# KPZ IN A MULTIDIMENSIONAL RANDOM GEOMETRY OF MULTIPLICATIVE CASCADES

I.F. BAILLEUL

Abstract. We show in this note how the one-dimensional KZP formula obtained by Benjamini and Schramm in [BS09] can be extended to a multidimensional setting.

## 1. Hausdorff dimension in a nested measure space

1.1. **Dimension.** Let $(S,\mathcal{S},\mu)$ be a measure space and suppose given a nested family of countable $\sigma$-algebras $\mathcal{S}_n = \sigma\big(A_n^i\,;\, i \geqslant 1\big)$, with $A_n^i \in \mathcal{S}$ disjoint up to $\mu$ null sets, and $\mu(A_n^i) > 0$ for each $i \geqslant 1$ and $n \geqslant 1$. Suppose further that $\epsilon_n := \sup \mu(A_n^i)$ decreases to 0 as $n$ goes to infinity. Given $s \geqslant 0$ and $\delta > 0$, set for any measurable $E \in \mathcal{S}$

$$\mathcal{H}_\delta^s(E) = \inf \sum \mu\big(A_{n_\alpha}^{i_\alpha}\big)^s,$$

where the infimum is over the set of coverings $E \subset \bigcup_{\alpha\in\mathcal{A}} A_{n_\alpha}^{i_\alpha}$ of $E$, indexed by a subset $\mathcal{A}$ of $\mathbb{N}^* \times \mathbb{N}^*$, and such that $\epsilon_{n_\alpha} \leqslant \delta$ for all $\alpha \in \mathcal{A}$. The quantity $\mathcal{H}_\delta^s(K)$ increases as $\delta$ decreases to 0. Set

$$\mathcal{H}^s(E) = \lim_{\delta\downarrow 0} \mathcal{H}_\delta^s(E).$$

Like in the usual definition of the Hausdorff dimension of a set, it is easy to see that if

- $\mathcal{H}^{s_0}(E) < \infty$ then $\mathcal{H}^t(K) = 0$ for any $s_0 < t$,
- $\mathcal{H}^{s_0}(E) = \infty$ then $\mathcal{H}^s(K) = 0$ for any $s < s_0$,

so it makes sense to define the dimension $\zeta_\mu(E)$ of $E$ as $\sup\{s \geqslant 0\,;\, \mathcal{H}^s(E) = \infty\} = \inf\{t \geqslant 0\,;\, \mathcal{H}^t(E) = 0\}$. *As $\mathcal{H}^1$ coincides with $\mu$, it follows that $\zeta_\mu(E) \leqslant 1$, for any $E \in \mathcal{S}$.* So only sets with null $\mu$-measure have a dimension smaller than 1.

**Open question.** Let us work in the space $S = \mathcal{C}\big([0,1],\mathbb{R}\big)$, with its Borel $\sigma$-algebra and Wiener measure. Define $A_n^{(j,k)}$ as $\Big\{\omega \in \mathcal{C}\big([0,1],\mathbb{R}\big)\,;\, \omega\big((j+1)2^{-n}\big) - \omega\big(j2^{-n}\big) \in \big[k2^{-n}, (k+1)2^{-n}\big)\Big\}$, for $0 \leqslant j \leqslant 2^n - 1$ and $k \in \mathbb{Z}$, and set $\mathcal{S}_n = \sigma\big(A_n^{(j,k)}\,;\, 0 \leqslant j \leqslant 2^n-1,\ k \in \mathbb{Z}\big)$. Let us call ***Wiener-Hausdorff dimension*** the above dimension of a measurable subset of $\mathcal{C}\big([0,1],\mathbb{R}\big)$. *Compute the Wiener-Hausdorff dimension of the set of $\alpha$-Hölder continuous paths, for $\alpha \geqslant \frac{1}{2}$.*

1.2. **Frostman lemma.** If $(S,\mathcal{S})$ is $\mathbb{R}^d$ with its Borel $\sigma$-algebra, and $\mathcal{S}_n$ is the $\sigma$-algebra generated by the dyadic cubes of side $2^{-n}$, then the above definition of dimension coincides with the usual Hausdorff dimension, up to a multiplicative constant $\frac{1}{d}$; see section 2.4, Chap. 2, in [Fal03]. We adopt the above definition of dimension for the sequel. Like its classical counterpart, the above set function $\mathcal{H}^s(\cdot)$ can be shown to be an $\big(\mathbb{R}_+ \cup \{\infty\}\big)$-valued measure

*Date*: April 15, 2011.

on $\big(\mathbb{R}^d, \mathsf{Bor}(\mathbb{R}^d)\big)$. The Euclidean background will not appear anymore except under the form of the nested family $(\mathcal{S}_n)_{n\geqslant 0}$.

Given two points $x, y \in \mathbb{R}^d$, define the *ball* $B(x,y)$ as the smallest dyadic cube containg $x$ and $y$, and define their "distance" as $\mu\big(B(x,y)\big)$. Define accordingly the ball $B_r(x) = \{y \in \mathbb{R}^d\,;\, \mu\big(B(x,y)\big) \leqslant r\}$. Working exactly as in theorem 4.10 and proposition 4.11 in [Fal03], one can prove the following proposition.

Proposition **1.** *For any Borel set $E$ with $0 < \mathcal{H}^s(E) < \infty$, there exists a constant $c$ and a compact set $K \subset E$ with $\mathcal{H}^s(K) > 0$ such that*

$$\mathcal{H}^s\big(K \cap B_r(x)\big) \leqslant cr^s$$

*for all $x \in \mathbb{R}^d$ and $r > 0$.*

It follows classically that the following version of Frostman lemma holds in our setting. Given any non-negative measure $\nu$ on $\big(\mathbb{R}^d, \mathsf{Bor}(\mathbb{R}^d)\big)$, define its $s$-energy as

$$I_s(\nu) = \iint \frac{\nu(dx)\nu(dy)}{\mu\big(B(x,y)\big)^s}.$$

Theorem **2.** *If $E$ is a Borel set with $0 < \mathcal{H}^s(E)$, then there exists a non-negative measure $\nu$ with support in (a compact subset of) $E$ such that $I_t(\nu) < \infty$, for all $t < s$. This is in particular the case if $s < \zeta_\mu(E)$.*

**Remark.** The work [RV10] contains in section 5.1 a similar, though different, notion of dimension in a metric measure space.

## 2. A dimension-free KPZ formula

Let $\mathcal{D}_n = \bigcup_{k=1}^{2^{dn}} A_k^n$ be the dyadic "partition" of the unit cube of $\mathbb{R}^d$ by closed dyadic cubes of side length $2^{-n}$. Given $m < n$, each $A_k^n$ is a subset of a unique $A_{k(m)}^m$. Let $W$ be a positive real-valued random variable with $\mathbb{E}[W] = 1$, and let $\big\{(W_i^n)_{i=1}^{2^{nd}}\,;\, n \geqslant 1\big\}$ be an iid sequence of random variables with common law the law of $W$. Define the measure $\mu_n$ by its density $w_n(x)$ with respect to Lebesgue measure. It is constant, equal to $\prod_{m=0}^n W_{k(m)}^m$, on each $A_k^n$. We adopt as in [BS09] the notation $\ell$ for $\mu\big([0,1]^d\big)$. It has expectation no greater than 1.

Proposition **3.** *Almost-surely, the measures $\mu_n$ converge weakly to some random measure $\mu$, which does not charge any dyadic hyperplane. It is almost-surely non-null if $\mathbb{E}[W \log W] < d$.*

Proof – The proof works exactly as in the 1-dimensional proof, with $2^d$ independent copies of $\ell$ rather than only two. $\triangleright$

The next result generalizes Benjamini and Schramm's result [BS09] obtained in a one-dimensional setting.

Theorem **4.** *Let $E$ be any Borel set of $[0,1]^d$. Denote by $\zeta_0$ its dimension as defined above using Lebesgue measure, and let $\eta$ be its dimension using the random measure $\mu$. Suppose that $\mathbb{E}[W \log W] < d$, and $\mathbb{E}[W^{-s}] < \infty$, for all $s \in [0,1)$. Then $\zeta$ is almost-surely a constant and satisfies the identity*

$$2^{\zeta_0} = \frac{2^\zeta}{\mathbb{E}[W^\zeta]}$$

The above conditions are satisfied by an exponential of Gaussian with a small enough variance.

PROOF – The proof mimicks word by word the proof of [BS09]. Write $|A|$ for the Lebesgue measure of a Borel set $A$. Set, for $s \in [0,1]$, $\phi(s) = s - \ln_2 \mathbb{E}[W^s]$. Note that since the notion of dimension introduced in section 1 is no greater than 1 the function $\phi$ is an increasing homeomorphism from $[0,1]$ to itself.

**a)** Lemma 3.3 becomes here: $\mathbb{E}\big[\mu\big(B(x,y)\big)^s\big] \leqslant \big|B(x,y)\big|^{\phi(s)}$, for all $x, y \in [0,1]^d$.
Note that the balls $B(x,y)$ are always dyadic balls; suppose the given ball belongs to $\mathcal{D}_n$, so $\big|B(x,y)\big| = 2^{-nd}$. Then, we have by the independence in the construction of $\mu$

$$\mathbb{E}\big[\mu\big(B(x,y)\big)^s\big] = 2^{-nd}\,\mathbb{E}[W^s]^{nd}\,\mathbb{E}[\ell^s] \leqslant \{2^{-nd}\}^{\phi(s)} = \big|B(x,y)\big|^{\phi(s)},$$

as $0 \leqslant s \leqslant 1$, so $\mathbb{E}[\ell^s] \leqslant \mathbb{E}[\ell]^s = 1$. It follows directly that we have almost-surely $\phi(\zeta) \leqslant \zeta_0$.

**b)** The proof that $\phi(\zeta) \geqslant \zeta_0$, theorem 3.5, works identically, replacing the usual energy of a measure by its above modification, and using the version of Frostman lemma provided in theorem 2. A straightforward adaptation of the proof that $\mathbb{E}[\ell^{-s}] < \infty$ if $\mathbb{E}[W^{-s}] < \infty$, given in [BS09], gives the same result in our setting. Note also that a different choice of Hölder coefficient is needed to prove that the sequence $\nu_n\big([0,1]\big)$ is uniformly bounded in some $\mathbb{L}^p$. ▷

Note that the above theorem does not come as a surprise and should actually hold on much more general state spaces than $[0,1]^d$. It should be interesting in particular to investigate what happens on random trees like Galton-Watson trees, and tree-like objects like random fractals.

## References


[BS09] I. Benjamini and O. Schramm. KPZ in one dimensional random geometry of multiplicative cascades. *Comm. Math. Phys.*, 289(2):653–662, 2009.

[Fal03] K. Falconer. *Fractal geometry*. John Wiley & Sons Inc., Hoboken, NJ, second edition, 2003. Mathematical foundations and applications.

[RV10] R. Rhodes and V. Vargas. Multidimensional multifractal random measures. *Electron. J. Probab.*, 15:no. 9, 241–258, 2010.



Statistical Laboratory, Center for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB, UK

*E-mail address*: i.bailleul@statslab.cam.ac.uk

*URL*: http://www.statslab.cam.ac.uk/~ismael/