4. Flows driven by rough paths

Guide for this section

We have seen in part I of the course that a $\mathcal{C}^1$-approximate flow on a Banach space E defines a unique flow $\varphi = \big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ on E such that the inequality

$$\big\|\varphi_{ts} - \mu_{ts}\big\|_\infty \leqslant c|t-s|^a \tag{4.1}$$

holds for some positive constants c and $a>1$, for all $0\leqslant s\leqslant t\leqslant T$ sufficiently close. The construction of φ is actually quite explicit, for if we denote by $\mu_{\pi_{ts}}$ the composition of the maps $\mu_{t_{i+1}t_i}$ along the times t_i of a partition π_{ts} of an interval $[s,t]$, the map μ_{ts} satisfies the estimate

$$\big\|\varphi_{ts} - \mu_{\pi_{ts}}\big\|_\infty \leqslant \frac{2}{1-2^{1-a}}\, c_1^2\, T\, \big|\pi_{ts}\big|^{a-1}, \tag{4.2}$$

where c_1 is the constant that appears in the definition of a $\mathcal{C}^1$-approximate flow

$$\big\|\mu_{tu}\circ\mu_{us} - \mu_{ts}\big\|_{\mathcal{C}^1} \leqslant c_1|t-s|^a. \tag{4.3}$$

It follows in particular from equation (4.1) that if μ depends continuously on some metric space-valued parameter λ, with respect to the $\mathcal{C}^0$-topology, and if identity (4.3) holds uniformly for λ moving in a bounded set say, then φ depends continuously on λ, as a uniform limit of continuous functions.

The point about the machinery of $\mathcal{C}^1$-approximate flows is that they actually pop up naturally in a number of situations, under the form of a local in time description of the dynamics under study; nothing else than a kind of Taylor expansion. This was quite clear in exercice 1 on the ordinary controlled differential equation

$$dx_t = V_i(x_t)\, dh^i_t, \tag{4.4}$$

with $\mathcal{C}^1$ real-valued controls $h^1,\dots,h^\ell$ and $\mathcal{C}^2_b$ vector fields $V_1,\dots,V_\ell$ in $\mathbb{R}^d$. The 1-step Euler scheme

$$\mu_{ts}(x) = x + \big(h^i_t - h^i_s\big)V_i(x)$$

defines in that case a $\mathcal{C}^1$-approximate flow which has the awaited Taylor-type expansion, in the sense that one has

$$f\big(\mu_{ts}(x)\big) = f(x) + \big(h^i_t - h^i_s\big)\big(V_i f\big)(x) + O\big(|t-s|^{>1}\big) \tag{4.5}$$

for any function f of class $\mathcal{C}^2_b$; but μ fails to be a flow. Its associated flow is not only a flow, it also satisfies equation (4.5) as a consequence of identity (4.1).

We shall proceed in a very similar way to give some meaning and solve the rough differential equation on flows

$$d\varphi = V\,dt + \mathrm{F}^{\otimes}\mathbf{X}(dt), \tag{4.6}$$

where V is a Lipschitz continuous vector field on E and $\mathrm{F} = \big(V_1,\dots,V_\ell\big)$ is a collection of sufficiently regular vector fields on E, and $\mathbf{X}$ is a Hölder p-rough path over

$\mathbb{R}^\ell$. A solution flow to equation (4.6) will be defined as a flow on E with a uniform Taylor-Euler expansion of the form

$$f\big(\varphi_{ts}(x)\big) = f(x) + \sum_{|I|\leqslant[p]} X^I_{ts}\big(V_I f\big)(x) + O\big(|t-s|^{>1}\big), \tag{4.7}$$

where $I = \big(i_1,\dots,i_k\big) \in [\![1,\ell]\!]^k$ is a multi-index with size $k \leqslant [p]$, and X^I_{ts} stands for the coordinates of $\mathbf{X}_{ts}$ in the canonical basis of $T^{[p],1}_\ell$. The vector field V_i is seen here as a 1^{st}-order differential operator, and $V_I = V_{i_1}\cdots V_{i_k}$ as the k^{th}-order differential operator obtained by applying successively the operators V_{i_n}.

For $V=0$ and $\mathbf{X}$ the (weak geometric) p-rough path canonically associated with an $\mathbb{R}^\ell$-valued $\mathcal{C}^1$ control h, with $2\leqslant p<3$, equation (4.7) becomes

$$f\big(\varphi_{ts}(x)\big) = f(x) + \big(h^i_t - h^i_s\big)\big(V_i f\big)(x) + \left(\int_s^t\int_s^r dh^j_u\, dh^k_r\right)\big(V_jV_kf\big)(x) + O\big(|t-s|^{>1}\big), \tag{4.8}$$

which is nothing else than Taylor formula at order 2 for the solution to the ordinary differential equation (4.4) started at x at time s. Condition (4.7) is a natural analogue of (4.8) and its higher order analogues.

There is actually a simple way of constructing a map μ_{ts} which satisfies the Euler expansion (4.7). It can be defined as the time 1 map associated with an ordinary differential equation constructed form the V_i and their brackets, and where $\mathbf{X}_{ts}$ appears as a parameter under the form of its logarithm. That these maps μ_{ts} form a $\mathcal{C}^1$-approximate flow will eventually appear as a consequence of the fact that the time 1 map of a differential equation formally behaves as an exponential map, in some algebraic sense.

The notationally simpler case of flows driven by weak geometric Hölder p-rough paths, with $2\leqslant p<3$, is first studied in section 4.1 before studying the general case in section 4.2. The latter case does not present any additional conceptual difficulty, so a reader which who would like to get the core ideas can read section 4.1 only. The two sections have been written with almost similar words on purpose.

4.1. **A warm up: working with weak geometric Hölder p-rough paths, with $2\leqslant p<3$.** Let V be a $\mathcal{C}^2_b$ vector field on E and $V_1,\dots,V_\ell$ be $\mathcal{C}^3_b$ vector fields on E. Let $\mathbf{X} = (X,\mathbb{X})$ be a Hölder weak geometric p-rough path over $\mathbb{R}^\ell$, with $2\leqslant p<3$. Let μ_{ts} be the well-defined time 1 map associated with the ordinary differential equation

$$\dot y_u = (t-s)V(y_u) + \left(X^i_{ts}V_i + \frac{1}{2}\,\mathbb{X}^{jk}_{ts}\big[V_j,V_k\big]\right)(y_u), \quad 0\leqslant u\leqslant 1; \tag{4.9}$$

it associates to any $x\in E$ the value at time 1 of the solution of the above equation started from x; it is well-defined since V and the V_i are in particular globally Lipschitz. It is a direct consequence of classical results on ordinary differential equations, and of the definition of the topology on the space of Hölder weak geometric p-rough paths, that the maps μ_{ts} depend continuously on $\big((s,t),\mathbf{X}\big)$ in the uniform topology, and that

$$\big\|\mu_{ts} - \mathrm{Id}\big\|_{\mathcal{C}^2} = o_{t-s}(1). \tag{4.10}$$

Also, considering y_u as a function of x, it is elementary to see that one has the estimate

$$\big\|y_u - \mathrm{Id}\big\|_{\mathcal{C}^1} \leqslant c\big(1 + \|\mathbf{X}\|^3\big)|t-s|^{1/p}, \quad 0 \leqslant u \leqslant 1, \tag{4.11}$$

for some constant depending only on V and the V_i.

4.1.1. *From Taylor expansions to flows driven by rough paths.* The next proposition shows that μ_{ts} has precisely the kind of Taylor-Euler expansion property that we expect from a solution to a rough differential equation, as described in the introduction to that part of the course.

PROPOSITION **1.** *There exists a positive constant c, depending only on V and the V_i, such that the inequality*

$$\Big\|f\circ\mu_{ts} - \Big\{f + (t-s)Vf + X^i_{ts}\big(V_i f\big) + \mathbb{X}^{jk}_{ts}\big(V_j V_k f\big)\Big\}\Big\|_\infty \leqslant c\Big(1 + \|\mathbf{X}\|^3\Big)\,\|f\|_{\mathcal{C}^3}\,|t-s|^{\frac{3}{p}} \tag{4.12}$$

holds for any $f \in \mathcal{C}^3_b$.

The proof of this proposition and the following one are based on the following elementary identity, obtained by applying twice the identity

$$f(y_r) = f(x) + (t-s)\int_0^r (Vf)(y_u)\,du + X^i_{ts}\int_0^r \big(V_i f\big)(y_u)\,du + \frac{1}{2}\,\mathbb{X}^{jk}_{ts}\int_0^r \Big(\big[V_j,V_k\big]f\Big)(y_u)\,du,$$

first to f, then to $Vf, V_i f$ and $\big[V_j,V_k\big]f$ inside the integrals. One has

$$\begin{aligned}
f\big(\mu_{ts}(x)\big) &= f(x) + (t-s)\int_0^1 (Vf)\big(y_u\big)du + X^i_{ts}\int_0^1 \big(V_i f\big)\big(y_{s_1}\big)ds_1 + \frac{1}{2}\,\mathbb{X}^{jk}_{ts}\int_0^1 \Big(\big[V_j,V_k\big]f\Big)\big(y_u\big)du \\
&= f(x) + (t-s)\big(Vf\big)(x) + (t-s)\int_0^1 \big\{(Vf)\big(y_u\big) - (Vf)(x)\big\}du \\
&\quad + X^i_{ts}\big(V_i f\big)(x) + (t-s)X^i_{ts}\int_0^1\int_0^{s_1} \big(VV_i f\big)\big(y_{s_2}\big)\,ds_2 ds_1 \\
&\qquad + \frac{1}{2}\,X^{i'}_{ts}X^i_{ts}\big(V_{i'}V_i f\big)(x) + X^i_{ts}X^{i'}_{ts}\int_0^1\int_0^{s_1} \Big\{\big(V_{i'}V_i f\big)\big(y_{s_2}\big) - \big(V_{i'}V_i f\big)(x)\Big\}\,ds_2 ds_1 \\
&\qquad + \frac{1}{2}\,X^i_{ts}\mathbb{X}^{jk}_{ts}\int_0^1\int_0^{s_1} \Big(\big[V_j,V_k\big]V_i f\Big)\big(y_{s_2}\big)\,ds_2 ds_1 \\
&\quad + \frac{1}{2}\,\mathbb{X}^{jk}_{ts}\Big(\big[V_j,V_k\big]f\Big)(x) + \frac{1}{2}\,\mathbb{X}^{jk}_{ts}\int_0^1 \Big\{\Big(\big[V_j,V_k\big]f\Big)\big(y_u\big) - \Big(\big[V_j,V_k\big]f\Big)(x)\Big\}\,du.
\end{aligned}$$

Note that since the Hölder p-rough path $\mathbf{X}$ is assumed to be weak geometric, the symmetric part of $\mathbb{X}_{ts}$ is equal to $\frac{1}{2}X_{ts}\otimes X_{ts}$, so one has

$$f\big(\mu_{ts}(x)\big) = f(x) + (t-s)(Vf)(x) + X^i_{ts}\big(V_i f\big)(x) + \mathbb{X}^{jk}_{ts}\big(V_j V_k f\big)(x) + \epsilon^f_{ts}(x), \tag{4.13}$$

where the remainder ϵ^f_{ts} is defined by the formula

$$
\begin{aligned}
\epsilon^f_{ts}(x) := (t-s)\int_0^1 \big\{(Vf)(y_u) - (Vf)(x)\big\}du + (t-s)X^i_{ts}\int_0^1\int_0^{s_1} \big(VV_if\big)\big(y_{s_2}\big)\,ds_2ds_1 \\
+ X^i_{ts}X^{i'}_{ts}\int_0^1\int_0^{s_1}\Big\{\big(V_{i'}V_if\big)\big(y_{s_2}\big) - \big(V_{i'}V_if\big)(x)\Big\}\,ds_2ds_1 \\
+ \frac{1}{2}\,X^i_{ts}\mathbb{X}^{jk}_{ts}\int_0^1\int_0^{s_1}\Big(\big[V_j,V_k\big]V_if\Big)\big(y_{s_2}\big)\,ds_2ds_1 \\
+ \frac{1}{2}\,\mathbb{X}^{jk}_{ts}\int_0^1\Big\{\Big(\big[V_j,V_k\big]f\Big)(y_u) - \Big(\big[V_j,V_k\big]f\Big)(x)\Big\}\,du.
\end{aligned}
$$

Proof of proposition 1 – It is elementary to use estimate (4.11) and the regularity assumptions on the vector fields V, V_i to see that the remainder ϵ^f_{ts} is bounded above by a quantity of the form $c\big(1+\|\mathbf{X}\|^3\big)\,\|f\|_{\mathcal{C}^3}\,|t-s|^{\frac{3}{p}}$, for some constant depending only on V and the V_i. ▷

A further look at formula (4.29) and estimate (4.11) also make it clear that

$$
\Big\|\epsilon^f_{ts}\Big\|_{\mathcal{C}^1} \leqslant c\big(1+\|\mathbf{X}\|^3\big)|t-s|^{\frac{3}{p}}, \tag{4.14}
$$

for a constant c depending only on V and the V_i. This is the key remark for proving the next proposition.

Proposition **2.** *The family* $\big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ *forms a* $\mathcal{C}^1$*-approximate flow.*

It will be convenient in the following proof to slightly abuse notations and write $V_I(x)$ for $\big(V_I\mathrm{Id}\big)(x)$, for any multi-index I and point x.

Proof – We first use formula (4.13) to write

$$
\mu_{tu}\big(\mu_{us}(x)\big) = \mu_{us}(x)+(t-u)V\big(\mu_{us}(x)\big)+X^i_{tu}V_i\big(\mu_{us}(x)\big)+\mathbb{X}^{jk}_{tu}\big(V_jV_k\big)\big(\mu_{us}(x)\big)+\epsilon^{\mathrm{Id}\,;\,[p]}_{tu}\big(\mu_{us}(x)\big).
$$

We deal with the term $(t-u)V\big(\mu_{us}(x)\big)$ using estimate (4.11) and the Lipschitz character of V:

$$
\big|(t-u)V\big(\mu_{us}(x)\big) - (t-u)V(x)\big| \leqslant c\big(1+\|\mathbf{X}\|^3\big)\,|u-s|^{\frac{3}{p}}.
$$

The remainder $\epsilon^{\mathrm{Id}}_{tu}\big(\mu_{us}(x)\big)$ has a $\mathcal{C}^1$-norm bounded above by $c\big(1+\|\mathbf{X}\|^3\big)^2|t-u|^{\frac{3}{p}}$, by the remark preceeding proposition 2 and the $\mathcal{C}^1$-estimate (4.11) on μ_{us}. We develop $V_i\big(\mu_{us}(x)\big)$ to deal with the term $X^i_{tu}V_i\big(\mu_{us}(x)\big)$. As

$$
V_i\big(\mu_{us}(x)\big) = V_i(x) + (u-s)\big(VV_i\big)(x) + X^{i'}_{us}\big(V_{i'}V_i\big)(x) + \mathbb{X}^{jk}_{us}\big(V_jV_kV_i\big)(x) + \epsilon^{V_i}_{us}(x)
$$

we have

$$
X^i_{tu}V_i\big(\mu_{us}(x)\big) = X^i_{tu}V_i(x) + X^{i'}_{us}X^i_{tu}\big(V_{i'}V_i\big)(x) + \varepsilon^{V_i}_{tu,us}(x), \tag{4.15}
$$

where the remainder $\varepsilon^{V_i}_{tu,us}$ has $\mathcal{C}^1$-norm bounded above by

$$
\Big\|\varepsilon^{V_i}_{tu,us}\Big\|_{\mathcal{C}^1} \leqslant c\big(1+\|\mathbf{X}\|^3\big)\,|u-s|^{\frac{3}{p}}, \tag{4.16}
$$

for a constant c depending only on V and the V_n. Set

$$\varepsilon_{tu,us}(x) = \sum_{i=1}^{\ell} \varepsilon^{V_i}_{tu,us}(x).$$

The term $\mathbb{X}^{jk}_{tu}\big(V_jV_k\big)\big(\mu_{us}(x)\big)$ is simply dealt with writing

$$\mathbb{X}^{jk}_{tu}\big(V_jV_k\big)\big(\mu_{us}(x)\big) = \mathbb{X}^{jk}_{tu}\big(V_jV_k\big)(x) + \mathbb{X}^{jk}_{tu}\Big\{\big(V_jV_k\big)\big(\mu_{us}(x)\big) - \mathbb{X}^{jk}_{tu}\big(V_jV_k\big)(x)\Big\}, \tag{4.17}$$

and using estimate (4.11) and the $\mathcal{C}^1_b$ character of V_jV_k to see that the last term on the right hand side has a $\mathcal{C}^1$-norm bounded above by $c\big(1+\|\mathbf{X}\|^3\big)\,|u-s|^{\frac{3}{p}}$. All together, this gives

$$\begin{aligned}
\mu_{tu}\big(\mu_{us}(x)\big) &= \mu_{us}(x) + (t-u)V(x) + X^i_{tu}V_i(x) + X^{i'}_{us}X^i_{tu}\big(V_{i'}V_i\big)(x) + \mathbb{X}^{jk}_{tu}\big(V_jV_k\big)(x) + \varepsilon_{tu,us}(x) \\
&= x + (u-s)V(x) + X^i_{us}V_i(x) + \mathbb{X}^{jk}_{us}\big(V_jV_k\big)(x) + \epsilon^{\mathrm{Id}}_{us}(x) + (\cdots) \\
&= x + (t-s)V(x) + X^i_{ts}V_i(x) + \mathbb{X}^{jk}_{ts}\big(V_jV_k\big)(x) + \epsilon^{\mathrm{Id}}_{us}(x) + \varepsilon_{tu,us}(x) \\
&= \mu_{ts}(x) + \epsilon^{\mathrm{Id}}_{us}(x) + \varepsilon_{tu,us}(x),
\end{aligned}$$

so it follows from estimates (4.14) and (4.16) that μ is indeed a $\mathcal{C}^1$-approximate flow. ▷

The above proof makes it clear that one can take for constant c_1 in the $\mathcal{C}^1$-approximate flow property (**??**) for μ the constant $c\big(1+\|\mathbf{X}\|^3\big)$, for a constant c depending only on V and the V_i.

Recalling proposition 1 describing the maps μ_{ts} in terms of Euler expansion, the following definition of a solution flow to a rough differential equation is to be thought of as defining a notion of solution in terms of uniform Euler expansion

$$\Big\| f\circ\varphi_{ts} - \Big\{ f + X^i_{ts}V_if + \mathbb{X}^{jk}_{ts}V_jV_kf \Big\}\Big\|_\infty \leqslant c\,|t-s|^{>1}.$$

Definition 3. *A* **flow** $\big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ *is said to* **solve the rough differential equation**

$$d\varphi = V\,dt + \mathrm{F}^{\otimes}\,\mathbf{X}(dt) \tag{4.18}$$

if there exists a constant $a>1$ independent of $\mathbf{X}$ and two possibly $\mathbf{X}$-dependent positive constants δ and c such that

$$\big\|\varphi_{ts} - \mu_{ts}\big\|_\infty \leqslant c\,|t-s|^a \tag{4.19}$$

holds for all $0\leqslant s\leqslant t\leqslant T$ with $t-s\leqslant\delta$.

If for instance $\mathbf{X}$ is the weak geometric Hölder p-rough path canonically associated with an $\mathbb{R}^\ell$-valued piecewise smooth path h, it follows from exercice 1, and the fact that the iterated integral $\int_s^t\int_s^r dh_u\otimes dh_r$ has size $|t-s|^2$, that the solution flow to the rough differential equation

$$d\varphi = V\,dt + \mathrm{F}^{\otimes}\mathbf{X}(dt)$$

is the flow associated with the ordinary differential equation

$$\dot{y}_t = V(y_t)dt + V_i(y_t)\, dh^i_t.$$

The following well-posedness result follows directly from theorem **??** on $\mathcal{C}^1$-approximate flows and proposition 2.

THEOREM **4.** *The rough differential equation on flows*

$$d\varphi = V dt + \mathrm{F}^{\otimes}\, \mathbf{X}(dt)$$

has a unique solution flow; it takes values in the space of uniformly Lipschitz continuous homeomorphisms of E *with uniformly Lipschitz continuous inverses, and depends continuously on* $\mathbf{X}$*.*

PROOF – Applying theorem **??** on $\mathcal{C}^1$-approximate flows to μ we obtain the existence of a unique flow φ satisfying condition (4.34), for δ small enough; it further satisfies the inequality

$$\|\varphi_{ts} - \mu_{\pi_{ts}}\|_\infty \leqslant c\big(1 + \|\mathbf{X}\|^3\big)^2 T\, \big|\pi_{ts}\big|^{a-1}, \tag{4.20}$$

for any partition π_{ts} of $[s,t] \subset [0,T]$ of mesh $\big|\pi_{ts}\big| \leqslant \delta$, as a consequence of inequality (**??**). As this bound is uniform in (s,t), and for $\mathbf{X}$ in a bounded set of the space of weak geometric Hölder p-rough paths, and since each map $\mu_{\pi_{ts}}$ is a continuous function of $\big((s,t),\mathbf{X}\big)$, the flow φ depends continuously on $\big((s,t),\mathbf{X}\big)$.

To prove that φ is a homeomorphism, note that, with the notations of section **??**,

$$\Big(\mu^{(n)}_{ts}\Big)^{-1} = \mu^{-1}_{s_1 s_0} \circ \cdots \circ \mu^{-1}_{s_{2^n} s_{2^n-1}}, \qquad s_i = s + i2^{-n}(t-s),$$

can actually be written $\big(\mu^{(n)}_{ts}\big)^{-1} = \overline{\mu}_{s_{2^n} s_{2^n-1}} \circ \cdots \circ \overline{\mu}_{s_1 s_0}$, for the time 1 map $\overline{\mu}$ associated with the rough path $\mathbf{X}_{t-\bullet}$. As $\overline{\mu}$ enjoys the same properties as μ, the maps $\big(\mu^{(n)}_{ts}\big)^{-1}$ converge uniformly to some continuous map φ^{-1}_{ts} which satisfies by construction $\varphi_{ts} \circ \varphi^{-1}_{ts} = \mathrm{Id}$.

Recall that proposition **??** provides a uniform control of the Lipschitz norm of the maps φ_{ts}; the same holds for their inverses in view of the preceeding paragraph. We propagate this property from the set $\big\{(s,t) \in [0,T]^2\,;\, s \leqslant t,\ t-s \leqslant \delta\big\}$ to the whole of the $\big\{(s,t) \in [0,T]^2\,;\, s \leqslant t\big\}$ using the flow property of φ. ▷

REMARKS **5.** (1) **Friz-Victoir approach to rough differential equations.** *The continuity of the solution flow with respect to the driving rough path* $\mathbf{X}$ *has the following consequence, which justifies the point of view adopted by Friz and Victoir in their works. Suppose the Hölder weak geometric p-rough path* $\mathbf{X}$ *is the limit in the rough path metric of the canonical Hölder weak geometric p-rough paths* $\mathbf{X}^n$ *associated with some piecewise smooth* $\mathbb{R}^\ell$*-valued paths* $(x^n_t)_{0\leqslant t\leqslant T}$*. We have noticed that the solution flow* φ^n *to the rough differential equation*

$$d\varphi^n = V dt + \mathrm{F}^{\otimes}\mathbf{X}^n(dt)$$

is the flow associated with the ordinary differential equation

$$\dot y_u = V(y_u)du + V_i(y_u)\, d(x^n_u)^i.$$

As $\|\varphi^n - \varphi\|_\infty = o_n(1)$, *from the continuity of the solution flow with respect to the driving rough path, the flow* φ *appears in that case as a uniform limit of the elementary flows* φ^n. *A Hölder weak geometric p-rough path with the above property is called a Hölder geometric p-rough path; not all Hölder weak geometric p-rough path are Hölder geometric p-rough path* [14], *although there is little difference.*

(2) ***Time-inhomogeneous dynamics.*** *The above results have a straightforward generalization to dynamics driven by a time-dependent bounded drift* $V(s;\cdot)$ *which is Lipschitz continuous with respect to the time variable and* $\mathcal{C}^2_b$ *with respect to the space variable, uniformly with respect to time, and time-dependent vector fields* $V_i(s;\cdot)$ *which are Lipschitz continuous with respect to time, and* $\mathcal{C}^3_b$ *with respect to the space variable, uniformly with respect to time. We define in that case a* $\mathcal{C}^1$*-approximate flow by defining* μ_{ts} *as the time 1 map associated with the ordinary differential equation*

$$\dot y_u = (t-s)V(s;y_u) + X^i_{ts}V_i(s;y_u) + \mathbb{X}^{jk}_{ts}\big[V_j,V_k\big](y_u), \quad 0 \leqslant u \leqslant 1.$$

4.1.2. *Classical rough differential equations.* In the classical setting of rough differential equations, one is primarily interested in a notion of *solution path*, defined in terms of local Taylor-Euler expansion.

Definition **6.** *A* ***path*** $\big(z_s\big)_{0\leqslant s\leqslant T}$ *is said to* **solve the rough differential equation**

$$dz = V dt + \mathrm{F}\,\mathbf{X}(dt) \tag{4.21}$$

with initial condition x, *if* $z_0 = x$ *and there exists a constant* $a > 1$ *independent of* $\mathbf{X}$, *and two possibly* $\mathbf{X}$*-dependent positive constants* δ *and* c, *such that*

$$\Big| f(z_t) - \Big\{ f(z_s) + (t-s)(Vf)(z_s) + X^i_{ts}\big(V_i f\big)(z_s) + \mathbb{X}^{jk}_{ts}\big(V_jV_kf\big)(z_s) \Big\} \Big| \leqslant c\,\|f\|_{\mathcal{C}^3}\,|t-s|^a \tag{4.22}$$

holds for all $0 \leqslant s \leqslant t \leqslant T$, *with* $t-s \leqslant \delta$, *for all* $f \in \mathcal{C}^3_b$.

Theorem **7** (Lyons' universal limit theorem)**.** *The rough differential equation* (4.21) *has a unique solution path; it is a continuous function of* $\mathbf{X}$ *in the uniform norm topology.*

Proof – **a) Existence.** It is clear that if $\big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ stands for the solution flow to the equation

$$d\varphi = V dt + \mathrm{F}^{\otimes}\mathbf{X}(dt),$$

then the path $z_t := \varphi_{t0}(x)$ is a solution path to the rough differential equation (4.21) with initial condition x.

b) Uniqueness. Let agree to denote by $O_c(m)$ a quantity whose norm is bounded above by $c\,m$. Let α stand for the minimum of $\frac{3}{p}$ and the constant a

in definition 6, and let $y_\bullet$ be any other solution path. It satisfies by proposition 1 the estimate

$$\big|y_t - \varphi_{ts}(y_s)\big| \leqslant c|t-s|^\alpha.$$

Using the fact that the maps φ_{ts} are uniformly Lipschitz continuous, with a Lipschitz constant bounded above by L say, one can write for any $\epsilon > 0$ and any integer $k \leqslant \frac{T}{\epsilon}$

$$\begin{aligned} y_{k\epsilon} &= \varphi_{k\epsilon,(k-1)\epsilon}\big(y_{(k-1)\epsilon}\big) + O_c\big(\epsilon^\alpha\big) \\ &= \varphi_{k\epsilon,(k-1)\epsilon}\Big(\varphi_{(k-1)\epsilon,(k-2)\epsilon}\big(y_{(k-2)\epsilon}\big) + O_c\big(\epsilon^\alpha\big)\Big) + O_c\big(\epsilon^\alpha\big) \\ &= \varphi_{k\epsilon,(k-2)\epsilon}\big(y_{(k-2)\epsilon}\big) + O_{cL}\big(\epsilon^\alpha\big) + O_c\big(\epsilon^\alpha\big), \end{aligned}$$

and see by induction that

$$\begin{aligned} y_{k\epsilon} &= \varphi_{k\epsilon,(k-n)\epsilon}\big(y_{(k-n)\epsilon}\big) + O_{cL}\big((n-1)\epsilon^\alpha\big) + O_c\big(\epsilon^\alpha\big) \\ &= \varphi_{k\epsilon,0}(x) + O_{cL}\big(k\epsilon^\alpha\big) + o_\epsilon(1) \\ &= z_{k\epsilon} + O_{cL}\big(k\epsilon^\alpha\big) + o_\epsilon(1). \end{aligned}$$

Taking ϵ and k so that $k\epsilon$ converges to some $t \in [0,T]$, we see that $y_t = z_t$, since $\alpha > 1$.

The continuous dependence of the solution path $z_\bullet$ with respect to $\mathbf{X}$ is transfered from φ to $z_\bullet$. ▷

4.2. **The general case.** We have defined in the previous section a solution to the rough differential equation

$$d\varphi = V dt + \mathrm{F}^{\otimes}\mathbf{X}(dt),$$

driven by a weak geometric Hölder p-rough path, for $2 \leqslant p < 3$, as a flow with $(s,t;x)$−uniform Taylor-Euler expansion of the form

$$f\big(\varphi_{ts}(x)\big) = f(x) + (t-s)(Vf)(x) + X^i_{ts}\big(V_i f\big)(x) + \mathbb{X}^{jk}_{ts}\big(V_j V_k f\big)(x) + O\big(|t-s|^{>1}\big).$$

The definition of a solution flow in the general case will require from φ that it satisfies a similar expansion, of the form

$$f\big(\varphi_{ts}(x)\big) = f(x) + (t-s)(Vf)(x) + \sum_{|I|\leqslant[p]} X^I_{ts}\big(V_I f\big)(x) + O\big(|t-s|^{>1}\big). \tag{4.23}$$

As in the previous section, we shall obtain φ as the unique flow associated with some $\mathcal{C}^1$-approximate flow $\big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$, where μ_{ts} is the time 1 map associated with an ordinary differential equation constructed from the V_i and their brackets, and V and $\mathbf{X}_{ts}$. In order to avoid writing expressions with loads of indices (the $\mathbf{X}^I_{ts}$), I will first introduce in subsection 4.2.1 a coordinate-free way of working with rough paths and vector fields. A $\mathcal{C}^1$-approximate flow with the awaited Euler expansion will be constructed in subsection 4.2.2, leading to a general well-posedness result for rough differential equations on flows.

To make the crucial formula (4.29) somewhat shorter we assume in this section that $V = 0$. The reader is urged to workout by herself/himself the infinitesimal changes that have to be done in what follows in order to work with a non-null drift V. From hereon, the vector fields V_i are assumed to be of class $\mathcal{C}_b^{[p]+1}$. We denote by $\mathcal{C}_b^{[p]+1}(\mathrm{E},\mathrm{E})$ the set of $\mathcal{C}_b^{[p]+1}$ vector fields on E. We denote for by $\pi_k : T_\ell^\infty \to (\mathbb{R}^\ell)^k$ the natural projection operator and set $\pi_{\leqslant k} = \sum_{j\leqslant k} \pi_j$.

4.2.1. *Differential operators.* Let F be a continuous linear map from $\mathbb{R}^\ell$ to $\mathcal{C}_b^{[p]+1}(\mathrm{E},\mathrm{E})$ – one usually calls such a map a *vector field valued 1-form on* $\mathbb{R}^\ell$. For any $v \in \mathbb{R}^\ell$, we identify the $\mathcal{C}^{[p]+1}$ vector field $\mathrm{F}(v)$ on E with the first order differential operator

$$\mathrm{F}^\otimes(v) \,:\, g \in \mathcal{C}^1(\mathrm{E}) \mapsto (D.g)\big(\mathrm{F}(v)(\cdot)\big) \in \mathcal{C}^0(\mathrm{E});$$

in those terms, we recover the vector field $\mathrm{F}(v)$ as $\mathrm{F}^\otimes(v)\mathrm{Id}$. The map $\mathrm{F}^\otimes$ is extended to $T_\ell^{[p]+1}$ by setting

$$\mathrm{F}^\otimes(1) := \mathrm{Id} : \mathcal{C}^0(\mathrm{E}) \mapsto \mathcal{C}^0(\mathrm{E}),$$

and defining $\mathrm{F}^\otimes(v_1\otimes\cdots\otimes v_k)$, for all $1 \leqslant k \leqslant [p]+1$ and $v_1\otimes\cdots\otimes v_k \in (\mathbb{R}^\ell)^{\otimes k}$, as the k^{th}-order differential operator from $\mathcal{C}^k(\mathrm{E})$ to $\mathcal{C}^0(\mathrm{E})$, defined by the formula

$$\mathrm{F}^\otimes(v_1\otimes\cdots\otimes v_k) := \mathrm{F}^\otimes\big(v_1\big)\cdots\mathrm{F}^\otimes\big(v_k\big),$$

and by requiring linearity. So, we have the morphism property

$$\mathrm{F}^\otimes(\mathbf{e})\,\mathrm{F}^\otimes(\mathbf{e}') = \mathrm{F}^\otimes(\mathbf{e}\mathbf{e}') \tag{4.24}$$

for any $\mathbf{e},\mathbf{e}' \in T_\ell^{[p]+1}$ with $\mathbf{e}\mathbf{e}' \in T_\ell^{[p]+1}$. This condition on $\mathbf{e},\mathbf{e}'$ is required for if $\mathbf{e}' = v_1\otimes\cdots\otimes v_k$ with $v_i \in \mathbb{R}^\ell$, the map $\mathrm{F}^\otimes(\mathbf{e}')\mathrm{Id}$ from E to itself is $\mathcal{C}_b^{[p]+1-k}$, so $\mathrm{F}^\otimes(\mathbf{e})\,\mathrm{F}^\otimes(\mathbf{e}')$ only makes sense if $\mathbf{e}\mathbf{e}' \in T_\ell^{[p]+1}$. We also have

$$\Big[\mathrm{F}^\otimes(\mathbf{e}),\mathrm{F}^\otimes(\mathbf{e}')\Big] = \mathrm{F}^\otimes\big([\mathbf{e},\mathbf{e}']\big)$$

for any $\mathbf{e},\mathbf{e}' \in T_\ell^{[p]+1}$ with $\mathbf{e}\mathbf{e}'$ and $\mathbf{e}'\mathbf{e}$ in $T_\ell^{[p]+1}$. This implies in particular that $\mathrm{F}^\otimes(\Lambda)$ is actually a first order differential operator for any $\Lambda \in \mathfrak{g}_\ell^{[p]+1}$, that is a vector field. Note that for any $\Lambda \in \mathfrak{g}_\ell^{[p]+1}$ and $1 \leqslant k \leqslant [p]+1$, then $\Lambda^k := \pi_k(\Lambda)$ is an element of $\mathfrak{g}_\ell^{[p]}$, and the vector field $\mathrm{F}^\otimes\big(\Lambda^k\big)\mathrm{Id}$ is $\mathcal{C}_b^{[p]+1-k}$.

We extend $\mathrm{F}^\otimes$ to the unrestricted tensor space T_ℓ^∞ setting

$$\mathrm{F}^\otimes(\mathbf{e}) = \mathrm{F}^\otimes\big(\pi_{\leqslant [p]+1}\mathbf{e}\big) \tag{4.25}$$

for any $\mathbf{e} \in T_\ell^\infty$.

Consider as a particular case the map F defined for $u \in \mathbb{R}^\ell$ by the formula

$$\mathrm{F}(u) = u^i\,V_i(\cdot).$$

Using the formalism of this paragraph, an Euler expansion of the form

$$f\big(\varphi_{ts}(x)\big) = f(x) + \sum_{|I|\leqslant[p]} X_{ts}^I\big(V_I f\big)(x) + O\big(|t-s|^{>1}\big),$$

as in equation (4.23), becomes

$$f\big(\varphi_{ts}(x)\big) = \big(\mathrm{F}^\otimes\big(\mathbf{X}_{ts}\big)f\big)(x) + O\big(|t-s|^{>1}\big).$$

4.2.2. *From Taylor expansions to flows driven by rough paths: bis.* Let $2 \leqslant p$ be given, together with a $\mathfrak{G}_\ell^{[p]}$-valued weak-geometric Hölder p-rough path $\mathbf{X}$, defined on some time interval $[0,T]$, and some continuous linear map F from $\mathbb{R}^\ell$ to the set $\mathcal{C}_b^{[p]+1}(\mathrm{E},\mathrm{E})$ of vector fields on E. For any $0 \leqslant s \leqslant t \leqslant T$, denote by $\mathbf{\Lambda}_{ts}$ the logarithm of $\mathbf{X}_{ts}$, and let μ_{ts} stand for the well-defined time 1 map associated with the ordinary differential equation

$$\dot{y}_u = \mathrm{F}^{\otimes}\big(\mathbf{\Lambda}_{ts}\big)(y_u), \quad 0 \leqslant u \leqslant 1. \tag{4.26}$$

This equation is indeed an ordinary differential equation since $\mathbf{\Lambda}_{ts}$ is an element of $\mathfrak{g}_\ell^{[p]}$. For $2 \leqslant p < 3$, it reads

$$\dot{y}_u = X_{ts}^i V_i(y_u) + \frac{1}{2}\left(\mathbb{X}_{ts}^{jk} + \frac{1}{2}X_{ts}^j X_{ts}^k\right)\big[V_j, V_k\big](y_u), \quad 0 \leqslant u \leqslant 1.$$

As the tensor $X_{ts} \otimes X_{ts}$ is symmetric and the map $(j,k) \mapsto \big[V_j, V_k\big]$ is antisymmetric, this equation actually reads

$$\dot{y}_u = X_{ts}^i V_i(y_u) + \frac{1}{2}\mathbb{X}_{ts}^{jk}\big[V_j, V_k\big](y_u),$$

which is nothing else than equation (4.9), whose time 1 map defined the $\mathcal{C}^1$-approximate flow we studied in section 4.1.1.

It is a consequence of classical results from ordinary differential equations, and the definition of the norm on the space of weak-geometric Hölder p-rough paths, that the solution map $(r,x) \mapsto y_r$, with $y_0 = x$, depends continuously on $\big((s,t),\mathbf{X}\big)$ in $\mathcal{C}^0$-norm, and satisfies the following basic estimate. The next proposition shows that μ_{ts} has precisely the kind of Taylor-Euler expansion property that we expect from a solution to a rough differential equation.

$$\big\|y_r - \mathrm{Id}\big\|_{\mathcal{C}^1} \leqslant c\Big(1 + \|\mathbf{X}\|^{[p]}\Big)|t-s|^{\frac{1}{p}}, \quad 0 \leqslant r \leqslant 1 \tag{4.27}$$

Proposition 8. *There exists a positive constant c, depending only on the V_i, such that the inequality*

$$\Big\|f \circ \mu_{ts} - \mathrm{F}^{\otimes}\big(\mathbf{X}_{ts}\big)f\Big\|_\infty \leqslant c\Big(1 + \|\mathbf{X}\|^{[p]}\Big)\,\|f\|_{\mathcal{C}^{[p]+1}}\,|t-s|^{\frac{[p]+1}{p}} \tag{4.28}$$

holds for any $f \in \mathcal{C}_b^{[p]+1}(E)$.

Recall $\mathrm{F}^{\otimes}(0)$ is the null map from $\mathcal{C}^0(E)$ to itself and $\pi_0\Lambda = 0$ for any $\Lambda \in \mathfrak{g}_\ell^{[p]}$. The proof of this proposition and the following one are based on the elementary identity (4.29) below, obtained by applying repeatedly the identity

$$\begin{aligned} f\big(y_r\big) &= f(x) + \int_0^r \Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}_{ts}\big)f\Big)(y_u)\,du \\ &= f(x) + \sum_{k_1=0}^{[p]+1}\int_0^r \Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}_{ts}^{k_1}\big)f\Big)(y_u)\,du, \quad 0 \leqslant r \leqslant 1 \end{aligned}$$

together with the morphism property (4.24). The above sum over k_1 is needed to take care of the different regularity properties of the maps $\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{k_1}_{ts}\big)f$.

$$\begin{aligned} f\big(\mu_{ts}(x)\big) &= f(x) + \Big(\mathrm{F}^{\otimes}(\mathbf{\Lambda}_{ts})f\Big)(x) + \sum_{k_1+k_2\leqslant [p]+1} \int_0^1\int_0^{s_1} \Big(\mathrm{F}^{\otimes}(\mathbf{\Lambda}^{k_2}_{ts})\mathrm{F}^{\otimes}(\mathbf{\Lambda}^{k_1}_{ts})f\Big)\big(y_{s_2}\big)\,ds_2\,ds_1 \\ &= f(x) + \Big(\mathrm{F}^{\otimes}(\mathbf{\Lambda}_{ts})f\Big)(x) + \int_0^1\int_0^{s_1} \Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{\bullet 2}_{ts}\big)f\Big)\big(y_{s_2}\big)\,ds_2\,ds_1 \end{aligned}$$

We use here the notation $\bullet 2$ to denote the multiplication $\Lambda^{\bullet 2}_{ts} = \Lambda_{ts}\Lambda_{ts}$, not to be confused with the second level $\mathbf{\Lambda}^2_{ts}$ of $\mathbf{\Lambda}_{ts}$; the product is done here in T^{∞}_{ℓ}, and definition (4.25) used to make sense of $\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{\bullet 2}_{ts}\big)f$. Set

$$\Delta_n := \big\{(s_1,\dots,s_n)\in[0,T]^n\,;\, s_1\leqslant\cdots\leqslant s_n\big\},$$

for $2\leqslant n\leqslant [p]$, and write ds for $ds_n\dots ds_1$. Repeating $(n-1)$ times the above procedure in an iterative way, we see that

$$\begin{aligned} f\big(\mu_{ts}(x)\big) &= f(x) + \sum_{k=1}^{n-1}\frac{1}{k!}\Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{\bullet k}_{ts}\big)f\Big)(x) + \int_{\Delta_n}\Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{\bullet n}_{ts}\big)f\Big)\big(y_{s_n}\big)\,ds \\ &= f(x) + \sum_{k=1}^{n}\frac{1}{k!}\Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{\bullet k}_{ts}\big)f\Big)(x) + \int_{\Delta_n}\Big\{\Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{\bullet n}_{ts}\big)f\Big)\big(y_{s_n}\big) - \Big(\mathrm{F}^{\otimes}\big(\mathbf{\Lambda}^{\bullet n}_{ts}\big)f\Big)(x)\Big\}\,ds. \end{aligned}$$

Note that $\pi_j\mathbf{\Lambda}^{\bullet n}_{ts} = 0$, for all $j\leqslant n-1$, and

$$\pi_{\leqslant [p]}\left(\sum_{k=1}^{[p]}\frac{1}{k!}\mathbf{\Lambda}^{\bullet k}_{ts}\right) = \mathbf{X}_{ts};$$

also $\pi_{\leqslant [p]}\Big(\mathbf{\Lambda}^{\bullet [p]}_{ts}\Big) = \big(X^1_{ts}\big)^{\otimes [p]}$ is of size $|t-s|^{\frac{[p]}{p}}$. We separate the different terms in the above identity according to their size in $|t-s|$; this leads to the following expression for $f\big(\mu_{ts}(x)\big)$.

(4.29)

$$\begin{aligned} &f(x) + \left(\mathrm{F}^{\otimes}\Big(\pi_{\leqslant [p]}\Big\{\sum_{k=1}^{n}\frac{1}{k!}\,\Lambda^{\bullet k}_{ts}\Big\}\Big)f\right)(x) + \int_{\Delta_n}\Big\{\Big(\mathrm{F}^{\otimes}\big(\pi_{\leqslant [p]}\mathbf{\Lambda}^{\bullet n}_{ts}\big)f\Big)\big(y_{s_n}\big) - \Big(\mathrm{F}^{\otimes}\big(\pi_{\leqslant [p]}\mathbf{\Lambda}^{\bullet n}_{ts}\big)f\Big)(x)\Big\}\,ds \\ &+ \left(\mathrm{F}^{\otimes}\Big(\pi_{[p]+1}\Big\{\sum_{k=1}^{n}\frac{1}{k!}\,\mathbf{\Lambda}^{\bullet k}_{ts}\Big\}\Big)f\right)(x) + \int_{\Delta_n}\Big\{\Big(\mathrm{F}^{\otimes}\Big(\pi_{[p]+1}\mathbf{\Lambda}^{\bullet n}_{ts}\Big)f\Big)\big(y_{s_n}\big) - \Big(\mathrm{F}^{\otimes}\Big(\pi_{[p]+1}\mathbf{\Lambda}^{\bullet n}_{ts}\Big)f\Big)(x)\Big\}\,ds \end{aligned}$$

We denote by $\epsilon^{f\,;\,n}_{ts}(x)$ the sum of the two terms involving $\pi_{[p]+1}$ in the above line, made up of terms of size at least $|t-s|^{\frac{[p]+1}{p}}$. Note that for $n=[p]$, the integral term in the first line involves $\pi_{\leqslant [p]}\Big(\mathbf{\Lambda}^{[p]}_{ts}\Big) = \big(X^1_{ts}\big)^{\otimes [p]}$ and the increment $y_{s_n}-x$, of size $|t-s|^{\frac{1}{p}}$, by estimate (4.27), so this term is of size $|t-s|^{\frac{[p]+1}{p}}$; we include it in $\epsilon^{f\,;\,[p]}_{ts}(x)$.

PROOF OF PROPOSITION 8 – Applying the above formula with $n=[p]$, we get the identity

$$f\big(\mu_{ts}(x)\big) = \Big(\mathrm{F}^{\otimes}\big(\mathbf{X}_{ts}\big)f\Big)(x) + \epsilon_{ts}^{f\,;\,[p]}(x).$$

It is clear on the formula for $\epsilon_{ts}^{f\,;\,[p]}(x)$ that its absolute value is bounded above by a constant multiple of $\Big(1+\|\mathbf{X}\|^{[p]}\Big)|t-s|^{\frac{[p]+1}{p}}$, for a constant depending only on the data of the problem and f as in (4.28). ▷

A further look at formula (4.29) makes it clear that if $2\leqslant n\leqslant [p]$, and f is $\mathcal{C}_b^{n+1}$, the estimate

$$\Big\|\epsilon_{ts}^{f\,;\,n}\Big\|_{\mathcal{C}^1} \leqslant c\Big(1+\|\mathbf{X}\|^{[p]}\Big)\,\|f\|_{\mathcal{C}^{n+1}}|t-s|^{\frac{[p]+1}{p}}, \tag{4.30}$$

holds as a consequence of formula (4.27), for a constant c depending only on the V_i.

PROPOSITION **9.** *The family of maps $\big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ is a $\mathcal{C}^1$-approximate flow.*

PROOF – As the vector fields V_i are of class $\mathcal{C}_b^{[p]+1}$, with $[p]+1\geqslant 3$, the identity

$$\big\|\mu_{ts}-\mathrm{Id}\big\|_{\mathcal{C}^2} = o_{t-s}(1)$$

holds as a consequence of classical results on ordinary differential equations; we turn to proving the $\mathcal{C}^1$-approximate flow property (**??**). Recall X_{ts}^m stans for $\pi_m\mathbf{X}_{ts}$. We first use for that purpose formula (4.29) to write

$$\begin{aligned}\mu_{tu}\big(\mu_{us}(x)\big) &= \Big(\mathrm{F}^{\otimes}\big(\mathbf{X}_{tu}\big)\mathrm{Id}\Big)\big(\mu_{us}(x)\big) + \epsilon_{tu}^{\mathrm{Id}\,;\,[p]}\big(\mu_{us}(x)\big) \\ &= \mu_{us}(x) + \sum_{m=1}^{[p]}\Big(\mathrm{F}^{\otimes}\big(X_{tu}^m\big)\mathrm{Id}\Big)\big(\mu_{us}(x)\big) + \epsilon_{tu}^{\mathrm{Id}\,;\,[p]}\big(\mu_{us}(x)\big).\end{aligned} \tag{4.31}$$

We splitted the function $\mathrm{F}^{\otimes}\big(\mathbf{X}_{tu}\big)\mathrm{Id}$ into a sum of functions $\mathrm{F}^{\otimes}\big(X_{tu}^m\big)\mathrm{Id}$ with different regularity properties, so one needs to use different Taylor expansions for each of them. One uses (4.30) and inequality (4.27) to deal with the remainder

$$\Big\|\epsilon_{tu}^{\mathrm{Id}\,;\,[p]}\big(\mu_{us}(x)\big)\Big\|_{\mathcal{C}^1} \leqslant c\Big(1+\|\mathbf{X}\|^{[p]}\Big)^2|t-u|^{\frac{[p]+1}{p}}.$$

To deal with the term $\Big(\mathrm{F}^{\otimes}\big(X_{tu}^m\big)\mathrm{Id}\Big)\big(\mu_{us}(x)\big)$, we use formula (4.29) with $n=[p]-m$ and $f=\mathrm{F}^{\otimes}\big(X_{tu}^m\big)\mathrm{Id}$. Writing ds for $ds_{[p]-m}\dots ds_1$, we have

$$\Big(\mathrm{F}^{\otimes}\big(X_{tu}^m\big)\mathrm{Id}\Big)\big(\mu_{us}(x)\big) = \Big(\mathrm{F}^{\otimes}\big(X_{tu}^m\big)\mathrm{Id}\Big)(x) + \left(\mathrm{F}^{\otimes}\left(\Big\{\pi_{\leqslant[p]}\sum_{k=1}^{[p]-m}\frac{1}{k!}\mathbf{\Lambda}_{us}^{\bullet k}\Big\}\,X_{tu}^m\right)\mathrm{Id}\right)(x) + \epsilon_{us}^{\star\,;\,p-m}(x). \tag{4.32}$$

The notation $\star$ in the above identity stands for the $\mathcal{C}_b^{[p]+2-m}$ function $\mathrm{F}^{\otimes}\big(X_{tu}^m\big)\mathrm{Id}$; it has $\mathcal{C}^1$-norm controlled by (4.30). The result follows directly from (4.31) and (4.32) writing

$$\mu_{us}(x) = \Big(\mathrm{F}^{\otimes}\big(\mathbf{X}_{us}\big)\mathrm{Id}\Big)(x) + \epsilon_{us}^{\mathrm{Id}\,;\,[p]}(x),$$

and using the identities $\exp\big(\mathbf{\Lambda}_{us}\big) = \mathbf{X}_{us}$ and $\mathbf{X}_{ts} = \mathbf{X}_{us}\mathbf{X}_{tu}$ in $T_\ell^{[p]}$. ▷

Definition **10.** *A* **flow** $(\varphi_{ts}\,;\, 0\leqslant s\leqslant t\leqslant T)$ *is said to* **solve the rough differential equation**

$$d\varphi = \mathrm{F}^{\otimes}\,\mathbf{X}(dt) \tag{4.33}$$

if there exists a constant $a>1$ *independent of* $\mathbf{X}$ *and two possibly* $\mathbf{X}$*-dependent positive constants* δ *and* c *such that*

$$\|\varphi_{ts}-\mu_{ts}\|_\infty \leqslant c\,|t-s|^a \tag{4.34}$$

holds for all $0\leqslant s\leqslant t\leqslant T$ *with* $t-s\leqslant\delta$*.*

This definition can be equivalently reformulated in terms of uniform Taylor-Euler expansion of the form

$$f\big(\varphi_{ts}(x)\big) = f(x) + \sum_{|I|\leqslant[p]} X^I_{ts}\big(V_I f\big)(x) + O\big(|t-s|^{>1}\big).$$

The following well-posedness result follows directly from theorem **??** and proposition 9; its proof is identical to the proof of theorem 7, without a single word to be changed, except for the power of $\|\mathbf{X}\|$ in estimate (4.20), which needs to be taken as $[p]+1$ instead of 3.

Theorem **11.** *The rough differential equation*

$$d\varphi = \mathrm{F}^{\otimes}\,\mathbf{X}(dt)$$

has a unique solution flow; it takes values in the space of uniformly Lipschitz continuous homeomorphisms of E with uniformly Lipschitz continuous inverses, and depends continuously on $\mathbf{X}$*.*

Remarks 5 on Friz-Victoir's approach to rough differential equations and time-inhomogeneous dynamics also hold in the general setting of this section.

4.3. **Exercice on flows driven by rough paths. 12. Local Lipschitz continuiuty of φ with respect to $\mathbf{X}$.** Use the result proved in exercice 5 to prove that the solution flow to a rough differential equation driven by $\mathbf{X}$ is a locally Lipschitz continuous function of $\mathbf{X}$, in the uniform norm topology.

13. Taylor expansion of solution flows. Let $V_1,\dots,V_\ell$ be $\mathcal{C}^{[p]+1}_b$ vector fields on a Banach space E, and $\mathbf{X}$ be a weak geometric Hölder p-rough path over $\mathbb{R}^\ell$, with $2\leqslant p$. Set $\mathrm{F}=\big(V_1,\dots,V_\ell\big)$. The solution flow to the rough differential equation

$$d\varphi = \mathrm{F}^{\otimes}\mathbf{X}(dt)$$

enjoys, by definition, a uniform Taylor-Euler expansion property, expressed either by writing

$$\big\|\varphi_{ts}-\mu_{ts}\big\|_\infty \leqslant c|t-s|^a$$

for the $\mathcal{C}^1$-approximate flow $\big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ contructed in section 4.2.2, or by writing

$$\Bigg\|f\circ\varphi_{ts} - \sum_{|I|\leqslant[p]} X^I_{ts}V_I f\Bigg\|_\infty \leqslant c|t-s|^a.$$

What can we say if the vector fields V_i are actually more regular than $\mathcal{C}_b^{[p]+1}$?

Assume $N \geqslant [p]+2$ is given and the V_i are $\mathcal{C}_b^N$. Let $\mathbf{Y}$ be the canonical lift of $\mathbf{X}$ to a $\mathfrak{G}_\ell^N$-valued weak geometric Hölder N-rough path, given by Lyons' extension theorem proved in exercice 7. Let $\Theta_{ts} \in \mathfrak{g}_\ell^N$ stand for $\log \mathbf{Y}_{ts}$. For any $0 \leqslant s \leqslant t \leqslant 1$, let ν_{ts} be the time 1 map associated with the ordinary differential equation

$$\dot{z}_u = \mathrm{F}^{\otimes}\big(\Theta_{ts}\big)(z_u), \quad 0 \leqslant u \leqslant 1.$$

a) Prove that ν_{ts} enjoys the following Euler expansion property. For any $f \in \mathcal{C}_b^{N+1}$ we have

$$\big\| f \circ \nu_{ts} - \mathrm{F}^{\otimes}\big(\mathbf{Y}_{ts}\big) f \big\|_\infty \leqslant c|t-s|^{\frac{N+1}{p}}, \tag{4.35}$$

where the contant c depends only on the V_i and $\mathbf{X}$.

b) Prove that $\big(\nu_{ts}\big)_{0 \leqslant s \leqslant t \leqslant 1}$ is a $\mathcal{C}^1$-approximate flow.

c) Prove that φ_{ts} satisfies the high order Euler expansion formula (4.35).

14. Perturbing the signal or the dynamics? Let $2 \leqslant p$ be given and $V_1, \dots, V_\ell$ be $\mathcal{C}_b^{[p]+1}$ vector fields on E. Let $\mathbf{X}$ be a weak geometric Hölder p-rough path over $\mathbb{R}^\ell$, and $\mathbf{a} \in \mathfrak{g}_\ell^{[p]}$ be such that $\pi_j \mathbf{a} = 0$ for all $j \leqslant [p]-1$. Write it

$$\mathbf{a} = \sum_{|I|=[p]} a^I \mathbf{e}_{[I]},$$

where $\big(e_1, \dots, e_\ell\big)$ stand for the canonical basis of $\mathbb{R}^\ell$, and for $I = \big(i_1, \dots, i_k\big)$,

$$\mathbf{e}_{[I]} = \Big[e_{i_1}, \big[\dots, \big[e_{i_{k-1}}, e_{i_k}\big] \dots\Big]$$

in $T_\ell^{[p]}$. The $\mathbf{e}_{[I]}$'s form a basis of $\mathfrak{g}_\ell^{[p]}$ with $\pi_n \mathbf{e}_{[I]} = 0$ if $n \neq |I|$. Recall the definition of $\exp : T_\ell^{[p],0} \to T_\ell^{[p],1}$ and its reciprocal log.

a) Show that one defines a weak geometric Hölder p-rough path $\overline{\mathbf{X}}$ over $\mathbb{R}^\ell$ setting

$$\overline{\mathbf{X}}_{ts} = \exp\Big(\log \mathbf{X}_{ts} + (t-s)\mathbf{a}\Big).$$

b) Show that the solution flow to the rough differential equation

$$d\psi = \mathrm{F}^{\otimes}\overline{\mathbf{X}}(dt)$$

coincides with the solution flow to the rough differential equation

$$d\varphi = V dt + \mathrm{F}^{\otimes}\mathbf{X}(dt),$$

where the vector field V is defined by the formula

$$V = a^I V_{[I]}.$$

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