Regularity structures for quasilinear singular SPDEs

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Abstract. We prove the well-posed character of a regularity structure formulation of the quasilinear generalized (KPZ) equation and give an explicit form for a renormalized equation in the full subcritical regime. Convergence results for the solution of the regularized renormalized equation are obtained in regimes that cover the spacetime white noise case.

1 – Introduction

Denote by $\mathbf{T}$ the one dimensional torus. We consider the one dimensional space periodic quasilinear generalized (KPZ) equation

$$\big(\partial_t - a(u)\partial_x^2\big)u = f(u)\xi + g(u)(\partial_x u)^2, \tag{1.1}$$

for regular enough functions a, f, g, where a takes values in a compact interval of $(0,\infty)$ and ξ is a random spacetime distribution – with main example spacetime white noise. The initial condition $u_0 \in C^{0+}(\mathbf{T}) := \bigcup_{\mu>0} C^\mu(\mathbf{T})$ is given. Following [3, 5] set

$$L^{a(v_0)} := a(v_0)\partial_x^2$$

for a smooth function v_0 on $\mathbf{T}$ and we rewrite equation (1.1) under the form

$$\big(\partial_t - L^{a(v_0)} + c\big)u = f(u)\xi + g(u)(\partial_x u)^2 + cu + \big(a(u) - a(v_0)\big)\partial_x^2 u \tag{1.2}$$

for a large positive constant c. We consider (1.2) as a 'perturbation' of the non-translation invariant generalized (KPZ) equation

$$\big(\partial_t - L^{a(v_0)} + c\big)u = f(u)\xi + g(u)(\partial_x u)^2 + cu.$$

Below we will set the scene to reformulate equation (1.2) in a regularity structure where it takes the form

$$\begin{aligned}\boldsymbol{u} &= \mathsf{P}_{<2}\big(Q^{a(v_0),c}u_0\big)\\ &\quad + \mathsf{K}^{a(v_0),c,\mathsf{M}}\Big(\mathsf{Q}_{\le 0}\Big\{F(\boldsymbol{u})\,\zeta + G(\boldsymbol{u})(\boldsymbol{D}\boldsymbol{u})^2 + c\boldsymbol{u} + \big\{A(\boldsymbol{u}) - A(\mathsf{P}_{<2}(v_0))\big\}\boldsymbol{D}^2\boldsymbol{u}\Big\}\Big).\end{aligned} \tag{1.3}$$

The operator $\mathsf{P}_{<2}$ stands for the canonical lift operator of a spacetime/spatial function to the part of the polynomial regularity structure spanned by monomials of homogeneity less than 2, and the operator $Q^{a(v_0),c}u_0$ is the free propagation of the initial condition u_0 under the non-translation invariant operator $(\partial_t - L^{a(v_0)} + c)$. The operator $\mathsf{K}^{a(v_0),c,\mathsf{M}}$ is the model dependent integration operator on modelled distributions intertwined to $(\partial_t - L^{a(v_0)} + c)$ via the reconstruction operator. The operator $\mathsf{Q}_{\le 0}$ projects on elements of nonpositive homogeneity, and the operator $\boldsymbol{D}$ is a natural derivative operator on a space of modelled functions.

We will see in Theorem *15* that given any admissible model M on our regularity structure, equation (1.3) has a unique solution over a model-dependent time interval $(0, t_0(\mathsf{M}))$, in an appropriate class of modelled distributions. This analytical statement holds in the full subcritical range provided the model is part of the data. Such a statement was already proved by Gerencsér & Hairer in [16] in a different setting. However their choice of formulation for (1.1) did not allow them to write down in the full subcritical range the renormalized equation satisfied by the reconstruction of the model dependent solution $\boldsymbol{u}$ of (1.3) when the noise is smooth and one uses an appropriate admissible model. The spacetime white noise regime is in particular out of range of their result. Working with an appropriate choice of model M that is the natural analogue in our setting of the BHZ renormalized model from [8] we are able to give in Theorem *1* below a renormalized equation in the full subcritical regime. Denote by ε

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a positive regularization parameter and by $\xi^\varepsilon \in C^\infty(\mathbf{R}\times\mathbf{T})$ an ε-regularized noise ξ. Denote by M^ε the BHZ renormalized model associated with ξ^ε and the operator $(\partial_t - L^{a(v_0)} + c)$, and denote by u^ε the M^ε-reconstruction of the solution $\boldsymbol{u}^\varepsilon$ of equation (1.3) with M^ε in place of M. (The model M^ε is described precisely in Section 4.3.2.) The function u^ε is defined on a time interval $[0, t_0(\mathsf{M}^\varepsilon))$. Our main results take a conditional form involving two assumptions. Assumption 1 is stated in Section 4.3.2 and assumes the convergence of the natural BHZ model associated with the non-translation invariant operator $(\partial_t - L^{a(v_0)} + c)$. There is no doubt that it holds true but we refrain from describing here the modifications of Chandra & Hairer's work [11] needed to extend their result to our non translation-invariant setting.

1 – *Theorem. Take $v_0 = e^{t_0\partial_x^2}u_0$. Under Assumption 1 there exist continuous functions $\mathfrak{F}^a((\tau^{\boldsymbol{p}})^*) \in C(\mathbf{R}^3)$ and*

$$\ell^\varepsilon_{a(v_0)}(\cdot, \tau^{\boldsymbol{p}}) \in C(\mathbf{R}\times\mathbf{T})$$

indexed by an infinite set of symbols $\{\tau^{\boldsymbol{p}} \in \mathbb{B}^-_\circ\}$, such that the solution u^ε to

$$\big(\partial_{x_0} - a(u^\varepsilon)\partial_x^2\big)u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon + g(u^\varepsilon)(\partial_x u^\varepsilon)^2 + \sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}^-_\circ} \frac{\ell^\varepsilon_{a(v_0)}(\cdot,\tau^{\boldsymbol{p}})}{S(\tau^{\boldsymbol{p}})}\,\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon, \partial_x u^\varepsilon, v_0\big) \tag{1.4}$$

starting from $u_0 \in C^{0+}(\mathbf{T})$ converges in $C([0,t_0)\times\mathbf{T})$ for a positive random time t_0 in probability as $\varepsilon \to 0$.

Let us emphasize that this conditional convergence result holds in the whole subcritical regime $\alpha > 0$ where $\alpha - 2$ is the expected regularity of the noise ξ. The sum over $\tau^{\boldsymbol{p}}$ in (1.4) is called the 'counterterm' and the functions $\ell^\varepsilon_{a(v_0)}(\cdot,\tau^{\boldsymbol{p}})$ are non-local functionals of the function $a(v_0(\cdot))$. Note that the functions $\mathfrak{F}^a((\tau^{\boldsymbol{p}})^*)$ also depend on v_0. One can give a simpler representation of the counterterm when the functions $\ell^\varepsilon_{a(v_0)}(\cdot,\tau^{\boldsymbol{p}})$ can be traded off for a local functional of $a(v_0(\cdot))$ – meaning that $\ell^\varepsilon_{a(v_0)}(z,\tau^{\boldsymbol{p}})$ can be replaced by a function of $a(v_0(z))$. This is the content of Assumption 2 stated in Section 4.4.

2 – *Theorem. Under Assumption 1 and 2 there exist continuous functions $\chi^a_\tau \in C(\mathbf{R})$, $\mathfrak{F}_{\tau^*} \in C(\mathbf{R}^2)$ and $\ell^\varepsilon_{(\cdot)}(\tau) \in C(\mathbf{R})$ all three indexed by a finite set of symbols $\{\tau \in \mathbb{B}^{-0}_\circ\}$, such that the third term of the right hand side of (1.4) is of the form*

$$\sum_{\tau\in\mathbb{B}^{-0}_\circ} \frac{\ell^\varepsilon_{a(u^\varepsilon)}(\tau)}{S(\tau)}\,\chi^a_\tau(u^\varepsilon)\,\mathfrak{F}_{\tau^*}(u^\varepsilon, \partial_x u^\varepsilon) + O(1), \tag{1.5}$$

for a term $O(1)$ uniform in ε.

Above, the functions $\chi^a_\tau(\cdot)$ are polynomial functions of a and its derivatives and the coefficient $S(\tau)$ stands for a positive τ-dependent integer. Note that appart from the $O(1)$ term in (1.5), which we can discard in the renormalized equation, the counterterm is independent of v_0. Proposition *25* gives a fine description of the functions $\ell^\varepsilon_{a(u^\varepsilon(\cdot))}$ when the noise ξ is Gaussian, centered and stationary, and the regularization procedure obtained by convolution with a symmetric kernel. We show in Section 4.5 that Assumption 2 holds in particular for the quasilinear generalized (KPZ) equation driven by a spacetime white noise.

The first works on quasilinear singular stochastic PDEs by Otto & Weber [23], Furlan & Gubinelli [14] and Bailleul, Debussche, & Hofmanová [3] all three investigated the generalized (PAM) equation in the regime where the noise is $(\alpha - 2)$ regular and $\alpha > 2/3$. Interestingly each of these works used a different method: A variant of regularity structures in [23], a variant of paracontrolled calculus based on the use of the paracomposition operator for [14], and the initial form of paracontrolled calculus in [3]. On the paracontrolled side Bailleul & Mouzard [5] extended the high order paracontrolled calculs toolbox to deal with the paracontrolled equivalent of equation (1.3) in the spacetime white noise regime $\alpha > 2/5$. On the regularity structures side Otto & Weber deepened their framework in their works [21, 22] with Sauer & Smith, dedicated to the study of the equation with linear additive forcing

$$\partial_t u - a(u)\partial_x^2 u = \xi. \tag{1.6}$$

They obtained in particular in [22] an explicit form of the renormalized equation for (1.6). Our general formula for the counterterm in the renormalized equation generalizes theirs. The algebraic machinery behind their approach was further analysed by Linares, Otto & Tempelmayr in [19]. This series of works culminated very recently with the construction in [20] of the analogue in their framework of the BHZ renormalized models for equation (1.6), for a large class of random noises in the full subcritical regime. Meanwhile Gerencsér & Hairer provided in [16] an analysis of a regularity structure counterpart of equation (1.1), in the full subcritical regime. Their method allowed for an analysis of the renormalized equation only in the regime $\alpha > 1/2$. By implementing some tricky integration by parts-type formulas Gerencsér was able in [15] to obtain the renormalized equation for the special case of equation (1.6) from the analysis of [16] in the spacetime white noise regime $\alpha > 2/5$.

Theorem 1 extends these results and deals with the quasilinear generalized (KPZ) in the full subcritical regime. The reader familiar with regularity structures will see that our arguments extend immediately to coupled systems of generalized (KPZ) equations. Such a generalization is left to the reader and we concentrate here on the renormalized equation.

Dealing with quasilinear singular SPDEs rather than semilinear singular SPDEs requires a twist that appears in the form of an infinite dimensional ingredient. It is related in our formulation (1.3) to the fact that our structure needs to be stable by the operator $\mathcal{I}_{(0,2)} := \boldsymbol{D}^2\mathcal{I}$. In the previous works using regularity structures this infinite dimensional feature appeared under the form of a one parameter family of heat kernels or abstract integration operators. Our regularity structure is different from the regularity structures used in these works. Its T-space $T = \bigoplus_{\beta\in A} T_\beta$ will have infinite dimensional homogeneous spaces T_β whose basis elements will be the usual trees associated with the generalized (KPZ) equation, with an additional integer decoration p on each edge accounting for how many times the operator $\mathcal{I}_{(0,2)}$ is applied to this edge. The same infinite dimensional ingredient appeared in Bailleul & Mouzard's work [5] in a paracontrolled setting.

We set the scene in Section 2, where the function spaces we work with are introduced together with our regularity structure. We introduced in particular a non-classical spacetime elliptic operator to define our parabolic spaces. For reader's convenience some properties of its heat kernel are proved in Appendix A. Section 3 is dedicated to proving that equation (1.3) is locally well-posed in the full subcritical regime. The analysis of the renormalized equation problem is done in Section 4, where we give in particular in Section 4.3 an explicit description of the functions χ^a_τ. We take profit all the way of the fact that a number of known results in the usual setting of regularity structures have direct analogues in our setting. As in [16] this explains the relatively short size of the present work.

Notations *– We denote by* $\mathbf{R}$ *the set of real numbers and by* $\mathbf{N}$ *the set of integers. We denote by* $z = (t,x) \in \mathbf{R}^2$ *a generic spacetime variable, for which we set*

$$\|z\|_{\mathfrak{s}} := |t|^{1/2} + |x|.$$

We also set for $\mathbf{k} = (k_1, k_2) \in \mathbf{N}^2$

$$|\mathbf{k}|_{\mathfrak{s}} := 2k_1 + k_2,$$

and

$$\partial_z^{\mathbf{k}} := \partial_t^{k_1}\partial_x^{k_2}.$$

Given a function denoted by a lowercase letter we will use the corresponding capital letter for its lift as a function on a space of modelled distributions. By convention the product $\prod_{i=1}^0$ *is equal to* 1.

2 – The setting

We introduce in this section the functional setting and the regularity structure in which we set the study of equation (1.3).

2.1 – Function spaces. The following basic facts are proved in Appendix A.2.

– The fundamental solution $Q_t^{a(v_0),0}(x,y)$ of the operator $\partial_t - L^{a(v_0)}$ satisfies the estimate
$$\big|\partial_t^n\partial_x^k Q_t^{a(v_0),0}(x,y)\big| \leq \frac{c_0 e^{c_0 t}}{t^{(1+k+2n)/2}}\,\exp\left(-c_1\frac{|x-y|^2}{t}\right) \tag{2.1}$$
for any $k+2n\leq 2$, for some positive constants c_0, c_1 depending only on $\inf a > 0$, $\|a\|_{C^1}$, and $\|v_0\|_{C^\mu(\mathbf{T})}$ for any fixed $\mu > 0$.

– Define the spacetime elliptic operator
$$\mathcal{L}^{a(v_0)} := \big(\partial_t - L^{a(v_0)}\big)(\partial_t + \partial_x^2) = \partial_t^2 - a(v_0)\partial_x^4 - \big(a(v_0)-1\big)\partial_t\partial_x^2.$$
We introduce the additional variable $\theta > 0$ and consider the parabolic operator
$$\partial_\theta - \mathcal{L}^{a(v_0)}$$
on functions of $(\theta, z)\in(0,\infty)\times\mathbf{R}^2$. The fundamental solution $\mathcal{Q}_\theta^{a(v_0),0}(\cdot,\cdot)$ of $\partial_\theta - \mathcal{L}^{a(v_0)}$ satisfies the estimate
$$\big|\partial_z^{\mathbf{k}}\mathcal{Q}_\theta^{a(v_0),0}\big((t,x),(s,y)\big)\big| \leq \frac{C_0e^{C_0\theta}}{\theta^{(3+|\mathbf{k}|_{\mathfrak{s}})/4}}\exp\left\{-C_1\left(\frac{|t-s|^2}{\theta}+\frac{|x-y|^{4/3}}{\theta^{1/3}}\right)\right\} \tag{2.2}$$
for any $|\mathbf{k}|_{\mathfrak{s}}\leq 4$, for some positive constants C_0, C_1 depending only on $\inf a > 0$, $\|a\|_{C^1}$, and $\|v_0\|_{C^\mu(\mathbf{T})}$ for any fixed $\mu > 0$.

Recall we denote by $\alpha-2$ the spacetime Hölder regularity of the noise ξ in equation (1.1). We will consider initial conditions $u_0\in C^\mu(\mathbf{T})$ with any $\mu\in(0,\alpha)$ and choose later $v_0 = e^{t_0\partial_x^2}u_0$, for some small $t_0\in(0,1]$. Since the family of functions $\{e^{t\partial_x^2}u_0\}_{t\in(0,1]}$ have t-uniform μ-Hölder estimates depending only on $\|u_0\|_{C^\mu(\mathbf{T})}$, the constants c_0, c_1, C_0, C_1 above can be chosen to depend only on $\inf a > 0$, $\|a\|_{C^1}$, and $\|u_0\|_{C^\mu}$. Therefore, all the proportional constants appearing sometime implicitly in some inequalities below are independent of t_0. We will choose later t_0 sufficiently small to prove the local well-posedness of equation (1.1). For a bounded continuous functions f on $\mathbf{R}^2$ set for $|\mathbf{k}|_{\mathfrak{s}}\leq 4$,
$$\big(\partial^{\mathbf{k}}\mathcal{Q}_\theta^{a(v_0),0}f\big)(z) := \int_{\mathbf{R}^2}\partial_z^{\mathbf{k}}\mathcal{Q}_\theta^{a(v_0),0}(z,z')f(z')dz'.$$
We use the operators $\mathcal{Q}_\theta^{a(v_0),0}$ to define the full scale of anisotropic parabolic Hölder spaces.

Definition – For $\beta<0$, define $\mathcal{C}_{\mathfrak{s}}^\beta(a(v_0))$ as the completion of the set of bounded continuous functions f on $\mathbf{R}^2$ under the norm
$$\|f\|_{\mathcal{C}_{\mathfrak{s}}^\beta(a(v_0))} := \sup_{0<\theta\leq 1}\theta^{-\beta/4}\big\|\mathcal{Q}_\theta^{a(v_0),0}f\big\|_{L^\infty(\mathbf{R}^2)}.$$
For $\beta>0$, define $\mathcal{C}_{\mathfrak{s}}^\beta$ as the classical parabolic α-Hölder space, that is, $f\in\mathcal{C}_{\mathfrak{s}}^\beta$ if $\partial_z^{\mathbf{k}}f$ exists and is bounded for any $\mathbf{k}\in\mathbf{N}^2$ with $|\mathbf{k}|_{\mathfrak{s}}<\beta$, and $\partial_z^{\mathbf{k}}f$ with $|\mathbf{k}|_{\mathfrak{s}} = \lfloor\beta\rfloor$ is $(\beta-\lfloor\beta\rfloor)$-Hölder continuous with respect to the parabolic norm $\|\cdot\|_{\mathfrak{s}}$.

The following property is used in the proof of the reconstruction theorem, Theorem *8* below. Its proof is given in Theorem *37* of Appendix A.

3 – Proposition. For any $\beta'<\beta<0$, the embedding $\mathcal{C}_{\mathfrak{s}}^\beta(a(v_0))\subset\mathcal{C}_{\mathfrak{s}}^{\beta'}(a(v_0))$ is compact.

Next we consider the Schauder estimate for the resolvent of $\partial_t - L^{a(v_0)}$ in the space $\mathcal{C}_{\mathfrak{s}}^\alpha(a(v_0))$. With an eye on the heat kernel estimates (2.1) and (2.2) pick a positive constant $c>\max\{c_0, C_0\}$

and write $Q_t^{a(v_0),c} := e^{-ct} Q_t^{a(v_0),0}$ and $\mathcal{Q}_\theta^{a(v_0),c} := e^{-c\theta}\mathcal{Q}_\theta^{a(v_0),0}$. Then the operators $c - \mathcal{L}^{a(v_0)}$ and $\partial_t - L^{a(v_0)} + c$ have inverses of the forms

$$\big(c - \mathcal{L}^{a(v_0)}\big)^{-1} f = \int_0^\infty \mathcal{Q}_\theta^{a(v_0),c} f \, d\theta = \int_0^1 \mathcal{Q}_\theta^{a(v_0),c} f\, d\theta + \mathcal{Q}_1^{a(v_0),c} \circ (c - \mathcal{L}^{a(v_0)})^{-1} f$$

and

$$\big((\partial_t - L^{a(v_0)} + c)^{-1} g\big)(t) = \int_{-\infty}^t Q_{t-s}^{a(v_0),c} g(s)\, ds.$$

For any given bounded continuous function f on $\mathbf{R}^2$, one can write the resolvent operator of the parabolic operator $\partial_t - L^{a(v_0)} + c$ in terms of the spacetime elliptic operator $c - \mathcal{L}^{a(v_0)}$. Indeed, setting $g = (c - \mathcal{L}^{a(v_0)})^{-1} f$ and $h = (\partial_t + \partial_x^2) g$, we have

$$(\partial_t - L^{a(v_0)} + c) h = \mathcal{L}^{a(v_0)} g + ch = -f + c(g+h),$$

thus

$$\begin{aligned}\big(\partial_t - L^{a(v_0)} + c\big)^{-1} f &= -h + c\big(\partial_t - L^{a(v_0)} + c\big)^{-1}(g+h)\\ &= -(\partial_t + \partial_x^2)(c - \mathcal{L}^{a(v_0)})^{-1} f + c\big(\partial_t - L^{a(v_0)} + c\big)^{-1}(1 + \partial_t + \partial_x^2)(c - \mathcal{L}^{a(v_0)})^{-1} f.\end{aligned}$$

Set

$$K^{a(v_0),c} f := -\int_0^1 (\partial_t + \partial_x^2)\mathcal{Q}_\theta^{a(v_0),c} f\, d\theta =: \int_0^1 K_\theta^{a(v_0),c} f\, d\theta$$

and

$$R^{a(v_0),c} f := K_1^{a(v_0),c}(c - \mathcal{L}^{a(v_0)})^{-1} f + c\big(\partial_t - L^{a(v_0)} + c\big)^{-1}(1 + \partial_t + \partial_x^2)(c - \mathcal{L}^{a(v_0)})^{-1} f,$$

so one has the decomposition

$$\big(\partial_t - L^{a(v_0)} + c\big)^{-1} f = K^{a(v_0),c} f + R^{a(v_0),c} f. \tag{2.3}$$

The letter 'R' in $R^{a(v_0),c}$ is chosen for 'remainder', which is justified by the regularizing properties of this operator stated in the next statement.

4 – Theorem. Let $\beta \in (-2,0) \setminus \{-1\}$. The map $K^{a(v_0),c}$ sends $\mathcal{C}_\mathfrak{s}^\beta(a(v_0))$ into $\mathcal{C}_\mathfrak{s}^{\beta+2}$ and the map $R^{a(v_0),c}$ sends $\mathcal{C}_\mathfrak{s}^\beta(a(v_0))$ into $\mathcal{C}_\mathfrak{s}^{2+} := \bigcup_{\mu>0} \mathcal{C}_\mathfrak{s}^{2+\mu}$.

Proof – By the semigroup property and the Gaussian estimate (2.2) one has

$$\big\|\partial_z^{\mathbf{k}} K_\theta^{a(v_0),c} f\big\|_{L^\infty} \le \sup_z \big\|\partial_z^{\mathbf{k}} K_{\theta/2}^{a(v_0),c}(z,\cdot)\big\|_{L^1} \big\|\mathcal{Q}_{\theta/2}^{a(v_0),c} f\big\|_{L^\infty} \lesssim \theta^{-(2+|\mathbf{k}|_\mathfrak{s})/4}\theta^{\beta/4}\, \|f\|_{\mathcal{C}_\mathfrak{s}^\beta(v_0)}$$

for any $|\mathbf{k}|_\mathfrak{s} \le 2$ and any $\theta \in (0,1]$. Therefore,

$$\big\|\partial_z^{\mathbf{k}} K^{a(v_0),c} f\big\|_{L^\infty} \lesssim \|f\|_{\mathcal{C}_\mathfrak{s}^\beta(v_0)}$$

for any $|\mathbf{k}|_\mathfrak{s} < \beta + 2$. For $|\mathbf{k}|_\mathfrak{s} = \lfloor \beta + 2 \rfloor$, the Hölder estimate of $\partial^{\mathbf{k}} K^{a(v_0),c} f$ follows from that of $\partial_x K_{\theta/2}^{a(v_0),c}$ ((A.9) of Theorem *33*). If $\|z - z'\|_\mathfrak{s} < 1$, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}&\big|\partial_z^{\mathbf{k}} K^{a(v_0),c} f(z) - \partial_z^{\mathbf{k}} K^{a(v_0),c} f(z')\big|\\ &\le \int_0^1 \big\|\partial_z^{\mathbf{k}} K_{\theta/2}^{a(v_0),c}(z,\cdot) - \partial_z^{\mathbf{k}} K_{\theta/2}^{a(v_0),c}(z',\cdot)\big\|_{L^1} \big\|\mathcal{Q}_{\theta/2}^{a(v_0),c} f\big\|_{L^\infty} d\theta\\ &\lesssim \|z - z'\|_\mathfrak{s}^{\beta+2-|\mathbf{k}|_\mathfrak{s}-\varepsilon} \int_0^{\|z-z'\|_\mathfrak{s}^4} \theta^{-(4-\varepsilon)/4} d\theta + \|z - z'\|_\mathfrak{s}^{\beta+2-|\mathbf{k}|_\mathfrak{s}+\varepsilon} \int_{\|z-z'\|_\mathfrak{s}^4}^1 \theta^{-(4+\varepsilon)/4} d\theta\\ &\lesssim \|z - z'\|_\mathfrak{s}^{\beta+2-|\mathbf{k}|_\mathfrak{s}}.\end{aligned}$$

The case $\|z - z'\|_\mathfrak{s} \ge 1$ can be treated by a similar way. For $R^{a(v_0),c}$ note that $(c - \mathcal{L}^{a(v_0)})^{-1}$ maps $\mathcal{C}_\mathfrak{s}^\beta(v_0)$ into $\mathcal{C}_\mathfrak{s}^{2+}$. Indeed one has as above

$$\big\|\partial_z^{\mathbf{k}}\big(c - \mathcal{L}^{a(v_0)}\big)^{-1} f\big\|_{L^\infty} \le \int_0^1 \big\|\partial_z^{\mathbf{k}} \mathcal{Q}_\theta^{a(v_0),c} f\big\|_{L^\infty} d\theta + \int_0^\infty \big\|\partial_z^{\mathbf{k}} \mathcal{Q}_\theta^{a(v_0),c} \mathcal{Q}_1^{a(v_0),c} f\big\|_{L^\infty} d\theta$$

$$\lesssim \int_0^1 \theta^{-(2+|\mathbf{k}|_{\mathfrak{s}}-\beta)/4} \|f\|_{\mathcal{C}^\beta_{\mathfrak{s}}(v_0)}\, d\theta + \int_0^\infty e^{-(c-C_1)\theta} \|\mathcal{Q}_1^{a(v_0)} f\|_{L^\infty}\, d\theta$$
$$\lesssim \|f\|_{\mathcal{C}^\beta_{\mathfrak{s}}(a(v_0))}$$

for any $|\mathbf{k}|_{\mathfrak{s}} \leq 2$. The Hölder estimate of $\partial_z^{\mathbf{k}}(c - \mathcal{L}^{a(v_0)})^{-1} f$ is also obtained similarly. Since $K_1^{a(v_0)}$ maps $\mathcal{C}^{2+}_{\mathfrak{s}}$ into itself, the first term of the definition of $R^{a(v_0),c} f$ is in $\mathcal{C}^{2+}_{\mathfrak{s}}$. For the second term note that $(1+\partial_t+\partial_x^2)\big(c-\mathcal{L}^{a(v_0)}\big)^{-1} f \in \mathcal{C}^{0+}_{\mathfrak{s}}$. Since the inverse operator $\big(\partial_t - L^{a(v_0)} + c\big)^{-1}$ maps $\mathcal{C}^{0+}_{\mathfrak{s}}$ into $\mathcal{C}^{2+}_{\mathfrak{s}}$ by the Schauder estimate – see e.g. Theorem 5 of [13, Chapter 3], we see that the second term in the definitino of $R^{a(v_0),c} f$ is an element of $\mathcal{C}^{2+}_{\mathfrak{s}}$. ▷

We fix from now on a constant $c > \max\{c_0, C_0\}$ and omit the letter 'c' in $Q^{a(v_0)}$, $\mathcal{Q}^{a(v_0)}$, $K^{a(v_0)}$, $R^{a(v_0)}$ unless it needs to be emphasized.

2.2 – The regularity structure. We construct in this section the regularity structure associated with equation (1.3). It will be convenient, for notational purposes, to rewrite (1.3) under the form

$$\begin{aligned} \boldsymbol{u} &= \mathsf{P}_{<2}\big(Q_t^{a(v_0)} u_0\big) \\ &\quad + \mathsf{K}^{a(v_0),\mathsf{M}}\Big(\mathsf{Q}_{\leq 0}\Big\{F(\boldsymbol{u})\,\zeta_1 + \big\{G(\boldsymbol{u})(\boldsymbol{D}\boldsymbol{u})^2 + c\boldsymbol{u}\big\}\zeta_2 + \big\{A(\boldsymbol{u}) - A(\mathsf{P}_{<2}(v_0))\big\}(\boldsymbol{D}^2\boldsymbol{u})\,\zeta_3\Big\}\Big) \end{aligned} \tag{2.4}$$

with three 'noise' symbols $\zeta_1, \zeta_2, \zeta_3$ in the regularity structure. This will help us distinguish three different types of terms.

We first define a 'preparatory' collection of rooted decorated trees

$$\overline{\mathbb{B}} = \overline{\mathbb{B}}_\bullet \cup \overline{\mathbb{B}}_\circ$$

with node decorations $\{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^2}$ and $\{\zeta_l\}_{l\in\{1,2,3\}}$ and edge decorations $(\mathcal{I}_{\mathbf{n}})_{\mathbf{n}\in\mathbf{N}^2}$. Write $\mathcal{I} := \mathcal{I}_{\mathbf{0}}$ and define $\overline{\mathbb{B}}_\bullet$ and $\overline{\mathbb{B}}_\circ$ as the smallest sets such that

(a) $\overline{\mathbb{B}}_\bullet = \overline{\mathbb{B}}^1_\bullet \cup \overline{\mathbb{B}}^2_\bullet \cup \overline{\mathbb{B}}^3_\bullet$ with

$$\overline{\mathbb{B}}^1_\bullet := \Big\{X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}(\tau_i)\ ;\ \mathbf{k} \in \mathbf{N}^2,\ n \in \mathbf{N},\ \tau_1, \dots, \tau_n \in \overline{\mathbb{B}}_\circ\Big\},$$

$$\overline{\mathbb{B}}^2_\bullet := \Big\{X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i)\ ;\ \mathbf{k} \in \mathbf{N}^2, n \in \mathbf{N},\ \tau_1, \dots, \tau_n \in \overline{\mathbb{B}}_\circ,\ \mathbf{n}_i = 0 \text{ except at most two } \mathbf{n}_i = (0,1)\Big\},$$

$$\overline{\mathbb{B}}^3_\bullet := \Big\{X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i)\ ;\ \mathbf{k} \in \mathbf{N}^2, n \in \mathbf{N},\ \tau_1, \dots, \tau_n \in \overline{\mathbb{B}}_\circ,\ \mathbf{n}_i = 0 \text{ except at most one } \mathbf{n}_i = (0,2)\Big\},$$

This definition ensures in particular that $X^{\mathbf{k}} \in \overline{\mathbb{B}}_\bullet$. We assume that the product (called *tree product*) between $\mathcal{I}_{\mathbf{n}_i}(\tau_i)$ is commutative, which means that we consider non-planar trees.

(b) $\overline{\mathbb{B}}_\circ = \overline{\mathbb{B}}^1_\circ \cup \overline{\mathbb{B}}^2_\circ \cup \overline{\mathbb{B}}^3_\circ$ with

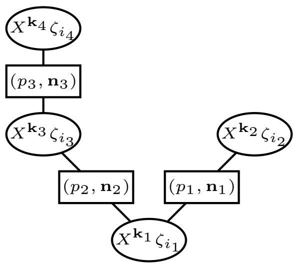
$$\overline{\mathbb{B}}^l_\circ := \big\{\zeta_l\,\sigma\ ;\ \sigma \in \overline{\mathbb{B}}^l_\bullet\big\}, \qquad l \in \{1,2,3\}.$$

The set $\overline{\mathbb{B}}_\bullet$ contains all trees necessary to describe $F(\boldsymbol{u})$, $G(\boldsymbol{u})(\boldsymbol{D}\boldsymbol{u})^2 + c\boldsymbol{u}$, and $\{A(\boldsymbol{u}) - A(\mathsf{P}_{<2}(v_0))\}(\boldsymbol{D}^2\boldsymbol{u})$ in (2.4). The set $\overline{\mathbb{B}}_\circ$ is a collection of trees in $\overline{\mathbb{B}}_\bullet$ multiplied by noise symbols ζ_l. As usual in a regularity structure setting we think of basis elements in $\overline{\mathbb{B}}$ as decorated trees. We define the homogeneity map $|\cdot| : \overline{\mathbb{B}} \to \mathbf{R}$ setting

$$|X^{\mathbf{k}}| := |\mathbf{k}|_{\mathfrak{s}}, \quad |\zeta_1| := \alpha - 2, \quad |\zeta_2| = |\zeta_3| := 0,$$
$$|\mathcal{I}_{\mathbf{n}}(\tau)| := |\tau| + 2 - |\mathbf{n}|_{\mathfrak{s}}, \quad |\tau_1 \cdots \tau_n| := \sum_{i=1}^n |\tau_i|.$$

Since the operator $\mathcal{I}_{(0,2)}$ does not change the homogeneity an infinite number of trees in $\overline{\mathbb{B}}$ have the same homogeneity. Modelled distributions we will treat will then involve infinite linear combinations of trees. To deal with such infinite sums it will be convenient to introduce a new set of symbols $\mathcal{I}^p_{\mathbf{n}}$, for $p \in \mathbf{N}$ and $\mathbf{n} \in \mathbf{N}^2$.

Let $\widehat{\mathbb{B}}$ the collection of ***rooted decorated trees*** with node decorations $\{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^2}$ and $\{\zeta_l\}_{l\in\{1,2,3\}}$ and the edge decorations $(\mathcal{I}^p_{\mathbf{n}})_{p\in\mathbf{N},\,\mathbf{n}\in\mathbf{N}^2}$. For example,



Define inductively the projection map

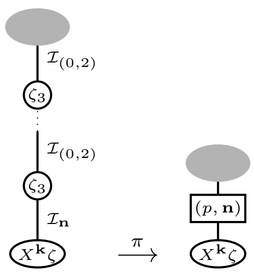
$$\pi : \overline{\mathbb{B}} \to \widehat{\mathbb{B}}$$

by the identity

$$\pi\Big(\mathcal{I}_{\mathbf{n}}(\zeta_3\mathcal{I}_{(0,2)})^p(\tau)\Big) = \mathcal{I}^p_{\mathbf{n}}\big(\pi(\tau)\big),$$

at each branches of the tree τ, for $\tau \in \overline{\mathbb{B}} \setminus \zeta_3\mathcal{I}_{(0,2)}(\overline{\mathbb{B}}_\circ)$. Define $\mathbb{B} := \pi(\overline{\mathbb{B}})$ as the image of π.

The letter $\mathbb{B}$ is chosen for 'basis'. Each element of $\mathbb{B}$ is then a rooted decorated tree with a further edge decoration $\boldsymbol{p} : E_\tau \to \mathbf{N}$, in addition to usual two decorations $\mathfrak{n} : N_\tau \to \mathbf{N}^2$ and $\mathfrak{e} : E_\tau \to \mathbf{N}^2$ considered in Bruned, Hairer & Zambotti's work [8]. The decoration $\boldsymbol{p}$ represents the number of consecutive operators $\zeta_3\mathcal{I}_{(0,2)}$.



We write $\tau^{\boldsymbol{p}}$ for a generic element of $\mathbb{B}$ when we have to emphasize its $\boldsymbol{p}$ decoration. Since $|\tau^{\boldsymbol{p}}| = |\tau^{\mathbf{0}}|$, an infinite number of trees in $\mathbb{B}$ have the same homogeneity. We will use the quantity

$$|\boldsymbol{p}| := \sum_{e\in E_\tau} \boldsymbol{p}(e)$$

to define the topology on the linear space spanned by $\mathbb{B}$. Set

$$\mathbb{B}_\circ := \pi(\overline{\mathbb{B}}_\circ).$$

The following subfamilies of elements of $\mathbb{B}$ will be useful in this and next section.

$$\begin{aligned}
&\mathbb{B}_\circ^- := \big\{\tau^{\boldsymbol{p}} \in \mathbb{B}_\circ\,;\, |\tau^{\boldsymbol{p}}| < 0\big\},\\
&\mathbb{B}^0 := \big\{\tau^{\mathbf{0}}\,;\, \tau^{\boldsymbol{p}} \in \mathbb{B}\big\},\\
&\mathbb{B}_\circ^{-0} := \mathbb{B}_\circ^- \cap \mathbb{B}^0,\\
&\mathbb{U} := \{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^2} \cup \{\mathcal{I}(\tau^{\boldsymbol{p}})\}_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ}.
\end{aligned}$$

The set $\mathbb{B}_\circ^{-0}$ is the index set in formula (1.5) for the counterterm in the renormalized equation. We denote by

$$\mathbb{B}_\beta := \Big\{\tau^{\boldsymbol{p}} \in \mathbb{B}\ ;\ |\tau^{\boldsymbol{p}}| = \beta\Big\}$$

the set of elements of $\mathbb{B}$ of homogeneity β. It is elementary to see that the set $A := \big\{|\tau^{\boldsymbol{p}}|\ ;\ \tau^{\boldsymbol{p}} \in \mathbb{B}\big\}$ is locally finite and $\min A = \alpha - 2$. Moreover the set $\mathbb{B}_\beta \cap \mathbb{B}^0$ is finite for each $\beta \in A$.

To complete the construction of a regularity structure we consider the collection $\mathbb{B}^+$ of all the elements

$$X_+^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}^{+,q_i}(\tau_i^{\boldsymbol{p}_i})$$

with $\mathbf{k} \in \mathbf{N}^2$, $n \in \mathbf{N}$, $\tau_i^{\boldsymbol{p}_i} \in \mathbb{B}_\circ$, $q_i \in \mathbf{N}$, and $\mathbf{n}_i \in \mathbf{N}^2$ such that $|\tau_i| + 2 - |\mathbf{n}_i| > 0$ for each i. The label '+' is to distinguish elements of $\mathbb{B}^+$ with those of $\mathbb{B}_\bullet$. We define the homogeneity map $|\cdot| : \mathbb{B}^+ \to \mathbf{R}_+$ by setting

$$\Big|X_+^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}^{+,q_i}(\tau_i^{\boldsymbol{p}_i})\Big| := |\mathbf{k}|_{\mathfrak{s}} + \sum_{i=1}^n \big(|\tau_i| + 2 - |\mathbf{n}_i|_{\mathfrak{s}}\big).$$

Pick a positive parameter m. For each $\beta \in A$ define $T_\beta^{(m)}$ as the completion of the linear space spanned by $\mathbb{B}_\beta$ under the norm defined by

$$\Big\| \sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\beta} c_{\tau^{\boldsymbol{p}}}\tau^{\boldsymbol{p}} \Big\|_{\beta,m}^2 := \sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\beta} |c_{\tau^{\boldsymbol{p}}}|^2\, m^{2|\boldsymbol{p}|}.$$

We define

$$T^{(m)} := \bigoplus_{\beta\in A} T_\beta^{(m)}$$

as the algebraic sum. Similarly, we define the space

$$T^{(m),+} := \bigoplus_{\beta\ge 0} T_\beta^{(m),+}$$

from the set $\mathbb{B}^+$, using the same notation $\|\cdot\|_{\beta,m}$ for the norms on $T^{(m)}$ and $T^{(m),+}$. By definition $T^{(m),+}$ is an algebra.

We define the two continuous linear operators

$$\Delta : T^{(m)} \to T^{(m)} \otimes T^{(m),+}$$

and

$$\Delta^+ : T^{(m),+} \to T^{(m),+} \otimes T^{(m),+}$$

by the identities

$$\begin{aligned}
\Delta\zeta_l &= \zeta_l \otimes X_+^{\mathbf{0}},\\
\Delta^{(+)} X_{(+)}^{\mathbf{k}} &= \sum_{\mathbf{k}'\le\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} X_{(+)}^{\mathbf{k}'} \otimes X_+^{\mathbf{k}-\mathbf{k}'},\\
\Delta^{(+)}\mathcal{I}_{\mathbf{n}}^{(+)}\tau &= \big(\mathcal{I}_{\mathbf{n}}^{(+)} \otimes \mathrm{Id}\big)\Delta\tau + \sum_{|\mathbf{k}|_{\mathfrak{s}} < |\tau| + 2 - |\mathbf{n}|_{\mathfrak{s}}} \frac{X_{(+)}^{\mathbf{k}}}{\mathbf{k}!} \otimes \mathcal{I}_{\mathbf{n}+\mathbf{k}}^+\tau
\end{aligned}$$

and the multiplicativity $\Delta^{(+)}(\tau_1\cdots\tau_n)=\prod_{i=1}^n\Delta^{(+)}\tau_i$. If $\tau\in\mathbb{B}_\circ^-$ one has similar identities for the operators $\mathcal{I}_{\mathbf{n}}^{(+),p}$

$$\Delta^{(+)}\mathcal{I}_{\mathbf{n}}^{(+),p}\tau=\big(\mathcal{I}_{\mathbf{n}}^{(+),p}\otimes\mathrm{Id}\big)\Delta\tau+\sum_{\mathbf{k}}\frac{X_{(+)}^{\mathbf{k}}}{\mathbf{k}!}\otimes\mathcal{I}_{\mathbf{n}+\mathbf{k}}^{+,p}\tau$$

since $\Delta\mathcal{I}_{(0,2)}\tau=(\mathcal{I}_{(0,2)}\otimes\mathrm{Id})\Delta\tau$ for τ with negative homogeneity. This definition of $\Delta^{(+)}$ turns it into an extension of the BHZ regularity structure for the semilinear generalized (KPZ) equation. The pair

$$\mathscr{T}^{(m)}:=\big((T^{(m)},\Delta),(T^{(m),+},\Delta^+)\big)$$

is a concrete regularity structure in the sense of [4]. Denote by $G^{(m),+}$ the set of all continuous algebra maps $g:T^{(m),+}\to\mathbf{R}$, that is, g is multiplicative with respect to the tree product and with respect to the product with polynomials. Then $G^{(m),+}$ is a topological group with respect to the convolution product $g*h:=(g\otimes h)\Delta^+$.

2.3 – Models and modelled distributions.

In what follows, we denote by $\mathsf{Q}_{<\gamma}$ the canonical projection from $T^{(m)}$ to the subspace $T_{<\gamma}^{(m)}:=\bigoplus_{\beta<\gamma}T_\beta^{(m)}$.

5 – *Definition. Given a positive parameter m a pair $\mathsf{M}=(\mathsf{g},\Pi)$ made up of a map $\mathsf{g}:\mathbf{R}^d\to G^{(m),+}$ and a linear map $\Pi:T^{(m)}\to\mathcal{C}_{\mathfrak{s}}^{-2}(a(v_0))$ is called* ***a model on*** $\mathscr{T}^{(m)}$ *if one has*

$$\big|\mathsf{g}_{z'z}(\tau^{\boldsymbol{p}})\big|\lesssim m^{|\boldsymbol{p}|}\|z'-z\|_{\mathfrak{s}}^{|\tau|}\qquad(\mathsf{g}_{z'z}:=\mathsf{g}_{z'}*\mathsf{g}_z^{-1}),$$

for all $\tau^{\boldsymbol{p}}\in\mathbb{B}^+$ and $z,z'\in\mathbf{R}^2$, and

$$\big|\mathcal{Q}_\theta^{a(v_0)}\big(\Pi_z^{\mathsf{g}}\sigma^{\boldsymbol{p}}\big)(z)\big|\lesssim m^{|\boldsymbol{p}|}\,\theta^{|\sigma|/4}\qquad(\Pi_z^{\mathsf{g}}:=(\Pi\otimes\mathsf{g}_z^{-1})\Delta),$$

for all $\sigma^{\boldsymbol{p}}\in\mathbb{B}$, $z\in\mathbf{R}^2$ and $\theta\in(0,1]$. The model M is said to be ***spatially periodic*** *if*

$$\mathsf{g}_{(z'+(0,1))\,(z+(0,1))}=\mathsf{g}_{z'z},\qquad\mathcal{Q}_\theta^{a(v_0)}\big(\Pi_{z+(0,1)}^{\mathsf{g}}(\cdot)\big)(z+(0,1))=\mathcal{Q}_\theta^{a(v_0)}\big(\Pi_z^{\mathsf{g}}(\cdot)\big)(z)$$

for any $z,z'\in\mathbf{R}^2$.

These conditions ensure that $\mathsf{g}_{z'z}$ and Π_z^{g} are continuous on the metric spaces $T_\beta^{(m)}$ and $T_\beta^{+,(m)}$ respectively under the norm $\|\cdot\|_{\beta,m}$, so the same analytical arguments as in [4] work to prove the results stated in this section. We record here for later use a straightforward adaptation of Proposition 2 and Lemma 12 in [4]. Recall from (2.3) the decomposition $\big(\partial_t-L^{a(v_0)}+c\big)^{-1}=K^{a(v_0)}+R^{a(v_0)}$.

6 – *Lemma. For any $\sigma^{\boldsymbol{p}}\in\mathbb{B}$, $z\in\mathbf{R}^2$, $\theta\in(0,1]$, and $\mathbf{k}\in\mathbf{N}^2$ such that $|\mathbf{k}|_{\mathfrak{s}}\le 4$,*

$$\Big|\big(\partial_z^{\mathbf{k}}\mathcal{Q}_\theta^{a(v_0)}\big)\big(\Pi_z^{\mathsf{g}}\sigma^{\boldsymbol{p}}\big)(z)\Big|\lesssim m^{|\boldsymbol{p}|}\,\theta^{(|\sigma|-|\mathbf{k}|_{\mathfrak{s}})/4}.$$

For any $\sigma^{\boldsymbol{p}}\in\mathbb{B}$ and $\mathbf{k}\in\mathbf{N}^2$ such that $|\mathbf{k}|_{\mathfrak{s}}<|\sigma|\wedge 0+2$,

$$\big(\partial_z^{\mathbf{k}}K^{a(v_0)}\big)\big(\Pi_z^{\mathsf{g}}\sigma^{\boldsymbol{p}}\big)(z)=\int_0^1\big(\partial_z^{\mathbf{k}}K_\theta^{a(v_0)}\big)\big(\Pi_z^{\mathsf{g}}\sigma^{\boldsymbol{p}}\big)(z)\,d\theta$$

converges for all $z\in\mathbf{R}^2$ and satisfies

$$\Big|\big(\partial_z^{\mathbf{k}}K^{a(v_0)}\big)\big(\Pi_z^{\mathsf{g}}\sigma^{\boldsymbol{p}}\big)(z)\Big|\lesssim m^{|\boldsymbol{p}|}.$$

7 – *Definition. Pick $-2<\eta\leq\gamma$. We denote by $\mathcal{D}^{\gamma,\eta}_m=\mathcal{D}^{\gamma,\eta}(T^{(m)};\mathsf{g})$ the set of functions $\boldsymbol{u}:\mathbf{R}^2\to T_{<\gamma}$ such that*

$$(\!(\boldsymbol{u})\!)_{\mathcal{D}^{\gamma,\eta}_m}:=\max_{\beta<\gamma}\sup_{s>0}\Big\{s^{\left(\frac{\beta-\eta}{2}\vee 0\right)}\sup_{|t|\geq s}\|\boldsymbol{u}(z)\|_{\beta,m}\Big\}<\infty,$$

$$\|\boldsymbol{u}\|_{\mathcal{D}^{\gamma,\eta}_m}:=\max_{\beta<\gamma}\sup_{s>0}\left\{s^{\frac{\gamma-\eta}{2}}\sup_{|t|,|t'|\geq s}\frac{\big\|\boldsymbol{u}(z')-\widehat{\mathsf{g}_{z'z}}\boldsymbol{u}(z)\big\|_{\beta,m}}{\|z'-z\|_{\mathfrak{s}}^{\gamma-\beta}}\right\}<\infty,$$

where t and t' represent the time variable part of z and z' respectively. Equipped with the norm

$$|\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_m}:=(\!(\boldsymbol{u})\!)_{\mathcal{D}^{\gamma,\eta}_m}+\|\boldsymbol{u}\|_{\mathcal{D}^{\gamma,\eta}_m},$$

the space $\mathcal{D}^{\gamma,\eta}_m$ is a Banach space. Moreover $\boldsymbol{u}$ is said to be spatially periodic if

$$\boldsymbol{u}(z+(0,1))=\boldsymbol{u}(z)$$

for any $z\in\mathbf{R}^2$.

Since g and $\mathsf{\Pi}$ are bounded linear operators on the spaces $T^{(m)}_\beta$ and $T^{+,(m)}_\beta$ respectively we can prove the reconstruction theorem for $\mathcal{D}^{\gamma,\eta}_m$ similarly to [4, Theorem 20].

8 – *Theorem. Let $-2<\eta\leq\gamma$ and $\gamma>0$. Let M be a model on $\mathscr{T}$ of growth factor $m>0$. There exists a unique continuous linear operator*

$$\mathsf{R}^{\mathsf{M}}:\mathcal{D}^{\gamma,\eta}(T^{(m)};\mathsf{g})\to\mathcal{C}^{\eta\wedge(\alpha-2)}_{\mathfrak{s}}(a(v_0))$$

such that the bound

$$\Big|\mathcal{Q}_\theta\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{v}-\mathsf{\Pi}^{\mathsf{g}}_z\boldsymbol{v}(z)\big)(z)\Big|\lesssim\big(|t|\vee\theta^{1/4}\big)^{\eta\wedge(\alpha-2)-\gamma}\,\theta^{\gamma/4}$$

holds uniformly over for any $\boldsymbol{v}\in\mathcal{D}^{\gamma,\eta}_m$ with unit norm and $z=(t,x)\in\mathbf{R}^2$. Moreover, if M and $\boldsymbol{v}$ are spatially periodic, then $\mathsf{R}^{\mathsf{M}}\boldsymbol{v}$ is also a spatially periodic distribution.

We say that a vector space $S=\bigoplus_{\beta\in A}S_\beta$ is a **sector** if each vector space S_β is a closed subspace of $T^{(m)}_\beta$ and $\Delta(S)\subset S\otimes T^{(m),+}$. Then

$$\beta_0:=\min\big\{\beta\in A\,;\,S_\beta\neq\{0\}\big\}$$

is called a regularity of S. We denote by $\mathcal{D}^{\gamma,\eta}_m(S)$ the set of the elements $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}_m$ taking values in a sector S. In particular we use the sectors

$$U\text{ and }T_\circ$$

spanned by $\mathbb{U}$ and $\mathbb{B}_\circ$, respectively. Since the minimum of the set $\big\{|\tau^{\boldsymbol{p}}|\,;\,\tau^{\boldsymbol{p}}\in\mathbb{U}\setminus\{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^2}\big\}$ is $\alpha>0$, for $-2<\eta\leq\gamma$ and $\gamma>0$, for any $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}_m(U)$, the reconstruction $\mathsf{R}^{\mathsf{M}}\boldsymbol{u}$ coincides with the $X^{\mathbf{0}}$-component of $\boldsymbol{u}$ and belongs to $\mathcal{C}^\alpha_{\mathfrak{s}}$ on any compact set of $(0,\infty)\times\mathbf{R}$.

The proper notion of admissible model in the present setting is captured by the following definition.

9 – *Definition. An **admissible model on $\mathscr{T}^{(m)}$** is a model $(\mathsf{g},\mathsf{\Pi})$ such that*

$$\mathsf{g}_z(X^{\mathbf{k}}_+)=z^{\mathbf{k}},\qquad\mathsf{\Pi}(X^{\mathbf{k}}\tau)(z)=z^{\mathbf{k}}(\mathsf{\Pi}\tau)(z),$$

and one has for all $\tau^{\boldsymbol{p}}\in\mathbb{B}^-_\circ$,

$$\mathsf{\Pi}(\mathcal{I}\tau^{\boldsymbol{p}})=K^{a(v_0)}(\mathsf{\Pi}\tau^{\boldsymbol{p}}).$$

An admissible model satisfies the identity

$$\mathsf{g}_z^{-1}\big(\mathcal{I}^+_{\mathbf{n}}\tau\big)=-\sum_{|\mathbf{k}|_{\mathfrak{s}}<|\tau|+2-|\mathbf{n}|_{\mathfrak{s}}}\frac{(-z)^{\mathbf{k}}}{\mathbf{k}!}\Big(\big(\partial_z^{\mathbf{n}+\mathbf{k}}K^{a(v_0)}\big)\big(\mathsf{\Pi}_z\tau\big)\Big)(z)$$

for any $\tau\in\mathbb{B}^-_\circ$ – see e.g. Proposition 15 of [4]. The proof of the multi-level Schauder estimates can be done along the same lines as in Hairer's original statement, Theorem *5.12* of [17] – see also the short proof given in Theorem 17 of [4]. The fact that the quantity $\mathcal{J}^{a(v_0)}(z)\tau^{\boldsymbol{p}}$ below is well-defined is a consequence of Lemma *6*. (We stated it explicitly to make that point clear.)

10 – Theorem. Let M *stand for an admissible model on* $\mathscr{T}^{(m)}$*. For any* $\tau^{\boldsymbol{p}} \in \mathbb{B}_\circ$ *set*

$$\mathcal{J}^{a(v_0)}(z)\tau^{\boldsymbol{p}} := \sum_{|\mathbf{k}|_{\mathfrak{s}}<|\tau^{\boldsymbol{p}}|\wedge 0+2} \frac{X^{\mathbf{k}}}{\mathbf{k}!}\,\partial_z^{\mathbf{k}} K^{a(v_0)}(\Pi_z^{\mathsf{g}}\tau^{\boldsymbol{p}})(z).$$

For $\boldsymbol{v} \in \mathcal{D}_m^{\gamma,\eta}(T_\circ,\mathsf{g})$ *with* $\gamma > 0$*, set*

$$\big(\mathcal{N}^{a(v_0)}\boldsymbol{u}\big)(z) := \sum_{|\mathbf{k}|_{\mathfrak{s}}<\gamma+2} \frac{X^{\mathbf{k}}}{\mathbf{k}!}\,\partial_z^{\mathbf{k}} K^{a(v_0)}\Big(\mathsf{R}^{\mathsf{M}}\boldsymbol{u} - \Pi_z^{\mathsf{g}}\boldsymbol{u}(z)\Big)(z).$$

For $\boldsymbol{v} \in \mathcal{D}_m^{\gamma,\eta}(T_\circ,\mathsf{g})$ *with* $\gamma > 0$*, set*

$$\big(\mathcal{K}^{a(v_0),\mathsf{M}}\boldsymbol{u}\big)(z) := \mathsf{Q}_{<2}\Big\{\big(\mathcal{I} + \mathcal{J}^{a(v_0)}(z)\big)\boldsymbol{u}(z) + \big(\mathcal{N}^{a(v_0)}\boldsymbol{u}\big)(z)\Big\}.$$

If $-2 < \eta$*, the map* $\mathcal{K}^{a(v_0),\mathsf{M}}$ *sends continuously* $\mathcal{D}_m^{\gamma,\eta}(T_\circ)$ *into* $\mathcal{D}_m^{2,(\eta+2)\wedge\alpha}(U)$*.*

Define $\mathcal{D}_m^{\gamma,\eta}(0,t_0)$ as the space of modelled distributions defined on $(0,t_0)\times\mathbf{T}$. It is defined as in Definition *7* with functions $\boldsymbol{u}$ only defined on $(0,t_0)\times\mathbf{T}$. Recall that we denote by $\mathsf{P}_{<2}$ the operator that lifts a smooth function on $(0,\infty)\times\mathbf{T}$ into the polynomial part of T of homogeneity strictly smaller than 2, so

$$\big(\mathsf{P}_{<2}f\big)(z) = \sum_{|\mathbf{k}|_{\mathfrak{s}}<2} \big(\partial_z^{\mathbf{k}} f\big)(z)\,\frac{X^{\mathbf{k}}}{\mathbf{k}!}.$$

Define

$$\mathcal{R}^{a(v_0),\mathsf{M}}\boldsymbol{u} := \mathsf{P}_{<2}\big(R^{a(v_0)}(\mathsf{R}^{\mathsf{M}}\boldsymbol{u})\big)$$

and

$$\mathsf{K}^{a(v_0),\mathsf{M}} := \mathcal{K}^{a(v_0),\mathsf{M}} + \mathcal{R}^{a(v_0),\mathsf{M}}.$$

11 – Theorem. Pick $\gamma > 0$ *and* $-2 < \eta$*. Then for any* $\kappa > 0$ *we have*

$$\big\|\!\big|\mathsf{K}^{a(v_0),\mathsf{M}}(\boldsymbol{u})\big|\!\big\|_{\mathcal{D}_m^{2,(\eta+2)\wedge\alpha-\kappa}(0,t_0)} \lesssim t_0^{\kappa/2}\,\|\!|\boldsymbol{u}|\!\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}.$$

We end this section by mentioning continuity results of some operations on modelled distributions. Below, the product $\tau\sigma$ between elements τ,σ of T is defined by the linear extension of tree product, as long as it belongs to T. The following results are variants of [17, Propositions *6.12*, *6.13*, *6.15* and *6.16*] so we omit the proofs here.

- Let S_1 and S_2 are sectors of regularities α_1 and α_2 respectively, and such that the product $S_1\times S_2 \to T^{(m)}$ is defined. Then for any $\boldsymbol{u}_i \in \mathcal{D}_m^{\gamma_i,\eta_i}(S_i)$ $(i=1,2)$, we have
$$\mathsf{Q}_{<\gamma}(\boldsymbol{u}_1\cdot\boldsymbol{u}_2) \in \mathcal{D}_m^{\gamma,\eta}$$
with $\gamma = (\gamma_1+\alpha_2)\wedge(\gamma_2+\alpha_1)$ and $\eta \in (\eta_1+\alpha_2)\wedge(\eta_2+\alpha_1)\wedge(\eta_1+\eta_2)$. Moreover, the mapping $(\boldsymbol{u}_1,\boldsymbol{u}_2)\mapsto \mathsf{Q}_{<\gamma}(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)$ is locally Lipschitz continuous.

- For any $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(U)$ and a function $h \in C^\kappa(\mathbf{R})$ with $\kappa \geq \max\{\gamma/\alpha, 1\}$, we define
$$H(\boldsymbol{u}) := \mathsf{Q}_{<\gamma}\left(\sum_{n=0}^{\infty} \frac{h^{(n)}(u_0)}{n!}(\boldsymbol{u} - u_0X^{\mathbf{0}})^n\right),$$
where u_0 is an $X^{\mathbf{0}}$-component of $\boldsymbol{u}$. Then $H(\boldsymbol{u}) \in \mathcal{D}_m^{\gamma,\eta}$, and the mapping $\boldsymbol{u}\mapsto H(\boldsymbol{u})$ is locally Lipschitz continuous.

- Define $\boldsymbol{D}$ as a linear operator on T such that
$$\boldsymbol{D}X^{(k_1,k_2)} := k_2X^{(k_1,k_2-1)}\mathbf{1}_{k_2>0}, \qquad \boldsymbol{D}\,\mathcal{I}_{\mathbf{n}}(\tau) := \mathcal{I}_{\mathbf{n}+(0,1)}(\tau).$$
Let $n\in\{1,2\}$. If $\gamma > n$, then $\mathcal{D}_m^{\gamma,\eta}(U) \ni \boldsymbol{u} \mapsto \boldsymbol{D}^n\boldsymbol{u} \in \mathcal{D}_m^{\gamma-n,\eta-n}$ is continuous.

3 – Local well-posedness

We prove in this section that the regularity structure formulation (1.3) of the quasilinear equation (1.1) is locally well-posed in time. We emphasize some elementary facts before stating and proving the well-posedness result in Theorem *15*.

12 – Lemma. Let $\mu \in [0,1]$ and $u_0 \in C^\mu(\mathbf{T}), v_0 \in C^{0+}(\mathbf{T})$. Denote by P_t either $e^{t\partial_x^2}$ or $Q_t^{a(v_0),c}$, with $c > 0$. For any $T > 0$, there exists a constant $C > 0$ depending only on T, and $\inf a$, $\|a\|_{C^1}$, $\|v_0\|_{C^{0+}(\mathbf{T})}$ when $P_t = Q_t^{a(v_0),c}$, such that

$$\|P_t u_0\|_{L^\infty(\mathbf{T})} \le C\|u_0\|_{L^\infty(\mathbf{T})} \tag{3.1}$$

holds for any $0 < t \le T$,

$$\left\|\partial_t^n \partial_x^k P_t u_0\right\|_{L^\infty(\mathbf{T})} \le C t^{(\mu-k-2n)/2}\|u_0\|_{C^\mu(\mathbf{T})} \tag{3.2}$$

holds for any $1 \le k + 2n \le 2$ and $0 < t \le T$, and

$$\left\|(P_t - \mathrm{Id})u_0\right\|_{L^\infty(\mathbf{T})} \le C t^{\mu/2}\|u_0\|_{C^\mu(\mathbf{T})} \tag{3.3}$$

holds for any $0 < t \le T$.

Proof – These are immediate consequences of (2.1). For (3.2) and (3.3), we have to decompose

$$\partial_t^n \partial_x^k P_t u_0(x) = \int_{\mathbf{R}} (\partial_t^n \partial_x^k P_t)(x,y)\big(u_0(y) - u_0(x)\big)\,dy + u_0(x)\int_{\mathbf{R}} \partial_t^n \partial_x^k P_t(x,y)\,dy,$$

and use the μ-Hölder continuity of u_0 for the first term, and the fact that $\int_{\mathbf{R}} P_t(x,y)dy = 1$ when $P_t = e^{t\partial_x^2}$ and $\int_{\mathbf{R}} P_t(x,y)dy = e^{-ct}$ when $P_t = Q_t^{a(v_0),c}$ for the second term. ▷

13 – Lemma. Let $\mu \in [0,1]$ and $u_0 \in C^\mu(\mathbf{T})$. Denote by P_t either of $e^{t\partial_x^2}$ or $Q_t^{a(v_0),c}$ with $c > 0$ and $v_0 \in C^{0+}(\mathbf{T})$. Then

$$\mathsf{P}_{<2}\big(P_t u_0(x)\big) = \sum_{|\mathbf{k}|_{\mathfrak{s}}<2} \frac{\big(\partial_z^{\mathbf{k}} P_t^{a(v_0)} u_0\big)(x)}{\mathbf{k}!} X^{\mathbf{k}} \in \mathcal{D}_m^{2,\mu}(0,\infty).$$

Proof – Write $f(t,x) = P_t u_0(x)$. By (3.1) and (3.2) of Lemma *12*, for any $0 < s < t_0$,

$$\sup_{s\le|t|<t_0} |f(t,x)| \lesssim \|u_0\|_{C^\mu(\mathbf{T})}, \qquad \sup_{s\le|t|<t_0} |\partial_x f(t,x)| \lesssim s^{(\mu-1)/2}\|u_0\|_{C^\mu(\mathbf{T})}.$$

These imply

$$(\!|\, \mathsf{P}_{<2} f \,|\!)_{\mathcal{D}_m^{2,\mu}(0,t_0)} \lesssim \|u_0\|_{C^\mu(\mathbf{T})}.$$

Next we consider $\|\cdot\|_{\mathcal{D}_m^{2,\eta}}$-bounds. Let $z = (t,x)$ and $z' = (t',x')$ be generic elements of $\mathbf{R}^2$. For the $X^{\mathbf{0}}$-component, by (3.2) of Lemma *12*,

$$\begin{aligned}
\sup_{s\le|t|,|t'|<t_0} &\big|f(z') - f(z) - \partial_x f(z)(x'-x)\big| \\
&\lesssim \Big(\sup_{s\le|t|<t_0} |\partial_t f(z)|\Big)\,|t'-t| + \Big(\sup_{s\le|t|<t_0} |\partial_x^2 f(z)|\Big)\,|x'-x|^2 \\
&\lesssim s^{(\mu-2)/2}\|z'-z\|_{\mathfrak{s}}^2 \|u_0\|_{C^\mu(\mathbf{T})}.
\end{aligned}$$

For the estimate of the $X^{(0,1)}$-component $\partial_x f(z') - \partial_x f(z)$, we consider the t-shift and x-shift separately. For the x-shift, by (3.2) of Lemma *12*,

$$\begin{aligned}
\sup_{s\le|t'|<t_0} \big|\partial_x f(t',x') - \partial_x f(t',x)\big| &\lesssim \Big(\sup_{s\le|t|<t_0} |\partial_x^2 f(z)|\Big)\,|x'-x| \\
&\lesssim s^{(\mu-2)/2}\,|x'-x|\,\|u_0\|_{C^\mu(\mathbf{T})}.
\end{aligned}$$

For the t-shift, we assume $t < t'$ without loss of generality. By (3.3) of Lemma *12*,

$$\begin{aligned}\sup_{s\le t<t'<t_0}\frac{\big|\partial_x f(t',x)-\partial_x f(t,x)\big|}{(t'-t)^{1/2}} &= \sup_{s\le t<t'<t_0}\frac{\big|\partial_x P_{t/2}(P_{t'-t}-\mathrm{Id})P_{t/2}u_0(x)\big|}{(t'-t)^{1/2}}\\ &= \sup_{s\le t<t'<t_0} t^{-1/2}\,\frac{\big\|(P_{t'-t}-\mathrm{Id})P_{t/2}u_0\big\|_{L^\infty(\mathbf{T})}}{(t'-t)^{1/2}}\\ &\lesssim \sup_{s\le t<t'<t_0} t^{-1/2}\,\|P_{t/2}u_0\|_{C^1(\mathbf{T})}\\ &\lesssim \sup_{s\le t<t'<t_0} t^{(\mu-2)/2}\,\|u_0\|_{C^\mu(\mathbf{T})}\lesssim s^{(\mu-2)/2}\,\|u_0\|_{C^\mu(\mathbf{T})}.\end{aligned}$$

By the above estimates we have

$$\|\mathsf{P}_{<2}f\|_{\mathcal{D}^{2,\mu}_m(0,t_0)}\lesssim \|u_0\|_{C^\mu(\mathbf{T})}.$$

▷

14 – Lemma. Let $\mu\in[0,1]$ and $t_0>0$. For any $u_0\in C^\mu(\mathbf{T})$, choose $v_0=e^{t_0\partial_x^2}u_0$. Then for any $\eta<\mu$,

$$\big|\!\big|\!\big|\mathsf{P}_{<2}\big\{(Q_t^{a(v_0),c}-e^{t_0\partial_x^2})u_0(x)\big\}\big|\!\big|\!\big|_{\mathcal{D}^{2,\eta}_m(0,t_0)}\lesssim t_0^{(\mu-\eta)/2}\,\|u_0\|_{C^\mu(\mathbf{T})}.$$

Proof – Write $g(t,x)=(Q_t^{a(v_0),c}-e^{t_0\partial_x^2})u_0(x)$. As shown in the proof of Lemma *13*, all terms in Definition *7* – $\|g(z)\|_{\beta,m}$ and $\|g(z')-\widehat{\mathsf{g}_{z'z}}g(z)\|_{\beta,m}/\|z'-z\|_{\mathfrak{s}}^{2-\beta}$ except $\|g(z)\|_{0,m}$ are bounded by $s^{(\mu-\beta)/2}$ in the region $s\le|t|,|t'|<t_0$. Since

$$s^{(\mu-\beta)/2}\le t_0^{(\mu-\eta)/2}s^{(\eta-\beta)/2},$$

we can see that they have the required $|\!|\!|\cdot|\!|\!|_{\mathcal{D}^{2,\eta}_m}$-estimate. It remains to consider $\|g(z)\|_{0,m}$. By (3.3) of Lemma *12*, one has

$$\|g(t)\|_{L^\infty(\mathbf{T})}\le\big\|(Q_t^{a(v_0),c}-\mathrm{Id})u_0\big\|_{L^\infty(\mathbf{T})}+\big\|(e^{t_0\partial_x^2}-\mathrm{Id})u_0\big\|_{L^\infty(\mathbf{T})}\lesssim t_0^{\mu/2}\,\|u\|_{C^\mu(\mathbf{T})},$$

which completes the proof. ▷

15 – Theorem. Let $\mu\in(0,\alpha)$ and $t_0>0$. For any $u_0\in C^\mu(\mathbf{T})$, choose $v_0=e^{t_0\partial_x^2}u_0$. Then for sufficiently small $t_0>0$, equation (2.4) has a unique solution $\boldsymbol{u}$ in the class $\mathcal{D}^{2,\mu}_m(0,t_0)$. The time t_0 can be chosen to be a lower semicontinuous function of M and u_0.

Proof – We find a solution by the Picard iteration. Let $\boldsymbol{v}_0=0$ and

$$\begin{aligned}\boldsymbol{u}_n&=\mathsf{P}_{<2}\big(Q^{a(v_0)}u_0\big)+\mathsf{K}^{a(v_0),\mathsf{M}}(\boldsymbol{v}_n),\\ \boldsymbol{v}_{n+1}&=\mathsf{Q}_{<\alpha}\Big\{F(\boldsymbol{u}_n)\zeta_1+\big(G(\boldsymbol{u}_n)(\boldsymbol{D}\boldsymbol{u}_n)^2+c\boldsymbol{u}\big)\zeta_2+\zeta_3\big\{A(\boldsymbol{u}_n)-A(\mathsf{P}_{<2}(v_0))\big\}\boldsymbol{D}^2\boldsymbol{u}_n\Big\}.\end{aligned}\tag{3.4}$$

We show that the sequence $(\boldsymbol{u}_n,\boldsymbol{v}_n)$ is well defined in the class

$$(\boldsymbol{u}_n,\boldsymbol{v}_n)\in\mathcal{D}^{2,\mu}_m(U)\times\mathcal{D}^{\alpha,2\mu-2}_m(T_\circ).$$

Before that, note that $\boldsymbol{u}_n$ is of the form

$$\boldsymbol{u}_n-\mathcal{I}\boldsymbol{v}_n\in\mathrm{Span}\{X^{\mathbf{k}}\}_{|\mathbf{k}|_{\mathfrak{s}}<2}.$$

Since $\boldsymbol{D}^2$ eliminates the polynomials of order less than 2, $\boldsymbol{D}^2\boldsymbol{u}_n$ is nothing but $\mathcal{I}_{(0,2)}\boldsymbol{v}_n$. Hence it is convenient to rewrite the iteration as

$$\boldsymbol{v}_{n+1}=\mathsf{Q}_{<\alpha}\Big\{F(\boldsymbol{u}_n)\zeta_1+\big(G(\boldsymbol{u}_n)(\boldsymbol{D}\boldsymbol{u}_n)^2+c\boldsymbol{u}\big)\zeta_2+\zeta_3\big\{A(\boldsymbol{u}_n)-A(\mathsf{P}_{<2}(v_0))\big\}\mathcal{I}_{(0,2)}\boldsymbol{v}_n\Big\}.$$

In what follows, C means the constant which is independent to t_0, u_0, and $(\boldsymbol{u}_n,\boldsymbol{v}_n)$, and whose value may change from one occurrence to the other. By the multi-level Schauder estimate from Theorem 11 we have

$$\begin{aligned} \| \boldsymbol{u}_{n+1} \|_{\mathcal{D}_m^{2,\mu}(0,t_0)} &\le \big\| \mathsf{P}_{<2}\big(Q^{a(v_0)} u_0 \big) \big\|_{\mathcal{D}_m^{2,\mu}(0,t_0)} + C t_0^{\kappa/2} \, \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \\ &\le C \Big(\| u_0 \|_{C^\mu(\mathbf{T})} + t_0^{\kappa/2} \, \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \Big), \end{aligned} \tag{3.5}$$

where $\kappa = \mu \wedge (\alpha - \mu) > 0$.

Next we consider $\boldsymbol{v}_{n+1}$. Since $\boldsymbol{u}_n$ takes values in the sector U, all $F(\boldsymbol{u}_n), G(\boldsymbol{u}_n), A(\boldsymbol{u}_n)$ are well-defined elements of $\mathcal{D}_m^{2,\mu}$. Since ζ has a homogeneity $\alpha - 2$,

$$F(\boldsymbol{u}_n)\zeta \in \mathcal{D}_m^{2+\alpha-2,\mu+\alpha-2}(T_\circ) \subset \mathcal{D}_m^{\alpha,2\mu-2}(T_\circ).$$

Since $\boldsymbol{D}\boldsymbol{u}_n \in \mathcal{D}_m^{1,\mu-1}(T, \mathsf{g})$ is in the sector of regularity $\alpha - 1$,

$$(\boldsymbol{D}\boldsymbol{u}_n)^2 \in \mathcal{D}_m^{\alpha,2\mu-2}(T_\circ),$$

and thus

$$G(\boldsymbol{u}_n)(\boldsymbol{D}\boldsymbol{u}_n)^2 \in \mathcal{D}_m^{\alpha,2\mu-2}(T_\circ).$$

Moreover, since $\mathcal{I}_{(0,2)}$ maps $\mathcal{D}_m^{\alpha,2\mu-2}(T_\circ)$ into $\mathcal{D}_m^{\alpha,2\mu-2}$ isometrically, one has

$$\| \boldsymbol{v}_{n+1} \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \le P\big(\| \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)} \big) + C \big\| A(\boldsymbol{u}_n) - A(\mathsf{P}_{<2}(v_0)) \big\|_{\mathcal{D}_m^{2,\eta}(0,t_0)} \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)},$$

where P is a polynomial whose coefficients are independent to t_0 and $(\boldsymbol{u}_n, \boldsymbol{v}_n)$, and η is a positive constant such that $\eta \in [2\mu - \alpha, \mu)$. To obtain a small factor from the second term of the right hand side, we decompose

$$A(\boldsymbol{u}_n) - A\big(\mathsf{P}_{<2}(v_0)\big) = A(\boldsymbol{u}_n) - A\big(\mathsf{P}_{<2}(Q^{a(v_0)} u_0)\big) + A\big(\mathsf{P}_{<2}(Q^{a(v_0)} u_0)\big) - A\big(\mathsf{P}_{<2}(v_0)\big).$$

Since A is locally Lipschitz as a mapping from $\mathcal{D}_m^{2,\eta}$ to itself

$$\begin{aligned} \big\| A(\boldsymbol{u}_n) - A\big(\mathsf{P}_{<2}(Q^{a(v_0)} u_0)\big) \big\|_{\mathcal{D}_m^{2,\eta}(0,t_0)} &\le C \big\| \boldsymbol{u}_n - \mathsf{P}_{<2}(Q^{a(v_0)} u_0) \big\|_{\mathcal{D}_m^{2,\eta}(0,t_0)} \\ &\le C \| \mathsf{K}^{a(v_0),\mathsf{M}} \boldsymbol{v}_n \|_{\mathcal{D}_m^{2,\eta}(0,t_0)} \\ &\le C t_0^{\kappa'/2} \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)}, \end{aligned}$$

where $\kappa' = (2\mu) \wedge \alpha - \eta > 0$, and $C = C\big(\|u_0\|_{C^\mu(\mathbf{T})}, \| \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)} \big)$ is a locally bounded function of $\|u_0\|_{C^\mu(\mathbf{T})}$ and $\| \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)}$. Moreover we have from Lemma *14* the estimate

$$\begin{aligned} \big\| A\big(\mathsf{P}_{<2}(Q^{a(v_0)} u_0)\big) - A\big(\mathsf{P}_{<2}(v_0)\big) \big\|_{\mathcal{D}_m^{2,\eta}(0,t_0)} &\le C \big\| \mathsf{P}_{<2}\big(Q^{a(v_0)} u_0 - v_0 \big) \big\|_{\mathcal{D}_m^{2,\eta}(0,t_0)} \\ &\le C t_0^{(\mu-\eta)/2} \, \|u_0\|_{C^\mu(\mathbf{T})}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \| \boldsymbol{v}_{n+1} \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} &\le P\big(\| \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)} \big) \\ &\quad + C\big(\|u_0\|_{C^\mu(\mathbf{T})}, \| \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)}, \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \big) \, t_0^\delta \, \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \end{aligned} \tag{3.6}$$

for small $\delta > 0$ and a some locally bounded function $C\big(\|u_0\|_{C^\mu(\mathbf{T})}, \| \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)}, \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \big)$.

By (3.5) and (3.6) we can find a small time $t_0 > 0$ and a large constant $M > 0$ such that

$$\| \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)} + \| \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \le M$$

for any $n \in \mathbf{N}$. By the local Lipschitz estimates of the operations in (3.4) (product, composition with smooth function, differentiation, and integration) we have the similar estimate

$$\begin{aligned} &\| \boldsymbol{u}_{n+1} - \boldsymbol{u}_n \|_{\mathcal{D}_m^{2,\mu}(0,t_0)} + \| \boldsymbol{v}_{n+1} - \boldsymbol{v}_n \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \\ &\qquad \le C(M) \, t_0^\delta \Big(\| \boldsymbol{u}_n - \boldsymbol{u}_{n-1} \|_{\mathcal{D}_m^{2,\mu}(0,t_0)} + \| \boldsymbol{v}_n - \boldsymbol{v}_{n-1} \|_{\mathcal{D}_m^{\alpha,2\mu-2}(0,t_0)} \Big) \end{aligned}$$

for some positive constant $C(M)$ and a small exponent $\delta > 0$. Hence we can choose t_0 smaller such that $(\boldsymbol{u}_n, \boldsymbol{v}_n)$ is a Cauchy sequence. The limit solves equation (2.4). Uniqueness also holds because of the local Lipschitz estimates. ▷

Otto, Sauer, Smith & Weber [22] and Linares, Otto & Tempelmayr [19] set up an analytic and an algebraic framework to deal with the quasilinear equation (1.6) with additive forcing. They use in particular a greedy index set for their local expansions and prove a local in time well-posedness result for that equation in the full sub-critical regime. The analysis of the present section shows that one can run this analysis within the variant of the usual regularity structure for the generalized (KPZ) equation described in Section 2. The present section can also be seen as a simple alternative to the somewhat convoluted approach of Gerencsér & Hairer [16].

4 – Renormalization matters

This section is dedicated to the analysis of the equation satisfied by the reconstruction of the solution $\boldsymbol{u}$ obtained in Theorem *15* – the so called renormalized equation. The first systematic treatment of this question in a semilinear setting was done by Bruned, Chandra, Chevyrev & Hairer in [7]. They relied on a morphism property satisfied by the coefficients u_τ of generic solutions to semilinear singular SPDEs, for some multi-pre-Lie structures. A deeper structure on the elements of BHZ regularity structures was unveiled by Bruned & Manchon in [9] and used by Bailleul & Bruned in [2] to simplify a lot the analysis of the renormalized equation. This structure is encoded in the $\star$ product introduced in Section 4.2. Its importance in the analysis of equation (1.3) is emphasized by Proposition *20*; it provides a basic morphism property – the counterpart here of the multi-pre-Lie morphism property used in [7]. We introduce in Section 4.3 the class of preparation maps – special linear maps from $T^{(m)}$ into itself, and their associated admissible models. A preliminary form of Theorem *1* follows from their properties in Proposition *22*. A special class of preparation maps is associated with the set of characters on $\mathbb{B}_\circ^-$. We show in Section 4.4 that working with the preparation map associated with the analogue in our setting of the BHZ character leads to Theorem *1*.

4.1 – Notations. We first fix some notations. In this section, we consider the set of *all* decorated trees

$$\overline{\mathbb{T}} = \overline{\mathbb{T}}_\bullet \cup \overline{\mathbb{T}}_\circ$$

since the operators we define below may not be closed in the smaller set $\mathbb{B}$. The sets $\overline{\mathbb{T}}_\bullet$ and $\overline{\mathbb{T}}_\circ$ are the smallest ones such that

$$\overline{\mathbb{T}}_\bullet := \Big\{ X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i) \ ; \ \mathbf{k} \in \mathbf{N}^2, \ n \in \mathbf{N}, \ \mathbf{n}_i \in \mathbf{N}^2, \ \tau_i \in \overline{\mathbb{T}}_\circ \Big\}$$

and

$$\overline{\mathbb{T}}_\circ := \Big\{ \zeta_l \, \sigma \ ; \ l \in \{1,2,3\}, \ \sigma \in \overline{\mathbb{T}}_\bullet \Big\}.$$

Similarly to Section 2.2 we assume that the tree product is commutative. For convenience we denote a generic element of $\overline{\mathbb{T}}$ by

$$X^{\mathbf{k}} \zeta_l \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i)$$

for $l \in \{1,2,3,4\}$ with the convention

$$\zeta_4 := X^{\mathbf{0}}.$$

The combinatorial symmetry factor $S(\tau)$ of the tree

$$\tau = X^{\mathbf{k}} \zeta_l \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i)^{\beta_i}$$

with $(\mathbf{n}_i, \tau_i) \neq (\mathbf{n}_j, \tau_j)$ for any $i \neq j$ is inductively defined by

$$S(\tau) := \mathbf{k}! \bigg(\prod_{i=1}^n S(\tau_i)^{\beta_i} \beta_i! \bigg).$$

We also define the map π similarly to what was done in Section 2.2 to introduce a further edge decoration $\boldsymbol{p}$ and set

$$\mathbb{T} := \pi(\overline{\mathbb{T}}).$$

The $\boldsymbol{p}$ decoration is used to deal with infinite sums. However it will also be convenient to use the set $\overline{\mathbb{T}}$ to deal with some operators defined similarly as in [7, 1]. The following identity will be useful later.

16 – Lemma. Let $\mathbb{S}$ be a finite set of $\mathbb{T}$ such that $\tau^{\mathbf{0}} \in \mathbb{S}$ if $\tau^{\boldsymbol{p}} \in \mathbb{S}$ and let $\{c_{\tau^{\boldsymbol{p}}}\}_{\tau^{\boldsymbol{p}}\in\mathbb{S}}$ be a family of real numbers. Then one has the identity

$$\sum_{\tau^{\boldsymbol{p}}\in\mathbb{S}} \frac{c_{\tau^{\boldsymbol{p}}}}{S(\tau^{\boldsymbol{p}})}\tau^{\boldsymbol{p}} = \sum_{\tau^{\mathbf{0}}\in\mathbb{S}} \frac{1}{S(\tau^{\mathbf{0}})} \sum_{\boldsymbol{p}\in\mathbf{N}^{E_\tau}} c_{\tau^{\boldsymbol{p}}}\tau^{\boldsymbol{p}}.$$

Note that $S(\tau^{\boldsymbol{p}})$ is smaller than or equal to $S(\tau^{\mathbf{0}})$ in general. The above identity comes from the order of the sums for trees and decorations. In the left hand side, each $\tau^{\boldsymbol{p}}$ is considered as a non-planar tree. In the right hand side, however, we fix a tree τ first and put a decoration $\boldsymbol{p}$ later, so $\tau^{\boldsymbol{p}}$ is rather considered as a planar tree. For example, the tree $\tau^{p,q} := \mathcal{I}^p(\zeta_1)\mathcal{I}^q(\zeta_1)$ is considered as the same as $\tau^{q,p}$ in the set $\mathbb{T}$, we have

$$\sum_{\tau^{p,q}=\mathcal{I}^p(\zeta_1)\mathcal{I}^q(\zeta_1)\in\mathbb{T}} \frac{c_{\tau^{p,q}}}{S(\tau^{p,q})}\tau^{p,q} = \sum_{p\in\mathbf{N}} \frac{c_{\tau^{p,p}}}{2}\tau^{p,p} + \sum_{p<q\in\mathbf{N}} c_{\tau^{p,q}}\tau^{p,q} = \frac{1}{S(\tau^{0,0})}\sum_{p,q\in\mathbf{N}} c_{\tau^{p,q}}\tau^{p,q}.$$

We denote by T the linear space spanned by $\mathbb{T}$ and by T^* its algebraic dual. For a fixed $m > 0$ and any $\tau^{\mathbf{0}} \in \mathbb{T}$ we define $\mathsf{T}_\tau^{(m)}$ as the completion of the linear space spanned by non-planar trees $\{\tau^{\boldsymbol{p}}\}_{\boldsymbol{p}}$ under the norm

$$\Big\|\sum_{\boldsymbol{p}} c_{\boldsymbol{p}}\tau^{\boldsymbol{p}}\Big\|_m^2 := \sum_{\boldsymbol{p}} |c_{\boldsymbol{p}}|^2 m^{2|\boldsymbol{p}|}.$$

We define

$$\mathsf{T}^{(m)} = \bigoplus_{\tau^{\mathbf{0}}\in\mathbb{T}} \mathsf{T}_\tau^{(m)}$$

as the algebraic sum. Setting

$$\big\langle \tau^{\boldsymbol{p}}, (\sigma^{\boldsymbol{q}})^*\big\rangle := S(\tau^{\boldsymbol{p}})\,\mathbf{1}_{\tau^{\boldsymbol{p}}=\sigma^{\boldsymbol{q}}}$$

for $\tau^{\boldsymbol{p}} \in \mathbb{T}$ and the dual element $(\sigma^{\boldsymbol{q}})^*$ of $\sigma^{\boldsymbol{q}} \in \mathbb{T}$, we can extend the duality relation between $\mathsf{T}^{(m)}$ and $\mathsf{T}^{*,(1/m)}$ to the completion of T^* under the norm $\|\cdot\|_{1/m}$.

4.2 – Coherence and morphism property for the $\star$ product.

We write $\bar\tau$ to mean a generic element of $\overline{\mathbb{T}}$. We denote by $\overline{\mathsf{T}}_{(\cdot)}$ the linear space spanned by $\overline{\mathbb{T}}_{(\cdot)}$ with $(\cdot) \in \{\emptyset, \circ, \bullet\}$, and by $\overline{\mathsf{T}}^*_{(\cdot)}$ its algebraic dual.

4.2.1 – Coherence property.

Let $\mathsf{c} = (\mathsf{c}_{\mathbf{k}})_{\mathbf{k}\in\mathbf{N}^2}$ and $\mathsf{c}' = (\mathsf{c}'_{\mathbf{k}})_{\mathbf{k}\in\mathbf{N}^2}$ be abstract variables. We introduce the operators $D'_{\mathbf{n}} := \partial_{\mathsf{c}'_{\mathbf{n}}}$, for $\mathbf{n} \in \mathbf{N}^2$, and set, for $\mathbf{k}_0 \in \big\{(1,0),(0,1)\big\}$ in the canonical basis of $\mathbf{N}^2$,

$$\partial^{\mathbf{k}_0} := \sum_{\mathbf{n}\in\mathbf{N}^2} \Big(\mathsf{c}_{\mathbf{n}+\mathbf{k}_0} D_{\mathbf{n}} + \mathsf{c}'_{\mathbf{n}+\mathbf{k}_0} D'_{\mathbf{n}}\Big).$$

The vector fields $\partial^{(1,0)}$ and $\partial^{(0,1)}$ commute so one defines unambiguously for $\mathbf{k} = (k_1,k_2) \in \mathbf{N}^2$ a $|\mathbf{k}|$-th order differential operator on functions of finitely many components of c and c' setting

$$\partial^{\mathbf{k}} := (\partial^{(1,0)})^{k_1}(\partial^{(0,1)})^{k_2}.$$

The following elementary relation is of crucial use in the proof of Proposition *20* below; its elementary proof is left to the reader.

17 – Lemma. For any $(\mathbf{k}_1,\dots,\mathbf{k}_n)\in(\mathbf{N}^2)^n$ *and* $\mathbf{m}\in\mathbf{N}^2$ *one has*

$$\sum_{\substack{(\mathbf{l}_1,\dots,\mathbf{l}_n)\in(\mathbf{N}^2)^n,\\ \mathbf{l}_1+\cdots+\mathbf{l}_n\le\mathbf{m}}}\binom{\mathbf{m}}{\mathbf{l}_1,\dots,\mathbf{l}_n}\prod_{j=1}^n\partial^{\mathbf{m}-\mathbf{l}_j}D_{\mathbf{k}_j-\mathbf{l}_j}=\Big(\prod_{j=1}^n D_{\mathbf{k}_j}\Big)\partial^{\mathbf{m}},\tag{4.1}$$

where

$$\binom{\mathbf{m}}{\mathbf{l}_1,\dots,\mathbf{l}_n}:=\frac{\mathbf{m}!}{\mathbf{l}_1!\cdots\mathbf{l}_n!}.$$

For any $\bar\tau\in\overline{\mathbb{T}}$ we define the function $\mathfrak{F}^a(\bar\tau^*)$ of the variables (c,c') as follows. Set

$$h(\mathsf{c}_{\mathbf{0}},\mathsf{c}'_{\mathbf{0}}):=a(\mathsf{c}_{\mathbf{0}})-a(\mathsf{c}'_{\mathbf{0}})$$

and

$$\begin{aligned}\mathfrak{F}^a(\zeta_1^*)(\mathsf{c},\mathsf{c}')&:=f(\mathsf{c}_{\mathbf{0}}),\\ \mathfrak{F}^a(\zeta_2^*)(\mathsf{c},\mathsf{c}')&:=g(\mathsf{c}_{\mathbf{0}})\,\mathsf{c}_{(0,1)}^2+c\,\mathsf{c}_{\mathbf{0}},\\ \mathfrak{F}^a(\zeta_3^*)(\mathsf{c},\mathsf{c}')&:=h(\mathsf{c}_{\mathbf{0}},\mathsf{c}'_{\mathbf{0}})\,\mathsf{c}_{(0,2)},\\ \mathfrak{F}^a(\zeta_4^*)(\mathsf{c},\mathsf{c}')&:=0,\end{aligned}\tag{4.2}$$

and for $\bar\tau=X^{\mathbf{k}}\zeta_l\prod_{i=1}^n\mathcal{I}_{\mathbf{n}_i}(\bar\tau_i)\in\overline{\mathbb{T}}$ set

$$\mathfrak{F}^a(\bar\tau^*)(\mathsf{c},\mathsf{c}'):=\bigg(\Big\{\partial^{\mathbf{k}}D_{\mathbf{k}_1}\dots D_{\mathbf{k}_n}\mathfrak{F}^a(\zeta_l^*)\Big\}\prod_{i=1}^n\mathfrak{F}^a(\bar\tau_i^*)\bigg)(\mathsf{c},\mathsf{c}').\tag{4.3}$$

With

$$\tau_1:=\mathcal{I}(\zeta_1)\mathcal{I}_{(0,1)}(\zeta)^2\zeta_2,\quad \tau_2:=\mathcal{I}(\zeta_1)^2\mathcal{I}_{(0,2)}(\zeta)\zeta_3$$

one has for instance

$$\begin{aligned}\mathfrak{F}^a(\tau_1^*)(\mathsf{c},\mathsf{c}')&=\Big\{D_{\mathbf{0}}D_{(0,1)}^2\mathfrak{F}^a(\zeta_2^*)\Big\}\mathfrak{F}^a(\zeta_1^*)^3=2g'(\mathsf{c}_{\mathbf{0}})f(\mathsf{c}_{\mathbf{0}})^3,\\ \mathfrak{F}^a(\tau_2^*)(\mathsf{c},\mathsf{c}')&=\Big\{D_{\mathbf{0}}^2D_{(0,2)}\mathfrak{F}^a(\zeta_3^*)\Big\}\mathfrak{F}^a(\zeta_1^*)^3=a^{(2)}(\mathsf{c}_{\mathbf{0}})f(\mathsf{c}_{\mathbf{0}})^3.\end{aligned}$$

We see on these definitions that $\mathsf{c}_{\mathbf{0}}$ and $\mathsf{c}'_{\mathbf{0}}$ are placeholders for u and v_0 in equation (1.2). The function $\mathfrak{F}^a$ vanishes outside $\overline{\mathbb{B}}$. Actually, if $\bar\tau\in\overline{\mathbb{T}}\setminus\overline{\mathbb{B}}$ then it has a node $v\in N_\tau$ such that a collection of all edges leaving from v contains either an edge $\mathcal{I}_{\mathbf{k}}$ with $\mathbf{k}\neq(0,0),(0,1),(0,2)$, or more than two edges $\mathcal{I}_{(0,1)}$, or more than one edges $\mathcal{I}_{(0,2)}$, then $\mathfrak{F}^a(\bar\tau^*)$ vanishes at v. By a similar argument, it is easy to check that $\mathfrak{F}^a\big((\tau^{\boldsymbol p})^*\big)(\mathsf{c},\mathsf{c}')$ with $\tau^{\boldsymbol p}\in\mathbb{B}_\circ^-$ are functions of $(\mathsf{c}_{\mathbf{0}},\mathsf{c}_{(0,1)})$ and $(\mathsf{c}'_{\mathbf{0}},\mathsf{c}'_{(0,1)})$ only. Furthermore since

$$\mathfrak{F}^a\big((\tau^{\boldsymbol p})^*\big)=h^{|\boldsymbol p|}\,\mathfrak{F}^a\big((\tau^{\mathbf{0}})^*\big)$$

from the definition, we have that

$$\bigg\|\mathfrak{F}^a\Big(\sum_{\boldsymbol p}c_{\boldsymbol p}(\tau^{\boldsymbol p})^*\Big)\bigg\|_{L^\infty(\mathcal{O}\times\mathbf{R}^2)}\le\big\|\mathfrak{F}^a\big((\tau^{\mathbf{0}})^*\big)\big\|_{L^\infty(\mathcal{O}\times\mathbf{R}^2)}\sum_{\boldsymbol p}\|h\|_{L^\infty(\mathcal{O})}^{|\boldsymbol p|}\,|c_{\boldsymbol p}|$$

for any domain $\mathcal{O}$ of $\mathbf{R}^2$. This means that $\mathfrak{F}^a$ maps $\mathsf{T}^{*,(m')}$ into $C_b(\mathcal{O}\times\mathbf{R}^2)$ if $\|h\|_{L^\infty(\mathcal{O})}<m'$.

18 – Proposition. The solution $\boldsymbol u=\sum u_{\mathbf{k}}X^{\mathbf{k}}+\sum u_{\tau^{\boldsymbol p}}\mathcal{I}(\tau^{\boldsymbol p})\in\mathcal{D}_m^{2,\mu}(0,t_0)$ *to equation* (2.4) *satisfies*

$$u_{\tau^{\boldsymbol p}}=\frac{1}{S(\tau^{\boldsymbol p})}\,\mathfrak{F}^a\big((\tau^{\boldsymbol p})^*\big)\big(u_{\mathbf{0}},u_{(0,1)},v_0,\partial_xv_0\big)\tag{4.4}$$

for all $\tau^{\boldsymbol p}\in\mathbb{B}_\circ^-$.

Proof – Given the definitions of the nonlinearities of $\boldsymbol u$ and $\mathsf{P}_{<2}(v_0)$, identity (4.4) is a direct encoding of the fixed point relation

$$\boldsymbol u\in\mathcal{I}\Big(\mathsf{Q}_{<0}\Big\{F(\boldsymbol u)\,\zeta_1+\big\{G(\boldsymbol u)(\boldsymbol D\boldsymbol u)^2+c\boldsymbol u\big\}\zeta_2+\big\{A(\boldsymbol u)-A(\mathsf{P}_{<2}(v_0))\big\}(\boldsymbol D^2\boldsymbol u)\,\zeta_3\Big\}\Big)+T_X$$

satisfied by $\boldsymbol{u}$, where $T_X := \mathrm{span}\{X^{\mathbf{k}}\}$. ▷

The analogue of identity (4.4) in the usual regularity structure setting was named '*coherence*' in [7].

4.2.2 – Star product. Following [1, Section 2], we introduce bilinear operators on $\overline{\mathsf{T}}$. Let $\uparrow_v^{\mathbf{n}} \bar\tau$ add $\mathbf{n}$ to the polynomial decoration of the vertex v of $\bar\tau$. For $\bar\sigma \in \overline{\mathbb{T}}_\circ$, $\bar\tau \in \overline{\mathbb{T}}$, and $\mathbf{n} \in \mathbf{N}^2$, set

$$\bar\sigma \curvearrowright_{\mathbf{n}} \bar\tau := \sum_{v\in N_\tau} \sum_{\mathbf{m}\in\mathbf{N}^2} \binom{\mathfrak{n}_v}{\mathbf{m}} \bar\sigma \curvearrowright^v_{\mathbf{n}-\mathbf{m}} (\uparrow_v^{-\mathbf{m}} \bar\tau),$$

where $\mathfrak{n}_v$ is the polynomial decoration at the node v, and $\curvearrowright^v_{\mathbf{n}-\mathbf{m}}$ grafts $\bar\sigma$ onto $\bar\tau$ at the node v with an edge of type $\mathcal{I}_{\mathbf{n}-\mathbf{m}}$. One has the following analogue of the Chapoton-Livernet universality result.

19 – Proposition. The space $\overline{\mathsf{T}}_\circ$ is freely generated by the symbols $\big(X^{\mathbf{k}}\zeta_l\big)_{\mathbf{k}\in\mathbf{N}^2, 1\leq l\leq 3}$ and the family of operations $(\curvearrowright_{\mathbf{n}})_{\mathbf{n}\in\mathbf{N}^2}$.

We define the $\star$ product as in [1, Section 2], defining first for $\bar\tau \in \overline{\mathbb{T}}$ and $B \subset N_{\bar\tau}$, the derivation map $\uparrow_B^{\mathbf{k}}$ by

$$\uparrow_B^{\mathbf{k}} \bar\tau = \sum_{\sum_{v\in B}\mathbf{k}_v = \mathbf{k}} \prod_{v\in B} \uparrow_v^{\mathbf{k}_v} \bar\tau,$$

then

$$\mathcal{I}_{\mathbf{n}}(\bar\sigma) \curvearrowright \bar\tau := \bar\sigma \curvearrowright_{\mathbf{n}} \bar\tau,$$

and

$$\Big(\prod_i \mathcal{I}_{\mathbf{n}_i}(\bar\sigma_i)\Big) \curvearrowright \bar\tau$$

by grafting each tree $\bar\sigma_i$ on $\bar\tau$ along the grafting operator corresponding to $\mathbf{n}_i$, independently of the others. Set finally for all $\bar\sigma = X^{\mathbf{k}} \prod \mathcal{I}_{\mathbf{n}_i}(\bar\sigma_i) \in \overline{\mathbb{T}}_\bullet$ and $\tau \in \overline{\mathbb{T}}$

$$\bar\sigma \star \bar\tau := \uparrow_{N_\tau}^{\mathbf{k}} \left(\prod_i \mathcal{I}_{\mathbf{n}_i}(\bar\sigma_i) \curvearrowright \bar\tau\right).$$

One has for instance

$$X^{\mathbf{k}} \zeta_l \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\bar\sigma_i) = \left(X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\bar\sigma_i)\right) \star \zeta_l. \tag{4.5}$$

One proves as in Section 3.3 of [9] that the $\star$ product is associative in the sense that

$$\bar\tau \star (\bar\sigma \star \bar\eta) = (\bar\tau \star \bar\sigma) \star \bar\eta$$

for any $\bar\tau, \bar\sigma \in \overline{\mathbb{T}}_\bullet$ and $\bar\eta \in \overline{\mathbb{T}}$. We also define the $\star$ operation on $\overline{\mathsf{T}}_\bullet^* \times \overline{\mathsf{T}}^*$ setting

$$\bar\sigma^* \star \bar\tau^* := (\bar\sigma \star \bar\tau)^*.$$

The following morphism property of $\mathfrak{F}^a$ with respect to the $\star$ product is crucial and plays the role played by pre-Lie morphisms in the original approach of Bruned, Chandra, Chevyrev & Hairer [7]. Its proof is a copy and paste of the proof of Proposition 2 in [1] based on identity (4.1).

20 – Proposition. One has

$$\mathfrak{F}^a\left(\Big\{X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\bar\sigma_i)\Big\}^* \star \bar\tau^*\right)(\mathsf{c}, \mathsf{c}') = \left(\Big\{\partial^{\mathbf{k}} D_{\mathbf{n}_1} ... D_{\mathbf{n}_n} \mathfrak{F}^a(\bar\tau^*)\Big\} \prod_{i=1}^n \mathfrak{F}^a(\bar\sigma_i^*)\right)(\mathsf{c}, \mathsf{c}'). \tag{4.6}$$

Note that the expansion formula (4.4) in our case only involves the $\mathbf{k} \in \{\mathbf{0}, (0,1)\}$ case of the general formula (4.6). We see on (4.5) that formula (4.6) is a generalization of the defining identity (4.3). The interest of formula (4.6) will appear below in Proposition *21* when we will look for a recursive formula for some quantities of the form $\mathfrak{F}^a\big(R(z)^*(\tau^{\boldsymbol{p}})^*\big)$, for a (spacetime dependent) linear map R^* on $\overline{\mathsf{T}}^*$.

4.3 – Strong preparation maps and their associated models. The objects introduced in this section are the building blocks of an inductive construction of a renormalization process.

4.3.1 – Preparation maps. For $\tau^{\boldsymbol{p}} \in \mathbb{T}$ denote by $|\tau|_{\zeta_1}$ the number of noise symbols ζ_1 that appear in τ. Recall from Bruned's work [6] that a ***preparation map*** is a linear map $R : \mathsf{T} \to \mathsf{T}$ such that for each basis vector $\tau^{\boldsymbol{p}} \in \mathbb{T}$ one has

$$\begin{aligned} &R(\zeta_l) = \zeta_l, \qquad R(X^{\mathbf{k}}\tau^{\boldsymbol{p}}) = X^{\mathbf{k}}R(\tau^{\boldsymbol{p}}) \;\text{ for all } \mathbf{k} \in \mathbf{N}^2, \\ &R(\mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}})) = \mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}}) \;\text{ for all } \mathbf{n} \in \mathbf{N}^2 \text{ and } q \in \mathbf{N}, \end{aligned} \tag{4.7}$$

and there exist *finitely many* $\tau_i^{\boldsymbol{p}_i} \in T$ and constants λ_i such that

$$R\tau^{\boldsymbol{p}} = \tau^{\boldsymbol{p}} + \sum_i \lambda_i \tau_i^{\boldsymbol{p}_i}, \quad \text{with} \quad |\tau_i^{\boldsymbol{p}_i}| \geq |\tau^{\boldsymbol{p}}| \quad \text{and} \quad |\tau_i^{\boldsymbol{p}_i}|_{\zeta_1} < |\tau^{\boldsymbol{p}}|_{\zeta_1},$$

and R is closed in $\mathbb{B}$ and satisfies the 'commutation' relation

$$(R \otimes \mathrm{Id})\,\Delta = \Delta R. \tag{4.8}$$

The role of R is to provide a definition of the product of two trees that have already been renormalized. Its use in Section 4.3.2 in the recursive definition of the actual analytical objects associated with decorated trees will make that point clear; see in particular (4.13). Accordingly the second and third identities of (4.7) account for the fact that there is no need, in the induction process that builds an admissible model, to 'renormalize' elements of the form $X^{\mathbf{k}}\tau^{\boldsymbol{p}}$ and $\mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}})$ if the element $\tau^{\boldsymbol{p}}$ has already been renormalized. We can think of a preparation map as generating a renormalization process in the same way as a vector fields generates a flow.

Denote by R^* the algebraic dual of the map R; it is defined by the identity

$$\big\langle R\sigma^{\boldsymbol{q}}, (\tau^{\boldsymbol{p}})^*\big\rangle = \big\langle \sigma^{\boldsymbol{q}}, R^*(\tau^{\boldsymbol{p}})^*\big\rangle.$$

It is elementary to see that identity (4.8) is equivalent to having the right derivation identity

$$R^*\left((\sigma^{\boldsymbol{q}})^* \star (\tau^{\boldsymbol{p}})^*\right) = (\sigma^{\boldsymbol{q}})^* \star (R^*(\tau^{\boldsymbol{p}})^*) \tag{4.9}$$

for all $\sigma^{\boldsymbol{q}} \in \mathbb{B}^+$ ($\mathbb{B}^+$ is regarded as a subset of $\mathbb{T}_\bullet$) and $\tau^{\boldsymbol{p}} \in \mathbb{B}$ – see e.g. Proposition 3 in [1]. A ***strong preparation map*** is a preparation map satisfying identity (4.9) *for all $\sigma^{\boldsymbol{q}} \in \mathbb{T}_\bullet$ and $\tau^{\boldsymbol{p}} \in \mathbb{T}$* – and not only for $\sigma^{\boldsymbol{p}} \in \mathbb{B}^+$ and $\tau^{\boldsymbol{p}} \in \mathbb{B}$.

Definition – (a) *A **spacetime dependent strong preparation map on** $\mathsf{T}^{(m)}$ is a continuous map*

$$R : \big(\mathbf{R}_+ \times \mathbf{T}\big) \times \mathsf{T}^{(m)} \to \mathsf{T}^{(m)}$$

satisfying the following properties for any fixed $z \in \mathbf{R}_+ \times \mathbf{T}$.

- *The map $R(z,\cdot) : \mathsf{T}^{(m)} \to \mathsf{T}^{(m)}$ is linear, closed in $T^{(m)}$, and satisfies (4.7).*
- *For any $\tau^{\mathbf{0}} \in \mathbb{T}$ there exist finitely many $\sigma_1^{\mathbf{0}}, \dots, \sigma_n^{\mathbf{0}} \in \mathbb{T}$ such that $|\sigma_i| > |\tau|$, $|\sigma_i|_{\zeta_1} < |\tau|_{\zeta_1}$, and*

$$\big(R(z,\cdot) - \mathrm{Id}\big)\mathsf{T}^{(m)}_\tau \subset \bigoplus_{i=1}^n \mathsf{T}^{(m)}_{\sigma_i}.$$

- *The map $R(z,\cdot)^*$ satisfies (4.9) for any $\sigma^{\boldsymbol{q}} \in \mathbb{T}_\bullet$ and $\tau^{\boldsymbol{p}} \in \mathbb{T}$.*

(b) *A **spacetime dependent renormalization character on** $\mathbb{B}^-_\circ$**, of growth factor** $m' > 0$**,** is a map*

$$\ell : \big(\mathbf{R}_+ \times \mathbf{T}\big) \times \mathbb{B}^-_\circ \to \mathbf{R}$$

which is continuous in $\mathbf{R}_+ \times \mathbf{T}$ and vanishes on the elements of the form

$$X^{\mathbf{k}}\tau^{\boldsymbol{p}} \;\; (\mathbf{k} \neq 0), \qquad \mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}}),$$

and such that for any $\tau^{\mathbf{0}} \in \mathbb{B}_\circ^{-0}$ *there exists a constant* $C(\tau)$ *such that*

$$\big|\ell(z,\tau^{\boldsymbol{p}})\big| \leq C(\tau)(m')^{|\boldsymbol{p}|} \tag{4.10}$$

for any $\boldsymbol{p} \in \mathbf{N}^{E_\tau}$ *and* $z \in \mathbf{R}_+ \times \mathbf{T}$.

One associates to a spacetime dependent character $\ell(z,\cdot)$ of growth factor m' the linear map

$$R_\ell(z)^*\big((\tau^{\boldsymbol{p}})^*\big) := (\tau^{\boldsymbol{p}})^* + \sum_{\sigma^{\boldsymbol{q}}\in\mathbb{B}_\circ^-} \frac{\ell(z,\sigma^{\boldsymbol{q}})}{S(\sigma^{\boldsymbol{q}})}\,(\tau^{\boldsymbol{p}})^* \star (\sigma^{\boldsymbol{q}})^* \tag{4.11}$$

for any $\tau^{\boldsymbol{p}} \in \mathbb{T}_\bullet$ and $R(z)^*((\tau^{\boldsymbol{p}})^*) = (\tau^{\boldsymbol{p}})^*$ for any $\tau^{\boldsymbol{p}} \in \mathbb{T}_\circ$. It can be easily checked that R_ℓ is a strong preparation map on $T^{(m)}$ with $m > m'$. The above definition corresponds to the usual definition of its dual described by the contraction of trees as in Corollary 4.5 of [6], so $R_\ell(z)$ is closed in $\mathbb{B}$. For the commutation relation (4.9) we use the associativity of the $\star$ product as in Proposition *4* of [1]. It remains to show that R_ℓ is bounded in $\mathsf{T}^{(m)}$. Actually since

$$\begin{aligned}\left\|\big(R_\ell(z)^* - \mathrm{Id}\big)\Big(\sum_{\boldsymbol{p}} c_{\boldsymbol{p}}(\tau^{\boldsymbol{p}})^*\Big)\right\|_{1/m} &\leq \sum_{\boldsymbol{p},\boldsymbol{q},\sigma} \frac{|\ell(z,\sigma^{\boldsymbol{q}})|}{S(\sigma^{\boldsymbol{q}})}\,|c_{\boldsymbol{p}}|\,\big\|\tau^{\boldsymbol{p}}\star\sigma^{\boldsymbol{q}}\big\|_{1/m} \\ \lesssim \sum_{\boldsymbol{p},\boldsymbol{q},\sigma} \frac{|\ell(z,\sigma^{\boldsymbol{q}})|}{S(\sigma^{\boldsymbol{q}})}\,|c_{\boldsymbol{p}}|\,|\boldsymbol{q}|\,m^{-|\boldsymbol{p}|-|\boldsymbol{q}|} &\leq \Big(\sum_{\boldsymbol{q},\sigma}\frac{|\ell(z,\sigma^{\boldsymbol{q}})|^2}{S(\sigma^{\boldsymbol{q}})^2}\,|\boldsymbol{q}|^2\,m^{-2|\boldsymbol{q}|}\Big)^{1/2}\left\|\sum_{\boldsymbol{p}} c_{\boldsymbol{p}}\tau^{\boldsymbol{p}}\right\|_{1/m},\end{aligned}$$

the map $R_\ell(z)^* : \mathsf{T}^{*,(1/m)} \to \mathsf{T}^{*,(1/m)}$ is continuous because of (4.10), so $R_\ell(z)$ sends continuously $\mathsf{T}^{(m)}$ into itself. The next statement is a direct corollary of Proposition *20* and identity (4.11).

21 – Proposition. Let $\mathcal{O}$ *be a domain of* $\mathbf{R}^2$ *and let* $\|h\|_{L^\infty(\mathcal{O})} < 1/m$. *Let* R *be a spacetime dependent strong preparation map on* $\mathsf{T}^{(m)}$. *For every* $z \in \mathbf{R}_+ \times \mathbf{T}$ *and* $(\mathsf{c}_0, \mathsf{c}_0', \mathsf{c}_{(0,1)}, \mathsf{c}_{(0,1)}') \in \mathcal{O}\times\mathbf{R}^2$ *one has*

$$\begin{aligned}&\mathfrak{F}^a\Big(R(z)^*\Big(X^{\mathbf{k}}\zeta_l\prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)^*\Big)(\mathsf{c},\mathsf{c}') \\ &\qquad\qquad = \Big(\Big\{\partial^{\mathbf{k}} D_{\mathbf{n}_1}\dots D_{\mathbf{n}_n}\mathfrak{F}^a\big(R(z)^*\zeta_l^*\big)\Big\}\prod_{i=1}^n \mathfrak{F}^a\big((\tau_i^{\boldsymbol{p}_i})^*\big)\Big)(\mathsf{c},\mathsf{c}').\end{aligned}$$

Proof – Writing

$$X^{\mathbf{k}}\zeta_l\prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i}) = \Big(X^{\mathbf{k}}\prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)\star\zeta_l$$

and using the right derivation property (4.9) – this is where we use the fact that the preparation map is ‘strong’, one gets

$$R^*(z)\bigg(\Big(\Big(X^{\mathbf{k}}\prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)\star\zeta_l\Big)^*\bigg) = \Big(X^{\mathbf{k}}\prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)^*\star\big(R(z)^*\zeta_l^*\big). \tag{4.12}$$

Identity (4.6) in Proposition *20* then yields the identity of the statement. ▷

4.3.2 – Admissible model associated with a preparation map. Fix a regularization parameter ε and denote by

$$\xi^\varepsilon =: \xi_1 \in C^\infty(\mathbf{R}\times\mathbf{T})$$

a regularized version of the spacetime white noise ξ and set

$$\xi_2 = \xi_3 = 1.$$

For any spacetime dependent strong preparation map R on $\mathsf{T}^{(m)}$ we define inductively the maps $\mathsf{\Pi}^{R,a(v_0)}$ and $\Pi^{R,a(v_0),\times}$ as follows. For $1 \leq l \leq 3$, set

$$\mathsf{\Pi}^{R,a(v_0)}\zeta_l = \Pi^{R,a(v_0),\times}\zeta_l := \xi_l,$$

and define

$$\begin{aligned} &\mathsf{\Pi}^{R,a(v_0)} = \Pi^{R,a(v_0),\times} \circ R, \quad \Pi^{R,a(v_0),\times}(\tau_1\tau_2) = \big(\Pi^{R,a(v_0),\times}\tau_1\big)\big(\Pi^{R,a(v_0),\times}\tau_2\big), \\ &\Pi^{R,a(v_0),\times}(\mathcal{I}^q_{\mathbf{n}}\tau) = \partial_z^{\mathbf{n}}\big(K^{a(v_0)} \circ \big(\partial_x^2 K^{a(v_0)}\big)^{\circ q}\big)\big(\mathsf{\Pi}^{R,a(v_0)}\tau\big). \end{aligned} \tag{4.13}$$

(The symbol $\circ$ stands here for the composition operator.) Here the operator $\partial_x^2 K^{a(v_0)}$ makes sense because $\mathsf{\Pi}^{R,a(v_0)}\tau$ constructed as above belongs to $\mathcal{C}^{0+}_{\mathfrak{s}}$. Note that $K^{a(v_0)}$ maps $\mathcal{C}^{0+}_{\mathfrak{s}}$ into $\mathcal{C}^{2+}_{\mathfrak{s}}$. As R is spacetime dependent the first identity in (4.13) reads

$$\big(\mathsf{\Pi}^{R,a(v_0)}\tau\big)(z) = \Pi^{R,a(v_0),\times}\big(R(z)\tau\big)(z),$$

for all z and all τ. It follows from this definition and the fact that we work with preparation maps R leaving fixed the elements of T of the form $\mathcal{I}^q_{\mathbf{n}}(\tau)$ that the map $\mathsf{\Pi}^{R,a(v_0)}$ satisfies the admissibility condition

$$\mathsf{\Pi}^{R,a(v_0)}(\mathcal{I}^q_{\mathbf{n}}\tau) = \partial_z^{\mathbf{n}}\Big(K^{a(v_0)} \circ (\partial_x^2 K^{a(v_0)})^{\circ q}\Big)(\mathsf{\Pi}^{R,a(v_0)}\tau).$$

(The notation $\mathcal{A}^{\circ q}$ stands for the q-fold iterate of an operator $\mathcal{A}$.) Define as well $\mathsf{g}^{R,a(v_0)}$ inductively from the identity

$$(\mathsf{g}_z^{R,a(v_0)})^{-1}\big(\mathcal{I}^{+,q}_{\mathbf{n}}\tau\big) = -\sum_{|\mathbf{k}|_{\mathfrak{s}}<|\tau|+2-|\mathbf{n}|_{\mathfrak{s}}} \frac{(-z)^{\mathbf{k}}}{\mathbf{k}!}\Big(\partial_z^{\mathbf{n}+\mathbf{k}}\big(K^{a(v_0)} \circ (\partial_x^2 K^{a(v_0)})^{\circ q}\big)(\mathsf{\Pi}_z^{R,a(v_0)}\tau)\Big)(z).$$

One can follow verbatim Section 7.1 of [2] and see that $\big(\mathsf{\Pi}^{R,a(v_0)}, \mathsf{g}^{R,a(v_0)}\big)$ *is a smooth admissible model* on $\mathscr{T}^{(m)}$ with a constant m coming from the operator norm of $\partial_x^2 K^{a(v_0)}$.

Among the renormalization characters, we are interested in the one $\ell^{\varepsilon}_{a(v_0)}(z,\tau^{\boldsymbol{p}})$ defined by the similar way to Section 6.3 of [8]. We denote by $R^{\varepsilon}_{a(v_0)}$ the strong preparation map defined by (4.11) with ℓ replaced by $\ell^{\varepsilon}_{a(v_0)}$. The associated model $M^{\varepsilon}_{a(v_0)}$ is called the BPHZ model. Note that, when $\ell^{\varepsilon}_{a(v_0)}$ has a growth factor $m' < m$, the BPHZ model $\mathsf{M}^{\varepsilon}_{a(v_0)}$ is a model on $\mathscr{T}^{(m)}$.

1 – *Assumption. There exists a character $\ell^{\varepsilon}_{a(v_0)}$ of growth factor $m' \in (0,m)$ for each ε (so the constant $C(\tau)$ in* (4.10) *may be ε-dependent) and the BPHZ renormalized model $\mathsf{M}^{\varepsilon}_{a(v_0)}$ is convergent as $\varepsilon \to 0$.*

We skip the proof of Assumption 1 in this paper but rather give some comments on what is involved in this assumption to end this section. This assumption essentially seems to be a consequence of Theorem 2.31 in [11] except that the integration operator $K^{a(v_0)}$ is not stationary in the space variable and $\partial_x^2 K^{a(v_0)}$ is too singular to be integrable around origin – the latter part would be solved by Lemma 28 below. The former part is not a serious problem but we need a slight modification of the 'negative twisted antipode' in [8]. When dealing with a kernel $K^{\lambda}(z,z')$ that depends only on $z-z'$, the character $\ell^{\varepsilon}_{\lambda}(\tau^{\boldsymbol{p}})$ is spacetime-independent and defined by the formula

$$\ell^{\varepsilon}_{\lambda}(\tau^{\boldsymbol{p}}) = h^{\varepsilon}_{\lambda}(S'_-\tau^{\boldsymbol{p}}),$$

where $h^{\varepsilon}_{\lambda} : \mathbb{B} \to \mathbf{R}$ is the non-renormalized expectation

$$h^{\varepsilon}_{\lambda}(\tau^{\boldsymbol{p}}) := \mathbb{E}\big[\mathsf{\Pi}^{\mathrm{Id},\lambda}\tau^{\boldsymbol{p}}(0)\big],$$

and $S'_- : \mathbf{R}[\mathbb{B}^-_{\circ}] \to \mathbf{R}[\mathbb{B}]$ is the negative twisted antipode – see Proposition 6.6 in [8] or Section 7 of [4] for its definition. Roughly speaking, the map S'_- is defined by the inductive relation

$$S'_-\tau^{\boldsymbol{p}} = -\tau^{\boldsymbol{p}} - \sum_{\sigma^{\boldsymbol{q}}}(S'_-\sigma^{\boldsymbol{q}})(\tau^{\boldsymbol{p}}/\sigma^{\boldsymbol{q}}),$$

where $\sigma^{\boldsymbol{q}}$ runs over subgraphs with negative homogeneity and $\tau^{\boldsymbol{p}}/\sigma^{\boldsymbol{q}}$ is a quotient graph obtained by identifying all the nodes of $\sigma^{\boldsymbol{q}}$ with its root, and hence $\ell_\lambda^\varepsilon(\tau^{\boldsymbol{p}})$ can be written as a sum of products of the form

$$\ell_\lambda^\varepsilon(\tau^{\boldsymbol{p}}) = -h_\lambda^\varepsilon(\tau^{\boldsymbol{p}}) - \sum_{\sigma^{\boldsymbol{q}}} \ell_\lambda^\varepsilon(\sigma^{\boldsymbol{q}})\, h_\lambda^\varepsilon(\tau^{\boldsymbol{p}}/\sigma^{\boldsymbol{q}}). \tag{4.14}$$

Behind the above splitting formula, the renormalization operator $\mathfrak{R}$ acting on kernels plays an important role – see Definition 10.15 of [17]. Let us take the example of a family $\{L_\varepsilon^i\}_{\varepsilon>0, i\in\{1,2,3\}}$ of kernels given for each i as limits of nice kernels $L^i = \lim_{\varepsilon\to 0} L_\varepsilon^i$, in the uniform norm weighted at the origin, and such that L^1 and L^3 are integrable but L^2 is not. Since we cannot expect the convolution $L_\varepsilon^1 * L_\varepsilon^2 * L_\varepsilon^3$ converges, instead we replace L_ε^2 by the distribution $\mathfrak{R}L_\varepsilon^2(z) = L_\varepsilon^2(z) - (\int L_\varepsilon^2(w)dw)\delta(z)$ and prove the convergence of the convolution

$$\begin{aligned} L_\varepsilon^1 * \mathfrak{R}L_\varepsilon^2 * L_\varepsilon^3(z) &= \int L_\varepsilon^1(z-z_1)L_\varepsilon^2(z_1-z_2)\big(L_\varepsilon^3(z_2) - L_\varepsilon^3(z_1)\big)dz_1dz_2 \\ &= L_\varepsilon^1 * L_\varepsilon^2 * L_\varepsilon^3(z) - \Big(\int L_\varepsilon^2(z')dz'\Big) L_\varepsilon^1 * L_\varepsilon^3(z) \end{aligned}$$

with the help of Lipschitz continuity of L_ε^3. In the last equality, the multiplication of the integral of L^2 and the convolution $L^1 * L^3$ corresponds to the second term of the right hand side of (4.14). However, if $\{L_\varepsilon^i\}_{\varepsilon>0, i\in\{1,2\}}$ actually depends both on z and z', we have to consider the renormalization

$$L_\varepsilon^1 * \mathfrak{R}L_\varepsilon^2 * L_\varepsilon^3(z,z') = \int L_\varepsilon^1(z,z_1)L_\varepsilon^2(z_1,z_2)\big(L_\varepsilon^3(z_2,z') - L_\varepsilon^3(z_1,z')\big)dz_1dz_2.$$

Since the integral of L^2 depends on z_1, it cannot be separated from L^1 and L^3 unlike the stationary case. So, when dealing with the kernel $K^{a(v_0)}(z,z')$ rather than with $K^\lambda(z-z')$, instead of (4.14) we have to subtract integrals of the form

$$\int_{\mathbf{R}^2} \ell_{a(v_0)}^\varepsilon(z,z_1,\sigma^{\boldsymbol{q}}) h_{a(v_0)}^\varepsilon(z,z_1,\tau^{\boldsymbol{p}}/\sigma^{\boldsymbol{q}})dz_1,$$

where z_1 is a spacetime variable associated with the root of $\sigma^{\boldsymbol{q}}$, and $h_{a(v_0)}^\varepsilon(z,z_1,\tau^{\boldsymbol{p}}/\sigma^{\boldsymbol{q}})$ is defined by the same way as $h_{a(v_0)}^\varepsilon(z,\tau^{\boldsymbol{p}}/\sigma^{\boldsymbol{q}})$ without taking an integral with respect to z_1.

4.4 – Renormalized equation. Denote by $\mathsf{R}_{a(v_0)}^\varepsilon$ the reconstruction map associated with $\mathsf{M}_{a(v_0)}^\varepsilon$. The proof of Theorem *20* in [2] works verbatim and gives in our setting the following result.

22 – Proposition. Let $\boldsymbol{u}^\varepsilon \in \mathcal{D}_m^{2,\mu}(0,t_0)$ *stand for the modelled distribution solution of* (2.4) *with respect to the model* $\mathsf{M}_{a(v_0)}^\varepsilon$. *One can choose* $t_0' < t_0$ *small enough for*

$$u^\varepsilon := \mathsf{R}_{a(v_0)}^\varepsilon(\boldsymbol{u}^\varepsilon)$$

to satisfy the bound

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,t_0')} \big\|a(u^\varepsilon) - a(v_0)\big\|_{C(\mathbf{T})} < \frac{1}{m}$$

and solve the 'renormalized' equation

$$\big(\partial_{x_0} - a(u^\varepsilon)\partial_x^2\big)u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon + g(u^\varepsilon)(\partial_x u^\varepsilon)^2 + \sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-} \frac{\ell_{a(v_0)}^\varepsilon(\cdot,\tau^{\boldsymbol{p}})}{S(\tau^{\boldsymbol{p}})} \mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon, \partial_x u^\varepsilon, v_0\big) \tag{4.15}$$

on $(0,t_0')\times\mathbf{T}$, *with initial condition* u_0. *The last term of* (4.15) *has a growth that is at most linear with respect to* $\partial_x u^\varepsilon$.

Proof – Denote by $R^{\varepsilon,*}_{a(v_0)}$ the dual of $R^{\varepsilon}_{a(v_0)}$. Theorem *9* of [1] yields that u^ε solves the equation

$$\big(\partial_{x_0} - L^{a(v_0)}\big)u^\varepsilon = f(u)\xi^\varepsilon + g(u)(\partial_x u^\varepsilon)^2 + \big(a(u^\varepsilon) - a(v_0)\big)\partial_x^2 u^\varepsilon + \sum_{l=1}^{4} \mathfrak{F}^a\Big((R^{\varepsilon,*}_{a(v_0)} - \mathrm{Id})\zeta_l^*\Big)\big(u^\varepsilon, \partial_x u^\varepsilon, v_0, \partial_x v_0\big).$$

Since $R^{\varepsilon,*}_{a(v_0)}\zeta_l^* \in \mathsf{T}^{*,(1/m)}$ by duality the term $\mathfrak{F}^a\Big((R^{\varepsilon,*}_{a(v_0)} - \mathrm{Id})\zeta_l^*\Big)$ is actually convergent in $C_b((0,t_0')\times\mathbf{T})$ by the remark before Proposition *18*. We have the right hand side of (4.15) from the definition of $R^{\varepsilon,*}_{a(v_0)}$.

It remains to check the last statement. Note that the BHZ characters satisfy

$$\ell^\lambda\big(X^{(0,1)}\sigma^{\boldsymbol{p}}\big) = 0 \tag{4.16}$$

for all λ. Since the only functions $\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)(\mathsf{c},\mathsf{c}')$ that depend on $\mathsf{c}'_{(0,1)}$ correspond to $\tau^{\boldsymbol{p}}$ of the form $X^{(0,1)}\sigma^{\boldsymbol{p}}$, the corresponding counterterms are null. Moreover no functions $\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)(\mathsf{c},\mathsf{c}')$ is of degree greater than 1 with respect to $\mathsf{c}_{(0,1)}$. Otherwise some $\tau^{\boldsymbol{p}}$ would have at least two ζ_2-type nodes from where exactly one edge $\mathcal{I}_{(0,1)}$ leaves. Since the minimal homogeneity among the trees

$$X^{\mathbf{k}}\zeta_2\mathcal{I}_{(0,1)}(\sigma)\prod_{i=1}^n \mathcal{I}(\sigma_i)$$

is $\big|\zeta_2\mathcal{I}(\zeta_1)\mathcal{I}_{(0,1)}(\zeta_1)\big| = 2\alpha - 1 > -1$, such $\tau^{\boldsymbol{p}}$ cannot have negative homogeneity. ▷

We finally consider the v_0-dependence of the counterterm of (4.16). Define inductively the function $\chi^a(\tau^{\boldsymbol{p}})(\mathsf{c}_{\mathbf{0}})$ by the relations

$$\begin{aligned} &\chi^a\big(\zeta_3\,\mathcal{I}_{(0,2)}(\tau^{\boldsymbol{p}})\big) = \chi^a(\tau^{\boldsymbol{p}}), \\ &\chi^a\Big(\zeta_3\,\mathcal{I}_{(0,2)}(\tau^{\boldsymbol{p}})\prod_{i=1}^n \mathcal{I}(\tau_i^{\boldsymbol{p}_i})\Big)(\mathsf{c}_{\mathbf{0}}) = a^{(n)}(\mathsf{c}_{\mathbf{0}})\prod_{i=1}^n \chi^a\big(\tau_i^{\boldsymbol{p}_i}\big)(\mathsf{c}_{\mathbf{0}}), \qquad \text{for } n\geq 1, \end{aligned} \tag{4.17}$$

and for $l\in\{1,2\}$

$$\chi^a\Big(\zeta_l\prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)(\mathsf{c}_{\mathbf{0}}) = \prod_{i=1}^n \chi^a\big(\tau_i^{\boldsymbol{p}_i}\big)(\mathsf{c}_{\mathbf{0}}). \tag{4.18}$$

We see on this definition that χ^a is a polynomial function of a and its derivatives. It is important to note that $\chi^a(\tau^{\boldsymbol{p}})$ does not depend on the $\boldsymbol{p}$-decoration of τ – rather it depends on the location of the ζ_3 vertices within $\tau^{\boldsymbol{p}}$. We denote by τ the non-$\boldsymbol{p}$-decorated tree associated with $\tau^{\boldsymbol{r}}$, so the symbols τ and $\tau^{\mathbf{0}}$ are used here interchangeably. As a shorthand notation we write

$$\chi^a_\tau := \chi^a(\tau).$$

With

$$\tau_1 := \mathcal{I}(\zeta_1)\mathcal{I}_{(0,1)}(\zeta)^2\zeta_2, \quad \tau_2 := \mathcal{I}(\zeta_1)^2\mathcal{I}_{(0,2)}(\zeta)\zeta_3$$

one has for instance

$$\chi^a(\tau_1)(\mathsf{c}_{\mathbf{0}}) = 1, \quad \chi^a(\tau_2)(\mathsf{c}_{\mathbf{0}}) = a^{(2)}(\mathsf{c}_{\mathbf{0}})\chi^a(\zeta_1)^2(\mathsf{c}_{\mathbf{0}}) = a^{(2)}(\mathsf{c}_{\mathbf{0}})$$

We also define the functions $\mathfrak{F}_{\tau^*}$ for $\tau\in\mathbb{B}^0$ by the same inductive relations as the functions $\mathfrak{F}^a(\tau^*)$ by replacing $\mathsf{c}_{\mathbf{0}}$-derivatives of $h(\mathsf{c}_{\mathbf{0}},\mathsf{c}'_{\mathbf{0}})$ of any order by the constant function equal to 1. Then the functions $\mathfrak{F}_{\tau^*}$ depend only on $\mathsf{c}_{\mathbf{0}}$. It is elementary to obtain the following identity by induction.

23 – Lemma. One has

$$\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(\mathsf{c}_{\mathbf{0}},\mathsf{c}_{(0,1)},\mathsf{c}'_{\mathbf{0}}\big) = \chi^a_\tau(\mathsf{c}_{\mathbf{0}})\,\big(a(\mathsf{c}_{\mathbf{0}}) - a(\mathsf{c}'_{\mathbf{0}})\big)^{|\boldsymbol{p}|}\,\mathfrak{F}_{\tau^*}(\mathsf{c}_{\mathbf{0}}). \tag{4.19}$$

The notion of admissible model on $\mathscr{T}^{(m)}$ is relative to an integration operator – $K^{a(v_0)}$ in Definition 9 for instance. For a positive parameter λ denote by

$$Z_t^\lambda(x) = Z^\lambda(t,x) = \mathbf{1}_{t>0}\frac{e^{-ct}}{\sqrt{4\pi\lambda t}}\exp\Big(-\frac{|x|^2}{4\lambda t}\Big)$$

the fundamental solution built from the constant coefficient parabolic operator $\partial_t - \lambda\partial_x^2 - c$. The naive admissible model on $\mathscr{T}$ associated with Z^λ and the smooth noise ξ^ε is the unique multiplicative model such that

$$\Pi_\lambda^\varepsilon\zeta_1 = \xi^\varepsilon, \qquad \Pi_\lambda^\varepsilon\zeta_l = 1 \quad (l\in\{2,3\}), \qquad \big(\Pi_\lambda^\varepsilon X^{\mathbf{k}}\big)(z) = z^{\mathbf{k}},$$

and

$$\Pi_\lambda^\varepsilon\big(\mathcal{I}_{\mathbf{n}}^p(\tau^{\boldsymbol{p}})\big) = \Big(\partial_z^{\mathbf{n}}Z^\lambda\circ(\partial_x^2Z^\lambda)^p\Big)\Pi_\lambda^\varepsilon\tau^{\boldsymbol{p}}.$$

The *BHZ character* $\ell_\lambda^\varepsilon(\cdot)$ on $\mathbb{B}_\circ^-$ is defined in that setting as

$$\ell_\lambda^\varepsilon(\tau^{\boldsymbol{p}}) := h_\lambda^\varepsilon(S'_-\tau^{\boldsymbol{p}}), \qquad h_\lambda^\varepsilon(\tau^{\boldsymbol{p}}) := \mathbb{E}\big[\Pi_\lambda^\varepsilon\tau^{\boldsymbol{p}}(0)\big],$$

where $S'_- : T^-\to\mathbf{R}[T]$ is the natural extension to our setting of the negative twisted antipode – see Proposition 6.6 in [8] or Section 7 of [4]for its definition in the usual BHZ setting.

2 – Assumption. For any $\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}$ there exist a constant $m>0$ and an ε-independent constant $C(\tau)$ such that

$$\big|\ell_{a(v_0)}^\varepsilon(z,\tau^{\boldsymbol{p}}) - \ell_{a(v_0(z))}^\varepsilon(\tau^{\boldsymbol{p}})\big| \le C(\tau)\,m^{|\boldsymbol{p}|}$$

for any $\boldsymbol{p}\in\mathbf{N}^{E_\tau}$ and $z\in\mathbf{R}_+\times\mathbf{T}$.

We check that the above assumption holds for some examples in the next section. Although elementary the next statement is the core fact to get the renormalized equation under the form (1.5) stated in Theorem *1*.

24 – Lemma. For $\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}$ the function

$$\lambda\mapsto\ell_\lambda^\varepsilon(\tau^{\mathbf{0}})$$

is analytic in any given bounded interval of $\mathbf{R}$ whose closure does not contain the point 0 and

$$\frac{1}{n!}\partial_\lambda^n\ell_\lambda^\varepsilon(\tau^{\mathbf{0}}) = \sum_{\boldsymbol{p}\in\mathbf{N}^{E_\tau},\,|\boldsymbol{p}|=n}\ell_\lambda(\tau^{\boldsymbol{p}}).$$

Proof – By an elementary computation we have

$$\partial_\lambda Z^\lambda(t,x) = t\Delta Z^1(\lambda t,x) = \int_0^t\int_{\mathbf{R}} Z^\lambda(t-s,x-y)\Delta Z^\lambda(s,y)\,dy\,ds. \tag{4.20}$$

Therefore once ∂_λ applies to one edge $\partial^{\mathbf{k}}Z^\lambda$ then this edge turns into a spacetime convolution $\partial^{\mathbf{k}}Z^\lambda * \Delta Z^\lambda$. ▷

Proof of Theorem 2 – It follows from Lemma *16*, Lemma *24* and (4.19) that, up to an ε-uniform $O(1)$ term, the counterterm in the renormalized equation (4.15) takes the simple form

$$\begin{aligned}
&\sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-}\frac{\ell_{a(v_0(\cdot))}^\varepsilon(\tau^{\boldsymbol{p}})}{S(\tau^{\boldsymbol{p}})}\,\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon,\partial_xu^\varepsilon,v_0\big)\\
&\qquad = \sum_{\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}}\frac{1}{S(\tau^{\mathbf{0}})}\sum_{\boldsymbol{p}\in\mathbf{N}^{E_\tau}}\ell_{a(v_0(\cdot))}^\varepsilon(\tau^{\boldsymbol{p}})\,\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon,\partial_xu^\varepsilon,v_0\big)\\
&\qquad = \sum_{\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}}\frac{1}{S(\tau^{\mathbf{0}})}\,\chi_\tau^a(u^\varepsilon)\,\mathfrak{F}_{\tau^*}\big(u^\varepsilon,\partial_xu^\varepsilon\big)\sum_{n=0}^\infty\big(a(u^\varepsilon)-a(v_0)\big)^n\sum_{|\boldsymbol{p}|=n}\ell_{a(v_0(\cdot))}^\varepsilon(\tau^{\boldsymbol{p}})
\end{aligned}$$

$$= \sum_{\tau^{\mathbf{0}}\in\mathbb{B}_{\circ}^{-0}} \frac{1}{S(\tau^{\mathbf{0}})}\, \chi_\tau^a(u^\varepsilon)\, \mathfrak{F}_{\tau^*}\big(u^\varepsilon, \partial_x u^\varepsilon\big)\, \ell^\varepsilon_{a(u^\varepsilon(\cdot))}(\tau^{\mathbf{0}}).$$

▷

Note that Theorem 2 holds for any noise satisfying assumption 1 – that is any noise to which Chandra & Hairer's result [11] applies. We finish that section by showing that the a priori diverging term $\ell^\varepsilon_{a(u^\varepsilon(\cdot))}$ in the counterterm takes a particularly nice form when the noise is Gaussian and the regularization operation appropriate. Recall that $|\tau|_{\zeta_1}$ denotes the number of ζ_1-type nodes that appear in τ. If we consider only the spatial regularization of the noise we have further simpler form by change of variables.

25 – Proposition. Assume that ξ is a stationary centered Gaussian noise and define

$$\xi^\varepsilon(t,x) = \big(\xi(t,\cdot) * \rho_\varepsilon\big)(x)$$

with even mollifier ρ_ε. Then $\ell^\varepsilon_\lambda(\tau^{\mathbf{0}}) = 0$ if $|\tau|_{\zeta_1}$ is odd. Otherwise,

$$\ell^\varepsilon_\lambda(\tau^{\mathbf{0}}) = \begin{cases} \lambda^{-\sharp N_\tau+1}\ell^\varepsilon_1(\tau^{\mathbf{0}}) & (\text{if } \xi(x) \text{ depends on only space}),\\ \lambda^{|\tau|_{\zeta_1}/2-\sharp N_\tau+1}\ell^\varepsilon_1(\tau^{\mathbf{0}}) & (\text{if } \xi(t,x) \text{ is white in time}).\end{cases}$$

Proof – Let $N_\circ$ and $N_\bullet$ be the collection of ζ_1-type nodes and ζ_2,ζ_3-type nodes respectively, and denote by $2a = \sharp N_\circ$ and $b = \sharp N_\bullet$. If $N_\bullet$ contains the root, the expectation of $\big(\Pi^\varepsilon_\lambda\tau\big)(0)$ is represented as the integral of the form

$$\int C_\varepsilon(z^1-z^2)\cdots C_\varepsilon(z^{2a-1}-z^{2a})(z^1)^{n_1}\cdots(z^{2a})^{n_{2a}}G^\lambda\big(z^1,\dots,z^{2a}\big)\,dz^1\cdots dz^{2a},$$

where $C_\varepsilon(z) = \mathbb{E}\big[\xi_\varepsilon(z)\xi_\varepsilon(0)\big]$ and

$$G^\lambda(z^1,\dots,z^{2a}) = \int \widetilde{G}^\lambda\big(z^1,\dots,z^{2a},w^1,\dots,w^{b-1}\big)\,dw^1\cdots dw^{b-1}$$

with $\widetilde{G}$ the product of polynomials $(w^i)^{m_i}$ and kernels $\partial^{e_{ij}}K^\lambda$. Note that the $\mathfrak{n}$-decorations m_i and $\mathfrak{e}$-decorations e_{ij} are $\mathbf{0}$ or $(0,1)$, so they are independent of the change of variables $z \mapsto z_\lambda := (\lambda t, x)$. Hence

$$G^\lambda\big(z^1,\dots,z^{2a}\big) = \lambda^{-b+1}G^1\big(z^1_\lambda,\dots,z^{2a}_\lambda\big)$$

by a scaling argument. If $\xi(x)$ depends only on the space argument then we have

$$\big(\Pi^\varepsilon_\lambda\tau\big)(0) = \lambda^{-2a-b+1}\big(\Pi^\varepsilon_1\tau\big)(0),$$

since $C_\varepsilon(x)$ does not depend on time. If $\xi(t,x)$ is white in time one has $C_\varepsilon(z) = \delta(t)\rho^{*2}_\varepsilon(x)$, which reduces the number of time components $t^1,\dots,t^{2a}$ of $z^1,\dots,z^{2a}$ in half and yields

$$\big(\Pi^\varepsilon_\lambda\tau\big)(0) = \lambda^{-a-b+1}\big(\Pi^\varepsilon_1\tau\big)(0).$$

We can perform similar computations when $N_\circ$ contains the root. ▷

In the setting of Lemma *25* the counterterm is of the form

$$\sum_{\tau\in\mathbb{B}_{\circ}^{-0}} \frac{\ell^\varepsilon_1(\tau)}{S(\tau)}\,\frac{\chi^a_\tau(u^\varepsilon)}{a(u^\varepsilon)^{\theta_\tau}}\,\mathfrak{F}_{\tau^*}(u^\varepsilon,\partial_x u^\varepsilon) \tag{4.21}$$

with an exponent θ_τ given in the statement of Proposition *25*. This situation applies in the example of equation (1.6) with a linear additive forcing.

4.5 – Examples. We consider in this section some examples satisfying Assumption 2. A couple of statements are needed before we can analyse in some details some examples in the next three subsections. Recall that the integration operator $K^{a(v_0)}$ is equal to the operator

$$\big(\partial_t - L^{a(v_0)} + c\big)^{-1}(\cdot) = \int_{-\infty}^t Q^{a(v_0)}_{t-s}(\cdot)ds$$

modulo an operator R sending $\mathcal{C}_{\mathfrak{s}}^{-2+}$ into $\mathcal{C}_{\mathfrak{s}}^{2+}$. Hence it is sufficient to consider the renormalization characters $\ell^{\varepsilon}_{a(v_0)}(\tau^{\boldsymbol{p}})$ by replacing the kernels $K^{a(v_0)}((t,x),(s,y))$ with $Q^{a(v_0)}_{t-s}(x,y)$.

We again replace each $Q^{a(v_0)}_{t-s}(x,y)$ with the heat kernel $Z^{a(v_0(\bullet))}(x-y)$ whose coefficient is frozen at the root and define the spacetime character $\ell^{\varepsilon}_{a(v_0(\bullet))}(\tau^{\boldsymbol{p}})$. As performed in Hairer's works [17, 18], we can associate each character $\ell^{\varepsilon}_{a(v_0)}(\tau^{\boldsymbol{p}})$ and $\ell^{\varepsilon}_{a(v_0(\bullet))}(\tau^{\boldsymbol{p}})$ with a directed graph called Feynman diagram whose edges are related with kernels $Q^{a(v_0)}$ and $Z^{a(v_0)}$ respectively and estimate it by using the singularity of each kernel around origin. To estimate the difference between $\ell^{\varepsilon}_{a(v_0)}(\tau^{\boldsymbol{p}})$ and $\ell^{\varepsilon}_{a(v_0(\bullet))}(\tau^{\boldsymbol{p}})$, we show that the difference between $Q^{a(v_0)}$ and $Z^{a(v_0)}$ is less singular than $Q^{a(v_0)}$ by the regularity of v_0 – see Proposition 26 below.

We use the notations from Section A.1 where we consider some properties of Gaussian-like kernels. This type of kernels appear in the construction of the fundamental solution $Q^{a(v_0)}$ of the operator $\partial_t - L^{a(v_0)} + c$ or more general operators, as described in Section A.2. Recall from Appendix A.1 the definition of the Gaussian kernel $\mathsf{G}_t^{(c,\zeta)}(x)$ with $d=1$, $\mathfrak{s}=1$, and $N=2$. Recall that Z^{λ} is the fundamental solution built from the operator $\partial_t - \lambda\partial_x^2 - c$.

26 – Lemma. Let $\mathfrak{K}$ be a compact subset of $(0,\infty)$. For any $\mu,\lambda\in\mathfrak{K}$ and $M,k\in\mathbf{N}$, there exists $c>0$ such that

$$\left|\partial_x^k Z_t^{\mu}(x) - \sum_{n<M}\frac{(\mu-\lambda)^n}{n!}t^n\partial_x^{2n+k}Z_t^{\lambda}(x)\right| \lesssim |\mu-\lambda|^M\,\mathsf{G}_t^{(c,-k)}(x).$$

Proof – By (4.20), $\partial_\lambda^n\partial_x^k Z_t^\lambda(x) = t^n\partial_x^{2n+k}Z_t^\lambda(x)$ is in the class $\mathsf{G}^{(-k)}$. ▷

The following result is essential in the proof of Assumption 2; we defer its proof to the end of Appendix A.2. For simplicity we write

$$b = a(v_0)$$

and we write $\mathsf{G}^{(\zeta)}_{\beta_1,\beta_2}(t,x,y)$ to mean a value of a function belonging to the class $\mathsf{G}^{(\zeta)}_{\beta_1,\beta_2}$ in the sense of Definition 30 with negative γ to guarantee its spacetime integrability.

27 – Proposition. If $v_0\in C^\alpha$ with $\alpha\in(0,1]$, for any $k\in\{0,1,2\}$ one has

$$\partial_x^k Q_t^b(x,y) = (\partial^k Z_t)^{b(y)}(x-y) + \mathsf{G}^{(\alpha-k)}_{\alpha,\alpha}(t,x,y) \tag{4.22}$$

$$= (\partial^k Z_t)^{b(x)}(x-y) + \mathsf{G}^{(\alpha-k)}_{\alpha,\alpha}(t,x,y). \tag{4.23}$$

If v_0 is a C^2 function, for any $k\in\{0,1,2\}$ one has

$$\partial_x^k Q_t^b(x,y) = (\partial^k Z_t)^{b(x)}(x-y) + b'(x)Y_t^{k,b(x)}(x-y) + \mathsf{G}^{(2-k)}_{1,1}(t,x,y), \tag{4.24}$$

where $\{Y_t^{k,\lambda}\}_{t>0,k\in\{0,1,2\},\lambda>0}$ belongs to $\mathsf{G}^{(1-k)}_{1,1}$ locally uniformly over λ. When k is even, respectively odd, the function $Y_t^k(\cdot)$ is odd, respectively even.

The iteration of $\partial^2 K^{a(v_0)}$ can be estimated properly by the following lemma.

28 – Lemma. For any $n\in\mathbf{N}$,

$$(\partial_1^2 Q^b)^{*n}(x,y) = \frac{1}{(n-1)!}(t\partial_x^2)^{n-1}Z_t^\lambda(x-y)|_{\lambda=b(x)} + \mathsf{G}^{(\alpha)}_{\alpha,\alpha}(t,x,y),$$

Proof – We only consider $n=1$. The general case is an easy extension. By the decompositions (4.22) and (4.23) of $\partial_1^2 Q^b$ and by Lemma 31, it is sufficient to consider the integral

$$\int_{(0,t)\times\mathbf{R}} (\partial^2 Z_{t-s})^{b(x)}(x-z)(\partial^2 Z_s)^{b(y)}(z-y)dz.$$

By the semigroup property of $Z_t^\lambda = e^{-ct}e^{\lambda t\Delta}$, the above integral is equal to

$$\int_0^t (\partial_t \partial_x^2 Z)^{b(x)(t-s)+b(y)s}(x-y)\,ds = \frac{\partial^2 Z_t^{b(y)} - \partial^2 Z_t^{b(x)}}{b(y)-b(x)} = \partial_\lambda \partial^2 Z_t^\lambda|_{\lambda=b(x)} + \mathsf{G}_{\alpha,\alpha}^{(\alpha)}$$
$$= t\partial_x^2 Z_t^\lambda|_{\lambda=b(x)} + \mathsf{G}_{\alpha,\alpha}^{(\alpha)}.$$

$\triangleright$

Equipped with the previous three statements we can now look at three examples where Assumption 2 can be proved to hold.

4.5.1 – Two dimensional parabolic Anderson model. In the slightly singular setting of the quasilinear parabolic Anderson model equation

$$\partial_t u - a(u)\Delta u = f(u)\xi$$

on a two dimensional torus, with space white noise ξ, one has $2/3 < \alpha < 1$, the only elements $\tau \in \mathbb{B}_\circ^{-0}$ with an even number of ζ_1 noises are the trees

$$\tau_1 = \zeta_1 \mathcal{I}(\zeta_1) = $$  , $$\tau_2 = \zeta_3 \mathcal{I}(\zeta_1)\mathcal{I}_{(0,2)}(\zeta_1) = $$  .

Here the thick line denotes the operator $\mathcal{I}$ and the double line denotes $\mathcal{I}_{(0,2)}$. The noise symbol ζ_1 is denoted by a white circle, while ζ_3 is denoted by a circled dot. The corresponding characters are

$$\ell^\varepsilon_{a(v_0)}((t,x),\tau_1) = \int_{(-\infty,t)\times\mathbf{R}} Q_{t-s}^{a(v_0)}(x,y)\,C^\varepsilon(x-y)\,dsdy,$$
$$\ell^\varepsilon_{a(v_0)}((t,x),\tau_2) = \int_{\{(-\infty,t)\times\mathbf{R}\}^2} Q_{t-s}^{a(v_0)}(x,y)\,\partial_x^2 Q_{t-s'}^{a(v_0)}(x,y')\,C^\varepsilon(y-y')\,dsds'dydy'.$$

By (4.23) of Proposition 26 we can replace $Q^{a(v_0)}(x,y)$ above by $Z^{a(v_0(x))}(x-y)$ up to integrable kernels. One has for instance

$$\int_{(-\infty,t)\times\mathbf{R}} \mathsf{G}_{0,0}^{(0+)}(t-s,x,y)\,C^\varepsilon(x-y)\,dsdy \sim \int_{-\infty}^t \mathsf{G}_{0,0}^{(0+)}(t-s,x,x)\,ds \sim \int_{-\infty}^t \frac{e^{\gamma(t-s)}}{(t-s)^{1-}}\,ds < \infty;$$

a similar estimate holds for $\ell^\varepsilon_{a(v_0)}((t,x),\tau_2)$. Recall that γ is negative in this section. One thus has

$$\ell^\varepsilon_\lambda(\tau_1) = \int_{(-\infty,t)\times\mathbf{R}} Z_{t-s}^\lambda(x-y)\,C^\varepsilon(x-y)\,dsdy,$$
$$\sim -\frac{1}{2\pi\lambda}\int_{\mathbf{R}} \log|y|\,C^\varepsilon(y)\,dy$$
$$\ell^\varepsilon_\lambda(\tau_2) = \int_{\{(-\infty,t)\times\mathbf{R}\}^2} Z_{t-s}^\lambda(x-y)\,\partial_x^2 Z_{t-s'}^\lambda(x-y')\,C^\varepsilon(y-y')\,dsds'dydy'$$
$$= -\frac{1}{\lambda}\int_{(-\infty,t)\times\mathbf{R}\times\mathbf{R}} Z_{t-s}^\lambda(x-y)\,\delta_0(x-y')\,C^\varepsilon(y-y')\,dsdydy'$$
$$\sim \frac{1}{2\pi\lambda^2}\int_{\mathbf{R}} \log|y|\,C^\varepsilon(y)\,dy.$$

The action of the characters acting on trees with nonzero $\boldsymbol{p}$-decorations can be estimated similarly using Lemma 28, showing that Assumption 2 holds in that case. Then formula (1.5) takes the form

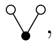
$$\Big(\ell^\varepsilon_{a(\cdot)}(\tau_1) f'f + \ell^\varepsilon_{a(\cdot)}(\tau_2) a'f^2\Big)(u^\varepsilon) = c^\varepsilon\left(\frac{f'f}{a} - \frac{a'f^2}{a^2}\right)(u^\varepsilon)$$

with a constant $c^\varepsilon = -\frac{1}{2\pi}\int_{\mathbf{R}} \log|y|\,C^\varepsilon(y)dy$. This matches with the previous works on the subject by Bailleul, Debussche & Hofmanová [3], Furlan & Gubinelli [14] and Otto & Weber [23].

4.5.2 – Quasilinear generalized (KPZ) equation with regularized noise. We work in this paragraph in the one dimensional space torus. Let ξ be the mildly singular case of a spacetime Gaussian noise of parabolic regularity $\alpha - 2$ with $\frac{1}{2} < \alpha < \frac{2}{3}$ and consider the quasilinear equation

$$\partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x^2 u).$$

Then the only elements $\tau \in \mathbb{B}_\circ^{-0}$ with an even number of noise symbols ζ_1 are the trees

$\tau_1 = \zeta_1\,\mathcal{I}(\zeta_1) =$ , $\quad \tau_2 = \zeta_3\,\mathcal{I}(\zeta_1)\mathcal{I}_{(0,2)}(\zeta_1) =$ , $\quad \tau_3 = \zeta_2\,\mathcal{I}_{(0,1)}(\zeta_1)^2 =$ , (4.25)

where the thin line denotes the operator $\mathcal{I}_{(0,1)}$ and the black dot denotes the symbol ζ_2. Since all of them have homogeneity $2\alpha - 2 > -1$, we can replace the kernel $Q^{a(v_0)}$ by $Z^{a(v_0)}$ up to integrable kernel $\mathsf{G}^{(\alpha)}$ by (4.23) of Proposition 26. Thus they satisfy Assumption 2 and the counterterm takes the form

$$\Big(\ell^\varepsilon_{a(\cdot)}(\tau_1)\,f'f + \ell^\varepsilon_{a(\cdot)}(\tau_2)\,gf^2 + \ell^\varepsilon_{a(\cdot)}(\tau_3)\,a'f\Big)(u^\varepsilon).$$

As mentioned in Gerencsér & Hairer's work [16], the renormalization constants are cancelled as follows. Actually

$$\begin{aligned}
\ell^\varepsilon_\lambda(\tau_1) &= -\int_{\mathbf{R}^2} C^\varepsilon(z) Z^\lambda(z)\,dz,\\
\ell^\varepsilon_\lambda(\tau_2) &= -\int_{(\mathbf{R}^2)^2} C^\varepsilon(z-z')\,\partial_x Z^\lambda(z)\partial_x Z^\lambda(z')\,dzdz' = -\int_{\mathbf{R}^2} C^\varepsilon(z)\,\overline{\partial_x Z^\lambda} * \partial_x Z^\lambda(z)\,dz,\\
\ell^\varepsilon_\lambda(\tau_3) &= -\int_{(\mathbf{R}^2)^2} C^\varepsilon(z-z') Z^\lambda(z)\,\partial_x^2 Z^\lambda(z')\,dzdz' = -\int_{\mathbf{R}^2} C^\varepsilon(z)\,\overline{Z^\lambda} * \partial_x^2 Z^\lambda(z)\,dz,
\end{aligned}$$

where $\overline{f}(z) := f(-z)$ for any function f and we assume that the function

$$C^\varepsilon(z) := \mathbb{E}\big[\xi_\varepsilon(z)\xi_\varepsilon(0)\big]$$

is even. As we see from the identity

$$\overline{\partial_x Z^\lambda} * \partial_x Z^\lambda(z) = -\overline{Z^\lambda} * \partial_x^2 Z^\lambda(z) = \frac{1}{2\lambda} Z^\lambda(|t|, x) + O(1),$$

that we have

$$\lambda \ell^\varepsilon_\lambda(\tau_2) = -\lambda \ell^\varepsilon_\lambda(\tau_3) = \ell^\varepsilon_\lambda(\tau_1) + O(1),$$

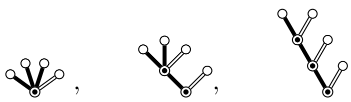
our formula matches with Gerencsér & Hairer's formula [16]

$$\ell^\varepsilon_{a(u^\varepsilon)}(\tau_1)\Big(f'f + \frac{gf^2}{a} - \frac{a'f}{a}\Big)(u^\varepsilon).$$

4.5.3 – Quasilinear generalized (KPZ) equation with space-time white noise. Let ξ be a spacetime Gaussian noise of parabolic regularity $\alpha - 2$ with $\frac{2}{5} < \alpha < \frac{1}{2}$ and consider the equation

$$\partial_t u - a(u)\partial_x^2 u = \xi.$$

Then the only elements $\tau \in \mathbb{B}_\circ^{-0}$ with an even number of noise symbols ζ_1 are the trees τ_2 from (4.25) together with the trees

 (4.26)

Since the last two trees have homogeneity $(4\alpha - 2)$ we can replace the kernel $Q^{a(v_0)}$ by $Z^{a(v_0)}$ as before. Note that Assumption 2 is not ensured at this stage since on the edge e whose lower node, respectively upper node, is associated with the spacetime variable (s, y), respectively (s', y'), the kernel $Q^{a(v_0)}_{s-s'}(y - y')$ is replaced by $Z^{a(v_0(y))}_{s-s'}(y - y')$ rahter than $Z^{a(v_0(x))}_{s-s'}(y - y')$. To show Assumption 2 we have to replace $Z^{a(v_0(y))}_{s-s'}(y - y')$ with $Z^{a(v_0(x))}_{s-s'}(y - y')$ up to a term of size $|x - y|$, which also smears the singularity of the Feynman diagram by α. Thus Assumption 2 holds for the trees in (4.26).

It turns out that the first tree of (4.26) is not involved in equation (1.5) because ξ is a centered Gaussian. Indeed, as we can here decompose

$$\ell^{\varepsilon}_{a(v_0)}\big(\cdot, [\text{tree}]\big) = -h^{\varepsilon}_{a(v_0)}\big(\cdot, [\text{tree}]\big) + 3h^{\varepsilon}_{a(v_0)}\big(\cdot, [\text{tree}]\big)h^{\varepsilon}_{a(v_0)}\big(\cdot, [\text{tree}]\big)$$

we see that the right hand side is zero because of Wick theorem for Gaussian random variables.

It remains to consider the tree τ_1. Since it has a homogeneity $2\alpha-2<-1$, it is not sufficient to replace $Q^{a(v_0)}$ by $Z^{a(v_0)}$. One sees however from (4.24) of Proposition 26 that we have

$$\begin{aligned}\ell^{\varepsilon}_{a(v_0)}\big((t,x), [\text{tree}]\big) = \int_{(\mathbf{R}^2)^2} \Big\{ & Z^{a(v_0(x))}_{t-s}(x-y)(\partial^2 Z_{t-s'})^{a(v_0(x))}(x-y') \\ & + a'(v_0(x))v_0'(x)Y^{a(v_0(x))}_{t-s}(x-y)(\partial^2 Z_{t-s'})^{a(v_0(x))}(x-y') \\ & + a'(v_0(x))v_0'(x)Z^{a(v_0(x))}_{t-s}(x-y)Y^{2,a(v_0(x))}_{t-s'}(x-y') \\ + \sum_{\gamma_1+\gamma_2\geq 0} & \mathsf{G}^{(\gamma_1)}_{0,0}(t-s,x,y)\mathsf{G}^{(\gamma_2)}_{0,0}(t-s',x,y')\Big\}\, C^{\varepsilon}(s-s',y-y')\, dsds'dydy'. \end{aligned} \tag{4.27}$$

The last term in (4.27) does not matter as one has the ε-uniform estimate

$$\begin{aligned}&\int_{(\mathbf{R}^2)^2} \mathsf{G}^{(\gamma_1)}_{0,0}(t-s,x,y)\mathsf{G}^{(\gamma_2)}_{0,0}(t-s',x,y')\, C^{\varepsilon}(s-s',y-y')\, dsds'dydy' \\ &\qquad\lesssim \int_{\mathbf{R}^2} \mathsf{G}^{(\gamma_1+\gamma_2-1)}_{0,0}(t-s,x,y)\, dsdy \lesssim \int_{-\infty}^{t} \frac{e^{\gamma(t-s)}}{(t-s)^{(1-(\gamma_1+\gamma_2))/2}} ds < \infty.\end{aligned}$$

Recall from Proposition 26 the parity of the functions Y. Although the second and third terms in (4.27) are not estimated as above, if we assume that the mollifier ρ_ε is an even function then C^ε is also an even function of its space argument, so the second and third terms in the right hand side of (4.27) vanish. In the end only the first term of (4.27) survives and Assumption 2 is satisfied with

$$\ell^{\varepsilon}_{\lambda}\big([\text{tree}]\big) = \int_{(\mathbf{R}^2)^2} Z^{\lambda}_{t-s}(x-y)\,(\partial^2 Z_{t-s'})^{\lambda}(x-y')\, C^{\varepsilon}(s-s',y-y')\, dsds'dydy'.$$

Eventually the counterterm takes the form

$$\Big\{ \ell^{\varepsilon}_{a(\cdot)}\big([\text{tree}]\big)\, a' + \ell^{\varepsilon}_{a(\cdot)}\big([\text{tree}]\big)\, a'a'' + \ell^{\varepsilon}_{a(\cdot)}\big([\text{tree}]\big)\, (a')^3 \Big\}(u^{\varepsilon}),$$

which matches Gerencsér's formula in Theorem 1.1 of [15]. We see on this formula the rule (4.17)-(4.18) in action.

In the case of the quasilinear generalized (KPZ) equation (1.1)

$$\partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2$$

driven by a one dimensional space time white noise ξ on the torus, the list of trees $\tau \in \mathbb{B}^{-0}_{\circ}$ with an even number of noise symbols ζ_1 contains, in addition to the previous trees, the trees τ_1, τ_3 from (4.25) and a number of other trees of homogeneity $4\alpha-2$. That Assumption 2 holds true for all the trees of homogeneity $4\alpha-2$ can be seen as for the trees of (4.26). The counterterms corresponding to τ_1 and τ_3 can be seen to satisfy Assumption 2 by a similar computation as in (4.27).

We note that the present analysis of equation (1.1) holds for a large classe of Gaussian spacetime noises of parabolic regularity $\alpha-2$, up to $\alpha>1/3$, as there are no other trees of homogeneity strictly smaller than -1 than those considered in the spacetime white noise case in that regularity range.

A – Appendix

We give in Appendix A.1 some elementary results on kernels that can be bounded by Gaussian-like kernels that were used in Section 4.5. In Appendix A.2 we prove some properties of the fundamental solutions of anisotropic parabolic operators following the arguments in [13, 12]. We believe that the results in this appendix are known but we could not find any suitable references. For the sake of generality for them we work on the space $\mathbf{R}^d$ and an anisotropic scaling $\mathfrak{s} = (\mathfrak{s}_j)_{j=1}^d \in \mathbf{N}^d$. Set

$$\begin{aligned}
&|\mathfrak{s}| := \textstyle\sum_{j=1}^d \mathfrak{s}_j,\\
&|k|_\mathfrak{s} := \textstyle\sum_{j=1}^d \mathfrak{s}_j k_j \text{ for } k = (k_j)_{j=1}^d \in \mathbf{N}_{>0}^d,\\
&\|x\|_\mathfrak{s} := \textstyle\sum_{j=1}^d |x_j|^{1/\mathfrak{s}_j} \text{ for } x = (x_j)_{j=1}^d \in \mathbf{R}^d,\\
&\partial_x^k := \textstyle\prod_{j=1}^d \partial_{x_j}^{k_j} \text{ for } k = (k_j)_{j=1}^d \in \mathbf{N}^d.
\end{aligned}$$

We consider the parabolic operator with time-independent coefficients

$$\partial_t - P(x, \partial_x) := \partial_t - \sum_{|k|_\mathfrak{s} \le N} a_k(x) \partial_x^k, \tag{A.1}$$

where N is an integer satisfying $N > \max_j \mathfrak{s}_j$.

A.1 – Gaussian kernels.

For $c > 0$ and $\zeta \in \mathbf{R}$, we write

$$\mathsf{G}_t^{(c,\zeta)}(x) := t^{(\zeta - |\mathfrak{s}|)/N} \exp\left\{ -c \sum_{j=1}^d \left(\frac{|x_j|^{N/\mathfrak{s}_j}}{t} \right)^{\mathfrak{s}_j/(N-\mathfrak{s}_j)} \right\}.$$

29 – Lemma. Let $\zeta_1, \zeta_2 \in \mathbf{R}$ and $c, c_1, c_2 > 0$.

(i) For any $\alpha \le \min_{1\le i\le d} \frac{N\mathfrak{s}_i}{N-\mathfrak{s}_i}$, one has

$$\|x-y\|_\mathfrak{s}^\alpha \, \mathsf{G}_t^{(c,\zeta_1)}(x,y) \lesssim \mathsf{G}_t^{(c',\zeta_1+\alpha)}(x,y)$$

for some $c' \in (0,c)$.

(ii) One has $\mathsf{G}_t^{(c,\zeta_1)}(x,y)\, \mathsf{G}_t^{(c,\zeta_2)}(x,y) = \mathsf{G}_t^{(2c,\zeta_1+\zeta_2-1)}(x,y)$.

(iii) If $c_1 < c_2$ and $\zeta_1, \zeta_2 > -N$, one has

$$\int_0^t \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_2,\zeta_1)}(x-y)\, \mathsf{G}_s^{(c_2,\zeta_2)}(y)\, dyds \le C\, \mathsf{G}_t^{(c_1,\zeta_1+\zeta_2+N)}(x).$$

(iv) If $c_1 < c_2$, $\zeta_1 > -N + |\mathfrak{s}|$, and $\zeta_2 > -N$,

$$\int_0^t \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_1,\zeta_1)}(x-y)\, \mathsf{G}_s^{(c_2,\zeta_2)}(y)\, dyds \le C\, \frac{\Gamma(\frac{\zeta_1 - |\mathfrak{s}| + N}{N})\Gamma(\frac{\zeta_2+N}{N})}{\Gamma(\frac{\zeta_1+\zeta_2-|\mathfrak{s}|+N}{N})}\, \mathsf{G}_t^{(c_1,\zeta_1+\zeta_2+N)}(x).$$

The constant C in items (iii) and (iv) is an absolute constant whose precise value has no importance.

Proof – The proof of the statements *(i)* and *(ii)* is elementary and left to the readers. For *(iii)* and *(iv)* we use the elementary inequality

$$\mathsf{G}_{t-s}^{(c,0)}(x-y)\mathsf{G}_s^{(c,0)}(y) \le t^{|\mathfrak{s}|/N}(t-s)^{-|\mathfrak{s}|/N} s^{-|\mathfrak{s}|/N} \mathsf{G}_t^{(c,0)}(x).$$

For *(iii)* we have

$$\int_0^t \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_2,\zeta_1)}(x-y)\mathsf{G}_s^{(c_2,\zeta_2)}(y)$$
$$= \int_0^t (t-s)^{(\zeta_1+|\mathfrak{s}|)/N} s^{(\zeta_2+|\mathfrak{s}|)/N} \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_1,0)}(x-y)\mathsf{G}_{t-s}^{(c_2-c_1,0)}(x-y)\mathsf{G}_s^{(c_1,0)}(y)\mathsf{G}_s^{(c_2-c_1,0)}(y)dyds$$
$$\le t^{|\mathfrak{s}|/N} \int_0^t (t-s)^{\zeta_1/N} s^{\zeta_2/N} \mathsf{G}_t^{(c_1,0)}(x) \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_2-c_1,0)}(x-y)\mathsf{G}_s^{(c_2-c_1,0)}(y)dyds.$$

Since the last integral with respect to y is bounded by

$$\min\{(t-s)^{-|\mathfrak{s}|/N}, s^{-|\mathfrak{s}|/N}\} \int_{\mathbf{R}^d} \mathsf{G}_1^{(c_2-c_1,0)}(y)dy \le C_{c_2-c_1}(t/2)^{-|\mathfrak{s}|/N}$$

with

$$C_c := \int_{\mathbf{R}^d} \mathsf{G}_1^{(c,0)}(x)dx,$$

we have

$$\int_0^t \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_2,\zeta_1)}(x-y)\mathsf{G}_s^{(c_2,\zeta_2)}(y) \le C_{c_2-c_1} 2^{|\mathfrak{s}|/N} \int_0^t (t-s)^{\zeta_1/N} s^{\zeta_2/N} ds\, \mathsf{G}_t^{(c_1,0)}(x)$$
$$\le C_{c_2-c_1} t^{(\zeta_1+\zeta_2+N)/N} \frac{\Gamma(\frac{\zeta_1+N}{N})\Gamma(\frac{\zeta_2+N}{N})}{\Gamma(\frac{\zeta_1+\zeta_2+N}{N})} \mathsf{G}_t^{(c_1,0)}(x).$$

For *(iv)* we have

$$\int_0^t \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_1,\zeta_1)}(x-y)\mathsf{G}_s^{(c_2,\zeta_2)}(y)\, dsdy$$
$$= \int_0^t (t-s)^{\zeta_1/N} s^{(\zeta_2+|\mathfrak{s}|)/N} \int_{\mathbf{R}^d} \mathsf{G}_{t-s}^{(c_1,0)}(x-y)\mathsf{G}_s^{(c_1,0)}(y)\mathsf{G}_s^{(c_2-c_1,0)}(y)dyds$$
$$\le t^{|\mathfrak{s}|/N} \int_0^t (t-s)^{(\zeta_1-|\mathfrak{s}|)/N} s^{\zeta_2/N} \mathsf{G}_t^{(c_1,0)}(x) \int_{\mathbf{R}^d} \mathsf{G}_s^{(c_2-c_1,0)}(y)dyds$$
$$\le C_{c_2-c_1} t^{(\zeta_1+\zeta_2+N)/N} \frac{\Gamma(\frac{\zeta_1-|\mathfrak{s}|+N}{N})\Gamma(\frac{\zeta_2+N}{N})}{\Gamma(\frac{\zeta_1+\zeta_2-|\mathfrak{s}|+N}{N})} \mathsf{G}_t^{(c_1,0)}(x).$$

▷

30 – *Definition. For $\zeta \in \mathbf{R}$, denote by $\mathsf{G}^{(\zeta)}$ the class of families of function $A = \{A_t\}_{t>0} \subset C(\mathbf{R}^d \times \mathbf{R}^d)$ satisfying*

$$|A_t(x,y)| \le Ce^{\gamma t}\, \mathsf{G}_t^{(c,\zeta)}(x-y)$$

for some positive constants C, γ, c. For $\beta_1, \beta_2 \in [0,1]$, denote by $\mathsf{G}_{\beta_1,\beta_2}^{(\zeta)}$ the class of families $A = \{A_t\}_{t>0} \in \mathsf{G}^{(\zeta)}$ satisfying

$$\big|A_t(x,y) - A_t(x',y)\big| \le Ce^{\gamma t}\, |x-x'|^{\kappa_1} \Big\{\mathsf{G}_t^{(c,\zeta-\kappa_1)}(x-y) + \mathsf{G}_t^{(c,\zeta-\kappa_1)}(x'-y)\Big\},$$
$$\big|A_t(x,y) - A_t(x,y')\big| \le Ce^{\gamma t}\, |y-y'|^{\kappa_2} \Big\{\mathsf{G}_t^{(c,\zeta-\kappa_2)}(x-y) + \mathsf{G}_t^{(c,\zeta-\kappa_2)}(x-y')\Big\}$$

for any $\kappa_1 \in [0,\beta_1)$ and $\kappa_2 \in [0,\beta_2)$.

For any families $A = \{A_t\}_{t>0}, B = \{B_t\}_{t>0} \subset C(\mathbf{R}^d \times \mathbf{R}^d)$, we define the spacetime convolution

$$(A * B)_t(x,y) := \int_{(0,t)\times\mathbf{R}} A_{t-s}(x,z)B_s(z,y)\, dsdz.$$

31 – *Lemma. Let $\zeta_1, \zeta_2 \in \mathbf{R}$, $\beta_1, \beta_2 \in [0,1]$, $A = \{A_t\}_{t>0} \in \mathsf{G}_{\beta_1,0}^{(\zeta_1)}$, $B = \{B_t\}_{t>0} \in \mathsf{G}_{0,\beta_2}^{(\zeta_2)}$.*

(i) *If* $\zeta_1, \zeta_2 > -N$, *then* $A * B \in \mathsf{G}^{(\zeta_1+\zeta_2+N)}_{\gamma_1,\gamma_2}$ *with* $\gamma_i = \min\{\beta_i, \zeta_i + N\}$ *for* $i \in \{1, 2\}$.

(ii) *Let* $\zeta_1 = -N$, $\zeta_2 > -N$, *and* $B \in \mathsf{G}^{(\zeta_2)}_{\delta,\beta_2}$ *with some* $\delta > 0$. *If*

$$\left| \int_{\mathbf{R}} A_t(x, y) dy \right| \lesssim t^{-1+\delta/N} \tag{A.2}$$

then $A * B \in \mathsf{G}^{(\zeta_2)}_{0,\gamma_2}$. *If in addition to* (A.2), A_t *is first differentiable with respect to the first variable*, $\partial_{x_i} A = \{\partial_{x_i} A_t\}_{t>0} \in \mathsf{G}^{(-N-\mathfrak{s}_i)}$ *and*

$$\left| \int_{\mathbf{R}} \partial_{x_i} A_t(x, y) dy \right| \lesssim t^{-1-\mathfrak{s}_i/N+\delta/N}, \qquad (1 \le i \le d) \tag{A.3}$$

then $A * B \in \mathsf{G}^{(\zeta_2)}_{\min\{\gamma_1,\delta\},\gamma_2}$.

(iii) *Let* $\zeta_1 > -N$, $\zeta_2 = -N$, *and* $A \in \mathsf{G}^{(\zeta_1)}_{\beta_1,\delta}$ *with some* $\delta > 0$. *Then* $A * B \in \mathsf{G}^{(\zeta_1)}_{\gamma_1,0}$, *respectively* $\mathsf{G}^{(\zeta_1)}_{\gamma_1,\gamma_2}$, *if a similar condition to* (A.2), *respectively* (A.2) *and* (A.3), *holds with the roles of first and second variables reversed.*

Proof – Item *(i)* follows from Lemma 29. To show item *(ii)* we decompose

$$(A * B)_t(x, y) = \int_{t/2}^{t} ds \int_{\mathbf{R}^d} A_{t-s}(x, z) B_s(z, y)\, dz + \int_0^{t/2} ds \int_{\mathbf{R}^d} A_{t-s}(x, z) B_s(z, y)\, dz.$$

Since the second term of the right hand side can be treated in the same way as *(i)*, we consider the first term. If (A.2) holds we can decompose the first term into

$$\begin{aligned}
&\left| \int_{t/2}^{t} ds \int_{\mathbf{R}^d} A_{t-s}(x, z) B_s(x, y)\, dz \right| + \left| \int_{t/2}^{t} ds \int_{\mathbf{R}^d} A_{t-s}(x, z) \big(B_s(z, y) - B_s(x, y)\big)\, dz \right| \\
&\lesssim e^{\gamma t} \int_{t/2}^{t} \left| \int_{\mathbf{R}} A_{t-s}(x, z) dz \right| \mathsf{G}^{(c,\zeta_2)}_s(x - y)\, ds \\
&\qquad + e^{\gamma t} \int_{t/2}^{t} ds \int_{\mathbf{R}^d} \mathsf{G}^{(c,-N)}_{t-s}(x - z) \|z - x\|^{\varepsilon}_{\mathfrak{s}} \big(\mathsf{G}^{(c,\zeta_2-\varepsilon)}_s(z - y) + \mathsf{G}^{(c,\zeta_2-\varepsilon)}_s(x - y)\big)\, dz \\
&\lesssim \mathsf{G}^{(c,\zeta_2)}_t(x - y)\, e^{\gamma t} \int_{t/2}^{t} (t - s)^{-1+\delta/N}\, ds \\
&\qquad + e^{\gamma t} \int_{t/2}^{t} \int_{\mathbf{R}^d} \mathsf{G}^{(c,\varepsilon-N)}_{t-s}(x - z) \big(\mathsf{G}^{(c,\zeta_2-\varepsilon)}_s(z - y) + \mathsf{G}^{(c,\zeta_2-\varepsilon)}_s(x - y)\big)\, dz ds \\
&\lesssim e^{\gamma t}\, \mathsf{G}^{(c',\zeta_2)}_t(x - y)
\end{aligned}$$

for any $\varepsilon \in (0, \delta)$ and $c' < c$. To show the Hölder estimate of $A * B$ for the first variable, it is sufficient to consider the difference between x and $x_h = x + (h, 0, \ldots, 0)$. Let $C_{t,s}(x, y) := \int_{\mathbf{R}^d} A_{t-s}(x, z) B_s(z, y) dz$ and we decompose

$$C_{t,s}(x_h, y) - C_{t,s}(x, y) = h \int_0^1 \int_{\mathbf{R}^d} \partial_{x_1} A_{t-s}(x_{\theta h}, z) B_s(z, y) dz d\theta$$

We decompose the right hand side as above and have

$$\begin{aligned}
|C_{t,s}(x_h, y) - C_{t,s}(x, y)| \lesssim e^{\gamma t} |h| \int_0^1 d\theta \bigg\{ &(t - s)^{-1-\mathfrak{s}_1/N+\delta/N} \mathsf{G}^{(c,\zeta_2)}_t(x_{\theta h} - y) \\
&+ \int_{\mathbf{R}^d} \mathsf{G}^{(c,\varepsilon-N-\mathfrak{s}_1)}_{t-s}(x_{\theta h} - z) \mathsf{G}^{(c,\zeta_2-\varepsilon)}_s(z - y) dz \bigg\}.
\end{aligned}$$

By interpolation between the above Gaussian estimate we have that $|C_{t,s}(x_h, y) - C_{t,s}(x, y)|$ is bounded above by $e^{\gamma t} |h|^{\tau}$ times

$$\int_0^1 \left\{(t-s)^{-1-\tau\mathfrak{s}_1/N+\delta/N}\mathsf{G}_t^{(c,\zeta_2)}(x_{\theta h}-y)+\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c,\varepsilon-N-\tau\mathfrak{s}_1)}(x_{\theta h}-z)\mathsf{G}_s^{(c,\zeta_2-\varepsilon)}(z-y)dz\right\}d\theta,$$

for any $\tau\in[0,1]$. If $\tau<\varepsilon/\mathfrak{s}_1$, the s-integral is finite and we have

$$\begin{aligned}|(A*B)_t(x_h,y)-(A*B)_t(x,y)| &\lesssim e^{\gamma t}\|x_h-x\|_{\mathfrak{s}}^{\tau\mathfrak{s}_1}\int_0^1\mathsf{G}_t^{(c',\zeta_2-\tau\mathfrak{s}_1)}(x_\theta-y)d\theta\\ &\lesssim e^{\gamma t}\|x_h-x\|_{\mathfrak{s}}^{\tau\mathfrak{s}_1}\big(\mathsf{G}_t^{(c',\zeta_2-\tau\mathfrak{s}_1)}(x_h-y)+\mathsf{G}_t^{(c',\zeta_2-\tau\mathfrak{s}_1)}(x-y)\big).\end{aligned}$$

This gives the required estimate since $\varepsilon<\delta$ is arbitrary. ▷

A.2 – Fundamental solution. First we consider the parabolic operator (A.1) when the coefficients a_k are x-independent constants. Then we write $P(\partial_x)=\sum_{|k|_{\mathfrak{s}}\le N}a_k\partial_x^k$.

32 – *Lemma. Assume there exists a constant $\delta>0$ such that the inequality*

$$\operatorname{Re}P(i\xi)=\operatorname{Re}\sum_{|k|_{\mathfrak{s}}\le N}a_k(i\xi)^k\le-\delta\|\xi\|_{\mathfrak{s}}^N \tag{A.4}$$

holds for any $\xi\in\mathbf{R}^d$. Then for any $\varepsilon>0$, $k\in\mathbf{N}^d$, and $n\in\mathbf{N}$, there exist positive constants C and c which depend only on $\mathfrak{s},N,A:=\max_k|a_k|,\delta,\varepsilon,k,n$ such that the fundamental solution $Z_t(x)$ of $\partial_t-P(\partial_x)$ satisfies

$$\big|\partial_t^n\partial_x^kZ_t(x)\big|\le Ce^{\varepsilon t}\mathsf{G}_t^{(c,-|k|_{\mathfrak{s}}-Nn)}(x) \tag{A.5}$$

for any $t>0$ and $x\in\mathbf{R}^d$. When $k=0$ and $n=0$, the constant C depends only on δ.

Proof – By definition $Z_t(x)$ is obtained as the Fourier inverse transform of the function $e^{tP(i\xi)}$ of $\xi\in\mathbf{R}^d$. Following the arguments in [13, Chapter 9, Section 2] we consider the bound of $e^{tP(i\xi-\eta)}$ for $\eta,\xi\in\mathbf{R}^d$. By the binomial theorem, we can expand

$$P(i\xi-\eta)=P(i\xi)+R(\xi,\eta),$$

where $R(\xi,\eta)$ is a linear combination of monomials $\xi^\ell\eta^m$ with $|\ell+m|_{\mathfrak{s}}\le N$ and $m\neq0$, and with coefficients depending only on $\{a_k\}$. For any $\varepsilon>0$, by Young's inequality we have

$$\begin{aligned}|R(\xi,\eta)|&\le A\sum_{\alpha\in\mathbf{N}_{>0},\,\beta\in\mathbf{N},\,\alpha+\beta\le N}\|\xi\|_{\mathfrak{s}}^\alpha\|\eta\|_{\mathfrak{s}}^\beta\\ &\le\varepsilon+\frac{\delta}{2}\|\xi\|_{\mathfrak{s}}^N+c'\|\eta\|_{\mathfrak{s}}^N\le\varepsilon+\frac{\delta}{2}\|\xi\|_{\mathfrak{s}}^N+c\sum_{j=1}^d|\eta_j|^{N/\mathfrak{s}_j},\end{aligned}$$

where c' and c are positive constants depending only on A,ε,δ. By the condition (A.4), we have

$$|e^{tP(i\xi-\eta)}|\le e^{t\operatorname{Re}P(i\xi)}e^{t|R(\xi,\eta)|}\le\exp\left\{t\left(\varepsilon-\frac{\delta}{2}\|\xi\|_{\mathfrak{s}}^N+c\sum_{j=1}^d|\eta_j|^{N/\mathfrak{s}_j}\right)\right\}.$$

By using the Cauchy's theorem we have

$$\begin{aligned}|Z_t(x)|&=\left|\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}e^{ix\cdot\xi}e^{tP(i\xi)}d\xi\right|=\left|\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}e^{ix\cdot(\xi+i\eta)}e^{tP(i\xi-\eta)}d\xi\right|\\ &\le\frac{e^{\varepsilon t}}{(2\pi)^d}\exp\left(-x\cdot\eta+ct\sum_{j=1}^d|\eta_j|^{N/\mathfrak{s}_j}\right)\int_{\mathbf{R}^d}\exp\left(-\frac{\delta t}{2}\|\xi\|_{\mathfrak{s}}^N\right)d\xi\end{aligned}$$

for any $\eta\in\mathbf{R}^d$. If we choose η_j as

$$\eta_j = (\operatorname{sgn} x_j)\left(\frac{|x_j|}{cp_j t}\right)^{1/(p_j-1)}, \tag{A.6}$$

where $p_j = N/\mathfrak{s}_j$, then

$$-x_j\eta_j + ct|\eta_j|^{p_j} = -\frac{p_j-1}{p_j}\left(\frac{|x_j|^{p_j}}{cp_j t}\right)^{1/(p_j-1)},$$

which becomes the argument of the exponential in (A.5). The integral over ξ becomes $Ct^{-|\mathfrak{s}|/N}$ with some constant C depending only on δ.

For the derivatives $\partial_x^k Z_t(x)$ we can derive the required estimate by a similar way from the identity

$$\partial_x^k Z_t(x) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{ix\cdot\xi}(i\xi)^k e^{tP(i\xi)}d\xi = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{ix\cdot(\xi+i\eta)}(i\xi-\eta)^k e^{tP(i\xi-\eta)}d\xi.$$

We decompose $(i\xi-\eta)^k$ into the linear combination of monomials $\xi^\ell\eta^m$ with $\ell+m=k$. The integral of $|\xi^\ell|\exp(-\frac{\delta t}{2}\|\xi\|_{\mathfrak{s}}^N)$ over ξ becomes the factor $Ct^{-(|\mathfrak{s}|+|\ell|_{\mathfrak{s}})/N}$. For the choice of η as in (A.6) we have

$$|\eta^m| = t^{-|m|_{\mathfrak{s}}/N}\prod_{j=1}^d\left(\frac{|x_j|}{cp_j t^{1/p_j}}\right)^{m_j/(p_j-1)}.$$

Since any powers of $|x_j|/t^{1/p_j}$ are absorbed in the exponential part of (A.5) and the factor $t^{-|m|_{\mathfrak{s}}/N}$ remains, we have the required estimate for $\partial_x^k Z_t(x)$. We have similar estimates for the time derivatives because $\partial_t^n Z_t(x) = (P(\partial_x))^n Z_t(x)$. ▷

Next we consider the x-dependent coefficients $a_k(x)$. We call a function $Q_t(x,y)$ defined on $t>0$ and $x,y\in\mathbf{R}^d$ a fundamental solution of $\partial_t - P(x,\partial_x)$ if it satisfies

- for fixed y, the function $(t,x)\mapsto Q_t(x,y)$ is C^1 in t and $\mathcal{C}_{\mathfrak{s}}^N$ in x, and satisfies $\big(\partial_t - P(x,\partial_x)\big)Q_t(x,y)=0$,
- for every $f\in C_b(\mathbf{R}^d)$,

$$\lim_{t\downarrow 0}\int_{\mathbf{R}^d} Q_t(x,y)f(y)dy = f(x).$$

There exists a unique fundamental solution under the assumptions of the following theorem – see e.g. Theorem *6* of [24], where only isotropic metric is considered but it is easy to modify the argument to anistropic case. See also Theorem *16* of [13, Chapter 1] for the case that all coefficients $a_k(x)$ are sufficiently regular.

33 – *Theorem. Assume the following conditions for the functions $a_k(x)$.*

– *There exists a constant constant $\delta>0$ such that the inequality*

$$\operatorname{Re} P(x,i\xi) \le -\delta\|\xi\|_{\mathfrak{s}}^N \tag{A.7}$$

holds for any $x,\xi\in\mathbf{R}^d$.

– *The functions a_k are in C^α for some $\alpha\in(0,1)$ and*

$$A := \max_{|k|_{\mathfrak{s}}\le N}\sup_{x\in\mathbf{R}^d}|a_k(x)| < \infty, \qquad H := \max_{|k|_{\mathfrak{s}}\le N}\sup_{x,y\in\mathbf{R}^d}\frac{|a_k(x)-a_k(y)|}{\|x-y\|_{\mathfrak{s}}^\alpha} < \infty.$$

Then for any $k\in\mathbf{N}^d$ and $n\in\mathbf{N}$ with $|k|_{\mathfrak{s}}+Nn\le N$, there exist positive constants C,c,γ which depend only on $\mathfrak{s},N,\delta,A,H,k,n$ such that the fundamental solution $Q_t(x,y)$ of $\partial_t - P(x,\partial_x)$ satisfies

$$\big|\partial_t^n\partial_x^k Q_t(x,y)\big| \le Ce^{\gamma t}\mathsf{G}_t^{(c,-|k|_{\mathfrak{s}}-Nn)}(x-y) \tag{A.8}$$

for any $t > 0$ *and* $x, y \in \mathbf{R}^d$. *Furthermore, for any* $k \in \mathbf{N}^d$ *with* $|k|_{\mathfrak{s}} \le N$ *and* $\beta \in (0,1)$, *there exist positive constants* C, c, γ *which depend only on* $\mathfrak{s}, N, \delta, A, H, k, \beta$ *such that, the Hölder estimate for the first variable*

$$\big|\partial_x^k Q_t(x,y) - \partial_x^k Q_t(x',y)\big| \le Ce^{\gamma t}\|x-x'\|_{\mathfrak{s}}^{\beta}\big(\mathsf{G}_t^{(c,-|k|_{\mathfrak{s}}-Nn)}(x-y) + \mathsf{G}_t^{(c,-|k|_{\mathfrak{s}}-Nn)}(x'-y)\big) \tag{A.9}$$

holds for any $t > 0$ *and* $x, x', y \in \mathbf{R}^d$.

We prove this theorem following [13, Chapter 9]. Let $L_t(x,y) := Z_t^y(x)$ be the fundamental solution of $\partial_t - P(y, \partial_x)$ for fixed y. We aim to construct the fundamental solution $Q_t(x,y)$ in the form

$$Q = L + L * \Phi. \tag{A.10}$$

with some family $\Phi = \{\Phi_t(x,y)\}$ of functions. We set

$$K_t(x,y) := (P(x,\partial_x) - \partial_t)L_t(x,y) = \big(P(x,\partial_x) - P(y,\partial_x)\big)Z_t^y(x-y).$$

Then $Q_t(x,y)$ satisfies $\big(\partial_t - P(x,\partial_x)\big)Q_t(x,y) = 0$ if and only if

$$\Phi = K + K * \Phi.$$

This implies that the formal solution Φ is given by the form

$$\Phi_t(x,y) = \sum_{m=1}^{\infty} K_t^{(m)}(x,y), \qquad K^{(m)} := K^{*m} = K^{(m-1)} * K. \tag{A.11}$$

The series (A.11) is actually absolutely convergent and we can obtain $Q_t(x,y)$ by the formula (A.10).

34 – Lemma. $\{\partial_t^n \partial_x^k L_t\}_{t>0}$ *is in the class* $\mathsf{G}_{1,\alpha}^{(-|k|_{\mathfrak{s}}-Nn)}$ *for any* $(n,k) \in \mathbf{N} \times \mathbf{N}^d$.

Proof – The Gaussian estimate and the Hölder estimate for the first variable immediately follow from Lemma 32. The Hölder estimate for the second variable comes from the analyticity of Z with respect to the coefficients and the Hölder regularity of $a_k(x)$. See Lemma 4 of [13, Chapter 9] for details. ▷

35 – Lemma. $\{K_t\}_{t>0}$ *is in the class* $\mathsf{G}_{\alpha,\alpha}^{(\alpha-N)}$.

Proof – Since $K_t(x,y) = \big(P(x,\partial_x) - P(y,\partial_x)\big)Z_t^y(x-y)$, we have

$$|K_t(x,y)| \lesssim \|x-y\|_{\mathfrak{s}}^{\alpha} e^{\varepsilon t}\mathsf{G}_t^{(c,-N)}(x-y) \lesssim e^{\varepsilon t}\mathsf{G}_t^{(c_1,\alpha-N)}(x-y) \tag{A.12}$$

for some $c_1 < c$, where $\|x-y\|_{\mathfrak{s}}^{\alpha}$ is absorbed in the exponential of $\mathsf{G}_t^{(c,-N)}$ instead of the reduction of the coefficient c. The Hölder estimate for both variables are obtained by a similar way. ▷

36 – Lemma. *One has* $\{\Phi_t\}_{t>0} \subset \mathsf{G}_{\alpha,\alpha}^{(\alpha-N)}$.

Proof – First we show the estimates

$$\big|K_t^{(m)}(x,y)\big| \le Ce^{\varepsilon t}\frac{B^m t^{m\alpha/N-1}}{\Gamma(\frac{m\alpha-|\mathfrak{s}|}{N})}\mathsf{G}_t^{(c',0)}(x-y) \tag{A.13}$$

for some constants $c', C, B > 0$ which depend only on $\mathfrak{s}, N, \delta, A, H, \varepsilon$. Let m_0 be the smallest integer m_0 such that $m_0\alpha > |\mathfrak{s}|$. Up to $m \le m_0$, (A.13) is inductively obtained by Lemma 29-(3). Indeed, starting from (A.12) we have

$$|K_t^{(m)}(x,y)| \lesssim e^{\varepsilon t}\big(\mathsf{G}^{(c_{m-1},(m-1)\alpha-N)} * \mathsf{G}^{(c_1,\alpha-N)}\big)_t(x) \lesssim e^{\varepsilon t}\mathsf{G}_t^{(c_m,m\alpha-N)}(x)$$

for some $c_m < c_{m-1}$. For $m > m_0$, we use Lemma 29-(4) to obtain

$$|K_t^{(m)}(x,y)| \le e^{\varepsilon t}\frac{B^{m-1}}{\Gamma(\frac{(m-1)\alpha-|\mathfrak{s}|}{N})}\big(\mathsf{G}^{(c',(m-1)\alpha-N)} * \mathsf{G}^{(c_1,\alpha-N)}\big)_t(x)$$
$$\le e^{\varepsilon t}\frac{B^{m-1}C\Gamma(\frac{\alpha}{N})}{\Gamma(\frac{m\alpha-|\mathfrak{s}|}{N})}\mathsf{G}_t^{(c',m\alpha-N)}(x).$$

Hence (A.13) holds with $B = C\Gamma(\frac{\alpha}{N})$. Summing up (A.13) over $m \ge 1$, we have

$$|\Phi_t(x,y)| \lesssim e^{\gamma t}t^{\alpha/N-1}\mathsf{G}_t^{(c',0)}(x-y) = e^{\gamma t}\mathsf{G}_t^{(c',\alpha-N)}(x-y).$$

for some C, c, γ. The Hölder estimates are obtained by applying Lemma 31-(1) into the formula.

$$\Phi = K + K * \Phi = K + \Phi * K.$$

▷

Proof of Theorem 33 – It is sufficient to apply Lemma 31 to the formula

$$\partial_x^k Q = \partial_x^k L + \partial_x L^k * \Phi.$$

to get $\partial_x^k Q \in \mathsf{G}_{\alpha,\alpha}^{(-|k|_\mathfrak{s})}$. If $|k|_\mathfrak{s} < N$, then Lemma 31-(1) is enough. If $|k|_\mathfrak{s} = N$, we use Lemma 31-(2) by noting that

$$\left|\int_{\mathbf{R}^d}\partial_x^\ell L_t(x,y)dy\right| = \lim_{z\to x}\left|\int_{\mathbf{R}^d}\big(\partial_x^\ell Z_t^y(x-y) - \partial_x^\ell Z_t^z(x-y)\big)dy\right|$$
$$\le Ce^{\varepsilon t}\int_{\mathbf{R}^d}\|x-z\|_\mathfrak{s}^\alpha \mathsf{G}_t^{(c,-|\ell|_\mathfrak{s})}(x-z)\,dz \lesssim e^{\varepsilon t}t^{\alpha/N-|\ell|_\mathfrak{s}}$$

for any $\ell \in \mathbf{N}^d\setminus\{0\}$. ▷

We return to the proof of Proposition 27.

Proof of Proposition 27 – We omit the proof of Hölder estimates because it is an easy modification. Recall that we can expand $\partial_x^k Q_t^b = \partial_x^k Q_t^b(x,y)$ into the series

$$\partial_x^k Q^b = \partial_1^k L + \partial_x^k L * \Phi = \partial_x^k L + \sum_{m=1}^\infty \partial_x^k L * K^{(m)}, \tag{A.14}$$

where $L_t(x,y) := Z^b(y)(x-y)$, $K_t^{(1)}(x,y) := (b(x)-b(y))\partial_x^2 L_t(x,y)$, and $K^{(m+1)} = K^{(1)} * K^{(m)}$ as in the proof of Theorem 33. (4.22) is obtained in the proof of Theorem 33. For (4.22), it is sufficient to use Lemma 26 with $M = 1$ to replace $b(y)$ with $b(x)$.

It remains to show (4.23). We expand each of the far right hand side of (A.14) as

$$\partial_x^k L_t(x,y) = (\partial_x^k Z_t)^{b(x)}(y-x) + t(\partial_x^{k+2}Z_t)^{b(x)}(y-x)(b(y)-b(x)) + \mathsf{G}_{1,1}^{(2-k)},$$
$$\sum_{m=1}^\infty \partial_x^k L * K^{(m)} = \partial_x L * K + \mathsf{G}_{1,1}^{(2-k)}$$

and compare the second term of the right hand side of the first equation and the first term of the right hand side of the second equation. We decompose

$$t(\partial_x^{k+2}Z_t)^{b(x)}(y-x)(b(y)-b(x)) + (\partial_x^k L * K)_t(x,y)$$
$$= \big(b(y)-b(x)\big)\int_{(0,t)\times\mathbf{R}}(\partial_z^k Z_{t-s})^{b(x)}(x-z)(\partial_z^2 Z_s)^{b(x)}(z-y)\,dsdz$$
$$+ \int_{(0,t)\times\mathbf{R}}(\partial_x^k Z_{t-s})^{b(z)}(x-z)\big(b(z)-b(y)\big)(\partial_z^2 Z_s)^{b(y)}(z-y)\,dsdz$$

$$= \int_{(0,t)\times\mathbf{R}} (\partial_x^k Z_{t-s})^{b(x)}(x-z)\big(b(z)-b(x)\big)(\partial_z^2 Z_s)^{b(x)}(z-y)\,dsdz$$
$$+ \int_{(0,t)\times\mathbf{R}} (\partial_x^k Z_{t-s})^{b(x)}(x-z)\big(b(z)-b(y)\big)\big((\partial_z^2 Z_s)^{b(y)}-(\partial_z^2 Z_s)^{b(x)}\big)(z-y)\,dsdz$$
$$+ \int_{(0,t)\times\mathbf{R}} \big((\partial_x^k Z_{t-s})^{b(z)}-(\partial_x^k Z_{t-s})^{b(x)}\big)(x-z)\big(b(z)-b(y)\big)(\partial_z^2 Z_s)^{b(y)}(z-y)\,dsdz$$
$$=: I_1(x,y)+I_2(x,y)+I_3(x,y).$$

For I_1 we further decompose

$$I_1(x,y) = b'(x)\int_{(0,t)\times\mathbf{R}} (\partial_x^k Z_{t-s})^{b(x)}(x-z)\,(z-x)\,(\partial_z^2 Z_s)^{b(x)}(z-y)\,dsdz$$
$$+ \int_{(0,t)\times\mathbf{R}} (\partial_x^k Z_{t-s})^{b(x)}(x-z)\big(b(z)-b(x)-b'(x)(z-x)\big)\partial_z^2 Z_s^{b(x)}(z-y)\,dsdz$$
$$=: b'(x)Y_t^{b(x)}(x-y)+J(x,y).$$

For J, by the integration by parts and Lemma 31, we have

$$J(x,y) = \int_{(0,t)\times\mathbf{R}} (\partial_x^{k+1} Z_{t-s})^{b(x)}(x-z)\big(b(z)-b(x)-b'(x)(z-x)\big)(\partial_z Z_s)^{b(x)}(z-y)\,dsdz$$
$$- \int_{(0,t)\times\mathbf{R}} (\partial_x^k Z_{t-s})^{b(x)}(x-z)\big(b'(z)-b'(x)\big)\partial_z Z_s^{b(x)}(z-y)\,dsdz$$

$$\lesssim \int_{(0,t)\times\mathbf{R}} \mathsf{G}_{t-s}^{(c,-1-k)}(x,z)|z-x|^2\mathsf{G}_s^{(c,-1)}(z,y)dsdz + \int_{(0,t)\times\mathbf{R}} \mathsf{G}_{t-s}^{(c,-k)}(x,z)|z-x|\mathsf{G}_s^{(c,-1)}(z,y)\,dsdz$$
$$\lesssim \mathsf{G}_t^{(c',2-k)}(x,y)$$

for some $c>c'>0$. We have that $Y_t^{k,\lambda} = -(x\partial^k Z^\lambda)*\partial^2 Z^\lambda \in \mathsf{G}_{1,1}^{(1-k)}$ by the similar argument. The proof of $I_2, I_3 \in \mathsf{G}_{1,1}^{(2-k)}$ are more direct because $b(z)-b(y)$ reduces the singularity of $\partial_z^2 Z^\lambda(z-y)$, while we use Lemma 31-(ii) for I_2 with $k=2$. ▷

In the rest of this appendix we prove Theorem *3* in the more general form stated in Theorem *37* below. Let $\mathcal{L} = \partial_t - P(x,\partial_x)$ be the parabolic operator satisfying the assumptions of Theorem *33*, and denote by $\Gamma_t(x,y)$ be its fundamental solution. For any bounded continuous function f on $\mathbf{R}^d$, we define

$$\big(\Gamma_t f\big)(x) := \int_{\mathbf{R}^d} \Gamma_t(x,y)f(y)dy.$$

It should be noted that Γ_t satisfies the semigroup property $\Gamma_t\Gamma_s f = \Gamma_{t+s}f$. Let then define for $\beta<0$ the space $\mathcal{C}_{\mathfrak{s}}^\beta(\mathcal{L})$ as the completion of the set of bounded continuous functions f on $\mathbf{R}^d$ under the norm

$$\|f\|_{\mathcal{C}_{\mathfrak{s}}^\beta(\mathcal{L})} := \sup_{0<t\le 1} t^{-\beta/N}\|\Gamma_t f\|_{L^\infty(\mathbf{R}^2)}.$$

37 – Theorem. For any $\beta'<\beta<0$, the embedding $\mathcal{C}_{\mathfrak{s}}^\beta(\mathcal{L}) \subset \mathcal{C}_{\mathfrak{s}}^{\beta'}(\mathcal{L})$ is compact.

Proof – We first remark the important inequalities

$$\|\Gamma_t f\|_{\mathrm{Lip}} \lesssim t^{(\beta-1)/N}\|f\|_{\mathcal{C}_s(\mathcal{L})},$$
$$\|\Gamma_t f - f\|_{L^\infty} \lesssim t^{1/N}\|f\|_{\mathrm{Lip}},$$

where Lip is the space of Lipschitz functions with respect to $\|\cdot\|_{\mathfrak{s}}$. They are easy consequences of the semigroup property and the Gaussian estimates of Γ_t. Let $\{f_n\}_{n\in\mathbf{N}}$ be a bounded sequence in $\mathcal{C}^\beta_\mathfrak{s}(\mathcal{L})$. Since

$$\|\Gamma_t f_n\|_{\mathrm{Lip}} \lesssim t^{(\beta-1)/N}$$

and

$$\big\|\Gamma_{t+s} f_n - \Gamma_t f_n\big\|_{L^\infty} \lesssim s^{1/N}\|\Gamma_t f_n\|_{\mathrm{Lip}} \lesssim s^{1/N} t^{(\beta-1)/N},$$

the family of functions $(t,x) \mapsto \Gamma_t f_n(x)$ is uniformly bounded and equicontinuous on compact sets of $(0,\infty) \times \mathbf{R}^d$. Hence by Ascoli-Arzelà theorem, there exists a subsequence $\{n(k)\}_{k\in\mathbf{N}^2}$ such that the locally uniform convergence limit $F_t(x) = \lim_{k\to\infty} \Gamma_t f_{n(k)}$ exists. Since F_t inherits the semigroup property and the above Hölder continuity with respect to t from $\Gamma_t f_n$, we have

$$\|F_{t+s} - F_t\|_{\mathcal{C}^{\beta-1}_\mathfrak{s}(\mathcal{L})} = \sup_{0<r\le 1} r^{-(\beta-1)/N} \|F_{t+s+r} - F_{s+r}\|_{L^\infty} \lesssim s^{1/N}.$$

This implies $\{F_t\}_{t>0}$ is a Cauchy sequence in $\mathcal{C}^\beta_\mathfrak{s}(\mathcal{L})$ as $t \to 0$. Setting $F = \lim_{t\to 0} F_t \in \mathcal{C}^{\beta-1}_\mathfrak{s}(\mathcal{L})$, we also have that $F = \lim_{k\to\infty} f_{n(k)}$ in $\mathcal{C}^{\beta-1}_\mathfrak{s}(\mathcal{L})$ by the standard 3ε argument. By interpolation this convergence also holds in $\mathcal{C}^{\beta'}_\mathfrak{s}(\mathcal{L})$ for any $\beta' < \beta$. $\rhd$

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