\big(\mathbf{1},\varepsilon(u,\cdot)u_{\tau\sigma\gamma},\gamma\big)+\mathsf{R}\big(d_0^{-1}d'(u)u_{\tau\sigma},u_\gamma,[\gamma]\big) \\ &\qquad + \mathsf{R}\big(\mathbf{1},d_0^{-1}d'(u)u_{\tau\sigma}u_\gamma,[\gamma]\big)+\mathsf{P}_{u_\sigma}\big(\varepsilon(u,\cdot)^\sharp\big)+\mathsf{\Pi}\big(\varepsilon(u,\cdot),u_{\tau\sigma}\big)\Big\} \\ &=: \widetilde{\mathsf{P}}_{v_{w\gamma}}\big([\gamma]\big)+v_w^\sharp \end{aligned}$$

with $\varepsilon(u,\cdot)^\sharp$ the remainder given by the nonlinear paracontrolled expansion of $\varepsilon(u,\cdot)$. Again, the remainder $v_w^\sharp$ satisfies the convergence condition with a similar computation. The reader is invited to check the other cases.

– A direct computation also shows that

$$\sum_{|w|=\alpha}(\!|w|\!)\,\|v_w^\sharp\|_{2\alpha+\beta_w}<\infty.$$

The remaining details are left to the reader.

• *We now prove that Φ is a contraction of $\mathcal{S}(u_0)$ if T small enough.* Pick $\widehat{u}^\sharp$ and $\widehat{v}^\sharp$ in $\mathcal{S}(u_0)$, with associated paracontrolled systems $\widehat{u}$ and $\widehat{v}$. Since both paracontrolled systems are in the solution space, the system

$$\widehat{z}:=\Phi(\widehat{u})-\Phi(\widehat{v})$$

is also paracontrolled by $\mathscr{T}$ at order 3 and its has all its remainders null at time 0. This fact will allow us to use the estimate (3.5) and gain a factor $T^{(\gamma'-\gamma)/2}$ when comparing the norms of such functions in two different parabolic Hölder spaces with respective exponents γ and γ'. From Proposition 9, we have

$$\widehat{z}_\varnothing=\sum_{|\tau|\leqslant3\alpha}\widetilde{\mathsf{P}}_{z_\tau}\tau+z^\sharp$$

with explicit formulas for the components $z_\tau=\Phi(\widehat{u})_\tau-\Phi(\widehat{v})_\tau$ of $\widehat{z}$. In the expansion of $\widehat{z}_\varnothing$ we need to control **(i)** the terms $\|z_\tau\|_{\beta_\tau}$ for $|\tau|=3\alpha$ and **(ii)** the terms $\|z_\tau^\sharp\|_{3\alpha+\beta_\tau-|\tau|}$ for $|\tau|\leqslant2\alpha$.

(i) We first consider the terms z_τ. For example, we need to control

$$\big\|f'(u)u_w-f'(v)v_w\big\|_{\beta_\tau},\qquad\text{with }|w|=2\alpha,\ \text{and }\tau=\mathscr{L}^{-1}(\zeta_w^{(1)}),$$

with $w \in \mathscr{W}$. This is done writing

$$\begin{aligned}\big\|f'(u)u_w - f'(v)v_w\big\|_{\beta_\sigma} &\lesssim \big\|\left(f'(u) - f'(v)\right)u_w\big\|_{\beta_\sigma} + \big\|f'(v)(u_w - v_w)\big\|_{\beta_\sigma} \\ &\lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\big\|f'(u) - f'(v)\big\|_\alpha \|u_w\|_{\beta_\sigma} + T^{\frac{\beta_w-\beta_\sigma}{2}}\big\|f'(v)\big\|_\alpha \|u_w - v_w\|_{\beta_w} \\ &\lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\left(\|f\|_{C_b^2}\big(1 + \|u\|_\alpha\big)\|u_w\|_{\beta_\sigma}\right)\|u - v\|_\alpha + T^{\frac{\beta_w-\beta_\sigma}{2}}\|f'(v)\|_\alpha \|u_w - v_w\|_{\beta_w} \\ &\lesssim \left\{T^{\frac{\alpha-\beta_\sigma}{2}}\left(\|f\|_{C_b^2}\big(1 + \|u\|_\alpha\big)\|u_w\|_{\beta_\sigma}\right) + T^{\frac{\beta_w-\beta_\sigma}{2}}\|f'(v)\|_\alpha\right\} \|\!|\widehat{u}^\sharp - \widehat{v}^\sharp|\!\|.\end{aligned}$$

Another example is

$$\big\|f^{(2)}(u)u_w u_{w'} - f^{(2)}(v)v_w v_{w'}\big\|_{\beta_\sigma}$$

with $|w| + |w'| = 2\alpha$ and $\sigma \in \mathscr{T}$ given by $\mathscr{L}^{-1}\left(\zeta^{(1)}_{ww'}\right)$. It is dealt with writing

$$\begin{aligned}&\big\|f^{(2)}(u)u_w u_{w'} - f^{(2)}(v)v_w v_{w'}\big\|_{\beta_\sigma} \\ &\lesssim \left\|\left(f^{(2)}(u) - f^{(2)}(v)\right)u_w u_{w'}\right\|_{\beta_\sigma} + \big\|f^{(2)}(v)(u_w u_{w'} - v_w v_{w'})\big\|_{\beta_\sigma} \\ &\lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\|f^{(2)}(u) - f^{(2)}(v)\|_\alpha \|u_w u_{w'}\|_{\beta_\sigma} \\ &\qquad\qquad + T^{\frac{\min(\beta_w,\beta_{w'})-\beta_\sigma}{2}}\big\|f^{(2)}(v)\big\|_{\beta_\sigma}\big\|u_w u_{w'} - v_w v_{w'}\big\|_{\min(\beta_w,\beta_{w'})} \\ &\lesssim \left(T^{\frac{\alpha-\beta_\sigma}{2}}\|f\|_{C_b^3}\|u_w u_{w'}\|_{\beta_\sigma} + T^{\frac{\min(\beta_w,\beta_{w'})-\beta_\sigma}{2}}\|f^{(2)}(v)\|_{\beta_\sigma}\right)\|\!|\widehat{u}^\sharp - \widehat{v}^\sharp|\!\|.\end{aligned}$$

All the other terms are dealed with using the following four inequalities.

- One has

$$\Big\|\varepsilon(u,\cdot)u_w - \varepsilon(v,\cdot)v_w\Big\|_{\beta_\sigma} \lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\big\|\varepsilon(u,\cdot) - \varepsilon(v,\cdot)\big\|_\alpha \|u_w\|_{\beta_\sigma} + T^{\frac{\beta_w-\beta_\tau}{2}}\|\varepsilon(v,\cdot)\|_\alpha \|u_w - v_w\|_{\beta_w}$$

  for $w \notin \mathscr{T}, |w| = 3\alpha$ and $\sigma \in \mathscr{T}$ given by $\mathscr{L}^{-1}\left(\zeta^{(2)}_w\right)$.
- One has

$$\begin{aligned}\big\|d_0^{-1}d'(u)u_w u_{w'} - d_0^{-1}d'(v)v_w v_{w'}\big\|_{\beta_\sigma} &\lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\big\|d_0^{-1}d'(u) - d_0^{-1}d'(v)\big\|_\alpha \|u_w u_{w'}\|_{\beta_\sigma} \\ &\quad + T^{\frac{\min(\beta_w,\beta_{w'})-\beta_\sigma}{2}}\big\|d_0^{-1}d'(v)\big\|_{\beta_\sigma}\big\|u_w u_{w'} - v_w v_{w'}\big\|_{\min(\beta_w,\beta_{w'})}\end{aligned}$$

  for $|w| + |w'| = 3\alpha$ and $\sigma \in \mathscr{T}$ given by $\mathscr{L}^{-1}\left(\zeta^{(2)}_{ww'}\right)$.
- One has

$$\begin{aligned}&\big\|d_0^{-1}d^{(2)}(u)u_w u_{w'} u_{w''} - d_0^{-1}d^{(2)}(v)v_w v_{w'} v_{w''}\big\|_{\beta_\sigma} \\ &\qquad \lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\big\|d_0^{-1}d^{(2)}(u) - d_0^{-1}d^{(2)}(v)\big\|_\alpha \|u_w u_{w'} u_{w''}\|_{\beta_\sigma} \\ &\qquad\quad + T^{\frac{\min(\beta_w,\beta_{w'},\beta_{w''})-\beta_\tau}{2}}\big\|d_0^{-1}d^{(2)}(v)\big\|_{\beta_\sigma}\big\|u_w u_{w'} u_{w''} - v_w v_{w'} v_{w''}\big\|_{\min(\beta_a,\beta_b,\beta_c)}\end{aligned}$$

  for $|w| + |w'| + |w''| = 3\alpha$ and $\sigma \in \mathscr{T}$ given by $\mathscr{L}^{-1}\left(\zeta^{(2)}_{ww'w''}\right)$.
- One has

$$\big\|d_\ell(u,\cdot)u_\tau - d_\ell(v,\cdot)v_\sigma\big\|_{\beta_\sigma} \lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\big\|d_\ell(u,\cdot) - d_\ell(v,\cdot)\big\|_\alpha \|u_\tau\|_{\beta_\sigma} + T^{\frac{\beta_\tau-\beta_\sigma}{2}}\big\|d_\ell(v,\cdot)\big\|_{\beta_\sigma}\big\|u_\tau - v_\tau\big\|_{\beta_\tau}$$

  for $|\tau| = \alpha$ and $\sigma \in \mathscr{T}$ given by $\mathscr{L}^{-1}(\zeta_{j,\tau})$.

There is only one case where we use the fact that $\varepsilon(u,\cdot)$ is small when T is small. We use it to estimate

$$\Big\|\varepsilon(u,\cdot)u_\tau - \varepsilon(v,\cdot)v_\tau\Big\|_{\beta_\sigma} \lesssim \big\|\big(\varepsilon(u,\cdot) - \varepsilon(v,\cdot)\big)u_\tau\big\|_{\beta_\sigma} + \big\|\varepsilon(v,\cdot)(u_\tau - v_\tau)\big\|_{\beta_\sigma}. \tag{3.6}$$

While we have indeed

$$\big\|\big(\varepsilon(u,\cdot)-\varepsilon(v,\cdot)\big)u_\tau\big\|_{\beta_\sigma} \lesssim T^{\frac{\alpha-\beta_\sigma}{2}}\big\|\varepsilon(u,\cdot)-\varepsilon(v,\cdot)\big\|_\alpha \|u_\tau\|_{\beta\sigma},$$

we do not gain a T-dependent fact using the regularity of $u_\tau - u_\sigma$ in the second term of the right hand side of inequality (3.6) since $\beta_\tau = \beta_\sigma$. We write instead

$$\begin{aligned}\big\|\varepsilon(v,\cdot)(u_\tau - v_\tau)\big\|_{\beta_\sigma} &\lesssim \|\varepsilon(v,\cdot)\|_{\beta_\tau}\|u_\tau - v_\tau\|_{\beta_\tau} \\ &\lesssim \Big(\big\|d(v)-d(u_0)\big\|_{\beta_\tau} + \big\|d(u_0)-d(\overline{u_0})\big\|_{\beta_\tau}\Big)\,\|u_\tau - v_\tau\|_{\beta_\tau} \\ &\lesssim \Big(T^{\frac{\beta_\tau-\alpha}{2}}\big\|d(v)-d(u_0)\big\|_{\alpha} + \big\|d(u_0)-d(\overline{u_0})\big\|_{\beta_\tau}\Big)\,\|u_\tau - v_\tau\|_{\beta_\tau}\end{aligned}$$

using estimate (3.5) to get the T-factor in the last line since $d(v)-d(u_0)$ is null at time 0. The factor $\|d(\overline{u_0})-d(u_0)\|_{\beta_\tau}$ is as small as we want for $\overline{u_0}$ close enough to u_0 in C^α. *This is the place where we choose $\overline{u_0}$ as a function of u_0.* (Note that it is only the regularity of u_0 as an element in C^α that matters here. We asked $u_0 \in C^{4\alpha}$ to treat the free propagation of the initial condition $\mathcal{F}u_0$ as a remainder term in the paracontrolled analysis and avoid the use of time weighted norms.)

(ii) This case is concerned with the remainder terms $\|z_\tau^\sharp\|_{3\alpha+\beta_\tau-|\tau|}$. Let us consider the case $|\tau| = 2\alpha$ for example. Then z_τ is written as

$$z_\tau = \sum_{\gamma\in\mathscr{T}_1} \widetilde{\mathsf{P}}_{z_{\tau\gamma}}([\gamma]) + z_\tau^\sharp.$$

In the case $\tau = \mathscr{L}^{-1}(\zeta_\sigma^{(1)})$ for $\sigma \in \mathscr{T}_1$, the computation at the begining of the proof gives the explicit remainder $z_\tau^\sharp$ with

$$\begin{aligned} z_\tau &= f'(u)u_\sigma - f'(v)v_\sigma \\ &= \widetilde{\mathsf{P}}_{f'(u)u_{\sigma\gamma}+f^{(2)}(u)u_\sigma u_\gamma - (f'(v)v_{\sigma\gamma}+f^{(2)}(v)v_\sigma v_\gamma)}([\gamma]) \\ &\quad + \Big\{\mathsf{R}\big(f'(u),u_{\sigma\gamma},[\gamma]\big) + \mathsf{R}\big(\mathbf{1},f'(u)u_{\sigma\gamma},\gamma\big) + \mathsf{R}\big(f^{(2)}(u)u_\sigma,u_\gamma,[\gamma]\big) \\ &\qquad + \mathsf{R}(\mathbf{1},f^{(2)}(u)u_\sigma u_\gamma,[\gamma]\big) + \mathsf{P}_{u_\sigma}\big(f'(u)^\sharp\big) + \Pi\big(f'(u),u_\sigma\big)\Big\} \\ &\quad - \Big\{\mathsf{R}\big(f'(v),v_{\sigma\gamma},[\gamma]\big) + \mathsf{R}\big(\mathbf{1},f'(v)v_{\sigma\gamma},\gamma\big) + \mathsf{R}\big(f^{(2)}(v)v_\sigma,v_\gamma,[\gamma]\big) \\ &\qquad + \mathsf{R}(\mathbf{1},f^{(2)}(v)v_\sigma v_\gamma,[\gamma]\big) + \mathsf{P}_{v_\sigma}\big(f'(v)^\sharp\big) + \Pi\big(f'(v),v_\sigma\big)\Big\} \\ &=: \widetilde{\mathsf{P}}_{z_{\tau\gamma}}\big([\gamma]\big) + z_\tau^\sharp \end{aligned}$$

with implicit sums over $\gamma \in \mathscr{T}_1$. From here, the analysis of the general terms $\widehat{z}_w$, with $w \in \mathscr{W}$, is similar or easier to the computations done in **(i)**. It is left to the reader.

We obtain the contacting character of the map Φ from the fact that the exponents β_w only take finitely many different values. ▷

Remarks – 1. *The quasilinear* (gPAM) *equation dealt with in* [6] *involved a space white noise ζ on the two-dimensional torus – as opposed to a spacetime noise as in the present work and the other works* [16, 31, 18]. *The fact that ζ depends only on space allows to write*

$$\mathscr{L}^{-1}(\zeta)(t) = L^{-1}(\zeta) - \int_t^\infty e^{-rL}(\zeta)\,dr$$

as a perturbation of $L^{-1}(\zeta)$ in a time weighted functional setting. Taking $Z_1 := L^{-1}(\zeta)$ one has indeed

$$\mathscr{L}^{-1}\big(L(Z_1)\big) \simeq Z_1,$$

up to a remainder term a time weighted functional space. It is that fact that allowed the authors of [6] *to work with a usual first order paracontrolled structure and avoid the infinite dimensional feature of the other approaches* [16, 31]. *Since $\mathscr{T}$ reduces to $\mathscr{T}_1$ in the two-dimensional setting this simplifies greatly the analysis. The fact that the term $\int_t^\infty e^{-rL}(\zeta)\,dr$ can be treated as a remainder term is specific to the* 2*-dimensional setting.*

2. *It is possible to use the results of* [4] *and the extended toolkit for the high order paracontrolled calculus from Section 2.2 to handle the analytic part of the study of the generalized* (KPZ) *equation.*

A – Basics on high order paracontrolled calculus

We recall in this appendix a number of results from [3, 4] that we use in this work. This should help the reader understanding the computations of Appendix B and their mechanics.

We first describe some approximation operators $\mathcal{P}_t$ and $\mathcal{Q}_t$ that we use in place of the usual Littlewood-Paley projectors $\sum_{j\leqslant n}\Delta_j$ and Δ_n, in which the heat semigroup plays the role of Fourier theory. The parabolic Hölder space are defined from these operators. We also recall the form of the space-time paraproduct and resonant operators that we use and give a number of the continuity estimates on different correctors/commutators and their iterated versions.

Recall that we denote by M a 3-dimensional closed Riemannian manifold and set

$$\mathcal{M} := [0,T]\times M,$$

for a finite positive time horizon T. We denote by $\rho(\cdot,\cdot)$ the parabolic distance on $\mathcal{M}$ and by $e=(\tau,x)$ a generic spacetime point. Denote by μ the Riemannian volume measure and define the parabolic measure

$$\nu := dt\otimes\mu.$$

Recall the reformulation (2.7) of equation (1.1), where the operator $L=-\sum_{i=1}^{\ell}V_i^2$ is a second order differential operator in Hörmander form.

A.1 Approximation operators and parabolic Hölder spaces

In the flat setting of the torus, we can use Fourier theory to approximate Schwartz distributions by smooth functions. We have

$$f=\lim_{n\to\infty}S_n(f)=\sum_{i\geqslant -1}\Delta_i(f)$$

with Δ_j the Paley-Littlewood projectors. Refer e.g. to [5] for basics on Littlewood-Paley theory. Using the heat semigroup, one has in a more general geometric framework

$$f=\lim_{t\to 0}P_t^{(b)}f=\int_0^1 Q_t^{(b)}f\,\frac{\mathrm{d}t}{t}+P_1^{(b)}f$$

where

$$Q_t^{(b)} := \frac{(tL)^b e^{-tL}}{(b-1)!}\quad\text{and}\quad -t\partial_t P_t^{(b)} := Q_t^{(b)}$$

with $P_0=\mathrm{Id}$. One can show that there exists a polynomial p_b of degree $(b-1)$ such that $P_t^{(b)}=p_b(tL)e^{-tL}$ and $p_b(0)=1$. The operators $Q_t^{(b)}$ and $P_t^{(b)}$ play the role of Paley-Littlewood projector and Fourier series, respectively. Indeed, if one works on the torus, then

$$\widehat{Q_t^{(b)}}(\lambda)=\frac{\left(t|\lambda|^2\right)^b}{(b-1)!}e^{-|\lambda|^2 t}\quad\text{and}\quad \widehat{P_t^{(b)}}(\lambda)=p_b\left(t|\lambda|^2\right)e^{-|\lambda|^2 t}$$

so we see that $Q_t^{(b)}$ localize in frequency around the annulus $|\lambda| \sim t^{-\frac{1}{2}}$ and $P_t^{(b)}$ localize in frequency on the ball $|\lambda| \lesssim t^{-\frac{1}{2}}$. Since the measure dt/t gives unit mass to each interval $[2^{-(i+1)}, 2^{-i}]$, the operator $Q_t^{(b)}$ is a multiplier that is approximately localized at 'frequencies' of size $t^{-\frac{1}{2}}$. However, this decomposition using a continuous parameter does not satisfy the perfect cancellation property $\Delta_i\Delta_j = 0$ for $|i-j|>1$, but the identity

$$Q_t^{(b)}Q_s^{(b)} = \left(\frac{ts}{(t+s)^2}\right)^b Q_{t+s}^{(2b)}$$

for any $s,t \in (0,1)$. The parameter b encodes a 'degree' of cancellation. In order to deal with time approximation, define for $m \in L^1(\mathbb{R})$, with support in $\mathbb{R}_+$, the convolution operator

$$m^\star(f)(\tau) := \int_0^\infty m(\tau-\sigma)f(\sigma)\mathrm{d}\sigma \quad \text{and} \quad m_t(\cdot) := \frac{1}{t}m\left(\frac{\cdot}{t}\right)$$

for $\tau \in \mathbb{R}$ and a positive scaling parameter t. Given $I = (i_1, \dots, i_n) \in \{1, \dots, \ell\}^n$, define the n^{th}-oder differential operator

$$V_I := V_{i_n} \dots V_{i_1}.$$

We say that a family $(\mathcal{Q}_t)_{t\in(0,1]}$ of operators is Gaussian if each the kernel of each $\mathcal{Q}_t$ is bounded pointwisely by the reference Gaussian kernel $\mathcal{G}_t$. (We do not recall its explicit expression here and refer the reader to Section 3.2 of [3]. It behaves as one expects.)

Definition – *Let $a \in [\![0,2b]\!]$. We define the* **standard collection** StGC^a **of operators with cancellation of order** a *as the family of operators*

$$\left(\left(t^{\frac{|I|}{2}}V_I\right)(tL)^{\frac{j}{2}}P_t^{(c)} \otimes \varphi_t^\star\right)_{t\in(0,1]}$$

where $a = |I| + j + 2k$, $c \in [\![1,b]\!]$ and φ a smooth function supported in $[2^{-1},2]$ with bounded first derivative by 1 such that

$$\int \tau^i\varphi(\tau)\mathrm{d}\tau = 0 \quad \textit{for every } 0 \leqslant i \leqslant k-1.$$

These operators are uniformly bounded in $L^p(\mathcal{M})$ for every $p \in [1,\infty]$, as functions of the parameter $t \in (0,1]$. We also set

$$\mathsf{StGC}^{[0,2b]} := \bigcup_{0\leqslant a\leqslant 2b} \mathsf{StGC}^a.$$

A standard family of operator $\mathcal{Q} \in \mathsf{StGC}^a$ can be seen as a bounded map $t \mapsto \mathcal{Q}_t$ from $(0,1]$ to the space of bounded linear operator on $L^p(\mathcal{M})$. Since $V_iV_j \neq V_jV_i$, the operators V_i do not commute with L so

$$V_IL^be^{-tL} \neq L^be^{-tL}V_I.$$

We introduce for the needs of the next proposition the notation

$$\Big(V_I\psi(L)\Big)^\bullet := \psi(L)V_I$$

for any holomorphic function ψ. This notation is *not* related to any notion of duality.

Proposition 11. *Consider $\mathcal{Q}^1 \in \mathsf{StGC}^{a_1}$ and $\mathcal{Q}^2 \in \mathsf{StGC}^{a_2}$ two standard collections with cancellation. Then for every $s,t \in (0,1]$, the composition $\mathcal{Q}_s^1 \circ \mathcal{Q}_t^{2\bullet}$ has a kernel pointwisely* *bounded by*

$$\left|K_{\mathcal{Q}_s^1\circ\mathcal{Q}_t^{2\bullet}}(e,e')\right| \lesssim \left\{\left(\frac{s}{t}\right)^{\frac{a_1}{2}}\mathbf{1}_{s<t} + \left(\frac{t}{s}\right)^{\frac{a_2}{2}}\mathbf{1}_{s\geqslant t}\right\}\mathcal{G}_{t+s}(e,e') \\ \lesssim \left(\frac{ts}{(s+t)^2}\right)^{\frac{a}{2}}\mathcal{G}_{t+s}(e,e') \tag{A.1}$$

with $a := \min(a_1, a_2)$.

Estimate (A.1) encodes a cancellation property that is in our setting the counterpart of the property $\Delta_i\Delta_j = 0$ for $|i-j| > 1$. We also need operators that are not in the standard form but still have a useful cancellation property.

Definition – *Let* $a \in [\![0, 2b]\!]$. *We define the collection* GC^{a} *of* **operators with cancellation of order** a *as the set of families of Gaussian operators* $\mathcal{Q}$ *such as the following property holds. For every* $s, t \in (0,1]$ *and every* $\mathsf{S} \in \mathsf{StGC}^{\mathsf{a}'}$ *with* $a < a' \leqslant 2b$, *the composition* $\mathcal{Q}_s \circ \mathsf{S}_{\mathsf{t}}^{\bullet}$ *has a kernel pointwisely bounded by*

$$\left|K_{\mathcal{Q}_s\circ\mathsf{S}_\mathsf{t}^\bullet}(e,e')\right| \lesssim \left(\frac{ts}{(t+s)^2}\right)^{\frac{a}{2}}\mathcal{G}_{t+s}(e,e').$$

Definition – *Given any* $\alpha \in (-3,3)$, *we define the* **parabolic Hölder spaces** $\mathcal{C}^\alpha(\mathcal{M})$ *as the set of distribution* $f \in \mathcal{D}'(\mathcal{M})$ *such that*

$$\|f\|_{\mathcal{C}^\alpha} := \left\|e^{-L}f\right\|_{L^\infty} + \sup_{\substack{\mathcal{Q}\in\mathsf{StGC}^k\\ |\alpha|<k\leqslant 2b}}\ \sup_{t\in(0,1]} t^{-\frac{\alpha}{2}}\|\mathcal{Q}_t f\|_{L^\infty} < \infty.$$

A.2 Parabolic paraproducts, correctors and commutators

The Paley-Littlewood decomposition can be used to describe a product as

$$\begin{aligned} fg &= \lim_{n\to\infty} S_n(f)S_n(g) \\ &= \sum_{i<j-2}\Delta_i(f)\Delta_j(g) + \sum_{|i-j|\leqslant 1}\Delta_i(f)\Delta_j(g) + \sum_{i>j+1}\Delta_i(f)\Delta_j(g) \\ &= \sum_i \Delta_{<i}(f)\Delta_i(g) + \sum_{|i-j|\leqslant 1}\Delta_i(f)\Delta_j(g) + \sum_i \Delta_i(f)\Delta_{<i}(g) \\ &= P^0_f g + \Pi^0(f,g) + P^0_g f. \end{aligned}$$

The paraproducts $P^0_f g$ and $P^0_g f$ are always well-defined unlike the resonant term $\Pi^0(f,g)$. We use here a slightly different identity

$$\begin{aligned} fg &= \lim_{t\to 0}\mathcal{P}_t^{(b)}\left(\mathcal{P}_t^{(b)}f\cdot\mathcal{P}_t^{(b)}g\right) \\ &= \int_0^1 \left\{\mathcal{Q}_t^{(b)}\big(\mathcal{P}_t^{(b)}f\cdot\mathcal{P}_t^{(b)}g\big) + \mathcal{P}_t^{(b)}\big(\mathcal{Q}_t^{(b)}f\cdot\mathcal{P}_t^{(b)}g\big) + \mathcal{P}_t^{(b)}\big(\mathcal{P}_t^{(b)}f\cdot\mathcal{Q}_t^{(b)}g\big)\right\}\frac{dt}{t} \\ &\quad + \mathcal{P}_1^{(b)}\left(\mathcal{P}_1^{(b)}f\cdot\mathcal{P}_1^{(b)}g\right), \end{aligned} \tag{A.2}$$

which corresponds to writing

$$fg = \lim_{n\to\infty} S_n\big(S_n(f)S_n(g)\big).$$

Since $\mathcal{P}_t^{(b)}$ plays the role of $\Delta_{<i}$ and $\mathcal{Q}_t^{(b)}$ the role of Δ_i we want to manipulate this expression to get terms of the following forms

$$\int_0^1 \mathcal{P}_t^{1\bullet}\left(\mathcal{Q}_t^1 f\cdot\mathcal{Q}_t^2 g\right)\frac{dt}{t}, \quad \text{or} \quad \int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{Q}_t^2 f\cdot\mathcal{P}_t^1 g\right)\frac{dt}{t}, \quad \text{and} \quad \int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{P}_t^1 f\cdot\mathcal{Q}_t^2 g\right)\frac{dt}{t},$$

where $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathsf{StGC}^c$ encode some cancellation, so $c > 0$, and $\mathcal{P}_1 \in \mathsf{StGC}^{[0,d]}$ can encode no cancellation. This is done using repeatedly the Leibniz rule $V_i(fg) = V_i(f)g + fV_i(g)$. We have for instance

$$\begin{aligned}\int_0^1 &\mathcal{P}_t^{(b)}\left(b^{-1}(tL)\mathcal{Q}_t^{(b-1)}f\cdot\mathcal{P}_t^{(b)}g\right)\frac{dt}{t}\\ &= b^{-1}\int_0^1 \mathcal{P}_t^{(b)}(tL)\left(\mathcal{Q}_t^{(b-1)}f\cdot\mathcal{P}_t^{(b)}g\right)\frac{dt}{t} - b^{-1}\int_0^1 \mathcal{P}_t^{(b)}\left(\mathcal{Q}_t^{(b-1)}f\cdot(tL)\mathcal{P}_t^{(b)}g\right)\frac{dt}{t}\\ &\quad - 2b^{-1}\sum_{i=1}^{\ell}\int_0^1 \mathcal{P}_t^{(b)}(\sqrt{t}V_i)\left(\mathcal{Q}_t^{(b-1)}f\cdot(\sqrt{t}V_i)\mathcal{P}_t^{(b)}g\right)\frac{dt}{t}\end{aligned}$$

where we 'take' some cancellation from $\mathcal{Q}_t^{(b)}$ to the other terms. Starting from identity (A.2) repeated use of this kind of decompositions allows to rewrite the product fg as

$$fg = \mathsf{P}_f g + \mathsf{\Pi}(f,g) + \mathsf{P}_g f,$$

where $\mathsf{P}_f g$ is a linear combination of terms of the form

$$\int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{P}_t^1 f\cdot\mathcal{Q}_t^2 g\right)\frac{dt}{t},$$

and $\mathsf{\Pi}(f,g)$ is a linear combination of terms of the form

$$\int_0^1 \mathcal{P}_t^{1\bullet}\left(\mathcal{Q}_t^1 f\cdot\mathcal{Q}_t^2 g\right)\frac{dt}{t},$$

with $\mathcal{Q}^1, \mathcal{Q}^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $\mathcal{P}^1 \in \mathsf{StGC}^{[0,2b]}$, up to the smooth term $\mathcal{P}_1^{(b)}\left(\mathcal{P}_1^{(b)}f\cdot\mathcal{P}_1^{(b)}g\right)$. All the details on this construction and the classical estimates on the paraproduct P and the resonant $\mathsf{\Pi}$ operators can be found in Section 4 of [3]. It is useful to introduce the conjugated paraproduct operator

$$\widetilde{\mathsf{P}}_f g = \mathscr{L}^{-1}\big(\mathsf{P}_f(\mathscr{L}g)\big)$$

for any functions/distributions f and g. One can show that $\widetilde{\mathsf{P}}_f g$ is given as a linear combination of operators of the form

$$\int_0^1 \widetilde{\mathcal{Q}}_t^{1\bullet}\left(\mathcal{P}_t^1 f\cdot\mathcal{Q}_t^2\right)\frac{dt}{t}$$

with $\widetilde{\mathcal{Q}}^1 \in \mathsf{GC}^{\frac{b}{4}-2}$, $\mathcal{Q}^2 \in \mathsf{StGC}^{\frac{b}{2}}$ and $\mathcal{P}^1 \in \mathsf{StGC}^{[0,2b]}$. The only difference is that $\widetilde{\mathcal{Q}}^1$ is not given by a standard form but still encodes some cancellation. This is however sufficient for $\widetilde{\mathsf{P}}$ to enjoy the same continuity properties as P. (See again Section 4 of [4].)

The study of semilinear singular SPDEs using paracontrolled calculus relies on a number of continuity estimate for different operators. We recall three of them here and refer the reader to [4] for a thorough account. Define the E-type operator

$$\mathsf{C}(a,b,c) := \mathsf{\Pi}\Big(\widetilde{\mathsf{P}}_a b, c\Big) - a\mathsf{\Pi}\big(b,c\big)$$

and its iterate

$$\mathsf{C}\Big((a,b),c,d\Big) := \mathsf{C}\Big(\widetilde{\mathsf{P}}_a b, c, d\Big) - a\mathsf{C}\Big(b,c,d\Big).$$

Proposition 12. *The following two facts hold true.*

- *Let $\alpha \in (0,1)$ and $\beta, \gamma \in (-3,3)$ such that*

$$\beta+\gamma < 0 \quad \textit{and} \quad 0 < \alpha+\beta+\gamma < 1.$$

  Then the corrector C has a unique extension as a continuous operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{C}^{\alpha+\beta+\gamma}$.

- *Let* $\alpha_1, \alpha_2 \in (0,1)$ *and* $\beta, \gamma \in (-3,3)$ *such that*
$$\alpha_1 + \beta + \gamma < 0, \quad \alpha_2 + \beta + \gamma < 0 \quad \textit{and} \quad 0 < \alpha_1 + \alpha_2 + \beta + \gamma < 1.$$
Then the iterated corrector C *has a unique extension as a continuous operator from* $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ *to* $\mathcal{C}^{\alpha_1+\alpha_2+\beta+\gamma}$.

Note that the Hölder regularity exponent of the first argument in the corrector C has to be less than 1 in the above statement. In order to gain more information from a regularity exponent in the interval $(1,2)$ one needs to consider the refined corrector given for any $e \in \mathcal{M}$ by
$$\mathsf{C}_{(1)}\Big(a,b,c\Big)(e) := \mathsf{C}\Big(a,b,c\Big)(e) - \sum_{i=1}^{\ell} \gamma_i\big(V_i a\big)(e)\Pi\Big(\widetilde{\mathsf{P}}_{\delta_i(e,\cdot)}b, c\Big)(e)$$
where the functions $\delta_i(\cdot)$ are given by
$$\delta_i(e,e') := \chi\big(d(x,x')\big)\langle V_i(x), \pi_{x,x'}\rangle_{T_xM}, \qquad e=(\tau,x),\ e'=(\tau',x'),$$
with χ a smooth non-negative function on $[0,+\infty)$ equal to 1 in a neighbourhood of 0 with $\chi(r)=0$ for $r \geqslant r_m$, the injectivity radius of the compact Riemannian manifold M, and $\pi_{x,x'}$ a tangent vector of T_xM of length $d(x,y)$ whose associated geodesic reaches y at time 1. The functions γ_i are defined from the identity
$$\nabla f = \sum_{i=1}^{\ell} \gamma_i (V_i f) V_i,$$
for all smooth real-valued functions f on M.

Proposition 13. *Let* $\alpha \in (1,2)$ *and* $\beta, \gamma \in (-3,3)$ *such that*
$$\alpha + \beta + \gamma > 0 \quad \textit{and} \quad \beta + \gamma < 0.$$
Then the operator $\mathsf{C}_{(1)}$ *has a unique extension as a continuous operator from* $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ *to* $\mathcal{C}^{\alpha+\beta+\gamma}$.

Set
$$\begin{aligned}
\mathsf{D}(a,b,c) &:= \Pi\big(\widetilde{\mathsf{P}}_a b, c\big) - \mathsf{P}_a\big(\Pi(b,c)\big),\\
\mathsf{R}(a,b,c) &:= \mathsf{P}_a\big(\widetilde{\mathsf{P}}_b c\big) - \mathsf{P}_{ab} c,\\
\mathsf{R}^{\circ}(a,b,c) &:= \mathsf{P}_a(\mathsf{P}_b c) - \mathsf{P}_{ab} c.
\end{aligned}$$

Proposition 14. *The following two facts hold true.*

- *Let* $\alpha, \beta, \gamma \in (0,3)$. *Then the commutator* D *is continuous from* $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ *to* $\mathcal{C}^{\alpha+\beta+\gamma}$.
- *Let* $\beta \in (0,1)$ *and* $\gamma \in (-3,3)$ *such that* $\beta + \gamma \in (-3,3)$. *Then the operators* R *and* R° *are continuous from* $L^{\infty} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ *to* $\mathcal{C}^{\beta+\gamma}$.
- *Let* $\alpha, \beta \in (0, 1/2)$ *and* $\gamma \in (-3,3)$. *Then the operator* R° *is continuous from* $\mathcal{C}^{\alpha} \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma}$ *to* $\mathcal{C}^{\alpha+\beta+\gamma}$.

We also need continuity estimates on iterates of the operator R°. However in this case the expansion rule isdifferent depending on which argument we expand.

Proposition 15. *The following two facts hold true.*

- *Let* $\alpha_1, \alpha_2 \in (0,1)$ *and* $\gamma \in (-3,3)$. *Then the operator*
$$\mathsf{R}^{\circ}\big((a_1,a_2),b,c\big) := \mathsf{R}^{\circ}\big(\widetilde{\mathsf{P}}_{a_1} a_2, b, c\big) - \mathsf{P}_{a_1}\mathsf{R}^{\circ}(a_2,b,c)$$
is continuous from $\mathcal{C}^{\alpha_1} \times \mathcal{C}^{\alpha_2} \times L^{\infty} \times \mathcal{C}^{\gamma}$ *to* $\mathcal{C}^{\alpha_1+\alpha_2+\gamma}$.

- *Let* $\beta_1, \beta_2 \in (0,1)$ *and* $\gamma \in (-3,3)$. *Then the operator*
$$\mathsf{R}^\circ\big(a,(b_1,b_2),c\big) := \mathsf{R}^\circ\big(a,\widetilde{\mathsf{P}}_{b_1}b_2,c\big) - \mathsf{R}^\circ(ab_1,b_2,c)$$
is continuous from $L^\infty \times \mathcal{C}^{\beta_1} \times \mathcal{C}^{\beta_2} \times \mathcal{C}^{\gamma}$ *to* $\mathcal{C}^{\beta_1+\beta_2+\gamma}$.

B – Correctors and commutators

In order to simplify the notation we write here $\|\cdot\|_\alpha$ for $\|\cdot\|_{\mathcal{C}^\alpha}$. The proofs of the corrector estimates follow the line of reasoning of similar estimates proved in [4]. Recall from Section 2.2.1 the definitions of the operators

$$\begin{aligned}
\mathsf{C}_L^-\Big(a_1,a_2,b\Big) &= \mathsf{P}_{L\widetilde{\mathsf{P}}_{a_1}a_2}b - a_1\mathsf{P}_{La_2}b,\\
\mathsf{C}_L^+\Big(a,b_1,b_1\Big) &= \mathsf{P}_{La}\Big(\widetilde{\mathsf{P}}_{b_1}b_2\Big) - b_1\mathsf{P}_{La}b_2,\\
\mathsf{C}_L\Big(a_1,a_2,b\Big) &= \Pi\Big(L\widetilde{\mathsf{P}}_{a_1}a_2,b\Big) - a_1\Pi\Big(La_2,b\Big).
\end{aligned}$$

Proof of Theorem 3 – We give here the details for the continuity estimate on C_L and explain how to adapt the proof for $\mathsf{C}_L^-, \mathsf{C}_L^+, \mathsf{C}_{V_i}, \mathsf{C}_{V_i}^-$ and $\mathsf{C}_{V_i}^+$.

We want to compute the regularity of $\mathsf{C}_L(a_1,a_2,b)$ using a family $\mathcal{Q}$ of StGC^r with $r > |\alpha_1+\alpha_2+\beta-2|$. Recall that a term $\Pi(La,b)$ can be written as a linear combination of terms of the form
$$\int_0^1 \mathcal{P}_t^{1\bullet}(\mathcal{Q}_t^1(tL)a\cdot\mathcal{Q}_t^2 b)\frac{dt}{t^2},$$
while $\widetilde{\mathsf{P}}_b a$ is a linear combination of terms of the form
$$\int_0^1 \widetilde{\mathcal{Q}}_t^{3\bullet}(\widetilde{\mathcal{Q}}_t^4 a\cdot\mathcal{P}_t^2 b)\frac{dt}{t}$$
with $\mathcal{Q}^1,\mathcal{Q}^2,\widetilde{\mathcal{Q}}^4 \in \mathsf{StGC}^{\frac{3}{2}}$, $\widetilde{\mathcal{Q}}^3 \in \mathsf{GC}^{\frac{3}{2}}$ and $\mathcal{P}^1,\mathcal{P}^2 \in \mathsf{StGC}^{[0,3]}$. For the terms where $\mathcal{P}^2 \in \mathsf{StGC}^{[1,3]}$, we already have the correct regularity since
$$\begin{aligned}
&\int_0^1\int_0^1 \mathcal{Q}_u\mathcal{P}_t^{1\bullet}\left(\mathcal{Q}_t^1(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\mathcal{P}_s^2a_1\cdot\widetilde{\mathcal{Q}}_s^4a_2\right)\cdot\mathcal{Q}_t^2b\right)\frac{ds}{s}\frac{dt}{t^2}\\
&\qquad\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\int_0^1\int_0^1\left(\frac{ut}{(t+u)^2}\right)^{\frac{r}{2}}\left(\frac{ts}{(s+t)^2}\right)^{\frac{3}{2}}s^{\frac{\alpha_1+\alpha_2}{2}}t^{\frac{\beta}{2}}\frac{ds}{s}\frac{dt}{t^2}\\
&\qquad\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\; u^{\frac{\alpha_1+\alpha_2+\beta-2}{2}}
\end{aligned}$$
using that $\alpha_1 \in (0,1)$. We only consider $\mathcal{P}^2 \in \mathsf{StGC}^0$ for the remainder of the proof. For all $e \in \mathcal{M}$, we have
$$\mathsf{C}_L(a_1,a_2,b)(e) = \Pi\left(L\widetilde{\mathsf{P}}_{a_1}a_2,b\right)(e) - a_1(e)\cdot\Pi(La_2,b)(e) = \Pi\left(L\widetilde{\mathsf{P}}_{a_1}a_2 - a_1(e)\cdot La_2,b\right)(e),$$
since Π is bilinear and $a_1(e)$ is a scalar. This yields that $\mathsf{C}_L(a_1,a_2,b)(e)$ is a linear combination of terms of the form
$$\int_0^1\int_0^1 \mathcal{P}_t^{1\bullet}\Bigg(\mathcal{Q}_t^1(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\big(\mathcal{P}_s^2a_1 - a_1(e)\big)\cdot\widetilde{\mathcal{Q}}_s^4a_2\right)\cdot\mathcal{Q}_t^2b\Bigg)(e)\,\frac{ds}{s}\frac{dt}{t^2}$$
using that $\displaystyle\int_0^1 L\widetilde{\mathcal{Q}}_s^{3\bullet}\widetilde{\mathcal{Q}}_s^4\frac{ds}{s} = L$ up to smooth terms. This gives $\big(\mathcal{Q}_u\mathsf{C}_L(a_1,a_2,b)\big)(e)$ as a linear combination of terms of the form

$$\int K_{\mathcal{Q}_u}(e,e')\mathcal{P}_t^{1\bullet}\Big(\mathcal{Q}_t^1(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\left(\mathcal{P}_s^2 a_1 - a_1(e')\right)\cdot\widetilde{\mathcal{Q}}_s^4 a_2\right)\cdot\mathcal{Q}_t^2 b\Big)(e')\,\frac{ds}{s}\frac{dt}{t^2}\nu(de')$$
$$= \int K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\Big(\mathcal{Q}_t^1(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\left(\mathcal{P}_s^2 a_1 - a_1(e'')\right)\cdot\widetilde{\mathcal{Q}}_s^4 a_2\right)\cdot\mathcal{Q}_t^2 b\Big)(e'')\frac{ds}{s}\frac{dt}{t^2}\nu(de')\nu(de'')$$
$$+\int\int_0^u K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\Big(a_1(e'')-a_1(e')\Big)\left(\mathcal{Q}_t^1(tL)a_2\cdot\mathcal{Q}_t^2 b\right)(e'')\frac{dt}{t^2}\nu(de')\nu(de'')$$
$$+\int\int_u^1 K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\Big(a_1(e'')-a_1(e')\Big)\left(\mathcal{Q}_t^1(tL)a_2\cdot\mathcal{Q}_t^2 b\right)(e'')\frac{dt}{t^2}\nu(de')\nu(de'')$$
$$=: A+B+C.$$

The term A is bounded using cancellations properties. We have

$$|A| = \int K_{\mathcal{Q}_u\mathcal{P}_t^{1\bullet}}(e,e')\Big(\mathcal{Q}_t^1(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\left(\mathcal{P}_s^2 a_1 - a_1(e')\right)\cdot\widetilde{\mathcal{Q}}_s^4 a_2\right)\cdot\mathcal{Q}_t^2 b\Big)(e')\frac{ds}{s}\frac{dt}{t^2}\nu(de')$$

$$\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\left(\int_0^u\int_0^1\left(\frac{st}{(s+t)^2}\right)^{\frac{3}{2}}(s+t)^{\frac{\alpha_1}{2}}s^{\frac{\alpha_2}{2}}t^{\frac{\beta}{2}}\frac{ds}{s}\frac{dt}{t^2}\right.$$
$$\left.+\int_u^1\int_0^1\left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}}\left(\frac{st}{(s+t)^2}\right)^{\frac{3}{2}}(s+t)^{\frac{\alpha_1}{2}}s^{\frac{\alpha_2}{2}}t^{\frac{\beta}{2}}\frac{ds}{s}\frac{dt}{t^2}\right)$$
$$\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\; u^{\frac{\alpha_1+\alpha_2+\beta-2}{2}},$$

using that $\alpha_1\in(0,1), \mathcal{P}^2\in\mathsf{StGC}^0$ and $(\alpha_1+\alpha_2+\beta-2)>0$.
For the term B, we have

$$|B| \lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\int_{e',e''}\int_0^u K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\rho(e',e'')^{\alpha_1}t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t^2}\nu(de')\nu(de'')$$
$$\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\int_0^u t^{\frac{\alpha_1+\alpha_2+\beta-2}{2}}\frac{dt}{t}$$
$$\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\; u^{\frac{\alpha_1+\alpha_2+\beta-2}{2}},$$

using again that $\alpha_1\in(0,1)$ and $(\alpha_1+\alpha_2+\beta-2)>0$.
Finally for C, we also use cancellations properties to get

$$|C| \lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\Big\{\int_{e',e''}\int_u^1 K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\Big|a_1(e)-a_1(e')\Big|t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t^2}\nu(de')\nu(de'')$$
$$+\int_{e',e''}\int_u^1 K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\Big|a_1(e')-a_1(e'')\Big|t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t^2}\nu(de')\nu(de'')\Big\}$$
$$\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\Big\{\int_{e',e''}\int_u^1 K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\rho(e,e')^{\alpha_1}t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t^2}\nu(de')\nu(de'')$$
$$+\int_{e',e''}\int_u^1 K_{\mathcal{Q}_u}(e,e')K_{\mathcal{P}_t^{1\bullet}}(e',e'')\rho(e',e'')^{\alpha_1}t^{\frac{\alpha_2+\beta}{2}}\frac{dt}{t^2}\nu(de')\nu(de'')\Big\}$$
$$\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\Big\{u^{\frac{\alpha_1}{2}}\int_u^1 t^{\frac{\alpha_2+\beta-2}{2}}\frac{dt}{t}+\int_u^1\left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}}t^{\frac{\alpha_1+\alpha_2+\beta-2}{2}}\frac{dt}{t}\Big\}$$
$$\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\; u^{\frac{\alpha_1+\alpha_2+\beta-2}{2}},$$

using that $\alpha_1\in(0,1)$ and $(\alpha_2+\beta-2)<0$. In the end, we have

$$\Big\|\mathcal{Q}_u\mathsf{C}_L(a_1,a_2,b)\Big\|_\infty \lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\; u^{\frac{\alpha_1+\alpha_2+\beta-2}{2}}$$

uniformly in $u \in (0,1]$, so the proof is complete for C_L. The proofs for $\mathsf{C}_L^<$ and $\mathsf{C}_L^>$ are then easy to obtain since $\mathsf{P}_{La}b$ has the same form as $\Pi(La,b)$. Indeed, $\mathsf{P}_{La}b$ is a linear combination of

$$\int_0^1 \mathcal{Q}_t^{1\bullet}\Big(\mathcal{P}_t^1(tL)a\cdot\mathcal{Q}_t^2 b\Big)\frac{dt}{t^2}$$

where $\mathcal{Q}^1,\mathcal{Q}^2 \in \mathsf{StGC}^{\frac{3}{2}}, \mathcal{P}^1 \in \mathsf{StGC}^{[0,3]}$ and we have $\big(\mathcal{P}_t^1(tL)\big)_{0<t\leqslant 1} \in \mathsf{StGC}^2$.

The proofs for $\mathsf{C}_{V_i}, \mathsf{C}_{V_i}^-$ and $\mathsf{C}_{V_i}^+$ also follow from the same argument and using the Leibniz rule as for the corrector C_∂ used in Section 3.3 of [4] to solve the generalised (KPZ) equation. ▷

Proof of Theorem 4 – For the continuity estimate of $\mathsf{C}_{L,(1)}$, we also want to compute the regularity using a family $\mathcal{Q}$ of StGC^r with $r > |\alpha_1+\alpha_2+\beta-2|$. Again a term $\Pi(La,b)$ can be written as a linear combination of terms of the form

$$\int_0^1 \mathcal{P}_t^{1\bullet}\big(\mathcal{Q}_t^1(tL)a\cdot\mathcal{Q}_t^2 b\big)\frac{dt}{t^2},$$

while $\widetilde{\mathsf{P}}_b a$ is a linear combination of terms of the form

$$\int_0^1 \widetilde{\mathcal{Q}}_t^{3\bullet}\big(\widetilde{\mathcal{Q}}_t^4 a\cdot\mathcal{P}_t^2 b\big)\frac{dt}{t},$$

with $\mathcal{Q}^1,\mathcal{Q}^2,\widetilde{\mathcal{Q}}^3,\widetilde{\mathcal{Q}}^4 \in \mathsf{StGC}^{\frac{3}{2}}$ and $\mathcal{P}^1,\mathcal{P}^2 \in \mathsf{StGC}^{[0,3]}$. For the terms where $\mathcal{P}^2 \in \mathsf{StGC}^{[2,3]}$, we already have the correct regularity since

$$\begin{aligned}\int_0^1\int_0^1 \mathcal{Q}_u\mathcal{P}_t^{1\bullet}&\Big(\mathcal{Q}_t^1(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\Big(\mathcal{P}_s^2 a_1\cdot\widetilde{\mathcal{Q}}_s^4 a_2\Big)\cdot\mathcal{Q}_t^2 b\Big)\frac{ds}{s}\frac{dt}{t^2}\\ &\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\int_0^1\int_0^1\left(\frac{ut}{(t+u)^2}\right)^{\frac{r}{2}}\left(\frac{ts}{(s+t)^2}\right)^{\frac{3}{2}} s^{\frac{\alpha_1+\alpha_2}{2}}t^{\frac{\beta}{2}}\frac{ds}{s}\frac{dt}{t^2}\\ &\lesssim \|a_1\|_{\alpha_1}\|a_2\|_{\alpha_2}\|b\|_\beta\, u^{\frac{\alpha_1+\alpha_2+\beta-2}{2}}\end{aligned}$$

using that $\alpha_1 \in (1,2)$ so we only consider $\mathcal{P}^2 \in \mathsf{StGC}^{[0,1]}$. For $\mathcal{P}^2 \in \mathsf{StGC}^0$, we control it using the term $a_1\Pi(La_2,b)$ as in the proof of the continuity estimate of C. We are left with

$$\int\mathcal{P}_t^{1\bullet}\left(\mathcal{Q}_t^1(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\left(\mathcal{P}_s^2\Big(a_1-d\big(\overline{u}_0(e)\big)^{-1}\sum_{i=1}^{\ell}(V_i a_1)(e)\delta_i(\cdot,e)\Big)\right)\cdot\widetilde{\mathcal{Q}}_s^4 a_2\right)\cdot\mathcal{Q}_t^2 b\right)(e)\frac{ds}{s}\frac{dt}{t^2}$$

with $\mathcal{P}^2 \in \mathsf{StGC}^1$. Then the result follows with the same proof using that $\mathcal{P}_s^2 1 = 0$ since it encodes some cancellation and the first order Taylor expansion

$$\left|a_1(e')-a_1(e)-d(\overline{u}_0)^{-1}\sum_{i=1}^{\ell}(V_i a_1)(e)\delta_i(e',e)\right| \lesssim \rho(e,e')^\alpha.$$

We let the reader prove the continuity resuls for $\mathsf{C}_{L,(1)}^-$ and $\mathsf{C}_{L,(1)}^+$; they can be proved by the same argument as above. ▷

Proof of Theorems 5-8 / Elements – We give the proof for the continuity estimate on L and $\mathsf{L}_{(1)}$. We let the reader adapt the proof from [4] for the iterated operators of L since it relies on the same argument. The same holds for $\mathsf{V}_i(a,b)$ and its first iteration.

We want to compute the regularity of $\mathsf{L}(a,b) = L\widetilde{\mathsf{P}}_a b - \mathsf{P}_a Lb$ using a family $\mathcal{Q} \in \mathsf{StGC}^r$ with $r > |\alpha+\beta-2|$. We write $\widetilde{\mathsf{P}}_a b$ and $\mathsf{P}_a b$ respectively as linear combination of

$$\int_0^1 \widetilde{\mathcal{Q}}_s^{3\bullet}\Big(\mathcal{P}_s^2 a\cdot\widetilde{\mathcal{Q}}_s^4 b\Big)\frac{ds}{s}\quad\text{and}\quad\int_0^1 \mathcal{Q}_t^{1\bullet}\big(\mathcal{P}_t^1 a\cdot\mathcal{Q}_t^2 b\big)\frac{dt}{t}$$

with $\mathcal{Q}^1, \mathcal{Q}^2, \widetilde{\mathcal{Q}}^4 \in \mathsf{StGC}^{\frac{3}{2}}, \widetilde{\mathcal{Q}}^3 \in \mathsf{GC}^{\frac{3}{2}}$ and $\mathcal{P}^1, \mathcal{P}^2 \in \mathsf{StGC}^{[0,3]}$. As done for C, we only have to consider $\mathcal{P}^1, \mathcal{P}^2 \in \mathsf{StGC}^0$ since the other terms already have the right regularity using that $\alpha \in (0,1)$. We consider a term

$$\int_0^1 L\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\mathcal{P}_s^2 a \cdot \widetilde{\mathcal{Q}}_s^4 b\right)\frac{ds}{s} - \int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{P}_t^1 a \cdot \mathcal{Q}_t^2(tL)b\right)\frac{dt}{t^2}.$$

We use that $\int_0^1 \mathcal{Q}_t^{1\bullet}\mathcal{Q}_t^2\frac{dt}{t} = \int_0^1 \widetilde{\mathcal{Q}}_s^{3\bullet}\widetilde{\mathcal{Q}}_s^4\frac{ds}{s} = \mathrm{Id}$, up to smooth term, to get

$$\int_0^1\int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\mathcal{P}_s^2 a\cdot\widetilde{\mathcal{Q}}_s^4 b\right) - \mathcal{P}_t^1 a\cdot\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\widetilde{\mathcal{Q}}_s^4 b\right)\frac{dt}{t^2}\frac{ds}{s}$$
$$= \int_0^1\int_0^1 \mathcal{Q}_t^{1\bullet}\left(\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\left(\mathcal{P}_s^2 a - \mathcal{P}_t^1 a(\cdot)\right)\cdot\widetilde{\mathcal{Q}}_s^4 b\right)\right)\frac{dt}{t^2}\frac{ds}{s}$$

where the variable of $\mathcal{P}_t^1 a(\cdot)$ is frozen as before, in the sense that $\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}$ does not act on it. Since $\alpha \in (0,1)$, we can use that for any e, e'

$$\left|\mathcal{P}_s^2 a(e') - \mathcal{P}_t^1 a(e)\right| \leqslant \left|\mathcal{P}_s^2 a(e') - a(e')\right| + |a(e') - a(e)| + \left|a(e) - \mathcal{P}_t^1 a(e)\right|,$$

to get

$$\int_0^1\int_0^1 \mathcal{Q}_u\mathcal{Q}_t^{1\bullet}\left(\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\left(\mathcal{P}_s^2 a - \mathcal{P}_t^1 a(\cdot)\right)\cdot\widetilde{\mathcal{Q}}_s^4 b\right)\right)\frac{dt}{t^2}\frac{ds}{s}$$
$$\lesssim \|a\|_\alpha\|b\|_\beta\int_0^1\int_0^1\left(\frac{tu}{(t+u)^2}\right)^{\frac{r}{2}}\left(\frac{st}{(s+t)^2}\right)^{\frac{3}{2}}(t+s)^\alpha s^\beta\frac{dt}{t^2}\frac{ds}{s}$$
$$\lesssim \|a\|_\alpha\|b\|_\beta\, u^{\frac{\alpha+\beta}{2}}$$

which complete the proof for $\mathsf{L}(a,b)$.

We finally prove the estimate for the refined commutator $\mathsf{L}_{(1)}(a,b)$ that is given for any $e \in \mathcal{M}$ by

$$\mathsf{L}_{(1)}(a,b)(e) = \big(L\widetilde{\mathsf{P}}_a b\big)(e) - \big(\mathsf{P}_a Lb\big)(e) - \sum_{i=1}^{\ell}\big(\mathsf{P}^{(i)}_{d(\overline{u}_0)^{-1}V_i a}b\big)(e).$$

where

$$\big(\mathsf{P}^{(i)}_a b\big)(e) = \int_{e',e''} K(e;e',e'')a(e')\left(\widetilde{\mathsf{P}}_{\delta_i(\cdot,e')}b\right)(e'')\nu(de')\nu(de''),$$

with K the kernel of the bilinear operator $(a,b)\mapsto \mathsf{P}_a b$. As in the proof of $\mathsf{C}_{L,(1)}$, we are left with

$$\int K_{\mathcal{Q}_t^{1\bullet}}(e,e')\Big\{\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\big(\mathcal{P}_s^2 a\cdot\widetilde{\mathcal{Q}}_s^4 b\big)$$
$$-\sum_{i=1}^{\ell}\big(\mathcal{P}_t^1(d(\overline{u_0})^{-1}V_i a)\big)(e')\cdot\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\big(\mathcal{P}_s^2\delta_i(\cdot,e')\cdot\widetilde{\mathcal{Q}}_s^4 b\big)\Big\}(e')\frac{dt}{t^2}\frac{ds}{s}\nu(de')$$
$$= \int K_{\mathcal{Q}_t^{1\bullet}}(e,e')\left(\mathcal{Q}_t^2(tL)\widetilde{\mathcal{Q}}_s^{3\bullet}\left(\mathcal{P}_s^2\Big(a - \sum_{i=1}^{\ell}\mathcal{P}_t^1\big(d(\overline{u}_0)a\big)(e')\big)\delta_i(\cdot,e')\Big)\cdot\widetilde{\mathcal{Q}}_s^4 b\right)\right)(e')\frac{dt}{t^2}\frac{ds}{s}\nu(de')$$

with $\mathcal{P}^1, \mathcal{P}^2 \in \mathsf{StGC}^1$. The result follows with the same proof using that $\mathcal{P}_s^2 1 = 0$ since it encodes some cancellation and the first order Taylor expansion for a. ▷

C – Paracontrolled expansion

We use in the body of the text the following variation on the high order paracontrolled expansion formula from [4], Theorem 4 therein.

Theorem 16. *Let $f:\mathbb{R}\to\mathbb{R}$ be a C^4 function and let u and v be respectively C^α and $C^{4\alpha}$ functions on $[0,T]\times\mathbb{T}^3$ with $\alpha\in(0,1)$. Then*

$$f(u)v = \mathsf{P}_{f'(u)v}u + \frac{1}{2}\Big\{\mathsf{P}_{f^{(2)}(u)v}u^2 - 2\mathsf{P}_{f^{(2)}(u)uv}u\Big\} + \frac{1}{3!}\Big\{\mathsf{P}_{f^{(3)}(u)v}u^3 - 3\mathsf{P}_{f^{(3)}(u)uv}u^2 + 3\mathsf{P}_{f^{(3)}(u)u^2v}u\Big\} + f_v(u)^\sharp$$

for some remainder $f_v(u)^\sharp\in\mathcal{C}^{4\alpha}$.

Proof – We need to prove that

$$R := vf(u) - \mathsf{P}_{vf'(u)}u - \frac{1}{2}\Big\{\mathsf{P}_{vf^{(2)}(u)}u^2 - 2\mathsf{P}_{vf^{(2)}(u)u}u\Big\} - \frac{1}{3!}\Big\{\mathsf{P}_{vf^{(3)}(u)}u^3 - 3\mathsf{P}_{vf^{(3)}(u)u}u^2 + 3\mathsf{P}_{vf^{(3)}(u)u^2}u\Big\}$$

is a 3α-Hölder function. Using that $\mathsf{P}_1 vf(u) = vf(u)$ up to smooth term and that $\mathsf{P}_a b$ is the sum of terms of the form

$$\int_0^1 \mathcal{Q}_t^{1\bullet}\big(\mathcal{Q}_t^2 a\cdot\mathcal{P}_t^1 b\big)\frac{dt}{t}$$

with $\mathcal{Q}^1,\mathcal{Q}^2\in\mathsf{StGC}^{\frac{3}{2}}$ and $\mathcal{P}^1\in\mathsf{StGC}^{[0,3]}$, R is a sum of terms of the form $\int_0^1\mathcal{Q}_t^{1\bullet}(r_t)\frac{dt}{t}$ with

$$r_t := \mathcal{Q}_t^2\Big(vf(u)\Big) - \mathcal{Q}_t^2\Big(vf'(u)\Big)\mathcal{P}_t^1(u) - \frac{1}{2}\mathcal{Q}_t^2\Big(vf^{(2)}(u)\Big)\mathcal{P}_t^1(u^2) + \mathcal{Q}_t^2\Big(vf^{(2)}(u)u\Big)\mathcal{P}_t^1(u) + \frac{1}{6}\mathcal{Q}_t^2\Big(vf^{(3)}(u)\Big)\mathcal{P}_t^1(u^3) + \frac{1}{2}\mathcal{Q}_t^2\Big(vf^{(3)}(u)u\Big)\mathcal{P}_t^1(u^2) - \frac{1}{2}\mathcal{Q}_t^2\Big(vf^{(3)}(u)u^2\Big)\mathcal{P}_t^1(u).$$

We need to get a bound on r_t in $L^\infty(\mathcal{M})$. We have for $e\in\mathcal{M}$

$$r_t(e) = \int_{\mathcal{M}^2} K_{\mathcal{Q}_t^2}(e,e')K_{\mathcal{P}_t^1}(e,e'')\Big\{\Big(vf(u)\Big)(e') - \Big(vf'(u)\Big)(e')u(e'') - \frac{1}{2}\Big(vf^{(2)}(u)\Big)(e')u^2(e'') + \Big(vf^{(2)}(u)u\Big)(e')u(e'') + \frac{1}{6}\Big(vf^{(3)}(u)\Big)(e')u^3(e'') + \frac{1}{2}\Big(vf^{(3)}(u)u\Big)(e')u^2(e'') - \frac{1}{2}\Big(vf^{(3)}(u)u^2\Big)(e')u(e'')\Big\}\nu(de')\nu(de'').$$

Using a Taylor expansion for f, we have

$$r_t(e) = \int_{[0,1]^4} f^{(4)}\Big(u(e'') + s_4s_3s_2s_1\big(u(e')-u(e'')\big)\Big)s_3s_2s_1\big(u(e')-u(e'')\big)^4\,ds_4ds_3ds_2ds_1 + v(e')\Big(f(u(e'')) + u(e'')f'(u(e'')) + \frac{1}{2}u^2(e'')f^{(2)}(u(e'')) + \frac{1}{3!}u^3(e'')f^{(3)}(u(e''))\Big) = (1) + (2).$$

For the first term, we have

$$(1) \leqslant \|u\|_\alpha^4\, t^{\frac{4\alpha}{2}}$$

and for the second term

$$(2) \leqslant \|u\|_{L^\infty}\|v\|_{4\alpha}\, t^{\frac{4\alpha}{2}}$$

which allows us to conclude. ▷

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• I. Bailleul – Univ. Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
E-mail: ismael.bailleul@univ-rennes1.fr

• A. Mouzard – Univ. Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
E-mail: antoine.mouzard@ens-rennes.fr