Paracontrolled calculus

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Abstract. At the same time that Hairer introduced his theory of regularity structures, Gubinelli, Imkeller and Perkowski developed paracontrolled calculus as an alternative playground where to study a number of singular, classically ill-posed, stochastic partial differential equations, such as the 2 or 3-dimensional parabolic Anderson model equation (PAM)

$$\partial_t u = \Delta u + u\zeta,$$

the Φ_3^4 equation of stochastic quantization

$$\partial_t u = \Delta u - u^3 + \zeta,$$

or the one dimensional KPZ equation

$$\partial_t u = \Delta u + (\partial_x u)^2 + \zeta,$$

to name but a few examples. In each of these equations, the letter ζ stands for a space or time/space white noise who is so irregular that we do not expect any solution u of the equation to be regular enough for the nonlinear terms, or the product $u\zeta$, in the equations to make sense on the sole basis of the regularizing properties of the heat semigroup. Like Hairer's theory of regularity structures, paracontrolled calculus provides a setting where one can make sense of such a priori ill-defined products, and finally give some meaning and solve some singular partial differential equations. We present here an overview of paracontrolled calculus, from its initial form to its recent extensions.

1 Introduction

Starting with T. Lyons' work on controlled differential equation [15], it is now well-understood that the construction of a robust approximation theory for continuous time stochastic systems, such as stochastic differential equations or stochastic partial differential equations, requires a twist in the notion of noise that allows to treat the resolution of such equations in a two step process.

(a) Enhance the noise into an enriched object that lives in some space of analytic objects – this is a purely *probabilistic* step;

(b) given *any* such object $\widehat{\zeta}$ in this space, one can introduce a $\widehat{\zeta}$-dependent Banach space $\mathcal{S}\big(\widehat{\zeta}\big)$ such that the equation makes sense for the unknown in $\mathcal{S}\big(\widehat{\zeta}\big)$, and it can be solved uniquely by a *deterministic* analytic argument, such as the contraction principle, which gives the continuity of the solution as a function $\widehat{\zeta}$.

These two steps are very different in nature and require totaly different tools. Hairer's theory of regularity structures [13] provides undoubtedly the most complete picture for the study of a whole class of singular stochastic partial differential equations (PDEs) from the above point of view – the class of the so-called singular

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subcritical parabolic stochastic PDEs, of which the 2 or 3-dimensional parabolic Anderson model equation (PAM)

$$\partial_t u = \Delta u + u\zeta,$$

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are typical examples. It comes with a very rich algebraic structure and an entirely new setting that are required to give flesh to the guiding principle that a solution should be described by the datum at each point in space-time of its high order 'jet' in a basis given by the elements of the enhanced noise. Regularity structures are introduced as a tool for describing these jets. At the same time that Hairer built his theory, Gubinelli, Imkeller and Perkowski implemented in [10] this idea of giving a local/global description of a possible solution in a different way, using the language of paraproducts and avoiding the introduction of any new setting, but providing only a first order description of the objects under study. This is what is called the *first order paracontrolled calculus*. While this kind of approach may seem far from being as powerful as Hairer's machinery, the first order paracontrolled approach to singular stochastic PDEs has been successful in recovering and extending a number of results that can be proved within the setting of regularity structures, on the parabolic Anderson model and Burgers equations [10, 2, 3, 9], the KPZ equation [12], the scalar Φ^4_3 equation [6], the stochastic Navier-Stokes equation [18, 19, 20], or the study of the continuous Anderson Hamiltonian [8], to name but a few.

The recent works [3, 4] extend the scope of paracontrolled calculus to a class of equations that is much closer to Hairer's class of singular subcritical parabolic stochastic PDEs, by providing an analytic setting where one can do an arbitrary high order paracontrolled analysis of some equation, such as advertised in point (b) above. The development of point (a) within this setting is presently under investigation.

We describe in this introductory paper the basics of paracontrolled calculus, such as it has evolved from its inception in [10] by Gubinelli, Imkeller and Perkowski to its more elaborate version developed in the works [2, 3], and finally [4]. No previous knowkedge of stochastic PDEs or even PDEs is needed to grasp the core of the story. For informations on regularity structures, see the gentle introductions [14, 7] by Hairer and Chandra-Weber. As we shall not comment on it, note here that all this story can be told in an unbounded manifold setting at the price of working with weighted functional spaces.

2 The product problem and its solution for controlled ODEs

2.1 The product problem

The common feature of all the above equations is the presence of terms in the equation that are not expected to make sense even for the most optimistic analyst. This is fundamentally linked to the fact that we cannot generically multiply a distribution by anything else than a smooth function. Even in the more restrictive setting of Hölder spaces, Bony showed in [5] that one can define the product of two Hölder functions, with possibly negative regularity exponents α and β, if and only if $\alpha+\beta$ is positive. Denote by $(\alpha-2)$ the (possibly parabolic) Hölder regularity exponent of the noise in any of the above equations; we have $\alpha = \big(2-\frac{d}{2}\big)$ for a space white noise in dimension d, and $\alpha = 1-\frac{d}{2}$ for a time/space white noise in dimension d. In each case, one expects the solution u to be of Hölder parabolic regularity α as a consequence of the regularising properties of the heat semigroup. This gives u of regularity $2-\frac{d}{2}$ for the (PAM) equation, of regularity $-\frac{1}{2}$ for the Φ^4_3 equation, of regularity $\frac{1}{2}$ for the KPZ equation. These regularities are not sufficient to make sense of the terms $u\zeta, u^3$ and $(\partial_x u)^2$ respectively, on a purely analytic basis.

There is actually no need to work with PDEs to see this problem appearing, and it is already here when one tries to make sense of, and solve, an ordinary controlled differential equation in $\mathbf{R}^d$

$$\dot{x}_t = V(x_t)\dot{h}_t, \tag{2.1}$$

where V is an $\mathbf{R}^d$-valued one form on $\mathbf{R}^\ell$, say, and h is some α-Hölder $\mathbf{R}^\ell$-valued control defined on some time interval $[0,T]$. One expects indeed x to be α-Hölder as well since $\dot{h}_t$ will be $(\alpha-1)$-Hölder regular, which will make the product $V(x_t)\dot{h}_t$ well defined only if $(2\alpha-1)$ is positive. While one can indeed set the equation in the setting of Young integrals for α-Hölder controls, with $\alpha > \frac{1}{2}$, there is no 'classical' analytic theory that can give any sensible meaning to equation (2.1), as testified by T. Lyons' no-go theorem [16].

2.2 Controlled ODEs

One of T. Lyons' deep insight in his theory of *rough paths* [15] is that one needs to *enrich the notion of control* in order to make sense of the equation and give conditions under which it can be solved uniquely. The enhancement of the control h consists here in *assuming* that we are *given a priori* all the 'iterated integrals'[2]

$$\int_s^t\int_s^u dh_r\otimes dh_u,$$

for all $0\leqslant s\leqslant t\leqslant T$, or simply the products

$$dh_r\otimes dh_u;$$

the sensible point here is that these 'integrals' *cannot* be defined by any analytic means from the sole data of h. One assumes in addition some natural algebraic and analytic conditions on the a priori given quantities. In a probabilistic setting where the control is random, one typically constructs these iterated integrals by using stochastic calculus, as *limits in probability* of elementary quantities.

On the other hand, one expects naively that, whatever the meaning of the integral below, the following first order Taylor expansion formula will hold for any solution to the controlled equation (2.1)

$$x_t - x_s = \int_s^t V(x_u)dh_u = V(x_s)\big(h_t-h_s\big) + O(h_t-h_s)^2.$$

So, in the end, we do not want to define the product $V(x_t)\dot{h}_t$ for any path x but only for *paths whose increments look like, at small time scales, the increments of the driving control*, such as implied by the above Taylor formula. One says that the *path x is controlled by h*. This requirement, and the a priori datum of the 'iterated integrals', happen to be sufficient to make sense of the equation, when $\frac{1}{3}<\alpha\leqslant\frac{1}{2}$, as a fixed point equation in the space of paths whose increments locally look like

[2]The tensor product is only here to deal with the multidimensional character of the control h.

the increments of the control. The naive and formal second order Taylor expansion formula

$$x_t - x_s = \int_s^t V(x_u)dh_u = V(x_s)\big(h_t - h_s\big) + \big(D_{x_s}V\big)V(x_s)\int_s^t\int_s^u dh_r \otimes dh_u + O(h_t - h_s)^3$$

provides indeed an explicit and sufficiently fine description of $x_t - x_s$, up to a remainder of size $|t-s|^{3\alpha>1}$, to get back x uniquely from the datum of its increments.[3] It happens then to be able to make sense an integral

$$\int_0^{\cdot} V(x_s)dh_s,$$

for a path x controlled by h, and to solve the equation (2.1) as a fixed point for some functional from the space of controlled paths to itself. Using fixed point procedures has the extraordinary benefit to provide a solution path that depends continuously on the enhanced control $(dh, dh \otimes dh)$ – see e.g. [15, 17, 1].

2.3 General fixed point scheme

Hairer's theory of regularity structures and paracontrolled calculus both approach the problem of giving sense to an ill-posed stochastic PDE with the same strategy as Lyons in his theory of rough paths and rough differential equations.

(a) Enhance the noise into an enriched object that lives in some space of analytic objects – this is a purely *probabilistic* step;

(b) given *any* such object $\widehat{\zeta}$ in this space, one can introduce a $\widehat{\zeta}$-dependent Banach space $\mathcal{S}\big(\widehat{\zeta}\,\big)$ such that the equation makes sense for the unknown in $\mathcal{S}\big(\widehat{\zeta}\,\big)$, and it can be solved uniquely by a *deterministic* analytic argument, such as the contraction principle, which gives the continuity of the solution as a function $\widehat{\zeta}$.

Note that the enhancement $\widehat{\zeta}$ of the noise and the ansatz solution space $\mathcal{S}\big(\widehat{\zeta}\,\big)$ both depend on the equation under study.

3 Paraproducts as a tool for local comparison

3.1 The Fourier picture

The above mentioned solution space $\mathcal{S}\big(\widehat{\zeta}\,\big)$ for a given singular equation is made up of functions/distributions that locally look like some reference function(s) / distribution(s). Bony's notion of paraproduct [5] provides a very efficient tool for constructing in a multidimensional setting some functions/distributions that look like another reference function/distribution. Roughly speaking, this bilinear operator $\Pi_f g$ provides an object obtained as a modulation of the high frequencies of g by the low frequencies of f, which justifies that $\Pi_f g$ should indeed look like g at small scales where it has its possibly wild oscillations. In the model case of space-dependent functions on the d-dimensional torus, recall that any distribution f can be described as an infinite sum of smooth functions f_i whose Fourier transform $\widehat{f}_i$

[3] Indeed, if one has $x_t - x_s = \mu_{ts} + O\big(|t-s|^{>1}\big)$, for some known quantities μ_{ts}, then the path x is determined uniquely from its initial condition and the knowledge of μ.

is essentially equal to the restriction of $\widehat{f}$ on a compact annulus depending on i. A product of two distributions f and g can thus always be written formally as

$$\begin{aligned} fg = \sum f_i g_j &= \sum_{i\leqslant j-2} f_i g_j + \sum_{|i-j|\leqslant 1} f_i g_j + \sum_{j\leqslant i-2} f_i g_j \\ &=: \Pi_f(g) + \Pi(f,g) + \Pi_g(f). \end{aligned} \tag{3.1}$$

The term $\Pi_f(g)$ is called the *paraproduct of f and g*, and the term $\Pi(f,g)$ is called the *resonant term*. The paraproduct is always well-defined for f and g in Hölder spaces, with possibly negative indices α and β respectively, while the resonant term only makes sense if $\alpha+\beta$ is positive.

3.2 The heat semigroup approach

It is a non-trivial task to give a robust analogue of these operators on a manifold M, or even a measures metric space, where no Fourier analysis can be used. We used instead in [2, 3, 4] heat semigroup technics to make sense of some paraproduct and resonant operators on the parabolic space $\mathcal{M} := [0,T]\times M$, that have the very same analytic properties as its basic counterpart. We use a semigroup associated with an operator L that plays the role of the Laplacian Δ in the above equations, on which we only require some Gaussian type estimates for its heat kernel and possibly on its 'derivatives'. To set the stage, we work here in a closed manifold M, with a possibly sub-elliptic operator L in Hörmander form

$$L = \sum_{i=1}^{\ell} V_i^2,$$

for some smooth vector fields V_i on M. Doubling and Alhfors regularity properties for the metric measured ambiant space are typically required when working on more general settings than a closed manifold – see [2, 3, 4] for more details. The scale of space and parabolic Hölder spaces $(\mathcal{C}^\alpha)_{\alpha\in\mathbf{R}}$ can be defined purely in terms of the semigroup associated with L.

To be a little more specific, given a real-valued integrable function ϕ on $\mathbf{R}$, set

$$\phi_t(\cdot) := \frac{1}{t}\,\phi\Big(\frac{\cdot}{t}\Big);$$

the family $(\phi_t)_{0<t\leqslant 1}$ is uniformly bounded in $L^1(\mathbf{R})$. We also define the "convolution" operator $\phi^\star$ associated with ϕ via the formula

$$\phi^\star(f)(\tau) := \int_0^\infty \phi(\tau-\sigma)f(\sigma)d\sigma.$$

Given an integer $b\geqslant 1$, we define a special family of operators on $L^2(M)$ setting

$$Q_t^{(b)} := \gamma_b^{-1}(tL)^b e^{-tL} \qquad \text{and} \qquad -t\partial_t P_t^{(b)} = Q_t^{(b)},$$

with $\gamma_b := (b-1)!$; so $P_t^{(b)}$ is an operator of the form $p_b(tL)e^{-tL}$, for some polynomial p_b of degree $b-1$, with value 1 in 0. The operator $Q_t^{(b)}$ can be thought of as an intrinsic replacement for the Littlewood-Paley projectors that depend only on L, in a Fourier-free setting. For defining the time/space counterpart of these operators, choose arbitrarily a smooth real-valued function φ on $\mathbf{R}$, with support in $\big[\frac{1}{2},2\big]$, unit integral and such that for every integer $k=1,..,b$

$$\int \tau^k\varphi(\tau)\,d\tau = 0.$$

(This kind of condition ensures some crucial cancellation property; see [3].) Set

$$\mathcal{P}_t^{(b)} := P_t^{(b)} \otimes \varphi_t^\star \qquad \text{and} \qquad \mathcal{Q}_t^{(b)} := -t\partial_t \mathcal{P}_t^{(b)}.$$

The operators $\mathcal{P}_t$ weakly tend to the identity on $L^p_0(\mathcal{M})$ (the set of functions $f \in L^p(\mathcal{M})$ with time-support included in $[0,\infty)$), $p \in [1,\infty)$, and the set of functions $f \in C^0(\mathcal{M})$ with time-support included in $[0,\infty)$, as t goes to 0; so we have the following **Calderón reproducing formula**. For every continuous function $f \in L^\infty(\mathcal{M})$ with time-support in $[0,\infty)$, then

$$f = \int_0^1 \mathcal{Q}_t^{(b)}(f)\,\frac{dt}{t} + \mathcal{P}_1^{(b)}(f). \tag{3.2}$$

Noting that the measure $\frac{dt}{t}$ gives unit mass to intervals of the form $\left[2^{-i-1}, 2^{-i}\right]$, and considering the operator $\mathcal{Q}_t^{(b)}$ as a kind of multiplier roughly localized at frequencies of size $t^{-\frac{1}{2}}$, Calderón's formula appears as nothing else than a continuous time analogue of the Littlewood-Paley decomposition of f, with $\frac{dt}{t}$ in the role of the counting measure. Building on Calderon's formula, and using iteratively the Leibniz rule for the differentiation operators V_i or ∂_τ, we have the following decomposition decomposition

$$fg = \sum_{\mathcal{I}_b} a_{k,\ell}^{I,J} \int_0^1 \Big(\mathcal{A}_{k,\ell}^{I,J}(f,g) + \mathcal{A}_{k,\ell}^{I,J}(g,f)\Big)\,\frac{dt}{t} + \sum_{\mathcal{I}_b} b_{k,\ell}^{I,J} \int_0^1 \mathcal{B}_{k,\ell}^{I,J}(f,g)\,\frac{dt}{t},$$

where

- $\mathcal{I}_b$ is the set of all tuples (I,J,k,ℓ) with the tuples I,J and the integers k,ℓ satisfying the constraint
$$\frac{|I|+|J|}{2} + k + \ell = \frac{b}{2};$$
- $a_{k,\ell}^{I,J}, b_{k,\ell}^{I,J}$ are bounded sequences of numerical coefficients;
- for $(I,J,k,\ell) \in \mathcal{I}_b$, $\mathcal{A}_{k,\ell}^{I,J}(f,g)$ has the form
$$\mathcal{A}_{k,\ell}^{I,J}(f,g) := \mathcal{P}_t^{(b)}\Big(t^{\frac{|I|}{2}+k} V_I \partial_\tau^k\Big)\Big(\mathcal{S}_t^{(b/2)} f \cdot \big(t^{\frac{|J|}{2}+\ell} V_J \partial_\tau^\ell\big)\mathcal{P}_t^{(b)} g\Big),$$
- for $(I,J,k,\ell) \in \mathcal{I}_b$, $\mathcal{B}_{k,\ell}^{I,J}(f,g)$ has the form
$$\mathcal{B}_{k,\ell}^{I,J}(f,g) := \mathcal{S}_t^{(b/2)}\Big(\Big\{\big(t^{\frac{|I|}{2}+k} V_I \partial_\tau^k\big)\mathcal{P}_t^{(b)} f\Big\} \cdot \Big\{\big(t^{\frac{|J|}{2}+\ell} V_J \partial_\tau^\ell\big)\mathcal{P}_t^{(b)} g\Big\}\Big),$$

for some operators $\mathcal{S}_t^{(b/2)}$ with nice properties – different occurences means different operators.

Definition. *Given f in $\bigcup_{s\in(0,1)} \mathcal{C}^s$ and $g \in L^\infty(\mathcal{M})$, we define the* **paraproduct** $\Pi_g^{(b)} f$ *by the formula*

$$\Pi_g^{(b)} f := \int_0^1 \left\{ \sum_{\mathcal{I}_b;\,\frac{|I|}{2}+k>\frac{b}{4}} a_{k,\ell}^{I,J}\, \mathcal{A}_{k,\ell}^{I,J}(f,g) + \sum_{\mathcal{I}_b;\,\frac{|I|}{2}+k>\frac{b}{4}} b_{k,\ell}^{I,J}\, \mathcal{B}_{k,\ell}^{I,J}(f,g) \right\} \frac{dt}{t},$$

and the **resonant term** $\Pi^{(b)}(f,g)$ *by the formula*

$$\Pi^{(b)}(f,g) :=$$
$$\int_0^1 \left\{ \sum_{\mathcal{I}_b;\, \frac{|I|}{2}+k \leqslant \frac{b}{4}} a_{k,\ell}^{I,J} \Big(\mathcal{A}_{k,\ell}^{I,J}(f,g) + \mathcal{A}_{k,\ell}^{I,J}(g,f) \Big) + \sum_{\mathcal{I}_b;\, \frac{|I|}{2}+k = \frac{|J|}{2}+\ell = \frac{b}{4}} b_{k,\ell}^{I,J}\, \mathcal{B}_{k,\ell}^{I,J}(f,g) \right\} \frac{dt}{t}.$$

With these notations, Calderón's formula becomes

$$fg = \Pi_g^{(b)}(f) + \Pi_f^{(b)}(g) + \Pi^{(b)}(f,g) + \Delta_{-1}(f,g)$$

with the "low-frequency part"

$$\Delta_{-1}(f,g) := \mathcal{P}_1^{(b)} \left(\mathcal{P}_1^{(b)} f \cdot \mathcal{P}_1^{(b)} g \right).$$

The integer-valued parameter 'b' in the formulas is here tuned on demand, for technical purposes; it is not crucial to understand its use, and we write in the sequel Π for $\Pi^{(b)}$, for a well-chosen, sufficiently big, parameter b.

4 Controlled distributions/functions and the product problem

4.1 The basics

Let a reference distribution Z in some parabolic Hölder space $\mathcal{C}^\alpha$ be given. Let $\beta > 0$ be given. Let say momentarily that a pair of distributions $(f,g) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ is said to be paracontrolled by Z if

$$(f,g)^\sharp := f - \Pi_g(Z) \in \mathcal{C}^{\alpha+\beta}. \tag{4.1}$$

The distribution g is called the *derivative of* f *with respect to* Z, and one can prove that for any e, e' in the parabolic space

$$f(e) \simeq f(e) + g(e)\big(Z(e') - Z(e)\big)$$

up to a remainder term that is $(\alpha+\beta)$-Hölder as a function of the parabolic distance between e' and e; see [4]. The twist offered by this definition, as far as the multiplication problem is concerned, is best illustrated on the example of the parabolic Anderson model equation (PAM)

$$(\partial_t + L)u = u\zeta, \tag{4.2}$$

where one wants first to give sense to the product $u\zeta$. Take for Z the solution to the equation $(\partial_t + L)Z = \zeta$, with null initial condition, and a space noise ζ that is $(\alpha-2)$-Hölder. From purely analytic data, the product $u\zeta$ is meaningful only if $\alpha + (\alpha - 2) > 0$, that is $\alpha > 1$. For a distribution (u,u') controlled by Z, with $\beta = \alpha$, say, and 2α-Hölder remainder $(u,u')^\sharp$ in the decomposition (4.1), the formal manipulation

$$\begin{aligned} u\zeta &= \Pi_u(\zeta) + \Pi_\zeta(u) + \Pi(u,\zeta) \\ &= \Pi_u(\zeta) + \Pi_\zeta(u) + \Pi\big(\Pi_{u'}(Z),\zeta\big) + \Pi\big((u,u')^\sharp,\zeta\big) \\ &=: \Pi_u(\zeta) + \Pi_\zeta(u) + \mathsf{C}(Z,u',\zeta) + u'\,\Pi(Z,\zeta) + \Pi\big((u,u')^\sharp,\zeta\big), \end{aligned}$$

gives a decomposition of $u\zeta$ where the first two paraproduct terms are always well-defined, with known regularity, and where the last term makes sense provided $2\alpha + (\alpha - 2)$ is positive, that is $\alpha > \frac{2}{3}$. It happens that the corrector

$$\mathsf{C}(Z, u', \zeta) := \Pi\big(\Pi_{u'}(Z), \zeta\big) - u'\,\Pi(Z, \zeta)$$

can be proved to define an $\big(\alpha + \alpha + (\alpha - 2)\big)$-Hölder distribution if $\alpha > \frac{2}{3}$, although the resonant term $\Pi\big(\Pi_{u'}(Z), \zeta\big)$ is only well-defined on its own if $\alpha > 1$. So we see that the only undefined term in the decomposition of $u\zeta$ is the product $u'\,\Pi(Z, \zeta)$, where the resonant term $\Pi(Z, \zeta)$ does not make sense so far. What is gained in this analysis is the fact that this formal quantity $\Pi(Z, \zeta)$ does not depend on any potential solution of the equation, it depends only on the noise ζ, given the definition of Z.

If ever one can define by some purely probabilistic means the quantity $\Pi(Z, \zeta)$ as a random variable with values in the space of parabolic $(\alpha+\alpha-2)$-Hölder distributions, then we see that the product $u'\,\Pi(Z, \zeta)$ is actually well-defined since under the assumption that $\alpha+(2\alpha-2)$ is positive, that is $\alpha > \frac{2}{3}$. This purely probabilistic step of defining $\Pi(Z, \zeta)$ as a random variable is step **(a)** in the above general resolution scheme. Step **(b)** is provided here by setting the problem in the space of functions controlled by the reference function Z; we have just seen that defining the product $u\zeta$, or rather $(u, u')\zeta$, is indeed not a problem in this setting, and one can then set the (PAM) equation (4.2) as a fixed point problem in the above space of paracontrolled functions. Note here that defining $\Pi(Z, \zeta)$ may not be that obvious. The naive idea that consists in regularizing the noise ζ into ζ^ε by convolution with a smooth kernel, defining Z^ε accordingly, and take a limit of the well-defined quantity $\Pi\big(Z^\varepsilon, \zeta^\varepsilon\big)$ in any reasonable sense is indeed bound to fail. One needs indeed to substract to $\Pi\big(Z^\varepsilon, \zeta^\varepsilon\big)$ a diverging ε-dependent constant to see anything in the limit; this is the core phenomenon of renormalisation, that we do not touch upon here.

4.2 A pair of intertwined paraproducts and a Taylor formula

This is the basic scheme of the first order paracontrolled calculus invented by Gubinelli-Imkeller-Perkowski in [10]. To run it properly, one requires some continuity estimates for a commutator operator between the resolution operator $\mathcal{R}$ for the heat equation and the paraproduct. This continuity result happens to limit critically the analysis of equations by this method to a first order description, such as given in the above definition of a controlled function/distribution. In order to set the stage for an arbitrary high order paracontrolled expansion, we introduced in [3] a modified parabolic paraproduct $\widetilde{\Pi}$ characterized by the intertwining relation

$$\mathcal{R}(\Pi_f g) = \widetilde{\Pi}_f(\mathcal{R}g).$$

It happens to enjoy the same continuity properties as the operator Π.

Definition. *Let $\beta > 0$ be given. A pair of* **distributions** $(f, g) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ *is said to be* **paracontrolled by** Z *if*

$$(f, g)^\sharp := f - \widetilde{\Pi}_g(Z) \in \mathcal{C}^{\alpha+\beta}.$$

A corrector operator associated with this choice of definition for a paracontrolled function/distribution can be introduced and proved to have the same continuity properties as the above operator C, so the above scheme works equally well with the definition of a controlled function/distribution. The interest of introducing this modified paraproduct is clearly seen when one tries to solve the fixed point equation

$$u = \mathcal{R}(u\zeta),$$

assuming a null initial condition for u. Indeed, since

$$u\zeta = \Pi_u \zeta + (\cdots),$$

the above fixed point relation becomes

$$\begin{aligned} \widetilde{\Pi}_{u'} Z + (u,u')^\sharp &= \mathcal{R}\big(\Pi_u\zeta + (\cdots)\big) \\ &= \widetilde{\Pi}_u(\mathcal{R}\zeta) + \mathcal{R}(\cdots) \\ &= \widetilde{\Pi}_u Z + \mathcal{R}(\cdots), \end{aligned}$$

which allows a relatively elementary analysis. The use of an ansatz space where the comparison operator Π is used in place of $\widetilde{\Pi}$ does not allow such a straightforward analysis.

This notion of paracontrolled function provides somehow a first order Taylor expansion of a function in terms of a reference object. One can give similar expansion representations in terms of two or more reference objects. It happens then to be crucial, when working with nonlinear function of the unknown, to be able to give a description of $f(u)$ in terms of the reference objects given an initial description of u in those terms. The following Taylor expansion formula [4] provides in that respect the perfect tool for doing that; it is a generalisation of Bony's paralinearisation formula [5].

Theorem 1 (High order Taylor expansion). *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a C^4 function with bounded fourth derivative, and let u be a real-valued α-Hölder function on the parabolic space $\mathcal{M}$, with $0 < \alpha < 1$. Then*

$$\begin{aligned} f(u) = \Pi_{f'(u)}(u) &+ \frac{1}{2}\Big\{\Pi_{f^{(2)}(u)}(u^2) - 2\Pi_{f^{(2)}(u)u}(u)\Big\} \\ &+ \frac{1}{3!}\Big\{\Pi_{f^{(3)}(u)}(u^3) - 3\Pi_{f^{(3)}(u)u}(u^2) + 3\Pi_{f^{(3)}(u)u^2}(u)\Big\} + f(u)^\sharp \end{aligned}$$

for some remainder $f(u)^\sharp$ of parabolic Hölder regularity 4α.

One can actually give an arbitrary high order Taylor expansion formula, see [4].

5 Paracontrolled calculus

In its final form [4], and given as input a noise ζ and some initial condition, the resolution process of a typical singular parabolic equation

$$(\partial_t + L)u = f(u,\zeta), \tag{5.1}$$

involves the following elementary steps.

1. **Paracontrolled ansatz.** *The irregularity of the noise ζ, and the form of the equation, dictate the choice of a Banach solution space made up of functions/distributions of the form*

$$u = \sum_{i=1}^{k_0} \widetilde{\Pi}_{u_i} Z_i + u^\sharp, \tag{5.2}$$

for some reference functions/distributions Z_i that depend formally only on ζ, to be determined later; we have for instance $Z_1 = \mathcal{R}(\zeta)$, if the equation is affine with respect to ζ. The derivatives' u_i of u also need to satisfy such a structural equation, to order $(k_0 - 1)$, and their derivatives a structural equation of order $(k_0 - 2)$, and so on. One sees the above description (5.2) *of u as a paracontrolled Taylor expansion at order k_0 for it; denote by $\widehat{u}$ the datum of u and all its derivatives.*

2. **Right hand side.** *The use of a Taylor expansion formula,* Theorem 1, *and continuity results for some operators, allow to rewrite the right hand side $f(u,\zeta)$ of equation* (5.1) *in the canonical form*

$$f(u,\zeta) = \sum_{j=1}^{k_0} \Pi_{v_j} Y_j + (\sharp)$$

where $(\sharp)$ is some nice, in particular sufficiently regular, remainder and the distributions Y_j depend only on ζ and the Z_i.

3. **Fixed point.** *Denote by P the resolution of the free heat equation*

$$Pu_0 := (\tau, x) \mapsto \big(e^{-\tau L} u_0\big)(x).$$

Then the fixed point relation

$$\begin{aligned} u &= Pu_0 + \mathcal{R}\big(f(u,\zeta)\big) \\ &= Pu_0 + \sum_{j=1}^{k_0} \mathcal{R}\Big(\Pi_{v_j} Y_j\Big) + \mathcal{R}(\sharp) \\ &= Pu_0 + \sum_{j=1}^{k_0} \widetilde{\Pi}_{v_j} Z_j + \mathcal{R}(\sharp), \end{aligned}$$

imposes some consistency relations on the choice of the $Z_i = \mathcal{R}(Y_i)$ that determine them uniquely as a function of ζ and Z_1. Those expressions inside the Y_i's that do not make sense on a purely analytical basis are precisely those elements that need to be given as components of the enhanced distribution $\widehat{\zeta}$. Schauder estimates for $\mathcal{R}$ play a role in running the fixed point argument. Note that, strictly speaking, the fixed point relation is a relation on $\widehat{u}$ rather than u. We choose to emphasize that point by rewriting the equation under the form

$$(\partial_t + L)u = f\big(\widehat{u}, \widehat{\zeta}\big).$$

As expected, the elements that need to be added in $\widehat{\zeta}$ to ζ are those needed to make sense of the corresponding ill-defined products in the regularity structures setting. List the elements of $\widehat{\zeta}$ in non-decreasing order of regularity and consider them as a basis of a finite dimensional space. A renormalisation map is a linear map of the form

$$\mathscr{M} : \widehat{\zeta} \mapsto T\widehat{\zeta} - \Xi,$$

for some upper triangular constant matrix T, with a unit diagonal, and some possibly space-time dependent renormalisation functions/constants Ξ.

4. Symmetry group. *The role of the extra components of* $\widehat{\zeta}$ *in the dynamics is completely clarified by writing*

$$f(u,\zeta) = f\big(\widehat{u},\widehat{\zeta}\,\big) = f_0(\widehat{u},\zeta) + f_1\big(\widehat{u}\,\big)\widehat{\zeta} \tag{5.3}$$

as a sum of a continuous function f_0 *of* $\widehat{u}$ *and* ζ*, and a continuous function* f_1 *of* $\widehat{u}$ *and* $\widehat{\zeta}$*, that is linear with respect to* $\widehat{\zeta}$*. If* ζ *is a stochastic noise and* ζ^ε *stands for a regularized noise, with associated canonical enhancement* $\widehat{\zeta^\varepsilon}$*, and if a renormalisation procedure* $\mathscr{M}^\varepsilon$ *provides an enhanced distribution* $\mathscr{M}^\varepsilon\widehat{\zeta^\varepsilon}$ *converging in probability to some limit element in the space of enhanced distributions, then the solution to the well-posed equation*

$$(\partial_t + L)u^\varepsilon = f\big(u^\varepsilon,\zeta^\varepsilon\big) + f_1(u^\varepsilon)\big(\mathscr{M}^\varepsilon - \mathrm{Id}\big)\widehat{\zeta^\varepsilon}$$

converges in probability to the first component u *of the solution to the equation*

$$(\partial_t + L)u = f\big(\widehat{u},\widehat{\zeta}\,\big). \tag{5.4}$$

Decomposition (5.3) makes it clear how the renormalisation group acts on the equation as a symmetry group. Three ingredients are used to run the above scheme in any concrete situation.

(i) *The pair* $\big(\Pi,\widetilde{\Pi}\,\big)$ *of* **intertwined paraproducts** *introduced in* [3]. It is crucially used to define a continuous map Φ from $\mathcal{S}\big(\widehat{\zeta}\,\big)$ to itself. The use of an ansatz solution space where Π-operators would be used in place of $\widetilde{\Pi}$-operators would not produce a map from $\mathcal{S}\big(\widehat{\zeta}\,\big)$ to itself.

(ii) *A* **high order Taylor expansion formula** *generalizing Bony's paralinearization formula is used to give a paracontrolled Taylor expansion of a non-linear function of* u*, starting from a paracontrolled function* u*.*

(iii) **Continuity results.** The technical core of Gubinelli-Imkeller-Perkowski' seminal work [10] is a continuity result for the operator

$$\mathsf{C}(f,g\,;h) = \Pi\big(\Pi_f g,h\big) - f\Pi(g,h).$$

The work [4] introduces a number of other operators and prove their continuity. These operators are used crucially in analyzing the right hand side $f(u,\zeta)$ of the equation, step **2**.

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