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par

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Sujet :

FRONTIERE DE POISSON D'UNE DIFFUSION RELATIVISTE

Soutenue le 13 décembre 2006 devant la commission d'examen composée de :

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Résumé

Frontière de Poisson d'une diffusion relativiste

Cette thèse a pour objet l'étude du comportement asymptotique d'une diffusion définie sur l'espace/temps de Minkowski. Le pendant analytique de ce problème est la détermination de l'ensemble des fonctions bornées du noyau d'un certain opérateur différentiel d'ordre 2. Utilisant des méthodes probabilistes (équations différentielles stochastiques, couplage), on donne une description explicite de cet ensemble de fonctions. On donne dans le meme temps une toute autre démonstration de ce résultat, dans l'esprit de travaux sur les marches aléatoires, préexistant. On montre par ailleurs comment la géométrie de l'espace se reflète sur le comportement asymptotique de la diffusion. En un sens, une trajectoire (aléatoire) typique finit par se comporter comme un trajectoire de lumière.

Abstract

Poisson Boundary of a Relativistic Diffusion

In this thesis, we study the asymptotic behaviour of a diffusion defined on Minkowski's spacetime. The analytic counterpart of this problem is to determine the set of bounded functions belonging to the kernel of some second order differential operator. Using probabilistic methods (stochastic differential equations, coupling), one gives an explicit description of this set of functions. In the same time, one give a completely different proof of this result, in the spirit of preexisting works on random walks on groups. Besides, one shows how the geometry of spacetime reflects on the asymptotic behaviour of the diffusion. In some sense, a typical (random) trajectory eventually behaves as a light ray.

Mots-clef: probabilités, diffusion, opérateurs hypoelliptiques, équations différentielles stochastiques, couplage, espace/temps de Minkowski, groupes, marches aléatoires

Keywords: probability, diffusion, hypoelliptic operators, stohastic differential equations, coupling, Minkowski's spacetime, groups random walks

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Contents

Chapter 1

Introduction

L'objet de cette thèse est l'étude du comportement asymptotique d'un processus aléatoire défini sur l'espace/temps. On commence par motiver la description d'Einstein de l'espace/temps comme un espace lorentzien (**1**). Ces espaces admettent une famille naturelle de mouvements aléatoires continus, $\mathcal{C}^1$ par morceaux, qui fut décrite par Dudley, dans [Dud66]. Le processus que l'on étudie est l'unique processus $\mathcal{C}^1$ de cette famille (**2**). On donne au **3** une description de la méthode que l'on a employée pour décrire exactement le comportement asymptotique de ce processus. Le point **4** décrit comment on peut compactifier l'espace d'état du processus et voir que celui-ci converge vers un point du bord. Une toute autre explication du comportement asymptotique du processus est donnée au **5**, en termes algébriques.

1 – La géométrie de l'espace/temps – La mécanique classique comme la mécanique relativiste reposent toutes les deux sur le choix d'une classe de cartes lisses de l'espace/temps, pour lesquelles on postule la validité du principe suivant.

Principe de Relativité. *Les équations qui décrivent les lois de la nature en fonction des coordonnées dans l'espace/temps conservent leur forme dans n'importe laquelle des cartes privilégiées.*

Deux a priori sont à la base de la mécanique classique.

a) Il existe une carte dans laquelle une particule soumise à aucune force est animée d'un mouvement de translation uniforme.

b) Le temps est une notion absolue.

Identifions l'espace/temps à $\mathbb{R} \times \mathbb{R}^3$ à travers la carte donnée par le postulat **a)**. On préfère, dans un cadre physique, le terme ***référentiel*** au terme carte. Le Principe de Relativité et le postulat **b)** nous disent que les seuls changements de référentiels privilégiés sur $\mathbb{R} \times \mathbb{R}^3$ sont de la forme

$$(t, x) \in \mathbb{R} \times \mathbb{R}^3 \mapsto \big(t, t\,a + \varphi(x)\big) \in \mathbb{R} \times \mathbb{R}^3,$$

où φ est un isomorphisme de $\mathbb{R}^3$ et a un vecteur de $\mathbb{R}^3$. Cette classe de référentiels sur l'espace/temps forme la classe des ***référentiels galiléens***.

L'usage de cette classe présente des difficultés connues de longue date. En mécanique classique, on décrit toutes les interactions à l'aide de fonctions dites d'énergie d'interaction potentielle, qui dépendent des positions des particules mises en jeu. Dans tous les cas, les équations dynamiques qui en découlent décrivent des phénomènes où les interactions sont instantanées. Or l'expérience tend à montrer que non seulement il n'existe aucune interaction instantanée, mais qu'il existe en plus une vitesse maximale finie de propagation des interactions. On montre que cette vitesse est celle de la lumière dans le vide. On la note traditionnellement c. Le fait que c soit finie contredit de façon flagrante la loi usuelle de composition des vitesses, qui découle des postulats **a)** et **b)** [1]. D'autres difficultés plus subtiles sont inhérentes au modèle galiléen.

Il faut attendre 1905 et l'article d'Einstein intitulé “Sur l'électrodynamique des corps en mouvement”, [Ein05], pour voir proposée une alternative au modèle galiléen de l'espace/temps et voir disparaître la notion de temps absolu.

On peut résumer les choix d'Einstein aux deux postulats suivants :

1. Il existe un référentiel de l'espace/temps dans lequel la vitesse de la lumière (dans le vide) est finie.

2. Aucun interaction ne va plus vite que la lumière.

Identifions l'espace/temps à $\mathbb{R}\times\mathbb{R}^3$ à travers le référentiel du postulat 1. Le Principe de Relativité détermine (théoriquement) l'ensemble des référentiels admissibles sur l'espace/temps ; il postule en outre que la vitesse de la lumière est la même dans tous ces référentiels. On la note encore c. On peut donner de ce fait et du postulat 1 une formulation mathématique.

Notons A l'événement consistant en l'émission au temps t_A d'un signal se propageant à la vitesse de la lumière depuis un point de coordonnées (x_A, y_A, z_A) [2]. Notons B l'événement consistant en la réception du signal au temps t_B, à la position (x_B, y_B, z_B). La distance entre (x_A, y_A, z_A) et (x_B, y_B, z_B) est d'une part égale à $\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}$ et d'autre part à $c(t_B-t_A)$, puisque la vitesse du signal est constante, égale à c. On a donc

$$c^2(t_b-t_A)^2-\big((x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2\big)=0.$$

Si l'on repère les événements A et B dans un autre référentiel admissible par leur coordonnées (t'_A, x'_A, y'_A, z'_A) et (t'_B, x'_B, y'_B, z'_B), on doit aussi avoir

$$c^2(t'_b-t'_A)^2-\big((x'_B-x'_A)^2+(y'_B-y'_A)^2+(z'_B-z'_B)^2\big)=0,$$

puisque la vitesse de la lumière y est aussi constante, égale à c. En ces termes, le postulat 2 signifie que

$$c^2(t_b-t_A)^2-\big((x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_B)^2\big)>0$$

si, et seulement si,

$$c^2(t'_b-t'_A)^2-\big((x'_B-x'_A)^2+(y'_B-y'_A)^2+(z'_B-z'_B)^2\big)>0.$$

Aussi l'espace/temps $\mathbb{R}\times\mathbb{R}^3$ est-il naturellement muni de la forme quadratique

$$q(t,x,y,z)=c^2\,t^2-(x^2+y^2+z^2),$$

de signature $(1,-3)$. Elle permet d'associer à chaque point $A\in\mathbb{R}\times\mathbb{R}^3$ le (demi-)cône $\mathcal{C}^{>0}(A)$ de sommet A formé par les événements B influençables par A : $\mathcal{C}^{>0}(A)=\big\{B\in\mathbb{R}\times\mathbb{R}^3;\,q(B-A)>0\big\}$. Avec ces notations, la classe des référentiels admissibles est l'ensemble des $\mathcal{C}^\infty$-difféomorphismes f de $\mathbb{R}\times\mathbb{R}^3$ tels que

$$q(B-A)>0 \iff q\big(f(B)-f(A)\big)>0. \qquad (1.0.1)$$

On pose $c=1$ dans la suite, ce qui revient à prendre (ct,x,y,z) comme coordonnées sur $\mathbb{R}\times\mathbb{R}^3$.

Les trois types suivants de transformations vérifient cette condition.

[1] D'après le Principe de Relativité, cette vitesse c est la même dans tout référentiel en translation uniforme (par rapport au référentiel canonique donné par le postulat 1). Choisissons un tel référentiel, en translation uniforme dans la direction x, allant à vitesse v. Si l'on envoie depuis le centre du référentiel mobile un signal allant à la vitesse de la lumière et se propageant dans la direction x, le signal doit avoir la vitesse c, vu depuis le référentiel canonique (Principe de Relativité), alors que la loi de composition des vitesses nous dit que le signal a pour vitesse $c+v$. L'égalité $c=c+v$ ne peut avoir lieu que si $c=+\infty$.

[2] (t_A, x_A, y_A, z_A) sont les coordonnées du point A dans le référentiel donné par le postulat 1.

- Les translations de $\mathbb{R} \times \mathbb{R}^3$.
- Les dilatations $\xi \in \mathbb{R} \times \mathbb{R}^3 \mapsto \lambda\xi \in \mathbb{R} \times \mathbb{R}^3$, $\lambda \neq 0$.
- Les isométries de q : $SO(1,3) = \{g \in GL(\mathbb{R}^4) \;\; q(g(\xi)) = q(\xi), \forall\, \xi \in \mathbb{R} \times \mathbb{R}^3\}$.

$SO(1,3)$ est un groupe de matrices dont l'algèbre de Lie $so(1,3) = \left\{ \begin{pmatrix} 0 & {}^t\zeta \\ \zeta & M \end{pmatrix} ; \zeta \in \mathbb{R}^3, M \in so(3) \right\}$ est engendrée par deux sortes d'éléments :

- les ***boosts*** $E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & & & \\ 0 & & \mathbf{0_3} & \\ 0 & & & \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & & & \\ 1 & & \mathbf{0_3} & \\ 0 & & & \end{pmatrix}$ et $E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & \mathbf{0_3} & \\ 1 & & & \end{pmatrix}$, où $\mathbf{0_3}$ est l'application nulle de $\mathbb{R}^3$,
- les ***rotations spatiales infinitésimales*** : $E_{1,2} = \begin{pmatrix} 0 & & (0) & \\ & 0 & 1 & 0 \\ (0) & -1 & 0 & 0 \\ & 0 & 0 & 0 \end{pmatrix}$, $E_{1,3} = \begin{pmatrix} 0 & & (0) & \\ & 0 & 0 & 1 \\ (0) & 0 & 0 & 0 \\ & -1 & 0 & 0 \end{pmatrix}$ et $E_{2,3} = \begin{pmatrix} 0 & & (0) & \\ & 0 & 0 & 0 \\ (0) & 0 & 0 & 1 \\ & 0 & -1 & 0 \end{pmatrix}$.

On doit à Zeeman [Zee64] la preuve du fait qu'il n'y a pas d'autres transformations satisfaisant la relation (1.0.1) que les composées de transformations du type précédent.

THÉORÈME **1 (Zeeman, [Zee64]).** *Toute bijection de $\mathbb{R} \times \mathbb{R}^3$ satisfaisant la relation* (1.0.1) *quels que soient les points A et B est la composée d'un nombre fini de translations, de dilatations et d'isométries de la forme quadratique q*[3]*.*

On ne retient en pratique que le groupe des isométries affines de q, engendré par les translations et les isométries de q. On note $\mathcal{G}_0$ ce groupe. L'espace/temps, muni de sa forme quadratique q et de son groupe d'isométries affines, est noté $\mathbb{R}^{1,3}$.

2 – Mouvement aléatoire naturel – Ce groupe de transformations de l'espace/temps agit sur l'ensemble des courbes $\gamma : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{1,3}$, non nécessairement continues, en envoyant γ sur $\varphi \circ \gamma$, si $\varphi \in \mathcal{G}_0$. C'est dans ce fait que géométrie et probabilités se rencontrent.

Donnons-nous un mouvement aléatoire, défini par une famille $\{\mathbb{P}_\xi\}_{\xi \in \mathbb{R}^{1,3}}$ de probabilités donnant la loi d'une trajectoire issue de ξ. Que doit-on imposer à cette famille pour que l'image par $\varphi \in \mathcal{G}_0$ d'une trajectoire issue de ξ ait même loi qu'une trajectoire issue de $\varphi(\xi)$, quel que soit $\varphi \in \mathcal{G}_0$?

Intéressons-nous plus particulièrement aux courbes aléatoires continues dont la composante temporelle croît strictement dans un certain référentiel (donc dans tous).

Dans l'espace/temps euclidien où le groupe $\mathcal{G}_0$ est remplacé par le groupe $\mathcal{G}_{\mathrm{Eucl}}$ des isométries affines euclidiennes, le temps est le même dans tous les référentiels ; on le choisit comme paramétrage des courbes aléatoires, que l'on peut supposer commencer au temps 0. Cette simplification faite, on cherche quelles lois sur $\mathcal{C}(\mathbb{R}^{\geqslant 0}, \mathbb{R}^3)$ sont invariantes par l'action des isométries affines euclidiennes. On ne peut apporter de réponse simple à cette interrogation sans imposer quelques conditions sur la nature du mouvement aléatoire[4].

[3]Noter que ce théorème ne recquiert aucune régularité a priori de la part de f.

[4]Qu'on pense aux processus de Markov, aux processus non Markoviens comme ceux qui dépendent de tout leur passé, etc. Leurs structures sont très différentes.

On obtient une réponse satisfaisante si l'on se restreint aux diffusion de Feller-Dynkin[5]. Dans ce cadre, on lit sur le générateur L de la diffusion l'invariance des lois du mouvement par l'action des isométries :

$$\left\{\forall\,\varphi \in \mathcal{G}_{\mathrm{Eucl}}\,,\ \forall\,\overrightarrow{\xi} \in \mathbb{R}^3,\ \varphi_*(\mathbb{P}_{\overrightarrow{\xi}}) = \mathbb{P}_{\varphi(\overrightarrow{\xi})}\right\} \iff \left\{\forall\,\varphi \in \mathcal{G}_{\mathrm{Eucl}}\,, \forall\, f \in \mathcal{C}_c^{\infty}(\mathbb{R}^3)\,,\ L(f\circ\varphi) = (Lf)\circ\varphi\right\}.$$

L est un opérateur différentiel du second ordre, à coefficients continus. L'invariance de L par les translations nous dit que ses coefficients sont constants. En utilisant la transformation de Fourier et l'invariance de L par l'action des isométries euclidiennes, on montre alors que L doit être un multiple positif du Laplacien[6].

Théorème **2.** *Les seuls opérateurs différentiels d'ordre* 2 *sur* $\mathbb{R}^3$*, positifs, invariants par l'action des isométries affines euclidiennes sont les multiples positifs du laplacien.*

Soit $c \geqslant 0$. L'opérateur $c\triangle$ est le générateur du processus $\{w_{c\,t}\}_{t\geqslant 0}$, où $\{w_s\}_{s\geqslant 0}$ est un mouvement brownien sur $\mathbb{R}^3$.

D'autres mouvements aléatoires dans $\mathbb{R}^3$ ont une loi invariante par l'action des isométries.

- Changeons d'espace d'états et regardons l'espace tangent à $\mathbb{R}^3$: $T\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$. Les isométries euclidiennes agissent naturellement sur $T\mathbb{R}^3$ en envoyant un point $(\overrightarrow{\xi}, v)$ sur $(\varphi(\overrightarrow{\xi}), \varphi(v))$. On définit par exemple une diffusion sur $T\mathbb{R}^3$, invariante par cette action, en posant

  $$\overrightarrow{\xi}_s = \overrightarrow{\xi}_0 + \int_0^s w_r\, dr\ , \quad v_s = v_0 + w_r,$$

  où $\{w_r\}_{r\geqslant 0}$ est un mouvement brownien sur $\mathbb{R}^3$. Il en va de même si l'on remplace w par un mouvement brownien sur la sphère [7].

- A l'opposé des processus continus, un processus sautant à intervalles de temps constants, selon une loi radiale, a aussi une loi invariante par l'action des isométries. Un mélange de sauts et de trajectoires browniennes aura encore cette propriété.

La situation sur $\mathbb{R}^{1,3}$ diffère radicalement du cas euclidien. D'abord en ce que la notion de temps dépendant du référentiel, on ne peut l'évincer du problème comme on l'a fait dans le cas euclidien. Plus fondamentalement, on cherche à modéliser l'évolution dans l'espace/temps d'un phénomène aléatoire *réel*. Ce phénomène ne pouvant se propager plus vite que la lumière, il lui correspond une courbe $\{\xi_s\}_{s\geqslant 0} = \left\{(t(s),\, \overrightarrow{\xi}(s))\right\}_{s\geqslant 0}$ dans $\mathbb{R}^{1,3}$ ayant les deux propriétés

1. $t(s)$ est une fonction continue, strictement croissante de s,
2. $\forall\, 0 \leqslant r \leqslant s,\ \left|\overrightarrow{\xi}(s) - \overrightarrow{\xi}(r)\right| \leqslant t(s) - t(r)$([8]).

Cette courbe est dérivable en presque tout s. On va s'intéresser aux trajectoires *de classe* $\mathcal{C}^1$ correspondant à un phénomène se propageant à une vitesse strictement inférieure à celle de la lumière :

$$\forall\, 0 \leqslant r \leqslant s\ ,\ \left|\overrightarrow{\xi}(s) - \overrightarrow{\xi}(r)\right| < t(s) - t(r),$$

[5]Il s'agit des processus de Markov continus, dont le domaine du générateur contient l'ensemble des fonctions de classe $\mathcal{C}^\infty$, à support compact. Cette dénomination est celle adoptée par Rogers & Wiliams dans [RW00], Chap.3.

[6]La tranformée de Fourier transforme l'opérateur différentiel L en l'opérateur de multiplication par un polynôme de degré 2, qui ne doit dépendre que de la distance à l'origine, à cause de l'invariance de L par les isométries.

[7]Pour cette diffusion, $\{\xi_s\}_{s\geqslant 0}$ est récurrente en dimension 2 et transiente dès lors que la dimension est supérieure ou égale à 3. Anticipant sur les problèmes à venir, on peut montrer que cette diffusion sur $\mathbb{R}^3\times\mathbb{S}^2$ n'a pas d'autres fonctions harmoniques bornéees que les constantes.

[8]Ces propriétés, valables dans le référentiel canonique donné par le postulat 1, sont valables dans tout autre référentiel admissible.

soit

$$\forall\, s \geqslant 0\,,\ q\big(\dot{\xi}(s)\big) > 0,$$

si l'on note $\{\dot{\xi}(s)\}_{s\geqslant 0}$ la dérivée de $\{\xi(s)\}_{s\geqslant 0}$. Sous cette condition, on peut reparamétrer la courbe ξ de façon à avoir $q(\dot{\xi}_s) = 1$, quel que soit $s \geqslant 0$.

Notation – On note $\{\varepsilon_0, \varepsilon_1, ..., \varepsilon_d\}$ la base canonique de $\mathbb{R}^{1,3}$.

DEFINITION **3.** *L'hyperboloïde* $\big\{(t, \overrightarrow{\xi}) = \xi \in \mathbb{R}^{1,3};\ q(\xi) = 1\big\}$ *a deux composantes connexes. On note* $\mathbb{H}$ *celle qui correspond à* $t > 0$*. Tout point de* $\mathbb{H}\backslash\{\varepsilon_0\}$ *s'écrit de façon unique* $\big(\mathrm{ch}\rho, (\mathrm{sh}\rho)\sigma\big)$, $\rho > 0$, $\sigma \in \mathbb{S}^2$*. Le couple* (ρ, σ) *forme les **coordonnées polaires** du point* $\big(\mathrm{ch}\rho, (\mathrm{sh}\rho)\sigma\big)$.

La proposition suivante peut être prise comme définition de la géométrie de l'espace hyperbolique[9].

PROPOSITION **4.** *La restriction de* q *à chaque espace tangent est définie négative. Cela fait de* $\mathbb{H}$ *une variété riemannienne à courbure constante, égale à* -1 *:* $\mathbb{H}$ *est un modèle de l'espace hyperbolique de dimension* 3.

$\mathbb{H}$ étant une demi-sphère unité de la forme quadratique q, est invariante par l'action d'un sous-groupe $\mathcal{G}$ de $\mathcal{G}_0$, d'indice 4. Ce groupe $\mathcal{G}$ est appelé le ***groupe de Poincaré*** : c'est le produit semi-direct du groupe $SO_0(1,3)$ des isométries directes de q laissant $\mathbb{H}$ stable[10] et du groupe des translations de $\mathbb{R}^{1,3}$ (dont l'action est triviale sur l'ensemble des vecteurs vitesses). Toute isométrie de $\mathbb{H}$ provient de l'action sur $\mathbb{H}$ d'une isométrie de q ; $\mathcal{G}$ agit transitivement sur $\mathbb{H}$.

On note $\triangle^{\mathbb{H}}$ le Laplacien sur $\mathbb{H}$. C'est un opérateur invariant par les isométries de $\mathbb{H}$.

La courbe (aléatoire) $\{\dot{\xi}_s\}_{s\geqslant 0}$ sur $\mathbb{H}$ déterminant la courbe $\{\xi_s\}_{s\geqslant 0}$, via la relation

$$\xi_s = \xi_0 + \int_0^s \dot{\xi}_r\, dr,$$

on est ramené à determiner l'ensemble des mouvements aléatoires sur $\mathbb{H}$, continus, dont la loi est invariante par l'action de $\mathcal{G}$. Il n'existe pas plus que dans le cadre euclidien de description simple de cet ensemble. Si l'on se restreint à la classe des diffusions de Feller-Dynkin sur $\mathbb{H}$, on obtient une description analogue à celle du théorème **2**.

THÉORÈME **5.** *Les seuls opérateurs différentiels d'ordre* 2 *sur* $\mathbb{H}$*, invariants par l'action sur* $\mathbb{H}$ *des isométries de* q *sont les multiples positifs du laplacien* $\triangle^{\mathbb{H}}$.

Soit $c \geqslant 0$. L'opérateur $c\triangle^{\mathbb{H}}$ est le générateur du processus $\{w_{ct}\}_{t\geqslant 0}$, où $\{w_s\}_{s\geqslant 0}$ est un mouvement brownien sur $\mathbb{H}$.

D'autres mouvements aléatoires sur $\mathbb{H}$ ont une loi invariante par l'action de $\mathcal{G}$.

- On peut contruire sur $\mathbb{H}$ des courbes aléatoires de classe $\mathcal{C}^1$, dont la loi est invariante par l'action de $\mathcal{G}$: il suffit par exemple, partant d'un point donné $\dot{\xi}$, de choisir la direction U selon la probabilité uniforme sur $\mathbb{S}^2$ et de parcourir linéairement (ou pas !) l'unique géodésique définie par $\dot{\xi}$ et U.
- Abandonnant l'hypothèse de continuité, un processus de Poisson $\{\dot{\xi}_s\}_{s\geqslant 0}$ sur $\mathbb{H}$, de loi de saut radiale, définit un processus $\big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ sur $\mathbb{H}\times\mathbb{R}^{1,3}$ dont la loi est invariante par l'action de $\mathcal{G}$.

Comme le remarque Dudley, [Dud66], p.259, un tel processus pour les vitesses semble adéquat, si l'on songe à une particule gardant une vitesse constante entre deux chocs aléatoires la faisant changer de direction.

On peut en fait donner une description exhaustive de l'ensemble des processus de Markov càdlàg sur $\mathbb{H}$, donnés par des noyaux de transition $\big\{P_t(\dot{\xi}, .)\big\}_{t\geqslant 0, \dot{\xi}\in\mathbb{H}}$ vérifiant les conditions

[9] Voir [Ive92], Chap.4.

[10] C'est là une définition de $SO_0(1,3)$, qui est la composante connexe de Id dans $SO(1,3)$.

1. **Conservation de la masse** – $P_t(\dot{\xi}, \mathbb{H}) = 1$, quels que soient $t \geqslant 0$ et $\dot{\xi} \in \mathbb{H}$.
2. **Invariance par l'action de $\mathcal{G}$** – Pour toute isométrie g de $\mathbb{H}$, tout point $\dot{\xi} \in \mathbb{H}$, tout borélien A de $\mathbb{H}$ et tout temps $t \geqslant 0$,
$$P_t(g(\dot{\xi}), g(A)) = P_t(\dot{\xi}, A).$$

Rappelons qu'on note $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ la base canonique de $\mathbb{R} \times \mathbb{R}^3$; $\varepsilon_0 \in \mathbb{H}$.

DÉFINITION **6.** *Une **mesure** [11] invariante par toute isométrie de $\mathbb{H}$ laissant ε_0 stable est dite **radiale**.*

On définit sans équivoque la convolution de deux mesure radiales μ et ν en posant pour tout borélien A de $\mathbb{H}$
$$\mu \star \nu(A) = \int \nu(A_{\dot{\xi}})\mu(d\dot{\xi}),$$
où $A_{\dot{\xi}} = T_{\dot{\xi}}(A)$, et $T_{\dot{\xi}}$ est n'importe quelle isométrie de $\mathbb{H}$, choisie continûment en fonction de $\dot{\xi}$, qui envoie $\dot{\xi}$ sur ε_0. La propriété 2 permet de réécrire la propriété de Markov sous la forme
$$P_s \star P_t = P_{s+t}.$$
Aussi, chaque probabilité $P_t(\varepsilon_0, .)$ [12] est-elle infiniment divisible :
$$\forall\, n \geqslant 1, \quad P_t(\varepsilon_0, .) = P_{\frac{t}{n}}(\varepsilon_0, .) \star \cdots \star P_{\frac{t}{n}}(\varepsilon_0, .).$$

Karpelivich, Schur et Tutubalin ont donné dans [KTŠ59] une caractérisation de l'ensemble des probabilités radiales sur $\mathbb{H}$ infiniment divisibles, analogue au théorème de Lévy-Khinchin. Comme son analogue réel, cette caractérisation se formule en termes de transformée de Fourier. Le rôle des fonctions $e^{i\lambda\cdot}$, vecteurs propres du laplacien associé aux valeurs propres $-\lambda^2$, est joué par les fonctions sphériques
$$F_\lambda(\dot{\xi}) = \frac{\sin(\lambda\rho)}{\lambda \mathrm{sh}(\rho)},\ F_\lambda(\varepsilon_0) = 1,\ F_0(\dot{\xi}) = \frac{\rho}{\mathrm{sh}\rho}, \tag{1.0.2}$$
où ρ est la distance hyperbolique de $\dot{\xi}$ à ε_0 et λ un nombre complexe tel que $|\Im m(\lambda)| \leqslant 1$. La fonction F_λ est la seule fonction radiale vecteur propre de $\triangle^{\mathbb{H}}$, associé à la valeur propre $-(1+\lambda^2)$. La transformée de Fourier d'une probabilité radiale μ est définie pour $|\Im m(\lambda)| \leqslant 1$ par la formule
$$\widehat{\mu}(\lambda) = \int F_\lambda(\dot{\xi})\mu(d\dot{\xi}). \tag{1.0.3}$$

THÉORÈME **7 (Karpelevich, Schur, Tutubalin, [KTŠ59]).** *Une probabilité radiale est infiniment divisible si, et seulement si,*
$$\log \widehat{\mu}(\lambda) = (1+\lambda^2)\alpha - \int \big(1 - F_\lambda(\dot{\xi})\big)\mathbb{Q}(d\dot{\xi}),$$
où $\alpha \geqslant 0$ et $\mathbb{Q}$ est une mesure positive, radiale, telle que
$$\int \frac{\rho^2}{1+\rho^2}\mathbb{Q}(d\dot{\xi}) < \infty.$$

Notons $\{\dot{\xi}_s\}_{s\geqslant 0}$ le processus de Markov càdlàg dont $\{P_s\}_{s\geqslant 0}$ est la famille des noyaux de transition. La caractérisation de Karpelevich, Schur et Tutubalin des noyaux de transition P_s admet un pendant dynamique dans lequel $\{\dot{\xi}_s\}_{s\geqslant 0}$ est décrit comme la solution d'une équation différentielle stochastique conduite par un mouvement brownien et un processus de Poisson. Cette description dynamique de $\{\dot{\xi}_s\}_{s\geqslant 0}$ permet de voir le processus comme la "composée" d'un mouvement brownien sur $\mathbb{H}$ et d'un processus de saut radial sur $\mathbb{H}$; elle est détaillée dans la première partie de l'appendice **Dudley's results**, p.88.

[11] Il est sous-entendu ici comme dans la suite que l'on considère une mesure Borélienne, σ-finie.

[12] Qui détermine $P_t(\dot{\xi}, .)$, quel que soit $\dot{\xi}$ grâce à la propriété 2 et à la transitivité de l'action de $\mathcal{G}$ sur $\mathbb{H}$.

3 – Comportement asymptotique – Si le comportement asymptotique de $\{\xi_s\}_{s\geqslant 0}$ est difficile à appréhender avec les seules informations sur $\{\dot{\xi}_s\}_{s\geqslant 0}$ données par le théorème de Karpelevich, Schur et Tutubalin, on peut tout de même obtenir une bonne description du comportement asymptotique de $\{\dot{\xi}_s\}_{s\geqslant 0}$ à l'aide de la seule propriété 2 d'invariance.

THÉORÈME **8 (Dudley, [Dud66], Theorem** 9.2, **p.**261, **[Dud73], Theorem** 3, **p.**3552**).**
- *Le processus $\{\dot{\xi}_s\}_{s\geqslant 0}$ est transient.*
- *Notons (ρ_s, σ_s) les coordonnées polaires de $\dot{\xi}_s$. La direction σ_s de $\dot{\xi}_s$ converge lorsque s tend vers l'infini.*

On donne une preuve de ces théorèmes dans l'appendice **Dudley's results**, p.88. Il est difficile de pousser plus loin l'investigation sur le comportement asympotique du processus $\big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ sur $\mathbb{H}\times\mathbb{R}^{1,3}$, sous les seules conditions 1 et 2, et Dudley laisse les choses en l'état en 1974([13]).

On notera dorénavant $u_s = (\dot{\xi}_s, \xi_s) \in \mathbb{H}\times\mathbb{R}^{1,3}$.

On regarde ce processus comme une variable aléatoire définie sur un espace mesurable $(\Omega, \mathcal{F})$, à valeurs dans $\mathcal{C}(\mathbb{R}^{\geqslant 0}, \mathbb{H}\times\mathbb{R}^{1,3})$. Cet espace mesurable porte une famille de probabilités $\{\mathbb{P}_{u_0}\}_{u_0\in\mathbb{H}\times\mathbb{R}^{1,3}}$, donnant les lois de $\{u_s\}_{s\geqslant 0}$ lorsque les trajectoires sont issues des points $u_0 \in \mathbb{H}\times\mathbb{R}^{1,3}$. On peut définir des applications mesurables $\theta_t : \Omega \to \Omega$, $t > 0$, telles que

$$u_s\big(\theta_t(\omega)\big) = u_{s+t}(\omega),$$

quels que soient $s \geqslant 0, t > 0, \omega \in \Omega$. Ces opérateurs θ_t sont appelés des ***shifts***.

D'un point de vue formel, c'est dans la tribu

$$\tau\big(\{u_s\}\big) = \bigcap_{t>0} \sigma(u_s; s \geqslant t)$$

que se trouvent les informations sur le comportement asymptotique de la diffusion $\{u_s\}_{s\geqslant 0}$. Cette tribu s'appelle la ***tribu asymptotique du processus*** $\{u_s\}_{s\geqslant 0}$.

La tribu asymptotique contient une sous-tribu intéressante : ***la tribu invariante***. Elle est formée de l'ensemble des évènements de $\tau\big(\{u_s\}\big)$ invariants par les shifts θ_t :

$$Inv\big(\{u_s\}\big) = \big\{Z \in \tau\big(\{u_s\}\big)\,;\ \forall\, t > 0,\ \theta_t^{-1} Z = Z\big\}.$$

PROBLÈME. *Peut-on trouver un variable aléatoire Y, à valeurs dans un certain espace (un ensemble discret, une variété, une espace de Banach...) telle que, sous chaque probabilité $\mathbb{P}_{u_0}$*

- *les tribus $\tau\big(\{u_s\}\big)$ (ou $Inv\big(\{u_s\}\big)$) et $\sigma(Y)$ coïncident à des ensembles $\mathbb{P}_{u_0}$-mesure nulle près,*
- *la loi de Y est connue ?*

Ce travail de thèse apporte une réponse à ces deux problèmes lorsque le processus $\{\dot{\xi}_s\}_{s\geqslant 0}$ sur $\mathbb{H}$ est un mouvement brownien. Le théorème de Karpelevitch, Tutubalin et Schur donne l'ordre de généralité de ce choix : on n'examine que les processus continus répondant aux exigences de conservation de la masse et d'invariance par l'action de $\mathcal{G}$. On gagne en contre-partie tous les outils du calcul stochastique.

Le premier bénéfice d'un tel choix est la redémonstration quasi-immédiate du théorème **8** de Dudley sur le comportement asymptotique de $\{\dot{\xi}_s\}_{s\geqslant 0}$. On montre en effet que ses coordonnées polaires $(\rho_s, \theta_s)_{s\geqslant 0}$ vérifient les équations

[13] A la recherche de conditions plus générales encore, Dudley abandonne dans l'article [Dud73] l'hypothèse d'invariance des noyaux de transition par les isométries de $\mathbb{H}$ et donne des conditions plus faibles sous lesquelles la direction σ_s du mouvement $\dot{\xi}_s$ sur $\mathbb{H}$ converge (que $\{\dot{\xi}_s\}_{s\geqslant}$ soit on non transient).

$$\rho_s = \rho_0 + w^\rho_s + \int_0^s \coth(\rho_r)\,dr,$$
$$\sigma_s = \Sigma\left(\int_0^s \frac{dr}{\mathrm{sh}\rho_r}\right),$$

où w^ρ est un mouvement brownien réel et Σ un mouvement brownien sur la sphère $\mathbb{S}^2$, indépendant de w^ρ. On lit sur ces équations que $\rho_s \geqslant \rho_0 + w^\rho_s + s$, tend vers l'infini plus vite que $\frac{s}{2}$, et que donc le changement de temps $\int_0^s \frac{dr}{\mathrm{sh}\rho_r}$ converge lorsque $s \to +\infty$.

Second bénéfice : on peut reformuler le problème précédent en termes analytiques, faisant intervenir le générateur différentiel de $\big\{(\dot\xi_s, \xi_s)\big\}_{s\geqslant 0}$.

$$Lf(\dot\xi, \xi) = \frac{\triangle^{\mathbb{H}}_{\dot\xi} f}{2} + \partial_\xi f(\dot\xi).$$

$\partial_\xi f(\dot\xi)$ y désigne la différentielle de f par rapport à ξ, dans la direction $\dot\xi$.

Problème Bis. *Déterminer l'ensemble des fonctions bornées h, définies sur $\mathbb{R}^{>0} \times \big(\mathbb{H} \times \mathbb{R}^{1,3}\big)$ (resp. $\mathbb{H} \times \mathbb{R}^{1,3}$), telles que $(\partial_s + L)h = 0$ (resp. $Lh = 0$).*[14]

Cette reformulation est effectuée dans la section **2.2.1**. On y montre qu'une fonction bornée vérifiant l'équation $(\partial_s + L)h = 0$ est indépendante de s ; elle vérifie donc l'équation $Lh = 0$. Aussi n'y a-t-il qu'un seul problème analytique. D'un point de vue probabiliste, cela signifie que toute l'information sur le comportement asymptotique de la diffusion $\{u_s\}_{s\geqslant 0}$ est contenue dans la tribu invariante $Inv\big(\{u_s\}\big)$.

Définition **9.** *Une fonction $h : \mathbb{H} \times \mathbb{R}^{1,3} \to \mathbb{R}$, de classe $\mathcal{C}^2$, telle que $Lf = 0$, est dite L-**harmonique**.*

L'objectif de la partie 2 est de **décrire l'ensemble des fonctions L-harmoniques bornées**. Cette description est donnée au théorème **11**.

La stratégie – Rappelons une définition.

Définition **10.** *Une **fonction L-harmonique positive** h est dite **minimale** si les seules fonctions h' L-harmoniques, positives, telles que $0 \leqslant h' \leqslant h$, sont les fonctions λh, avec $0 \leqslant \lambda \leqslant 1$.*

On va trouver un ensemble $\{h^\alpha\}_{\alpha\in A}$ de fonctions L-harmoniques minimales, indexé par un certain ensemble A, et une probabilité $\nu_\mathbf{1}$ sur A, tels que la fonction $\mathbf{1}$, identiquement égale à 1, est le barycentre des fonctions h^α :

$$\mathbf{1} = \int h^\alpha \nu_\mathbf{1}(d\alpha). \tag{1.0.4}$$

Utilisant des résultats de Choquet sur la représentation intégrale dans les convexes compacts, on déduit de l'identité (1.0.4) que toute fonction L-harmonique bornée est de la forme

$$h(.) = \int h^\alpha(.)H(\alpha)\nu_\mathbf{1}(d\alpha),$$

où H est une fonction sur A, borélienne, bornée.

[14]Remarquer qu'on ne suppose pas h de classe $\mathcal{C}^2$. Une fonction h vérifiant la relation $Lh = 0$, au sens des distributions, est automatiquement de classe $\mathcal{C}^2$. On règle ces questions de régularité dans la section **2.2.1**.

Quelles sont ces fonctions h^α **?** – La convergence de $\{\sigma_s\}_{s\geqslant 0}$ vers un point limite $\sigma_\infty \in \mathbb{S}^2$ nous fournit une identité du type (1.0.4). La variable σ_∞ admet sous chaque probabiblité $\mathbb{P}_{u_0}$, $u_0 \in \mathbb{H}\times\mathbb{R}^{1,d}$, une densité $\sigma \in \mathbb{S}^2 \to h^\sigma(u_0)$ par rapport à la probabilité uniforme $d\sigma$ sur $\mathbb{S}^2$, strictement positive. Pour chaque $\sigma \in \mathbb{S}^2$, la fonction $u \in \mathbb{H}\times\mathbb{R}^{1,3} \mapsto h^\sigma(u)$ est L-harmonique, et

$$\forall\, u \in \mathbb{H}\times\mathbb{R}^{1,3}, \quad 1 = \int h^\sigma(u)\, d\sigma.$$

Mais les fonctions h^σ sont-elles extrêmales ?

Soit φ une rotation de $\mathbb{R}^3$. Pour $\dot\xi \in \mathbb{H}$, de coordonnées polaires (ρ,σ), on note $\varphi(\dot\xi)$ le point de $\mathbb{H}$ de coordonnées polaires $\big(\rho,\varphi(\sigma)\big)$. Cette transformation est une isométrie de $\mathbb{H}$. Utilisant la relation

$$h^{\varphi(\sigma)}\big((\dot\xi,\xi)\big) = h^\sigma\big((\varphi^{-1}(\dot\xi),\varphi^{-1}(\xi))\big),$$

valable quels que soient $\varphi \in O(\mathbb{R}^3)$, $\sigma\in\mathbb{S}^2$, $(\dot\xi,\xi)\in\mathbb{H}\times\mathbb{R}^{1,3}$, et l'invariance de L par l'action de φ^{-1} sur $\mathbb{H}\times\mathbb{R}^{1,3}$, on voit que l'une des fonctions h^σ est minimale si, et seulement si, toutes le sont. On peut donc se contenter de voir si la fonction $h^{\varepsilon_1}(.)$ est minimale ou non[15]. On peut reformuler la définition d'une fonctions harmonique minimale sous la forme

($\bigstar$) : $h^{\varepsilon_1}(.)$ *est minimale si, et seulement si, les seules fonctions h, bornées, vérifiant l'égalité $L(h^{\varepsilon_1}h) = 0$, sont les constantes.*

On peut construire une diffusion $\big\{(\dot\xi_s,\xi_s)\big\}_{s\geqslant 0}$ sur $\mathbb{H}\times\mathbb{R}^{1,3}$ dont le générateur infinitésimal est

$$f \mapsto \frac{L(h^{\varepsilon_1}f)}{h^{\varepsilon_1}}.$$

Cette diffusion a même loi que la diffusion originelle, conditionnée par l'évènement (de probabilité nulle) $\{\sigma_\infty = \varepsilon_1\}$. On parle de diffusion conditionnée ou de ***h^{ε_1}-processus***[16]. On notera $\mathbb{P}^{\varepsilon_1}$ sa loi.

Dans ce cadre, le lien mentionné plus haut entre fonctions harmoniques bornées et tribu invariante donne de la phrase ($\bigstar$) une version probabiliste.

($\bigstar'$) : $h^{\varepsilon_1}(.)$ *est extrêmale si, et seulement si, la diffusion conditionnée a une tribu invariante triviale.*

Un argument simple de martingale montre que la tribu invariante est, sous chaque probabilité $\mathbb{P}^{\varepsilon_1}_{u_0}$, $u_0\in\mathbb{H}\times\mathbb{R}^{1,3}$, engendrée par les variables aléatoires de la forme

$$\lim_{s\to+\infty} f(u_s)$$

où f est une fonction borélienne, bornée, allant de $\mathbb{H}\times\mathbb{R}^{1,3}$ dans $\mathbb{R}$([17]). En somme, on montre que la fonction h^{ε_1} n'est pas extrêmale si l'on trouve une fonction qui converge (presque sûrement) le long des trajectoires de la diffusion conditionnée vers une variable aléatoire dont la loi n'est pas triviale.

On choisit un autre paramétrage de $\mathbb{H}$ que les coordonnées polaires pour examiner le comportement de la diffusion conditionnée. L'application

$$\psi : \big(y,(x_1,...,x_2)\big) \in \mathbb{R}^{>0}\times\mathbb{R}^2 \mapsto \left(\frac{|x|^2+y^2+1}{2y}, \frac{|x|^2+y^2-1}{2y}, \frac{x_1}{y}, \cdots, \frac{x_2}{y^2}\right) \in \mathbb{H}$$

est une isométrie entre l'espace $\mathbb{R}^{>0}\times\mathbb{R}^2$, muni de la métrique riemannienne

$$U,V\in\mathbb{R}^3,\ (y,x)\in\mathbb{R}^{>0}\times\mathbb{R}^2, \quad (U,V)_{(y,x)} = \frac{\langle U,V\rangle_{\mathrm{Eucl}}}{y^2},$$

[15] Rappelons qu'on note $\{\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ la base canonique de $\mathbb{R}\times\mathbb{R}^3$.
[16] On rappelle dans l'appendice **h-transform, 4.2**, p.94, le lien entre conditionnement et h-processus.
[17] Cette limite existe bien $\mathbb{P}^{\varepsilon_1}_{u_0}$-presque sûrement, quel que soit $u_0\in\mathbb{H}\times\mathbb{R}^{1,3}$.

et $(\mathbb{H}, q)$.

On utilise le demi-espace $\mathbb{R}^{>0} \times \mathbb{R}^2$ comme paramétrage de $\mathbb{H}$. Dans ces coordonnées, la fonction $h^{\varepsilon_1}\big((y,x),\xi\big)$ est un multiple de y^2 et la diffusion conditionnée est régie par un système d'équations différentielles stochastiques.

$$\begin{aligned} dy_s &= y_s dw_s^y + \frac{3}{2} y_s ds, \\ dx_s &= y_s dw_s^x, \\ d\xi_s &= \psi\big((y_s, x_s)\big), \end{aligned}$$

où w^y et w^x sont deux mouvements browniens réels indépendants. On montre que la fonction

$$R^{\varepsilon_1}\big((\dot{\xi},\xi)\big) = q(\xi, \varepsilon_0 + \varepsilon_1)$$

converge presque sûrement le long des trajectoires de la diffusion conditionnée vers une variable aléatoire $R_\infty^{\varepsilon_1}$ ayant une loi non triviale. La fonction L-harmonique h^{ε_1} n'est donc pas extrêmale.

On retrouve à cette occasion un résultat de Dufresne identifiant la loi de l'intégrale $\int_0^\infty e^{w_u - cu}\, du$, où w est un mouvement brownien réel et c une constante > 0.

La variable aléatoire $R_\infty^{\varepsilon_1}$ admet sous $\mathbb{P}_u^{\varepsilon_1}$ une densité $\ell \in \mathbb{R} \mapsto h_\ell^{\varepsilon_1}(u)$ par rapport à la mesure de Lebesgue. Les fonctions $u \in \mathbb{H} \times \mathbb{R}^{1,3} \mapsto h_\ell^{\varepsilon_1}(u)$, $\ell \in \mathbb{R}$, sont toutes $L^{h^{\varepsilon_1}}$-harmoniques. On a donc une famille $\{h^{\varepsilon_1} h_\ell^{\varepsilon_1}\}_{\ell \in \mathbb{R}}$ de fonctions L-harmoniques. L'une de ces fonctions $h^{\varepsilon_1} h_\ell^{\varepsilon_1}$ est-elle minimale ?

A la différence de h^{ε_1}, qui est > 0, chacune de ces fonctions $h^{\varepsilon_1} h_\ell^{\varepsilon_1}$ s'annule sur le fermé $\big\{(\dot{\xi},\xi) \in \mathbb{H} \times \mathbb{R}^{1,3}\,;\ \ell \leqslant q(\xi, \varepsilon_0 + \varepsilon_1)\big\}$. Etant donné $\ell \in \mathbb{R}$, on définit donc l'opérateur $L^{h^{\varepsilon_1} h_\ell^{\varepsilon_1}} f = \frac{L(h^{\varepsilon_1} h_\ell^{\varepsilon_1} f)}{h^{\varepsilon_1} h_\ell^{\varepsilon_1}}$ sur l'ouvert complémentaire $\big\{(\dot{\xi},\xi) \in \mathbb{H} \times \mathbb{R}^{1,3}\,;\ \ell > q(\xi, \varepsilon_0 + \varepsilon_1)\big\}$, ainsi que le $h^{\varepsilon_1} h_\ell^{\varepsilon_1}$-processus correspondant.

On montre que les fonctions $L^{h^{\varepsilon_1} h_\ell^{\varepsilon_1}}$-harmoniques, bornées, sont constantes (*i.e.* $h^{\varepsilon_1} h_\ell^{\varepsilon_1}$ est minimale) en utilisant un couplage des trajectoires du $h^{\varepsilon_1} h_\ell^{\varepsilon_1}$-processus.

Couplage – L'idée ici mise en oeuvre est simple. Oublions un instant le $h^{\varepsilon_1} h_\ell^{\varepsilon_1}$-processus et considérons une diffusion[18] sur $\mathbb{R}^n$, de générateur noté L, définie sur un espace mesurable $(\Omega, \mathcal{F})$. La formule d'Itô montre que pour une fonction L-harmonique bornée h, le processus $\{h(u_s)\}_{s \geqslant 0}$ est une martingale bornée. On a donc

$$h(u_0) = \mathbb{E}_{u_0}\big[h(u_T)\big],$$

pour tout temps d'arrêt T. Aussi, étant donnés deux points u_0, u_1, supposons avoir trouvé

- une probabilité $\mathbb{P}_{u_0,u_1}$ sur $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F})$, deux processus $\{u_s\}_{s \geqslant 0}$ et $\{u'_s\}_{s \geqslant 0}$, de lois respectives $\mathbb{P}_{u_0}$ et $\mathbb{P}_{u_1}$, sous $\mathbb{P}_{u_0,u_1}$,
- deux temps aléatoires T et T' tels que T (resp. T') est un temps d'arrêt par rapport à la filtration engendrée par $\{u_s\}_{s \geqslant 0}$ (resp. $\{u'_s\}_{s \geqslant 0}$), et $u_T = u'_{T'}$, $\mathbb{P}_{u_0,u_1}$-presque sûrement.

Alors, on aura

$$h(u_0) = \mathbb{E}_{u_0}[h(u_T)] = \mathbb{E}_{u_0,u_1}[h(u_T)] = \mathbb{E}_{u_0,u_1}[h(u'_{T'})] = \mathbb{E}_{u_1}[h(u'_{T'})] = h(u_1).$$

La difficulté dans ce genre d'argument réside dans la construction même du couplage (la probabilité $\mathbb{P}_{u_0,u_1}$), qui doit nous assurer que les trajectoires $\{u_s\}_{s \geqslant 0}$ et $\{u'_s\}_{s \geqslant 0}$ se rencontrent, ce qui n'a rien d'immédiat dès lors que l'on quitte $\mathbb{R}$.

[18] De Feller-Dynkin.

La situation est tout de même assez bonne pour que l'on puisse réaliser le couplage de mouvements browniens sur une variété à courbure de Ricci positive[19] ! Sous l'impulsion de W. Kendall et Cranston (entre autres) l'étude de tels couplages pour des diffusions elliptiques a donné de nombreux résultats intéressants[20].

Dans notre cas, l'utilisation de cette méthode est a priori embarrassée par la non ellipticité des opérateurs L, $L^{h^{\varepsilon_1}}$ et $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$ et les complications que le cadre non elliptique[21] apportent. Peu d'études ont été menées dans le cadre hypoelliptique[22]. Ben Arous, Cranston et W. Kendall examinent dans [BACK95] la diffusion plane $(w_s, \int_0^s w_r\, dr)$, où w est un mouvement brownien réel, ainsi que la diffusion (W_s, A_s) sur $\mathbb{R}^3$, où W est un mouvement brownien plan et A_s son aire de Lévy stochastique. W. Kendall prolonge avec C. Price l'étude du premier exemple, dans [KP04], en considérant aussi des intégrales itérées du mouvement brownien. Cependant, à ce jour, aucun résultat général sur le couplage de diffusions hypoelliptiques ne semble avoir été obtenu.

Dans l'usage qu'on en fait ici, ce n'est pas tant le couplage en soi qui nous intéresse que son application à l'étude des fonctions harmoniques. Cela nous donne quelque latitude. Avant de coupler des trajectoires de la $h^{\varepsilon_1}h_\ell^{\varepsilon_1}$-diffusion, on montre qu'une fonction $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonique *bornée* ne dépend a priori que de deux coordonnées y et $\xi'^0 (= q(\xi, \varepsilon_0 + \varepsilon_1))$; on n'aura donc à coupler que ces deux coordonnées pour la $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-diffusion.

Cette propriété des fonctions $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmoniques provient de le propriété analogue pour les fonctions $L^{h^{\varepsilon_1}}$-harmoniques bornées. On obtient cette dernière en utilisant un couplage, ou plutôt un ***quasi-couplage***.

Profitant de ce que les fonctions $L^{h^{\varepsilon_1}}$-harmoniques bornées sont naturellement uniformément continues (section **2.2.2**), on montre que l'on peut trouver pour chaque $\varepsilon > 0$ non pas un couplage exact $(u_T = u'_{T'})$, mais un couplage assez bon pour que $|h(u_T) - h(u'_{T'})| \leqslant \varepsilon$, ce qui nous donne

$$\big|h(u_0) - h(u_1)\big| \leqslant \mathbb{E}_{u_0,u_1}\big[\big|h(u_T) - h(u'_{T'})\big|\big] \leqslant \varepsilon,$$

et donc $h(u_0) = h(u_1)$, puisque $\varepsilon > 0$ est arbitraire. On peut définir un tel couplage dès lors que u_0 et u_1 ont les mêmes deux coordonnées y et ξ'^0 (section **2.2.2**).

On remarquera au **2.2.2** que l'on se ramène à coupler deux trajectoires d'une diffusion plane de la forme $\big\{(z_s, I_0 + \int_0^s z_r\, dr)\big\}_{s\geqslant 0}$, où z est une diffusion sur $\mathbb{R}$, récurrente positive. Cette situation ressemble donc à celle étudiée par Ben Arous & al. Leur méthode, cependant, ne fonctionne pas dans ce cas. On résume notre approche dans l'appendice **4.4**, **Coupling of a hypoelliptic diffusion on** $\mathbb{R}^2$, p.100.

L'approche esquissée ci-dessus permet de donner une description de l'ensemble des fonctions harmoniques bornées, et par là, de la tribu invariante de la diffusion $\big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ sur $\mathbb{H} \times \mathbb{R}^{1,3}$.

THÉORÈME **11.** *1. Toute fonction L-harmonique, bornée, est de la forme*

$$\int_{\mathbb{S}^2\times\mathbb{R}} h^\sigma(.)h_\ell^\sigma(.)H(\sigma,\ell)\, d\sigma d\ell,$$

où H est une fonction borélienne, bornée, sur $\mathbb{S}^2 \times \mathbb{R}$.

2. Notons

$$\sigma_\infty = \lim_{s\to+\infty} \sigma_s$$

et

$$R_\infty^{\sigma_\infty} \equiv \lim_{s\to+\infty} q(\xi_s, \varepsilon_0 + \sigma_\infty).$$

Etant donné $u_0 \in \mathbb{H} \times \mathbb{R}^{1,3}$, les limites précédentes existent $\mathbb{P}_{u_0}$-presque sûrement et la tribu invariante $Inv(\{u_s\})$ coïncide avec la tribu $\sigma(\sigma_\infty, R_\infty^{\sigma_\infty})$, à des ensembles de $\mathbb{P}_{u_0}$-mesure nulle près.

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[19] Voir [Ken86].

[20] Consulter par exemple [Cra91], où Cranston utilise un couplage pour obtenir des estimations du gradient de fonctions harmoniques sur certaines variétés riemanniennes.

[21] C'est-à-dire non riemannien.

[22] Qui est delui des opérateurs L, $L^{h^{\varepsilon_1}}$ et $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$. On définit ce terme dans la section **2.2.1**.

DEFINITION **12.** *Pour des raisons historiques, on appelle* ***frontière de Poisson de l'opérateur*** L***, sur le domaine*** $\mathbb{H} \times \mathbb{R}^3$*, l'ensemble des fonctions* L*-harmoniques bornées.*

4 – Une description géométrique de la frontière de Poisson de L – Il est naturel de se demander si l'on peut donner de ce théorème une version géométrique. L'espace $\mathbb{H} \times \mathbb{R}^{1,3}$ est-il naturellement muni d'un bord à l'infini $\partial(\mathbb{H} \times \mathbb{R}^{1,3})$ tel que

- les trajectoires $\{u_s\}_{s \geqslant 0}$ convergent $\mathbb{P}_{u_0}$-presque sûrement vers un point $u_\infty \in \partial(\mathbb{H} \times \mathbb{R}^{1,3})$,
- les tribus $Inv\big(\{u_s\}\big)$ et $\sigma(u_\infty)$ coïncident à des ensembles de $\mathbb{P}_{u_0}$-mesure nulle près,

quel que soit $u_0 \in \mathbb{H} \times \mathbb{R}^{1,3}$?

Une telle situation arrive par exemple lorsqu'on étudie le mouvement brownien sur une variété de Cartan-Hadamard à courbure sectionnelle bornée et inférieure à une constante $-\varepsilon < 0$. Une telle variété $\mathbb{V}$, de dimension n, admet une compactification naturelle.

Des coordonnées polaires (ρ, σ), $\rho > 0, \sigma \in \mathbb{S}^2$, sont bien définies sur $\mathbb{V}$ dès lors que l'on choisit un point de base sur $\mathbb{V}$([23]). On identifie deux courbes $\{\gamma_t\}_{t \geqslant 0}$ et $\{\gamma'_t\}_{t \geqslant 0}$, quittant tout compact, et données en coordonnées polaires $\big\{(\rho_t, \sigma_t)\big\}_{t \geqslant 0}$, $\big\{(\rho'_t, \sigma'_t)\big\}_{t \geqslant 0}$, si les directions σ_t et σ'_t de γ_t et γ'_t convergent toutes les deux vers un même point σ_∞ de $\mathbb{S}^{n-1}$. Cette relation d'équivalence permet d'ajouter à $\mathbb{V}$ un bord à l'infini, $\partial\mathbb{V}$, homéomorphe à $\mathbb{S}^{n-1}$, où un point de $\partial\mathbb{V}$ correpond à $\sigma \in \mathbb{S}^{n-1}$ s'il est dans la même classe que la géodésique $\{(t, \sigma)\}_{t \geqslant 0}$, issue du point de base, avec pour vitesse initiale σ.

Toute géodésique s'avère converger vers un point du bord. Dans ce contexte, le mouvement brownien $\{w_s\}_{s \geqslant 0}$ sur $\mathbb{V}$ converge vers un point σ_∞ de $\partial\mathbb{V}$([24]). On doit à Anderson [And83] la première démonstration du fait que $\sigma(\sigma_\infty)$ et $Inv\big(\{w_s\}\big)$ coïncident à des ensembles de $\mathbb{P}_{w_0}$-mesure nulle près, quel que soit la condition initiale w_0 (il démontre en fait un résultat plus fort)[25]. En ce sens, le mouvement brownien sur $\mathbb{V}$ finit par se comporter comme une géodésique[26].

La situation dans notre cadre lorentzien diffère radicalement. La notion de distance entre deux points n'est même pas bien définie : que signifierait l'égalité

$$d(0, \varepsilon_0 + \varepsilon_1) = \inf \left\{ \int_0^1 \sqrt{|q(\dot\gamma_t, \dot\gamma_t)|}\, ds \ ;\ \gamma_0 = 0,\, \gamma_1 = \varepsilon_0 + \varepsilon_1 \right\} = 0,$$

alors que les deux points 0 et $\varepsilon_0 + \varepsilon_1$ dont distincts ?

Il nous faut tenir compte de la structure lorentzienne. Introduisons pour cela un peu de vocabulaire.

DÉFINITION **13.**

- *Une* ***courbe*** *de classe* $\mathcal{C}^1$, $\gamma : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{1,3}$, $s \mapsto \gamma_s$, *est dite* ***temporelle*** *si* $q(\dot\gamma_s) > 0$, *quel que soit* $s \geqslant 0$.
- *On dit que c'est une* ***courbe de lumière*** *si* $q(\dot\gamma_s) = 0$, *quel que soit* $s \geqslant 0$.
- *Etant donné un point* $\xi \in \mathbb{R}^{1,3}$, *on note* $\mathcal{C}^{>0}(\xi) = \big\{\zeta \in \mathbb{R}^{1,3};\ q(\zeta - \xi) > 0\big\}$ *le* ***futur (chronologique) de*** ξ *; c'est l'ensemble des points de l'espace/temps influençables par* ξ *à l'aide d'un signal allant strictement moins vite que la lumière.*
- *On note aussi* $\mathcal{C}^{\geqslant 0}(\xi) = \{\zeta \in \mathbb{R}^{1,3};\ q(\zeta - \xi) \geqslant 0\}$ *le* ***futur causal de*** ξ *; c'est l'ensemble des points de l'espace/temps influençables par* ξ.

[23]Il s'agit des coordonnées exponentielles.
[24]Voir par exemple Prat [Pra75], Kifer [Kif86], ou Ancona [Anc90].
[25]Ancona donne dans [Anc90] des généralisations de ce résultat.
[26]Le théorème 7.3 de [Anc90] donne une version quantifiée de ce fait.

- *De la même manière, on définit le* ***passé chronologique de*** ξ*,* $\mathcal{C}^{<0}(\xi)$*, comme* $\{\zeta \in \mathbb{R}^{1,3};\ q(\zeta-\xi) < 0\}$*, et le* ***passé causal de*** ξ*,* $\mathcal{C}^{\leqslant 0}(\xi)$*, comme* $\{\zeta \in \mathbb{R}^{1,3};\ q(\zeta-\xi) \leqslant 0\}$.[27]

Dans la suite, toutes les courbes considérées seront de classe $\mathcal{C}^1$.

Remarquons que deux points ξ et ξ' de $\mathbb{H}\times\mathbb{R}^{1,3}$ sont égaux si, et seulement si, ils ont même passé :

$$\{\xi' = \xi\} \iff \{\mathcal{C}^{<0}(\xi') = \mathcal{C}^{<0}(\xi)\} \iff \{\mathcal{C}^{\leqslant 0}(\xi') = \mathcal{C}^{\leqslant 0}(\xi)\}(^{28}).$$

Remarquons aussi que si $\{\gamma_t\}_{t\geqslant 0}$ est une courbe causale, les ensembles $\mathcal{C}^{\leqslant 0}(\gamma_t)$ croissent avec t :

$$\text{si } 0 \leqslant s \leqslant t,\ \mathcal{C}^{\leqslant 0}(\gamma_s) \subset \mathcal{C}^{\leqslant 0}(\gamma_t).$$

Au vu de cela, il peut sembler honnête de définir la relation d'équivalence suivante.

DÉFINITION **14.** *Deux* ***courbes causales*** $\{\gamma_t\}_{t\geqslant 0}$ *et* $\{\gamma'_t\}_{t\geqslant 0}$*, quittant tout compact, sont dites* ***équivalentes*** *si*

$$\bigcup_{t>0} \mathcal{C}^{\leqslant 0}(\gamma_t) = \bigcup_{t>0} \mathcal{C}^{\leqslant 0}(\gamma'_t).$$

En somme, on identifie deux points infiniment loins s'ils ont même passé. On appelle ***bord causal*** la classe d'équivalence de chemins obtenue. On le note $\partial_c(\mathbb{H}\times\mathbb{R}^{1,d})$.

On peut munir la réunion disjointe de $\mathbb{H}\times\mathbb{R}^{1,d}$ et de $\partial_c(\mathbb{H}\times\mathbb{R}^{1,d})$ d'une topologie qui coïncide sur $\mathbb{H}\times\mathbb{R}^{1,d}$ avec la topologie originelle et qui fasse de $\partial_c(\mathbb{H}\times\mathbb{R}^{1,d})$ le bord de $\mathbb{H}\times\mathbb{R}^{1,d}$.

On peut définir un tel bord sur une variété lorentzienne. Il est cependant loin d'être évident que l'on pourra donner une description de cet ensemble de classes d'équivalence en termes géométriques simples[29]. Cette construction souffre de plus de défauts théoriques déjà notés dans l'article originel [GKP72] de Geroch, Kronheimer et Penrose sur le sujet. On peut cependant mener à bien cette approche en utilisant une idée proche.

Seule la structure causale spécifiée par les cônes $\mathcal{C}^{\leqslant 0}(\xi)$, $\xi \in \mathbb{H}\times\mathbb{R}^{1,3}$, importe dans la construction précédente. Supposons donnée une autre variété $\mathbb{V}$ où les notions de courbe temporelle, causale, de cône passé (chronologique, causal)... sont bien définies, en des termes analogues à ce qu'on a vu sur $\mathbb{R}^{1,3}$. Imaginons avoir une application $\varphi : \mathbb{R}^{1,3} \to \mathbb{V}$, respectant ces structures :

$$\{\zeta \in \mathcal{C}^{\leqslant 0}(\xi)\} \iff \{\varphi(\zeta) \in \mathcal{C}^{\leqslant 0}(\varphi(\xi))\},$$

des équivalences analogues ayant lieu pour $\mathcal{C}^{<0}(\varphi(\xi))$, $\mathcal{C}^{\geqslant 0}(\varphi(\xi))$ et $\mathcal{C}^{>0}(\varphi(\xi))$.

Si $\varphi(\mathbb{R}^{1,3})$ est relativement compact, et φ (et $\mathbb{V}$) bien choisie, on pourra identifier le bord causal de $\mathbb{R}^{1,3}$ (ou de $\varphi(\mathbb{R}^{1,3})$) à une partie du bord de $\varphi(\mathbb{R}^{1,3})$ dans $\mathbb{V}$.

C'est une telle construction qui est faite dans la section **3.1**, où le rôle de $\mathbb{V}$ est joué par l'univers d'Einstein. Le bord (causal) de $\mathbb{R}^{1,3}$ s'identifie à un cylindre $\mathbb{S}^2\times[-\infty,+\infty]$ où l'on identifie $\mathbb{S}^2\times\{-\infty\}$ et $\mathbb{S}^2\times\{+\infty\}$ à un même point. On note $\mathbf{C}$ ce bord, et $\mathbf{p}$ le point $\mathbb{S}^2\times\{\pm\infty\}$. On montre que $\mathbf{C}$ s'identifie naturellement à une certaine classe d'équivalence de géodésiques de lumière. Le théorème suivant donne une description géométrique de la frontière de Poisson de L sur $\mathbb{H}\times\mathbb{R}^{1,3}$.

THÉORÈME **15.** *Soit* $u_0 \in \mathbb{H}\times\mathbb{R}^{1,3}$.

1. *Le processus* $\{\xi_s\}_{s\geqslant 0}$ *converge* $\mathbb{P}_{u_0}$*-presque sûrement vers un point* $\xi_\infty \in \mathbf{C}\backslash\mathbf{p}$*. En ce sens,* $\{\xi_s\}_{s\geqslant 0}$ *finit par se comporter comme une géodésique de lumière.*

2. *Les tribus* $Inv(\{u_s\})$ *et* $\sigma(\xi_\infty)$ *coïncident à des ensembles de* $\mathbb{P}_{u_0}$*-mesure nulle près.*

[27] On adopte la notation non traditionnelle $\mathcal{C}^{>0}(\xi)$, etc. pour rappeler qu'il s'agit là de véritables cônes affines de $\mathbb{R}^{1,3}$: $\forall\,\lambda > 0,\ \zeta \in \mathcal{C}^{>0}(\xi),\ \xi + \lambda(\zeta-\xi) \in \mathcal{C}^{>0}(\xi)$.

[28] On a des équivalences analogues avec les ensembles futurs.

[29] On entend par là montrer que ce bord est homéomorphe à telle ou telle variété.

5 – Une description algébrique de la frontière de Poisson – On a indiqué après le théorème **7** de Karpelevich, Schur et Tutubalin, que l'on peut décrire la partie $\{\dot{\xi}_s\}_{s\geqslant 0}$ d'un processus relativiste $\{(\dot{\xi}_s,\xi_s)\}_{s\geqslant 0}$ général sur $\mathbb{H}\times\mathbb{R}^{1,d}$ comme la solution d'une équation différentielle stochastique conduite par un mouvement brownien et un processus de Poisson. Le véritable cadre de cette interprétation est le groupe $SO_0(1,3)$ dont la (demi-)pseudo-sphère unité est un espace homogène, et le théorème de Karpelevich, Schur et Tutubalin le reflet d'un théorème décrivant tous les processus de Lévy sur $SO_0(1,3)$ (voir l'appendice **Dudley's results**, **1**, p.88).

Appelons $\{\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ la base canonique de $\mathbb{R}^{1,3}$. Trois champs de vecteurs, H_1,H_2,H_3, sur $SO_0(1,3)$, invariants par translation à gauche, y jouent un rôle particulier. Ils sont donnés par leur valeur en l'identité ; respectivement

- E_1, échangeant ε_0 et ε_1, envoyant ε_2 et ε_3 sur 0,
- E_2: échangeant ε_0 et ε_2, envoyant ε_1 et ε_3 sur 0,
- E_3: échangeant ε_0 et ε_3, envoyant ε_1 et ε_2 sur 0.[30]

Dans l'identité (1.0.5) comme dans tout le manuscrit, la convention de sommation d'Einstein est adoptée. Aussi,

$$H_j\circ dw_s^j=\sum_{j=1..3}H_j\circ dw_s^j.$$

Le mouvement brownien sur $\mathbb{H}$ apparaît comme la projection sur $\mathbb{H}\simeq SO_0(1,3)/SO_0(3)$ de la diffusion $\{g_s\}_{s\geqslant 0}$ sur $SO_0(1,3)$ solution de l'équation différentielle stochastique

$$dg_s=H_j\circ dw_s^j, \tag{1.0.5}$$

où les w^j sont des mouvements browniens indépendants[31]. Plus simplement, $\{g_s\varepsilon_0\}_{s\geqslant 0}$ est un mouvement brownien sur $\mathbb{H}$. Si l'on identifie $SO_0(1,3)$ et l'ensemble $\mathbb{O}^+\mathbb{H}$ des repères orthonormés directs de $\mathbb{H}$, les champs de vecteurs H_i sont les champs de vecteurs horizontaux canoniques sur $\mathbb{O}^+\mathbb{H}$, et la construction précédente du mouvement brownien sur $\mathbb{H}$ la construction de Eells et Elworthy.

Comment, en ces termes envisager la diffusion $\{(\dot{\xi}_s,\xi_s)\}_{s\geqslant 0}$ sur $\mathbb{H}\times\mathbb{R}^{1,d}$?

Fixons $s>0$. Suivant McKean, considérons g_s comme la limite (presque sûre) du produit

$$\prod_{i=1..n-1}\exp\Big(\big(w^j_{\frac{(i+1)s}{n}}-w^j_{\frac{(i+1)s}{n}}\big)E_j\Big),$$

lorsque $n\to+\infty$ (voir [McK69], Chap.4.8).

Notons $\mathbf{e}=(g,\xi)$ un point de $SO_0(1,3)\times\mathbb{R}^3$, et posons $\mathbf{e}_s=(g_s,\xi_s)$, avec $g_s\varepsilon_0=\dot{\xi}_s$. La description de McKean justifie moralement l'approximation $\overset{\bullet}{\simeq}$ ci-dessous.

$$\begin{aligned}\mathbf{e}_{s+ds}=(g_{s+ds},\xi_{s+ds})&\simeq\big(g_{s+ds},\xi_s+\dot{\xi}_s ds\big)\\ &\overset{\bullet}{\simeq}\Big(g_s\exp\big((w^j_{s+ds}-w^j_s)E_j\big),\xi_s+\dot{\xi}_s ds\Big)\end{aligned}$$

Si l'on munit $SO_0(1,3)\times\mathbb{R}^{1,3}$ du produit semi-direct

$$(g,\xi)(g',\xi')=(gg',\xi+g\xi'),$$

[30] Ces trois champs de vecteurs sont les boosts, déjà apparus au **1**.

[31] L'opérateur $\frac{1}{2}\big(H_1^2+H_2^2+H_3^2\big)$ est hypoelliptique. On peut aussi obtenir le mouvement brownien sur $\mathbb{H}$ en projetant sur $\mathbb{H}$ la diffusion ellitpique solution de l'équation $dg_s=V_j\circ dw_s^j+V_{jk}dw_s^{jk}$, où les champs de vecteurs V_{jk} sont les champs sur $SO_0(1,3)$ associés aux rotations infinitésimales $E_{1,2},E_{1,3},E_{2,3}$ de $\mathbb{R}^3$; ces matrices sont définies p.8.

l'identité précédente s'écrit

$$\mathbf{e}_{s+ds} = (g_s, \xi_s)\left(\exp\big((w^j_{s+ds} - w^j_s)E_j\big), \varepsilon_0 ds\right).$$

Notons $\widetilde{E}_j = (E_j, 0)$, $j = 1..d$, l'extension naturelle de E_j à $SO_0(1,3) \times \mathbb{R}^{1,3}$, et $\widetilde{A}_0 = (0, \varepsilon_0)$. Notant **exp** l'exponentielle sur le produit semi-direct $SO_0(1,3) \times \mathbb{R}^{1,3}$, on a, avec des notations claires,

$$\mathbf{exp}\big((dg, \dot\xi)\big) = (\mathrm{Id}, 0) + (dg, \dot\xi) + \frac{1}{2}\big(dg^2, \dot\xi + dg\dot\xi\big) + \cdots.$$

Aussi,

$$\begin{aligned}\mathbf{exp}\big((w^j_{s+ds} - w^j_s)\widetilde{E}_j + \widetilde{A}_0 ds\big) &= (\mathrm{Id}, 0) + \big((w^j_{s+ds} - w^j_s)\widetilde{E}_j, \varepsilon_0 ds\big) + \frac{1}{2}\Big(\big((w^j_{s+ds} - w^j_s)\widetilde{E}_j\big)^2, O\big((w^j_{s+ds} - w^j_s)ds\big)\Big) + \cdots \\ &\simeq \Big(\exp\big((w^j_{s+ds} - w^j_s)\widetilde{E}_j\big), \varepsilon_0 ds + o(ds)\Big).\end{aligned}$$

Ainsi,

$$\mathbf{e}_{s+ds} \simeq \mathbf{e}_s\,\mathbf{exp}\big((w^j_{s+ds} - w^j_s)\widetilde{E}_j + \widetilde{A}_0 ds\big)$$

Le processus $\{\mathbf{e}_s\}_{s\geqslant 0}$ sur $SO_0(1,3) \times \mathbb{R}^{1,3}$ doit donc, au vu du théorème de McKean, être solution de l'équation différentielle stochastique

$$d\mathbf{e}_s = V_j \circ dw^j_s + V_0 ds, \tag{1.0.6}$$

où $V_j(\mathbf{e}) = \mathbf{e}\widetilde{E}_j$ et $V_0(\mathbf{e}) = \mathbf{e}\widetilde{A}_0$. On vérife directement que cette équation est la même que le système

$$dg_s = V_i \circ dw^j_s, \quad d\xi_s = g^0_s ds.$$

On notera $\mathbb{P}_{\mathbf{e}_0}$ la loi de la diffusion solution de l'équation (1.0.6), correspondant à la condition initiale $\mathbf{e}_0 \in \mathcal{G}$.

On peut donc voir la diffusion $\big\{(\dot\xi_s, \xi_s)\big\}_{s\geqslant 0}$ sur $\mathbb{H} \times \mathbb{R}^{1,d}$ comme la projection sur $\mathbb{H} \times \mathbb{R}^{1,d}$ d'une diffusion sur le produit semi-direct $SO_0(1,3) \times \mathbb{R}^{1,3}$, invariante à gauche. Ce groupe n'est autre que le groupe de Poincaré $\mathcal{G}$ des isométries affines de la forme quadratique q([32]).

C'est un fait classique (et le théorème **113** le montrera) qu'étant donné $\mathbf{e}_0 = (g_0, \xi_0) \in \mathcal{G}$, les tribus $Inv\big(\{\dot\xi_s\}\big)$ et $Inv\big(\{g_s\}\big)$ coïncident à des événements de $\mathbb{P}_{\mathbf{e}_0}$-mesure nulle près ; il en va donc de même pour $Inv\big(\{(\dot\xi_s, \xi_s)\}\big)$ et $Inv\big(\{(g_s, \xi_s)\}\big)$. Aussi s'occupe-t-on de la diffusion $\big\{(g_s, \xi_s)\big\}_{s\geqslant 0} = \{\mathbf{e}_s\}_{s\geqslant 0}$ sur $\mathcal{G}$. Cette diffusion étant engendrée par des champs de vecteurs invariants par translation à gauche, la suite $\{\mathbf{e}_n\}_{n\geqslant 0}$ est une marche aléatoire sur $\mathcal{G}$. Le théorème suivant ramène l'étude du comportement asymptotique de la diffusion $\{\mathbf{e}_s\}_{s\geqslant 0}$ à celle de la marche aléatoire $\{\mathbf{e}_n\}_{n\geqslant 0}$. Il fait partie du folklore sur le sujet.

THÉORÈME **16.** *Soit* $\mathbf{e}_0 \in \mathcal{G}$. *Les tribus* $Inv\big(\{\mathbf{e}_s\}\big)$ *et* $Inv\big(\{\mathbf{e}_n\}\big)$ *coincident à des ensembles de* $\mathbb{P}_{\mathbf{e}_0}$*-mesure nulle près.*

• Sous l'impulsion de Furstenberg ([Fur63]), l'étude du comportement asymptotique des marches aléatoires sur des groupes s'est développée dans de nombreuses directions, pour venir prendre sa place naturelle dans l'étude des processus de Lévy sur des groupes. Poursuivant les travaux d'Azencott ([Aze70]), Raugi donne dans [Rau77] une description achevée, en termes algébriques, de l'ensemble des fonctions harmoniques bornées (pour la marche aléatoire) sur un groupe localement compact, sous certaine hypothèse de moment sur la loi des sauts ([Rau77], théorèmes 8.4 et 13.4). Sa méthode s'applique ici, et la partie **3.2.1** est consacréee à l'exposé d'une démonstration algébrique d'un analogue des théorèmes **11** et **15**, dans l'esprit des travaux de [Rau77]. Les notions d'algèbre nécessaires y sont rappelées.

[32] La composante connexe de l'identité, plus précisément.

Notons $g = NAK$ la décomposition d'Iwasawa d'un élément $g \in SO_0(1,3)$, associée au sous-groupe commutatif maximal $\mathcal{A} = \left\{ A = \mathrm{diag}\left(\begin{pmatrix} \mathrm{ch}t & \mathrm{sh}t \\ \mathrm{sh}t & \mathrm{ch}t \end{pmatrix}, \mathrm{Id}_2 \right) ; t \in \mathbb{R} \right\}$. Tout point (g, ξ) de $SO_0(1,3) \times \mathbb{R}^{1,3}$ s'écrit de façon unique

$$(g, \xi) = \big(N, r(\varepsilon_0 - \varepsilon_1)\big)\big(A, u(\varepsilon_0 + \varepsilon_1) + v\varepsilon_2 + w\varepsilon_3\big)(K, 0),$$

la mutliplication étant celle du produit semi-direct.

Théorème **17 (Raugi, [Rau77]).** *Soit $\varepsilon_0 = (g_0, \xi_0) \in SO_0(1,3) \times \mathbb{R}^{1,3}$, et $\{\mathbf{e}_n\}_{n \geqslant 0}$ la marche aléatoire sur $SO_0(1,3) \times \mathbb{R}^{1,3}$, issue de $\mathbf{e}_0$, $\mathbf{e}_n = (g_n, \xi_n)$. Notons $g_n = N_n A_n K_n$ la décomposition d'Iwasawa de g_n.*

1. *Dans la décomposition*

$$(g_n, \xi_n) = \big(N_n, r_n(\varepsilon_0 - \varepsilon_1)\big)\big(A_n, u_n(\varepsilon_0 + \varepsilon_1) + v_n\varepsilon_2 + w_n\varepsilon_3\big)(K_n, 0),$$

les suites $\{N_n\}_{n \geqslant 0}$ et $\{r_n\}_{n \geqslant 0}$ convergent $\mathbb{P}_{\mathbf{e}_0}$-presque sûrement.

2. *La tribu invariante $Inv\big(\{\mathbf{e}_n\}\big)$ est engendrée, sous $\mathbb{P}_{\mathbf{e}_0}$, par les variables aléatoires N_∞ et r_∞.*

La matrice N_∞ dépend d'un vecteur $h_\infty \in \mathbb{R}^2$. La correspondance entre les quantités algébriques (h_∞, r_∞), d'une part, et les quantités géométriques $(\sigma_\infty, R_\infty^{\sigma_\infty})$, d'autre part, est explicite.

Proposition **18.** *Le vecteur h_∞ est la projection stéréographique de la direction asymptotique $\sigma_\infty \in \mathbb{S}^2$ de $\{\dot{\xi}_s\}_{s \geqslant 0}$.*

On a $r_\infty = \frac{1 + |h_\infty|^2}{\sqrt{2}} R_\infty^{\sigma_\infty}$.

Aussi le résultat de Raugi démontre-t-il de manière totalement différente le théorème **11**.

Chapter 2

Asymptotic behaviour of the relativistic diffusion

2.1 The framework

2.1.1 Geometric preliminaries

Let $d \geqslant 2$. Note $\{\varepsilon_0, \varepsilon_1, ..., \varepsilon_d\}$ the canonical basis of $\mathbb{R} \times \mathbb{R}^d$, and $\left(\xi^0, \xi^1, ..., \xi^d\right)$ the coordinates of a point $\xi \in \mathbb{R} \times \mathbb{R}^d$ in this basis. The vector space $\mathbb{R} \times \mathbb{R}^d$ is endowed with a quadratic form

$$q(\xi) = \left(\xi^0\right)^2 - \left(\left(\xi^1\right)^2 + \cdots + \left(\xi^d\right)^2\right).$$

To distinguish the usual Euclidian space $\mathbb{R}^{1+d}$ from $(\mathbb{R} \times \mathbb{R}^d, q)$, we shall note $\mathbb{R}^{1,d}$ this latter space.

a) Isometries of q

• The set of affine isometries of $\mathbb{R}^{1,d}$, is well known. The set of direct linear isometries is a subgroup $SO(1,d)$ of $GL(\mathbb{R}^{1+d})$, with Lie algebra $so(1,d) = \left\{\begin{pmatrix} 0 & {}^t c \\ c & A \end{pmatrix}; c \in \mathbb{R}^d, A \in so(d)\right\}$. Any direct affine isometry φ of $\mathbb{R}^{1,d}$ can be uniquely written

$$\xi \in \mathbb{R}^{1,d}, \quad \varphi(\xi) = g(\xi) + b,$$

for an element $g \in SO(1,d)$ and $b \in \mathbb{R}^{1,d}$. Note $\varphi = (g, b)$. The group of direct affine isometries of $\mathbb{R}^{1,d}$ is the semi-direct product $SO(1,d) \times \mathbb{R}^{1,d}$, with product:

$$(g, b)(g', b') = (gg', b + gb').$$

DEFINITION **19.** *$(Id, 0)$'s component of the set of affine isometries of $\mathbb{R}^{1,d}$ is called the* ***Poincaré group****, and noted $\mathcal{G}$.*

Notation

- Write $SO_0(1,d) = \exp\big(so(1,d)\big)$; this is Id's component in $SO(1,d)$. The Poincaré group is equal to $SO_0(1,d) \times \mathbb{R}^{1,d}$.

DEFINITION **20.** *The unit pseudo-sphere of $\mathbb{R}^{1,d}$ has two components; we note $\mathbb{H} = \left\{\xi \in \mathbb{R}^{1,d};\, q(\xi) = 1,\ \xi^0 > 0\right\}$ that corresponding to positive ξ^0's.*

b) Models of $\mathbb{H}$

• One can write $\mathbb{H} = \{(\mathrm{ch}\rho, (\mathrm{sh}\rho)\sigma); \rho \geqslant 0, \sigma \in \mathbb{S}^{d-1}\}$. The pair (ρ, σ) is said to be the ***polar coordinates of the point*** $(\mathrm{ch}(\rho), (\mathrm{sh}\rho)\sigma) \in \mathbb{H}$.

The q-norm of the velocity of a curve $\gamma_s = (\mathrm{ch}(\rho_s), (\mathrm{sh}\rho_s)\sigma_s) \in \mathbb{H}$ is

$$q(\dot{\gamma}_s) = -(\dot{\rho}_s^2 + ||\dot{\sigma}_s||^2_{\mathrm{Eucl}}\mathrm{sh}^2\rho_s),$$

so q induces a Riemannian metric on $\mathbb{H}$, given in (ρ, σ) coordinates by the $(0, 2)$ tensor: $d\rho^2 + (\mathrm{sh}^2\rho)||d\sigma||^2_{Eucl}$. There coordinates are the Riemannian exponential coordinates associated with the point $\varepsilon_0 \in \mathbb{H}$. Given $\sigma \in \mathbb{S}^{d-1}$, the path $\{(\rho, \sigma)\}_{\rho\in\mathbb{R}}$ is a geodesic.

Note that these coordinates have bad behaviour at $\rho = 0$; they parametrize $\mathbb{H}\backslash\{\varepsilon_0\}$.

• Endow the half-space $\mathbb{R}^{>0,d-1} \equiv \{(y, x) \in \mathbb{R}^d\, ;\, y > 0\, ,\, x \in \mathbb{R}^{d-1}\}$, with the Riemannian metric defined at a point $(y, x) \in \mathbb{R}^{>0} \times \mathbb{R}^d$ by

$$(X, Y)_{(y,x)} = \frac{\langle X, Y\rangle_{\mathrm{Eucl}}}{y^2}.$$

One can check by a direct calculation that

PROPOSITION **21.** *The restriction to $\mathbb{H}$ of the following application is an isometry between $(\mathbb{H}, q)$ and the half-space $(\mathbb{R}^{>0,d-1}, (\cdot,\cdot)_\cdot)$.*

$$(\xi^0, \xi^1, ..., \xi^d) \in \mathbb{R}^{1,d} \overset{\psi^{-1}}{\longmapsto} \left(\frac{1}{\xi^0 - \xi^1}, \frac{\xi^2}{\xi^0 - \xi^1}, \cdots, \frac{\xi_d}{\xi^0 - \xi^1}\right) \in \mathbb{R}^{>0,d-1}. \tag{2.1.1}$$

Its inverse is given by the formula

$$(y, (x_1, ..., x_{d-1})) \in \mathbb{R}^{>0,d-1} \overset{\psi}{\mapsto} \left(\frac{|x|^2 + y^2 + 1}{2y}, \frac{|x|^2 + y^2 - 1}{2y}, \frac{x_1}{y}, ..., \frac{x_{d-1}}{y}\right) \in \mathbb{H}. \tag{2.1.2}$$

where $|x|$ is the Euclidian norm of $x \in \mathbb{R}^{d-1}$.

The image under ψ of the level hypersurface $y = constant$ is the intersection of the hyperplan $\xi^0 - \xi^1 = \frac{1}{y}$ with $\mathbb{H}$.

• **Halfspace coordinates associated with a basis of** $SO_0(1, d)$ – Given $g \in SO_0(1, d)$, we define coordinates on $\mathbb{H}$ by setting $(y, x) = \psi^{-1}\left(g^{-1}(\dot{\xi})\right)$. These coordinates are said to be the halfspace coordinates associated with g.

c) Description of the geodesics of $\mathbb{H}$, sphere at the infinity, action on this sphere

i) Adopt polar coordinates on $\mathbb{H}$. We see on definition **20** that $SO_0(1, d)$ is the set of direct isometries of $\mathbb{H}$ and that it acts transitively on it. It also acts transitively on the set

$$\{(\dot{\xi}, V) \in T\mathbb{H}\, ;\, \dot{\xi} \in \mathbb{H},\, V \in T_{\dot{\xi}}\mathbb{H},\, -q(V) = 1\}.$$

ii) As a non parametrized path, the geodesic $\{(\rho, \sigma)\}_{\rho\in\mathbb{R}}$ is the intersection of $\mathbb{H}$ with the 2-dimensional vector space $\langle\varepsilon_0, \sigma\rangle \subset \mathbb{R}^{1,d}$. The intersection of $\langle\varepsilon_0, \sigma\rangle$ with the null cone is the line $\mathbb{R}\,(\varepsilon_0 + \sigma)$. The geodesic $\{(\rho, \sigma)\}_{\rho\in\mathbb{R}}$ is said to have asymptotic direction $\sigma \in \mathbb{S}^{d-1}$. From ***i)***, if γ is a geodesic started from $\dot{\xi}$, in direction V, there exists some $g \in SO_0(1, d)$ such that

$$\dot{\xi} = g(\varepsilon_0) \text{ and } V = g(\varepsilon_1).$$

The isometry g sends the null line $\mathbb{R}(\varepsilon_0+\varepsilon_1)$ to another null line $\mathbb{R}(\varepsilon_0+\sigma)$. This line $\mathbb{R}(\varepsilon_0+\sigma)$ is the intersection of the null cone with the 2-dimensional vector space $\langle\varepsilon_0, V\rangle \subset \mathbb{R}^{1,d}$. So, the point $\sigma \in \mathbb{S}^{d-1}$ is uniquely determined by the geodesic γ. One says that γ has asymptotic direction σ.

The set of asymptotic directions is $\mathbb{S}^{d-1}$.

iii) In our framework, $\mathbb{H}$ is seen as a set of velocity vectors. As a translation does not change the speed of a path, and an isometry $g \in SO_0(1,d)$ sends the path γ to $g(\gamma)$, we define the action of an affine isometry $(g,b) \in \mathcal{G}$ on the set $\mathbb{S}^{d-1}$ of asymptotic direction by the formula

$$\sigma \in \mathbb{S}^{d-1}, \quad (g,b).\sigma = \sigma',$$

if $g(\mathbb{R}(\varepsilon_0+\sigma)) = \mathbb{R}(\varepsilon_0+\sigma')$.

2.1.2 Relativistic diffusion

a) Brownian motion on $\mathbb{H}$

i) Brownian motion on $\mathbb{H}$ is defined as the diffusion whose infinitesimal generator is half of the associated Beltrami-Laplace operator $\triangle^{\mathbb{H}}$ (indeed, the associated martingale problem has a unique solution). Define the d'Alembertian operator $\Box = \partial^2_{\xi^0} - \partial^2_{\xi^1} - \cdots - \partial^2_{\xi^{d-1}}$ on $\mathbb{R}^{1,d}$. One can show that (Helgason [Hel84], Theorem 3.5, p.257), given a smooth function f on $\mathbb{H}$, extended radially on a neighbourhood of $\mathbb{H}$ to a function $\widetilde{f}(rm) = f(m)$, $m \in \mathbb{H}, r > 0$, then

$$\triangle^{\mathbb{H}} f(m) = \Box \widetilde{f}(m).$$

The invariance of $\Box$ by the action of the isometries of $\mathbb{R}^{1,d}$ explains why $\triangle^{\mathbb{H}}$ is also invariant under the action of (the restrictions to $\mathbb{H}$ of) these isometries. Brownian motion on $\mathbb{H}$ has the same invariance: for any $\dot{\xi} \in \mathbb{H}$ and any isometry φ of $\mathbb{H}$, the image by φ of a Brownian motion started from $\dot{\xi}$ has the same law as a Brownian motion started from $\varphi(\dot{\xi})$.

Notation – Note $\mathbb{P}_{\dot{\xi}_0}$ the law of Brownian motion on $\mathbb{H}$, started from $\dot{\xi}_0$.

A more dynamical approach consists in looking at Brownian motion on $\mathbb{H}$ as the solution of a stochastic differential equation, generating a diffusion with generator $\frac{\triangle^{\mathbb{H}}}{2}$.

ii) **Using polar coordinates**

In (ρ,σ) coordinates (on $\mathbb{H}\backslash\{\varepsilon_0\}$), $\triangle^{\mathbb{H}}$ has the expression

$$\triangle^{\mathbb{H}} f = \partial^2_\rho f + (d-1)\coth(\rho)\partial_\rho f + \frac{1}{\operatorname{sh}^2\rho}\triangle^{\mathbb{S}^2} f,$$

where $\triangle^{\mathbb{S}^{d-1}}$ is the Laplacian on $\mathbb{S}^{d-1}$ (associated with its natural metric induced by the ambiant Euclidian space $\mathbb{R}^d$). A Brownian motion started from any point of $\mathbb{H}$, except ε_0, is thus located by its coordinates (ρ_s,σ_s). These coordinates are solutions of the stochastic differential equation

$$\rho_s = \rho_0 + w^\rho_s + \frac{d-1}{2}\int_0^s \coth(\rho_r)\,dr,$$
$$\sigma_s = \Sigma\left(\int_0^s \frac{du}{\operatorname{sh}^2\rho_u}\right),$$

where w^ρ is a real Brownian motion and Σ a Brownian motion on $\mathbb{S}^{d-1}$, independent of w^ρ.

The first part of the following proposition gives, in our framework, a more precise version of the general transience theorem **8** of Dudley. It already appears in [Pra75].

PROPOSITION **22 (Asymptotic behaviour of the speed $\{\dot{\xi}_s\}_{s\geqslant 0}$).** *1. Let $\dot{\xi}_0 \in \mathbb{H}$. Given $\varepsilon > 0$, there exists $\mathbb{P}_{\dot{\xi}_0}$-almost surely a constant $C(\omega)$ such that for all $s \geqslant 0$,*

$$\frac{d-1}{2}(1-\varepsilon)s - C(\omega) \leqslant \rho_s \leqslant \frac{d-1}{2}(1+\varepsilon)s + C(\omega).$$

2. Let $\dot{\xi}_0 \in \mathbb{H}$. The direction σ_s of $\dot{\xi}_s$ $\mathbb{P}_{\dot{\xi}_0}$-almost surely converges to a point $\sigma_\infty \in \mathbb{S}^{d-1}$.

3. The law of σ_∞, when the diffusion is started from ε_0, is the uniform probability on the sphere $\mathbb{S}^{d-1}$. When the diffusion is started from another point than ε_0, the law of σ_∞ is the image of the uniform probability on $\mathbb{S}^{d-1}$ by any isometry of $\mathbb{H}$ bringing ε_0 to the initial point of the trajectory.

$\lhd$ *1.* On the one hand, as coth$\geqslant$ 1, the usual comparison theorem on one dimensional stochastic differential equations, as Theorem 1.1 p.437 of [IW89], tells us that

$$\rho_s \geqslant \rho_0 + w_s^\rho + \frac{d-1}{2}s.$$

One the other hand, since

$$d\left(\rho_s - w_s^\rho - \frac{d-1}{2}s\right) = \frac{d-1}{2}(\text{cotan}(\rho_s) - 1)\, ds = (d-1)\frac{ds}{e^{2\rho_s} - 1}$$

and $\rho_s \geqslant \rho_0 + w_s^\rho + s \geqslant \frac{d-1}{4}s$, for s large enough (law of iterated logarithm, LIL), $\{\rho_s - w_s^\rho - \frac{d-1}{2}s\}_{s\geqslant 0}$ converges.

2. Is a consequence of point *1* since the integral $\int_0^s \frac{du}{\text{sh}^2\rho_u}$ converges as $s \to +\infty$.

3. Use an isometry to bring the random trajectory started from the point $\dot{\xi}_0$ to a trajectory started from ε_0. This process $\{\underline{\dot{\xi}}_s\}_{s\geqslant 0}$ is a Brownian motion since $\triangle^{\mathbb{H}}$ is invariant by the action of the isometries. For the same reason, the law of $\{\underline{\dot{\xi}}_s\}_{s\geqslant 0}$ is invariant by the action of any isometry fixing ε_0 (the rotations of $\mathbb{R}^d$); so, the law of $\underline{\sigma}_\infty \in \mathbb{S}^{d-1}$ is invariant by any rotation. The uniform probability is the only such probability on $\mathbb{S}^{d-1}$. $\rhd$

iii) Using half-space coordinates

Hyperbolic Laplacian is given in these coordinates by the formula

$$\frac{\triangle^{\mathbb{H}}}{2} = \frac{y^2}{2}(\partial_{x_1}^2 + \cdots + \partial_{x_{d-1}}^2 + \partial_y^2) - \frac{d-2}{2}y\partial_y.$$

So, the diffusion solving the equations

$$\begin{aligned} dy_s &= y_s dw_s^y - \frac{d-2}{2}y_s ds, \\ dx_s &= y_s dw_s^x, \end{aligned} \tag{2.1.3}$$

where w^y is a real Brownian motion independent of the $(d-1)$-dimensional Brownian motion w^x, is a Brownian motion on $\mathbb{H}$. The process $\{y_s\}_{s\geqslant 0}$ is explicit.

$$\begin{aligned} y_s &= y_0\, e^{w_s^y - \frac{d-1}{2}s} \\ x_s &= x_0 + \int_0^s y_r dw_r^x. \end{aligned} \tag{2.1.4}$$

We see on equations (2.1.3) that both y_s and x_s converge; the first to 0 and the second to a random point x_∞. The law of x_∞ is easy to identify.

Suppose the diffusion is started from a point $(y_0, x_0) \in \mathbb{R}^{>0,d-1}$. If one looks at the point $\dot{\xi}_s$ in exponential coordinates associated with (y_0, x_0), we saw in proposition **22** that the asymptotic direction σ_∞ of $\dot{\xi}_s$ has the uniform probability on $\mathbb{S}^{d-1}$ as law. So, the law of x_∞ is the image of the uniform probablity on $\mathbb{S}^{d-1}$ by the function which associates to a direction $V \in \mathbb{S}^{d-1}$ the point $x_\infty \in \mathbb{R}^2$ which is "at the end" of the geodesic started from (y_0, x_0) in the direction V. It has a good expression if one uses on $\mathbb{R}^{d-1}$ polar coordinates *centered at* x_0, and noted (r, α), $\alpha \in \mathbb{S}^{d-2}$. Note $\sigma(d\alpha)$ the uniform probability on $\mathbb{S}^{d-2}$.

$$\mathbb{P}_{(y_0,x_0)}\big(r(x_\infty) \in r + dr,\ \alpha(x_\infty) \in \alpha + d\alpha\big) = \frac{2y_0}{y_0^2 + r^2}\mathbf{1}_{r>0} dr\sigma(d\alpha).$$

This formula shows that the law of x_∞ has a smooth density with respect to Lebegue's measure dx on $\mathbb{R}^{d-1}$, depending smoothly on $\big((y_0, x_0), x\big) \in \big(\mathbb{R}^{>0} \times \mathbb{R}^{d-1}\big) \times \mathbb{R}^{d-1}$.

Remark – Back to polar coordinates, this implies that the law of σ_∞ has, under any $\mathbb{P}_{\dot{\xi}_0}$, a smooth density $h^\sigma(\dot{\xi}_0)$ with respect to the uniform probability on $\mathbb{S}^{d-1}$, depending smoothly on $(\dot{\xi}_0, \sigma) \in \mathbb{H} \times \mathbb{S}^{d-1}$.

This is a well known fact that, for a fixed $\sigma \in \mathbb{S}^{d-1}$, $h^\sigma(.)$ is a $\triangle^{\mathbb{H}}$-harmonic function. As h^σ depends only on $\dot{\xi}_0$, it is also an L-harmonic function.

b) Relativistic diffusion

Definition 23. *Let $\{\dot{\xi}_s\}_{s\geqslant 0}$ be a Brownian motion on $\mathbb{H}$, started from $\dot{\zeta}_0 \in \mathbb{H}$. Let $\zeta_0 \in \mathbb{R}^{1d}$. The* ***relativistic diffusion on*** $\mathbb{H} \times \mathbb{R}^{1,d}$*, started from $(\dot{\zeta}_0, \zeta_0)$, is the process $\{(\dot{\xi}_s, \xi_s)\}_{s\geqslant 0}$, where*

$$\xi_s = \zeta_0 + \int_0^s \dot{\xi}_r\, dr.$$

Notations

- We shall note $\mathbb{P}_{(\dot{\zeta}_0,\zeta_0)}$ the law of the relativistic diffusion started from $(\dot{\zeta}_0, \zeta_0) \in \mathbb{H} \times \mathbb{R}^{1,d}$.
- Given $s > 0$, we note $P_s\big((\dot{\zeta}_0, \zeta_0), .\big)$ the law of $(\dot{\xi}_s, \xi_s)$ under $\mathbb{P}_{(\dot{\zeta}_0,\zeta_0)}$.
- Finally, for $\xi \in \mathbb{R}^{1,d}$, we write $h_s(\xi) = s\xi$.

Proposition 24 **(Support of $P_s\big((\dot{\xi}_0, \xi_0), .\big)$).** *For any $(\dot{\xi}_0, \xi_0) \in \mathbb{H} \times \mathbb{R}^{1,d}$ and any $s > 0$, the support of $P_s\big((\dot{\xi}_0, \xi_0), .\big)$ is*

$$\Big\{(\dot{\xi}, \xi) \in \mathbb{H} \times \mathbb{R}^{1,d}; \xi \in \xi_0 + ConvHull\big(h_s(\mathbb{H})\big)\Big\},$$

where $ConvHull\big(h_s(\mathbb{H})\big)$ is a notation for the convex hull of $h_s(\mathbb{H}) \subset \mathbb{R}^{1,d}$.

$\lhd$ Fix $s > 0$.

1) As the Brownian motion on $\mathbb{H}$ is an elliptic diffusion, for any continuous path $\{\gamma_r\}_{0\leqslant r\leqslant s}$ in $\mathbb{H}$, and any $\varepsilon > 0$, we have

$$\mathbb{P}_{\dot{\xi}_0}\left(\sup_{0\leqslant r\leqslant s} d(\dot{\xi}_r, \gamma_r) \leqslant \varepsilon\right) > 0.$$

So, the support of $P_s\big((\dot{\xi}_0, \xi_0), .\big)$ is the closure of the set of points $(\dot{\zeta}, \zeta) \in \mathbb{H} \times \mathbb{R}^{1,d}$ of the form $\dot{\zeta} = \gamma_s$ and $\zeta = \zeta_0 + \int_0^s \gamma_r\, dr$, for a continuous path $\{\gamma_r\}_{0\leqslant r\leqslant s}$ in $\mathbb{H}$.

2) As the Brownian motion $\{\dot{\xi}_s\}_{s\geqslant 0}$ can go from one open set of $\mathbb{H}$ to any other one in an arbitrarily small amount of time with positive probability, we see that the heart of the proof is to show that $\big\{\int_0^s \gamma_r\, dr\,;\ \gamma:[0,s]\to\mathbb{H},\ \text{continuous}\big\} = ConvHull\big(h_s(\mathbb{H})\big)$.
3) Taking constant γ's, one sees that

$$ConvHull\big(h_s(\mathbb{H})\big) \subset \left\{\int_0^s \gamma_r\, dr\,;\ \gamma:[0,s] \xrightarrow{\mathcal{C}^0} \mathbb{H}\right\}$$

4) Reciprocally, the convexity of $ConvHull(\mathbb{H})$ implies that for any $n\geqslant 1$,

$$\frac{1}{n}\sum_{j=0}^{n-1}\gamma_{\frac{js}{n}} \in ConvHull(\mathbb{H}),$$

since each $\gamma_{\frac{js}{n}} \in \mathbb{H}\subset ConvHull(\mathbb{H})$. As $\frac{1}{n}\sum_{j=0}^{n-1}\gamma_{\frac{js}{n}} \underset{n\to+\infty}{\longrightarrow} \frac{1}{s}\int_0^s \gamma_r\, dr$, the result follows. ▷

***c)* The relativistic diffusion as the projection of a Brownian motion on a group**

• Following Eells and Elworthy, one usually constructs Brownian motion on a(n oriented) Riemannian manifold $(\mathbb{M},g)$ of dimension n as the projection of a singular diffusion on the bundle $\mathbb{OM}$ of direct orthonormal frames of $T\mathbb{M}$. Precisely, there exist canonical horizontal vector fields $\{H_i\}_{i=1..n}$ on $\mathbb{OM}$ such that the differential operator $\sum_{i=1}^n H_i^2$ on $\mathbb{OM}$ induces the Laplacian $\triangle^{\mathbb{M}}$ on $\mathbb{M}$, in the sense that if we denote by π the natural projection $\mathbb{OM}\to\mathbb{M}$, we have for all $f\in\mathcal{C}^2(\mathbb{M})$,

$$(\sum_{i=1}^n H_i^2) f\circ\pi = (\triangle^{\mathbb{M}} f)\circ\pi. \tag{2.1.5}$$

These vector fields are defined as follows. Note $(x,\mathbf{e}) = \big(x,(e_1,...,e_n)\big)$ a generic element of $\mathbb{OM}$. Given $j\in\{1,..,d\}$, one defines a motion $\{(x_s,\mathbf{e}_s)\}$ in $\mathbb{OM}$ by asking that $\frac{dx_s}{ds} = e_j(s)$, and $\mathbf{e}$ should be transported parallely along $\{x_s\}$. One defines a vector field H_j on $\mathbb{OM}$ looking at the infinitesimal motion of all the points, according to the preceding rule. These are the canonical horizontal vector fields.

• In our situation, $\mathbb{H}$ being the half unit pseudo-sphere of $\mathbb{R}^{1,d}$, a point in $\mathbb{OH}$ is an orthonormal frame of $\mathbb{R}^{1,d}$. So $\mathbb{OH}$ can be identified with the set of orthonormal bases $(g^0,g^1...,g^d)$ of $\mathbb{R}^{1,d}$, with $g^0\in\mathbb{H}$, and the natural projection is $g\in\mathbb{OM}\mapsto g^0\in\mathbb{H}$. The horizontal vector fields H_i are:

$$H_i(g) = gE_i, \quad i=1..d$$

where the E_i's are the matrices $E_1 = \begin{pmatrix} 0 & 1 & (0) \\ 1 & 0 & \cdots \\ 0 & \vdots & \mathbf{O}_{d-1}\end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 & (0) \\ 0 & 0 & & \cdots \\ 1 & \vdots & & \\ (0) & \vdots & & \mathbf{O}_{d-1}\end{pmatrix}$, and so on: E_i exchanges ε_0 and ε_i and sends the other basis vectors to 0.

These left invariant vector fields give rise to the left invariant differential operator $H_1^2+...+H_d^2$ on $O(1,d)$. If g_0 belongs to $SO_0(1,d)$ (that is if $\big(g_0^1,...,g_0^d\big)$ is a direct basis of $T_{g_0^0}\mathbb{H}$), the diffusion $\{g_s\}_{s\geqslant 0}$ on $O(1,d)$ solution of the following stochastic differential equation remains in $SO_0(1,d)$.

$$dg_s = g_sE_1\circ dw_s^1 + ... + g_sE_d\circ dw_s^d = g_sE_i\circ dw_s^i,$$

where Einstein conventions on summations are used, and where the w^i's are real independent Brownian motions.

By construction, $\{g_s^0\}_{s\geqslant 0}$ is a Brownian motion on $\mathbb{H}$.

So, the natural framework to construct the relativistic diffusion seems to be $SO_0(1,d)\times\mathbb{R}^{1,d}$, the equations of motion being

$$\begin{aligned} dg_s &= g_s E_i \circ dw_s^i, \\ d\xi_s &= g_s^0\, ds. \end{aligned} \tag{2.1.6}$$

One has a more synthetic view of these equations by seeing them as describing a motion on the Poincaré group $\mathcal{G}$.

Set $\widetilde{E}_i = (E_i, 0)$, for $i = 1..d$, $\widetilde{E}_0 = (0, \varepsilon_0)$, and define the left invariant vector fields V_i, $i = 0..d$, on $\mathcal{G}$, by the formulas

$$V_i\big((g,\xi)\big) = (g,\xi)\widetilde{E}_i, \; i = 0..d.$$

It is elementary to see that equations (2.1.6) are equivalent to

$$d\big((g_s,\xi_s)\big) = V_i\big((g_s,\xi_s)\big) \circ dw_s^i + V_0\big((g_s,\xi_s)\big)\, ds. \tag{2.1.7}$$

The differential generator of the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ on $\mathcal{G}$ is

$$\frac{1}{2}\sum_{i=1}^{d} V_i^2 + V_0. \tag{2.1.8}$$

This algebraic framework will be useful in the section **2.2.2** to establish a uniform continuity property of some functions. It will be fully used in section **3.2.1** where we study the random walk $\big\{(g_n,\xi_n)\big\}_{n\geqslant 0}$.

Notation – Note $\mathbb{P}_{(g_0,\xi_0)}$ the law of the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ on $\mathcal{G}$, started from (g_0,ξ_0).

We determine its suppport using the support theorem, as exposed in Theorem 8.2, in [IW89]. Recall its content.

Note $\|.\|_T$ the uniform norm on $\mathcal{C}\big([0,T],\mathcal{G}\big)$.

Theorem **25 (Stroock-Varadhan).** *If ϕ^i, $i = 1..d$, are piecewise smooth, continuous controls, and $\varphi\big(t;(g_0,\xi_0)\big)$ the solution to the equation*

$$d\big(\varphi(t)\big) = V_i\big(\varphi(t)\big)\phi_t^i dt + V_0\big(\varphi(t)\big)dt, \quad \varphi(0) = (g_0,\xi_0), \tag{2.1.9}$$

on $\mathcal{G}$, one has

$$\forall\, T>0,\; \mathbb{P}_{(g_0,\xi_0)}\big(\|(g_s,\xi_s) - \varphi\big(s;(g_0,\xi_0)\big)\|_T < \varepsilon \;\mid\; \|w-\phi\|_T < \delta\big) \to 1, \; \text{as } \delta \searrow 0. \tag{2.1.10}$$

Proposition **26 (Support of the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ on $\mathcal{G}$).** *Let $(g_0,\xi_0)\in\mathcal{G}$. The support of the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$, started from (g_0,ξ_0), is*

$$\big\{(g,\zeta)\in\mathcal{G}; q(\zeta-\xi_0)\geqslant 0\big\}.$$

$\lhd$ The inclusion $\subset$ is obvious since the path $\{\xi_s\}_{s\geqslant 0}$ is timelike. To establish the reverse inclusion, we can suppose $\xi_0 = 0$. We must understand why for any point $(\underline{g},\underline{\xi})\in\mathcal{G}$, with $q(\underline{\xi})>0$ and any product neighbourhood $\mathcal{V}_1\times\mathcal{V}_2$ of $(\underline{g},\underline{\xi})$, we can find smooth controls $\phi^1,...,\phi^d$, defined on an interval $[0,T]$, such that the path $\big\{\big(g(s),\xi(s)\big)\big\}_{0\leqslant s\leqslant T}$ solving the equation

$$d\big(g(s),\xi(s)\big) = V_i\big(\big(g(s),\xi(s)\big)\big)\phi^i(s)ds + V_0\big(\big(g(s),\xi(s)\big)\big)ds,\ s\in[0,T],\ (g(0),\xi(0)) = \big((\varepsilon_0,...,\varepsilon_{d+1}),0\big),$$

satisfies $(g_T,\xi_T)\in\mathcal{V}_1\times\mathcal{V}_2$.
Fix $(\underline{g},\underline{\xi})\in\mathcal{G}$.
1) First, find a smooth timelike curve $\gamma = \{\varphi(s)\}_{0\leqslant s\leqslant T}$ on $\mathbb{R}^{1,d}$, with $\dot\varphi(s) = \frac{d\varphi(s)}{ds}\in\mathbb{H}$, defined on an unprescribed interval $[0,T]$, such that $\varphi(0)=0$, $\dot\varphi(0)=\varepsilon_0$, $\dot\varphi(T)=\underline{g}^0$ and $\varphi(T)$ is not far from $\underline{\xi}$. Such a curve exists.
The parallel transport in $\mathbb{H}$ of $\varepsilon_1,...,\varepsilon_d$ along $\{\dot\varphi(s)\}_{s\in[0,T]}$ defines the lift $\{g_s^{\dot\varphi}\}_{s\in[0,T]}$ of $\{\dot\varphi(s)\}_{s\in[0,T]}$ to $SO_0(1,d)(\simeq\mathbb{O}\mathbb{H})$. It gives us a unique d-uple of smooth functions on $[0,T]$, $\big(\varphi_1(s),...,\varphi_d(s)\big)$ such that

$$d\big(\dot\varphi(s)\big) = (g_s^{\dot\varphi})^i\varphi_i(s)\,ds.$$

2) Solving the equation on $\mathcal{G}$

$$d\big(\psi(s)\big) = \sum_{i=1}^d V_i\big(\psi(s)\big)\varphi_i(s)ds + V_0\big(\psi(s)\big)ds,\ \psi(0) = (\varepsilon_0,...,\varepsilon_d),$$

provides a function ψ such that $\psi(T)$ is of the form

$$\begin{pmatrix}\dot\varphi(T) & g^1 & ... & g^d & \varphi(T)\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\underline{g}^0 & g^1 & ... & g^d & \varphi(T)\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

where $(g^1,...,g^d)$ is an orthonormal basis of $T_{\underline{g}^0}\mathbb{H}$. This basis has no reason to be near $\big(\underline{g}^1,...,\underline{g}^d\big)$. The geometry of $\mathbb{H}$ allows to change a little $\dot\varphi$ (*i.e.* the φ_i's) so that $\psi(T)$ should be near $\big(\underline{g},\underline{\xi}\big)$.
The following proposition can be seen as a statement on the geometry of homogeneous spaces; see the book of Berger, [Ber03], p.641.

PROPOSITION **27.** *Let $\dot\xi\in\mathbb{H}$ be given. We can obtain any orthonormal basis of $T_{\dot\xi}\mathbb{H}$ by parallel transport of a fixed orthonormal basis of $T_{\dot\xi}\mathbb{H}$ along loops contained in an arbitrarily small neighbourhood of $\dot\xi\in\mathbb{H}$.*

3) We take advantage of this fact to add to the path $\dot\varphi$ a small loop at its end along which one parallel transport maps $\big(g^1,...,g^d\big)$ to $\big(\underline{g}^1,...,\underline{g}^d\big)$. For $0<\varepsilon<1$, define $\dot\phi$ by $\dot\phi_s = \dot\varphi_{(1+\varepsilon)sT}$, for $s\leqslant\frac{1}{1+\varepsilon}$, and $\dot\phi_s$ describes the loop for $\frac{1}{1+\varepsilon}\leqslant s\leqslant 1$. If ε is small enough, $\phi(1)$ remains near $\underline{\xi}$. The lift of $\dot\phi$ provides the expected ϕ^i. ▷

2.2 The asymptotic behaviour of the diffusion

2.2.1 From probability to analysis

a) Analytical properties of the differential generator of the diffusion

From a semi-group viewpoint, the diffusions $\big\{(\dot\xi_s,\xi_s)\big\}_{s\geqslant 0}$ and $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ on $\mathbb{H}\times\mathbb{R}^{1,d}$ and $\mathcal{G}$ are caracterized by their differential generators, respectively

$$Lf(\dot\xi,\xi) = \frac{\triangle^{\mathbb{H}}f(\dot\xi,\xi)}{2} + \partial_\xi f_{|(\dot\xi,\xi)}(\dot\xi),\quad f\in\mathcal{C}_0^\infty(\mathbb{H}\times\mathbb{R}^{1,d}),$$

and

$$\widetilde{L}\widetilde{f} = \frac{1}{2}\sum_{i=1}^d V_i^2\widetilde{f} + V_0\widetilde{f},\quad \widetilde{f}\in\mathcal{C}_0^\infty(\mathcal{G}).$$

Both operators are linked. If the function $\widetilde{f}(g,\xi)$ depends only on g^0 and ξ,

$$\widetilde{L}\widetilde{f} = L\widetilde{f}. \tag{2.2.1}$$

DEFINITION **28.** *A differential operator L' is said to be **hypoelliptic** if for any distribution v such that $L'v$ is of class $\mathcal{C}^\infty$, v is itself of class $\mathcal{C}^\infty$.*

Hörmander gave a criterion to show that a large class of second order differential operators are hypoelliptic.

THEOREM **29 (Hörmander, [Hör67], [Koh73], p.**61**).** *A differential operator L' on a manifold of dimension n, in Hormander form : $L' = \sum_{i=1}^{r} V_i^2 + V$, where the V_i and V are smooth vector fields, is hypoelliptic if the Lie algebra generated by the V_i and V has rank n at every point.*

Notation — Let $Haar$ be a (left) Haar measure on $\mathcal{G}$. Write $\widetilde{L}^*$ the $\mathbb{L}^2(Haar)$ adjoint of $\widetilde{L}$.

The vector fields V_i being left invariant have null divergence, and

$$\int \widetilde{f}(V_i^2\widetilde{g})Haar = \int (V_i^2\widetilde{f})\widetilde{g}Haar, \quad \widetilde{f},\widetilde{g} \in \mathcal{C}_0^\infty(\mathcal{G}),$$

and $\widetilde{L}^* = \frac{1}{2}\sum_{i=1}^{d} V_i^2 - V_0$.

PROPOSITION **30.** *The differential operators $\widetilde{L} - \partial_s$ and $\widetilde{L}^* - \partial_s$ on $\mathcal{G}\times\mathbb{R}^{>0}$ are hypoelliptic.*

$\triangleleft$ The brackets between the $\widetilde{E}_i$ and $\widetilde{A}_0$ are given by the following relations, where we note, for $1 \leqslant p < q \leqslant d$, $\widetilde{E}_{pq} = (E_{pq}, 0)$, where $\exp(tE_{pq})$ is the rotation of angle $-t$ in the 2-dimensional vector space $\langle \varepsilon_p, \varepsilon_q\rangle$, and for $1 \leqslant i \leqslant d$, $\widetilde{A}_i = (0, \varepsilon_i)$.

$$[\widetilde{E}_i, \widetilde{E}_j] = \widetilde{E}_{ij},\ [\widetilde{E}_i, \widetilde{A}_0] = \widetilde{A}_i,\ [\widetilde{E}_i, \widetilde{A}_i] = \widetilde{A}_0.$$

As the family $\big\{\widetilde{E}_i, \widetilde{E}_{jk}, \widetilde{A}_\ell;\ i = 1..d,\ 1 \leqslant j < k \leqslant d,\ \ell = 0..d\big\}$ is a basis of the Lie algebra of $\mathcal{G}$, Hörmander's theorem ensures the hypoellipticity of $\widetilde{L} - \partial_s$ and $\widetilde{L}^* - \partial_s$. $\triangleright$

Notation — Note $\{\widetilde{P}_s\}_{s\geqslant 0}$ the semi-group of the process $\{(g_s, \xi_s)\}_{s\geqslant 0}$ on $\mathcal{G}$ and $\{P_s\}_{s\geqslant 0}$ that of the process $\big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ on $\mathbb{H}\times\mathbb{R}^{1,d}$.

• It follows from the hypoellipticity of $\partial_s - \widetilde{L}$ that for any bounded function $\widetilde{f}$ on $\mathcal{G}$, the function $\widetilde{P}_s\widetilde{f}\big((g,\xi)\big)$ is a smooth function of (s and) (g,ξ). From relation (2.2.1), $P_s f$ is a smooth function on $\mathbb{H}\times\mathbb{R}^{1,d}$ if f is any bounded Borel function on $\mathbb{H}\times\mathbb{R}^{1,d}$. We can write

$$\widetilde{P}_s\widetilde{f}(g,\xi) = \int_{\mathcal{G}} \widetilde{p}_s\big((g,\xi),(g',\xi')\big)\widetilde{f}(g',\xi')Haar\big(d(g',\xi')\big).$$

Proposition **30** ensures that $\widetilde{p}_s\big((g,\xi),(g',\xi')\big)$ is a smooth function of $\big(s,(g,\xi)\big)$ and $\big(s,(g',\xi')\big)$, separately.

Remark that the left invariance of the vector fields V_i implies that the functions $\widetilde{p}_s$ are left invariant:

$$\forall\, (\underline{g},\underline{\xi}) \in \mathcal{G}, \quad \widetilde{p}_s\big((\underline{g},\underline{\xi})(g,\xi),(\underline{g},\underline{\xi})(g',\xi')\big) = \widetilde{p}_s\big((g,\xi),(g',\xi')\big).$$

So, it is jointly continuous in $\big(s,(g,\xi),(g',\xi')\big)$.

b) Asymptotic behaviour of the diffusion

Notation – Up to now, we noted $(\dot{\xi}, \xi)$ a point of $\mathbb{H} \times \mathbb{R}^{1,d}$ and $\{(\dot{\xi}_s, \xi_s)\}_{s\geqslant 0}$ the relativistic diffusion. It will be shorter to note u a generic point of $\mathbb{H} \times \mathbb{R}^{1,d}$ and $u_s = (\dot{\xi}_s, \xi_s)$, in formulas.

Add an extra point ∂ to $\mathbb{H} \times \mathbb{R}^{1,d}$, and define Ω as the set of paths $\omega : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0} \times \big(\mathbb{H} \times \mathbb{R}^{1,d} \cup \{\partial\}\big)$ such that there exists an $\alpha \geqslant 0$ such that

- for $s < \alpha$, $\omega(s) = (s, \partial)$,
- for $s \geqslant \alpha$, $\omega(s) = (s, u_s)$, $u_s \in \mathbb{H} \times \mathbb{R}^{1,d}$.

Note $\mathcal{F}$ the σ-field generated by the evaluation maps $\omega \mapsto \omega(s)$. For $t \geqslant 0$, the map

$$\theta_t : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F}), \quad \theta_t(\omega)(s) = \omega(s+t),$$

is measurable.

Given $u \in \mathbb{H} \times \mathbb{R}^{1,d}$, one can define the law $\mathbb{P}_u$ of the relativistic diffusion started from u as a probability measure on $(\Omega, \mathcal{F})$ (charging the set of w's such that $\alpha = 0$). For $r \geqslant 0$, define $\mathbb{P}_{r,u}$ as the image of $\mathbb{P}_u$ by the measurable map

$$\omega = \{(s, u_s)\}_{s\geqslant 0} \,(^1) \longrightarrow \omega',$$

where

$$\begin{aligned} &\text{for } s < r, \omega'(s) = (s, \partial), \\ &\omega'(r) = (r, u), \\ &\text{for } s > r, \omega'(s) = (s, u_{s-r}). \end{aligned} \tag{2.2.2}$$

So, $\mathbb{P}_{0,u} = \mathbb{P}_u$. We shall simply write $\mathbb{P}_u$ for $\mathbb{P}_{0,u}$.

Tail and invariant σ-fields From a formal viewpoint, the information on the asymptotic behaviour of the process $\{u_s\}_{s\geqslant 0} = \big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ is contained in its **tail σ-algebra**

$$\tau\big(\{u_s\}\big) = \bigcap_{t>0} \sigma\big((\dot{\xi}_s, \xi_s); s \geqslant t\big).$$

This σ-algebra gets some real sense in a probabilized framework.

Among the events Z of $\tau\big(\{u_s\}\big)$, those that are invariant under the shift operators

$$\big\{Z \in \tau\big(\{u_s\}\big); \theta_t^{-1} Z = Z, \text{ for all } t > 0\big\}, \tag{2.2.3}$$

form a σ-algebra called the ***invariant σ-algebra***, and noted $Inv\big(\{u_s\}\big)$[2].

The problem of the determination of these σ-algebras takes some precise sense in a probabilized framework.

Recall we note $\mathbb{P}_{u_0}$ the law of the relativistic diffusion $\big\{(\dot{\xi}_s, \xi_s)\big\}_{s\geqslant 0}$ started from $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$.

PROBLEM. *Can one find a random variable Y, with values in some well known set (a discret set, a manifold, a Banach space...) such that under each $\mathbb{P}_{(\dot{\xi}_0, \xi_0)}$*

- *the σ-algebras $\tau\big(\{u_s\}\big)$ (or $Inv\big(\{u_s\}\big)$) and $\sigma(Y)$ coincide up to $\mathbb{P}_{(\dot{\xi}_0, \xi_0)}$-null sets,*
- *the law of Y is known ?*

This probabilistic problem has an analytic counterpart.

---

[1] u_s can be equal to ∂.

[2] Remark that $\{\theta_t^{-1} Z = Z\} \Longleftrightarrow \{\mathbf{1}_Z \circ \theta_t = \mathbf{1}_Z\}$.

c) Harmonic functions and Poisson boundary

DEFINITION **31.** • *Given a second order differential operator L' on some manifold M, a $\mathcal{C}^2$ real function f on $\mathbb{M}$ is said to be **L'-harmonic** if $L'f = 0$.*

- *For historical reasons, the set of bounded L'-harmonic functions on $\mathbb{M}$ is called the **Poisson boundary of** $(L', \mathbb{M})$.*

i) The following classical proposition establishes a bridge between the invariant σ-field, one the one hand, and the Poisson boundary of $(L, \mathbb{H} \times \mathbb{R}^{1,d})$, on the other hand [3].

PROPOSITION **32 (Correspondence Probability/Analysis).** *Any bounded L-harmonic function h on $\mathbb{H} \times \mathbb{R}^{1,d}$ is of the form*

$$h(u) = \mathbb{E}_u[X], \tag{2.2.4}$$

for some bounded $Inv(\{u_s\})$-measurable random variable X.

Reciprocally, for any such random variable X, the formula (2.2.4) *defines an L-harmonic bounded function.*

$\lhd$ Given a bounded L-harmonic function h, define the $Inv(\{u_s\})$-measurable random variable $X = \lim_{s\to+\infty} h(u_s)$, when it exists, 0 elsewhere.
For $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$, $\{h(u_s)\}_{s\geqslant 0}$ is, under $\mathbb{P}_{u_0}$, a bounded martingale, and $h(u_0) = \mathbb{E}_{u_0}[X]$.
Reciprocally, given an $Inv(\{u_s\})$-measurable bounded random variable X, the measurable bounded function $h : u \mapsto \mathbb{E}_u[X]$, satisfies the identities

$$P_s h(u) = \mathbb{E}_u[h(u_s)] = \mathbb{E}_u\left[\mathbb{E}_{u_s}[X]\right] = \mathbb{E}_u[X \circ \theta_s] = \mathbb{E}_u[X] = h(u).$$

So it follows from the point after proposition **30** that h is smooth and L-harmonic, since it does not depend on s. $\rhd$

Remarks

1. The function $u \mapsto \mathbb{E}_u[X]$ is still L-harmonic if the random variable X coincide, under each probability $\mathbb{P}_{u_0}$, with an $Inv(\{u_s\})$-measurable random variable, up to $\mathbb{P}_{u_0}$ null sets.
2. Given a compact K, set T_K the exit time from K, and $\mathcal{F}_{\geqslant T_K} = \sigma(u_s; s \geqslant T_K)$. It is clear form the proposition that the σ-algebras $Inv(\{u_s\})$ and $\bigcap \mathcal{F}_{\geqslant T_K}$, where the intersection is over all compacta, coincide under $\mathbb{P}_{u_0}$, up to $\mathbb{P}_{u_0}$-null sets.

ii) There also exists an analytic analogous of the tail σ-algebra.
For $s \geqslant 0$, set

$$\mathcal{F}^{u_\cdot}_{\leqslant s} = \sigma\big(u_r; u \leqslant s\big), \quad \mathcal{F}^{\cdot,u_\cdot}_{\leqslant s} = \sigma\big((r, u_r); r \leqslant s\big),$$
$$\mathcal{F}^{u_\cdot}_{\geqslant s} = \sigma\big(u_r; s \leqslant r\big), \quad \mathcal{F}^{\cdot,u_\cdot}_{\geqslant s} = \sigma\big((r, u_r); s \leqslant r\big).$$

For each $s \geqslant 0$, we have $\mathcal{F}^{u_\cdot}_{\geqslant s} \subset \mathcal{F}^{\cdot,u_\cdot}_{\geqslant s}$. Actually, this inclusion is an equality in the following sense.

LEMMA **33.** *Let $s_0 \geqslant 0$ and $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$ be given. For any $s \geqslant 0$, the two σ-fields $\mathcal{F}^{u_\cdot}_{\geqslant s}$ and $\mathcal{F}^{\cdot,u_\cdot}_{\geqslant s}$ coincide up to a class of $\mathbb{P}_{s_0,u_0}$-null sets.*

[3] See [Pin95], Theorem 1.2, p.436, for instance.

$\triangleleft$ There is no problem for $r < s_0$. Suppose $s \geqslant s_0$.
We have to see that we can recover the time process $\{(r, u_r)\}_{r \geqslant s}$ from the knowledge of $\{u_r\}_{r \geqslant s}$. Note (ρ_r, σ_r) the polar coordinates of $\dot{\xi}_r$. Since

$$d\rho_r = dw_r^\rho + \frac{d-1}{2}\coth(\rho_r)\, dr,$$

$d\langle \rho \rangle_r = dr$, and $\langle \rho \rangle_r - \langle \rho \rangle_s$ is an $\mathcal{F}^{u_\cdot}_{\geqslant s}$-measurable random variable, we can recover the time/space process $\{(r, u_r)\}_{r \geqslant s}$ from the observation of $\{u_r\}_{r \geqslant s}$. $\triangleright$

As $\tau\big(\{u_s\}\big) = \bigcap_{s \geqslant 0} \mathcal{F}^{u_\cdot}_{\geqslant s}$,

Proposition 34. *Given $s_0 \geqslant 0$ and $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$, the σ-fields $\tau\big(\{u_s\}\big)$ and $\bigcap_{s \geqslant 0} \mathcal{F}^{\cdot, u_\cdot}_{\geqslant s}$ coincide up to a class of $\mathbb{P}_{s_0, u_0}$-null sets.*

The σ-field $\bigcap_{s \geqslant 0} \mathcal{F}^{\cdot, u_\cdot}_{\geqslant s}$ is the invariant σ-field of the time/space diffusion under the action of the shifts θ_t, $t > 0$.

Theorem 35. *Any bounded $(\partial_s + L)$-harmonic function h on $\mathbb{R}^{\geqslant 0} \times \mathbb{H} \times \mathbb{R}^{1,d}$ is of the form $h(s, u) = \mathbb{E}_{(s,u)}[Y]$, for some bounded random variable Y such that Y is indistinguishable from a $\tau\big(\{u_s\}\big)$-measurable random variable under each $\mathbb{P}_{(s,u)}$, $s \geqslant 0, u \in \mathbb{H} \times \mathbb{R}^{1,d}$.*

Reciprocally, for any bounded $\tau\big(\{u_s\}\big)$-measurable random variable Y, the function $h(s, u) = \mathbb{E}_{(s,u)}[Y]$ is $(\partial_s + L)$-harmonic.

$\triangleleft$ Let $h : \mathbb{R} \times \mathbb{H} \times \mathbb{R}^{1,d} \to \mathbb{R}$ be a bounded $(\partial_s + \widetilde{L})$-harmonic function. Set $Y = \lim_{s \to +\infty} h(s, u_s)$, when this limit exists, 0 elsewhere. This is an $\bigcap_{s \geqslant 0} \mathcal{F}^{\cdot, u_\cdot}_{\geqslant s}$-measurable random variable.

Given $s_0 \geqslant 0, u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$, the process $\{h(s, u_s)\}_{s \geqslant s_0}$ is under $\mathbb{P}_{s_0,u_0}$ a bounded martingale; it converges $\mathbb{P}_{s_0,u_0}$-almost surely toward Y, and $h(s_0, u_0) = \mathbb{E}_{(s_0,u_0)}[Y]$. From proposition **34**, Y is under $\mathbb{P}_{s_0,u_0}$ indistinguishable from a $\tau\big(\{u_s\}\big)$-measurable random variable.

Reciprocally, given a $\tau\big(\{u_s\}\big)$-measurable random variable Y, $s_0 \geqslant 0$ and $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$, proposition **34** justifies the first following equality, where $s \geqslant s_0$.

$$\mathbb{E}_{s_0,u_0}[Y] = \mathbb{E}_{s_0,u_0}[Y \circ \theta_{s-s_0}] = \mathbb{E}_{s_0,u_0}\big[\mathbb{E}_{s_0,u_0}[Y \,|\, \mathcal{F}^{\cdot, u_\cdot}_{\leqslant s}]\big] = \mathbb{E}_{s_0,u_0}\big[\mathbb{E}_{s,u_s}[Y]\big].$$

So the function $h : (s_0, u_0) \in \mathbb{R}^{\geqslant 0} \times \mathbb{H} \times \mathbb{R}^{1,d} \mapsto \mathbb{E}_{s_0,u_0}[Y]$ satisfies the equation

$$h(s_0, u_0) = \mathbb{E}_{s_0,u_0}[h(s, u_s)], \tag{2.2.5}$$

for any $s \geqslant s_0 \geqslant 0$. We saw in proposition **30** that the operator $\partial_s + L$ is hypoelliptic. It follows from Hörmander's theorem that h is smooth and from the equation (2.2.5) that it is $(\partial_s + L)$-harmonic. $\triangleright$

iii) The preceding correspondence immediately gives some fruits.

Theorem 36. *Any bounded $(\partial_s + L)$-harmonic function on $\mathbb{H} \times \mathbb{R}^{1,d}$ does not depend on the time s, and so is L-harmonic.*

Corollary 37 (Tail and invariant σ-algebras). *For any $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$, the σ-algebras $\tau\big(\{u_s\}\big)$ and $Inv\big(\{u_s\}\big)$ coincide up to a class of $\mathbb{P}_{0,u_0}$-null sets.*

We give a proof of theorem **38** for bounded $(\partial_s + \widetilde{L})$-harmonic function. The preceding results on asymptotic σ-algebras hold true for the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ on $\mathcal{G}$, and its time/space version.

THEOREM **38.** *Any bounded $(\partial_s + \widetilde{L})$-harmonic function on $\mathbb{R}^{>0}\times\mathcal{G}$ does not depend on the time s, and so is $\widetilde{L}$-harmonic.*

To prove this theorem, we use a Harnack inequality, shown in theorem **71**. You should read the following proof after reading section **2.2.4**. See [Anc90], Corollaire 3.2, p.35, for the same result with the operator $\partial_s + \triangle$ on some good Riemannian manifold.

$\lhd$ Let $0<\varepsilon<1$ and h be a bounded $(\partial_s + \widetilde{L})$-harmonic function. For any $\mathbf{e}\in\mathcal{G}$, $r\geqslant 1$,

$$h(s+\varepsilon,\mathbf{e}) - h(s,\mathbf{e}) = \int \big\{\widetilde{p}_{r-\varepsilon}(\mathbf{e},\mathbf{e}') - \widetilde{p}_r(\mathbf{e},\mathbf{e}')\big\} h(s+r,\mathbf{e}')\, Haar(d\mathbf{e}'),$$

so

$$\big|h(s+\varepsilon,\mathbf{e}) - h(s,\mathbf{e})\big| \leqslant \|h\|_\infty \|\widetilde{p}_{r-\varepsilon}(\mathbf{e},.) - \widetilde{p}_r(\mathbf{e},.)\|_{\mathbb{L}^1(Haar)}. \tag{2.2.6}$$

Using the left invariance of $\widetilde{p}$, we see that

$$\|\widetilde{p}_{r-\varepsilon}(\mathbf{e},.) - \widetilde{p}_r(\mathbf{e},.)\|_{\mathbb{L}^1(Haar)} = \|\widetilde{p}_{r-\varepsilon}(\mathrm{Id},.) - \widetilde{p}_r(\mathrm{Id},.)\|_{\mathbb{L}^1(Haar)}.$$

The function $(t,\mathbf{e}')\in\mathbb{R}^{>0}\times\mathcal{G}\mapsto \widetilde{p}_t(\mathrm{Id},\mathbf{e}')$ is $(\partial_t - \widetilde{L}^*)$-harmonic. This hypoelliptic[4] operator is defined on the *product group* $\mathbb{R}\times\mathcal{G}$, and left invariant. Harnack's inequality stated in theorem **71**, and remark 1 following it, assert that there exists a constant C such that

$$\forall\, r\geqslant 1,\ \forall\,\mathbf{e}'\in\mathcal{G},\quad \widetilde{p}_{r-\varepsilon}(\mathrm{Id},\mathbf{e}') \leqslant C\,\widetilde{p}_r(\mathrm{Id},\mathbf{e}'). \tag{2.2.7}$$

So, inequality (2.2.7) means that if $\widetilde{P}_{r-\varepsilon}(\mathrm{Id},.)$ puts some mass m on some set, $\widetilde{P}_r(\mathrm{Id},.)$ puts at least some mass $\frac{m}{C}$ on that set. So, the variation distance between these two probabilities is $\leqslant 1-\frac{1}{C}<1$. This distance is equal to $\frac{1}{2}\int \|\widetilde{p}_{r-\varepsilon}(\mathrm{Id},.) - \widetilde{p}_r(\mathrm{Id},.)\|_{\mathbb{L}^1(Haar)}$.
Thus, for any $\mathbf{e}\in\mathcal{G}$,

$$\big\|\widetilde{p}_{r-\varepsilon}(\mathbf{e},.) - \widetilde{p}_r(\mathbf{e},.)\big\|_{\mathbb{L}^1(Haar)} = \big\|\widetilde{p}_{r-\varepsilon}(\mathrm{Id},.) - \widetilde{p}_r(\mathrm{Id},.)\big\|_{\mathbb{L}^1(Haar)} \leqslant 2(1-\frac{1}{C}) < 2.$$

Derriennic's $0-2$ law, [Der76], p.118, states that under the preceding condition

$$\big\|\widetilde{p}_{r-\varepsilon}(\mathbf{e},.) - \widetilde{p}_r(\mathbf{e},.)\big\|_{\mathbb{L}^1(Haar)} \underset{r\to+\infty}{\longrightarrow} 0.$$

So, the inequality (2.2.6) gives

$$\big|h(s+\varepsilon,\mathbf{e}) - h(s,\mathbf{e})\big| = 0.$$

As this is true for any $\varepsilon>0$, the conclusion follows. $\rhd$

d) How can one find the Poisson boundary of $(L, \mathbb{H}\times\mathbb{R}^{1,d})$?

Recall the definition of a minimal non negative L-harmonic function given in the Introduction.

DEFINITION **39.** *A non negative L-harmonic function is said to be* ***minimal*** *if any L-harmonic function g such that*

$$0\leqslant g\leqslant f$$

is proportional to f.

[4] See proposition **30**.

The following reformulation is immediate.

LEMMA **40.** *A non negative L-harmonic function h is minimal iff the only bounded functions f satisfying the relation $L(hf) = 0$ are the constants.*

i) We shall apply Martin's method ([Mar41]), as used in the study of the heat operator (see the book [Doo01] of Doob, Chapter XIX, for instance).

We are going to find a set $\{f^\alpha\}_{\alpha\in A}$ of L-harmonic functions, indexed by some set A, and a probability ν_1 on A, such that the function $\mathbf{1}$, identically equal to 1, can be written

$$\mathbf{1} = \int_A f^\alpha \, \nu_1(d\alpha) \tag{2.2.8}$$

Using Choquet's theorem on integral representation in convex compacta, we shall see (in **2.2.4**) that the identity (2.2.8) implies that any L-harmonic bounded function h is of the form

$$h = \int_A f^\alpha H(\alpha)\, \nu_1(d\alpha),$$

for a bounded Borel function H on A; H is uniquely determined by h, up to ν_1-null sets.

***ii)* What are these functions f^α ?** – The convergence of $\{\sigma_s\}_{s\geqslant 0}$ toward a random point $\sigma_\infty \in \mathbb{S}^{d-1}$, provides an identity of the form (2.2.8). We noticed in the remark following point **b)**, p.26, that σ_∞ has under $\mathbb{P}_u$ a density $h^\sigma(u)$ with respect to the uniform probability on $\mathbb{S}^{d-1}$, which is jointly conitnuous in (σ, u), and which is an L-harmonic function of u, for fixed $\sigma \in \mathbb{S}^{d-1}$. It satisfies the identity

$$\forall\, u \in \mathbb{H} \times \mathbb{R}^{1,d}, \quad \mathbf{1} = \int_{\mathbb{S}^{d-1}} h^\sigma(u)\, d\sigma.$$

But are these functions h^σ minimal ?

Let $\varphi \in O(\mathbb{R}^d)$. For $\dot{\xi} \in \mathbb{H}$, with polar coordinates (ρ, σ), note $\varphi(\dot{\xi})$ the point of $\mathbb{H}$ with polar coordinates $\big(\rho, \varphi(\sigma)\big)$. This application is an isometry of $\mathbb{H}$. Using the relation

$$h^{\varphi(\sigma)}\big((\dot{\xi}, \xi)\big) = h^\sigma\big(\varphi^{-1}(\dot{\xi}), \varphi^{-1}(\xi)\big), \tag{2.2.9}$$

and the invariance of L by the action of φ^{-1} on $\mathbb{H} \times \mathbb{R}^{1,d}$, one sees that one of the functions h^σ is minimal iff they are all minimal[5].

We choose to work with $\sigma = \varepsilon_1$. Using the halfspace coordinates on $\mathbb{H}$, the function $h^\sigma\big((y,x), \xi\big)$ is proportional to y^{d-1}.

The appendix **h-transform**, p.94, recalls how one can construct a diffusion on $\mathbb{H} \times \mathbb{R}^{1,d}$, with differential generator

$$f \to \frac{L(h^{\varepsilon_1} h)}{h^{\varepsilon_1}} \equiv L^{h^{\varepsilon_1}} h,$$

and give an interpretation of this diffusion in terms of the original one. This new diffusion is called the **conditioned diffusion**, or the h^{ε_1}**-process**. From an analytical viewpoint, $L^{h^{\varepsilon_1}}$ is hypoelliptic, so the link between bounded harmonic functions and invariant σ-algebra stated in Corollary **32** still holds.

LEMMA **41.** *h^{ε_1} is minimal iff the h^{ε_1}-process has a trivial invariant σ-algebra.*

[5] Note that the function h^σ depends only on $\dot{\xi}$.

Notation – Note $\mathbb{P}^{\varepsilon_1}_{u_0}$ the law of the conditioned diffusion, started from $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$.

Using halfspace coordinates on $\mathbb{H}$, the operator $L^{h^{\varepsilon_1}}$ has the expression

$$L^{h^{\varepsilon_1}} f = \frac{y^2}{2}(\partial_x^2 + \partial_y^2) f + \frac{d}{2} y \partial_y f + \partial_\xi f(\dot{\xi})(^6).$$

So, the evolution of the Markov process with generator $L^{h^{\varepsilon_1}}$ is determined by the following stochastic differential system.

$$\begin{aligned} dy_s &= y_s dw_s^y + \frac{d}{2} y_s ds, \\ dx_s &= y_s dw_s^x, \\ d\xi_s &= \dot{\xi}_s \, ds = \psi\big((y_s, x_s)\big) ds. \end{aligned} \tag{2.2.10}$$

where w^y is a real Brownian motion and w^x a $(d-1)$-dimensional Brownian motion independent of w^y.

We get a better insight of the evolution of ξ_s by changing coordinates in $\mathbb{R}^{1,d}$. Take the coordinates associated with the basis $\mathbf{g}' = \{-\varepsilon_1, \varepsilon_0 + \varepsilon_1, \varepsilon_2, ..., \varepsilon_d\}$.

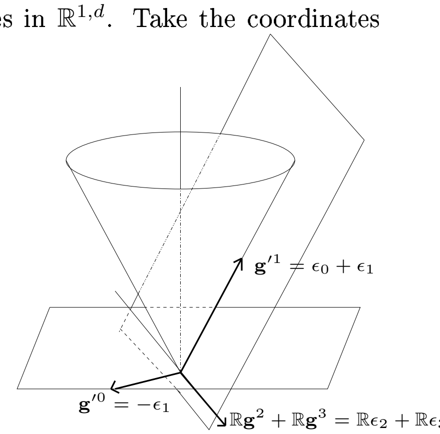
Note $(\xi'^0, \xi'^1, ..., \xi'^d)$ the coordinates of a point $\xi \in \mathbb{R}^{1,d}$ in the basis $\mathbf{g}'$.

In these coordinates, the equation

$$d\xi_s = \psi\big((y_s, x_s)\big) ds$$

takes the form

$$\begin{aligned} d\xi_s'^0 &= \frac{ds}{y_s}, \\ d\xi_s'^1 &= \frac{|x_s|^2 + y_s^2 + 1}{2y_s} ds, \\ d\xi_s'^j &= \frac{x_s^{j-1}}{y_s} ds, \; j = 2..d. \end{aligned} \tag{2.2.11}$$



Notation – From now on, we use $\mathbf{g}'$ coordinates on $\mathbb{R}^{1,d}$, and halfspace coordinates on $\mathbb{H}$. So, a generic point $u \in \mathbb{H} \times \mathbb{R}^{1,d}$ is denoted either $u = (\dot{\xi}, \xi)$ when no coordinates are needed, or $u = \big((y, x), (\xi'^0, ..., \xi'^d)\big)$.

The equation for y_s has explicit solution:

$$y_s = y_0 \, e^{w_s^y + \frac{d-1}{2} s}.$$

It comes from the first equation of (2.2.11) that the process $\{\xi_s'^0\}_{s \geqslant 0}$ converges $\mathbb{P}^{\varepsilon_1}_{u_0}$-almost surely to $\xi_0'^0 + \frac{1}{y_0} \int_0^\infty e^{-w_r^y - \frac{d-1}{2} r} \, dr$. This random variable has a non trivial law. So, h^{ε_1} is not minimal (as well as all the h^σ's). Note

$$R_\infty^{\varepsilon_1} = \xi_0'^0 + \frac{1}{y_0} \int_0^\infty e^{-w_r^y - \frac{d-1}{2} r} \, dr.$$

We shall see, in the next section, that $R_\infty^{\varepsilon_1}$ has, under any $\mathbb{P}^{\varepsilon_1}_u$, a smooth density $h_\cdot^{\varepsilon_1}(u)$ with respect to Lebesgue's measure:

$$u \in \mathbb{H} \times \mathbb{R}^{1,d}, a < b, \quad \mathbb{P}^{\varepsilon_1}_u (R_\infty^{\varepsilon_1} \in [a, b[) = \int_a^b h_\ell^{\varepsilon_1}(u) \, d\ell,$$

[6] Here and in the sequel, we use the notation ∂_x^2 for $\partial_{x_1}^2 + ... + \partial_{x_{d-1}}^2$.

and that any $h_\ell^{\varepsilon_1}(.)$, $\ell \in \mathbb{R}$, is an $L^{h^{\varepsilon_1}}$-harmonic function of u: $\frac{L(h^{\varepsilon_1} h_\ell^{\varepsilon_1})}{h^{\varepsilon_1}} = 0$. So, $h^{\varepsilon_1} h_\ell^{\varepsilon_1}$ is L-harmonic.

As $\frac{1}{y_s} = q(\dot{\xi}_s, \varepsilon_0 + \varepsilon_1)$, $R_\infty^{\varepsilon_1}$ has a coordinate free expression

$$R_\infty^{\varepsilon_1} = \lim_{s \to +\infty} q(\xi_s, \varepsilon_0 + \varepsilon_1).$$

iii) Note $\mathbb{P}_u^\sigma$ the law of the h^σ-process, started from $u \in \mathbb{H} \times \mathbb{R}^{1,d}$. Let $g \in SO_0(1,d)$ be a basis of $\mathbb{R}^{1,d}$ of the form $\{\varepsilon_0, \sigma, g_{\geqslant 2}\}$([7]). Using g-halfspace coordinates on $\mathbb{H}$[8] and $\{-g_1, g_0 + g_1, g_1, g_2, ..., g_d\}$-coordinates on $\mathbb{R}^{1,d}$, the h^σ-process is the solution of the system (2.2.10), (2.2.11). So, the random variable

$$R^\sigma(\xi_s) = q(\xi_s, \varepsilon_0 + \sigma)$$

$\mathbb{P}_u^\sigma$-almost surely converges, whatever $u \in \mathbb{H} \times \mathbb{R}^{1,d}$. Its limit R_∞^σ has, under $\mathbb{P}_u^\sigma$, a smooth density $h_\cdot^\sigma(u)$ with respect to Lebesgue's measure on $\mathbb{R}$.

$$\mathbb{P}_u^\sigma\big(R_\infty^\sigma \in [a,b[\big) = \int_a^b h_\ell^\sigma(u)\, d\ell.$$

We show that, given $\sigma \in \mathbb{S}^{d-1}$, $\ell \in \mathbb{R}$, $h_\ell^\sigma(.)$ is an L^{h^σ}-harmonic function; so $h^\sigma h_\ell^\sigma$ is an L-harmonic function.

The main result of this section is the following theorem.

THEOREM **42.** *The L-harmonic non negative functions $h^\sigma h_\ell^\sigma$, $\sigma \in \mathbb{S}^{d-1}$, $\ell \in \mathbb{R}$, are minimal, and for any $u \in \mathbb{H} \times \mathbb{R}^{1,d}$*

$$1 = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} h^\sigma h_\ell^\sigma(u)\, d\sigma d\ell.$$

It is enough to consider the case $\sigma = \varepsilon_1$.

Organization of the end of this chapter – We show, in section **2.2.2**, that bounded $L^{h^{\varepsilon_1}}$-harmonic functions depend only on y and ξ'^0, using a coupling argument.

$L^{h^{\varepsilon_1} h_\ell^{\varepsilon_1}}$-harmonic bounded functions share the same property. We use another coupling, in section **2.2.3**, to show that $L^{h^{\varepsilon_1} h_\ell^{\varepsilon_1}}$ has no non constant bounded harmonic functions, and prove theorem **42**.

2.2.2 h^{ε_1}-process and bounded $L^{h^{\varepsilon_1}}$-harmonic functions

a) Dufresne's integral

The operator $L^{h^{\varepsilon_1}}$ has the expression

$$L^{h^{\varepsilon_1}} f = \frac{y^2}{2}(\partial_x^2 + \partial_y^2) f + \frac{d}{2} y \partial_y f + \partial_\xi f(\dot{\xi});$$

as L, it is a hypoelliptic operator on $\mathbb{H} \times \mathbb{R}^{1,d}$. Indeed, suppose $L^{h^{\varepsilon_1}} f = \frac{L(h^{\varepsilon_1} f)}{h^{\varepsilon_1}}$ is of class $\mathcal{C}^\infty$, then $L(h^{\varepsilon_1} f)$ is $\mathcal{C}^\infty$ (since h^{ε_1} is $\mathcal{C}^\infty$), so $h^{\varepsilon_1} f$ is smooth; since h^{ε_1} *is positive*, f is smooth.

Similarly, $L^{h^{\varepsilon_1}} - \partial_t$ is an hypoelliptic operator on $\mathbb{R}^{>0} \times \big(\mathbb{H} \times \mathbb{R}^{1,d}\big)$.

For the h^{ε_1}-process $\big\{\big((y_s, x_s), \xi_s'\big)\big\}_{s \geqslant 0}$, started from $\big((y,x), \xi'\big)$,

$$y_s = y\, e^{w_s^y + \frac{d-1}{2} s},$$

and

$$\xi_s'^0 = \xi'^0 + \frac{1}{y} \int_0^s e^{-w_r^y - \frac{d-1}{2} r}\, dr$$

[7]The notation $g_{\geqslant 2}$ is for a $(d-1)$-uple $\{g_2, ..., g_d\}$.

[8]See 2.1.1, **b)**.

converges $\mathbb{P}_u^{\varepsilon_1}$-almost surely as $s \to +\infty$, whatever $u \in \mathbb{H} \times \mathbb{R}^{1,d}$.

As $L^{h^{\varepsilon_1}}$ is hypoelliptic, corollary **32** says that, given $\ell \in \mathbb{R}$, the function

$$h^{\varepsilon_1}_{\geqslant \ell} : u \in \mathbb{H} \times \mathbb{R}^{1,d} \mapsto \mathbb{P}_u^{\varepsilon_1}(R_\infty \geqslant \ell),$$

is smooth and $L^{h^{\varepsilon_1}}$-harmonic.

One sees on the expression

$$h^{\varepsilon_1}_{\geqslant \ell}(u) = \mathbb{P}_u^{\varepsilon_1}\left(\xi'^0 + \frac{1}{y}\int_0^{+\infty} e^{-w_r^y - \frac{d-1}{2}r}\, dr \geqslant \ell\right) = \mathbb{P}_u^{\varepsilon_1}\left(\int_0^{+\infty} e^{-w_r^y - \frac{d-1}{2}r}\, dr \geqslant y\big(\ell - \xi'^0\big)\right)$$

that $h^{\varepsilon_1}_{\geqslant \ell}$ is identically equal to 1 on the half space $\big\{(\dot{\xi}, \xi)\,;\, \ell - \xi'^0 \leqslant 0\big\}$. Set

$$G(t) = \mathbb{P}\left(\int_0^{+\infty} e^{-w_r^y - \frac{d-1}{2}r}\, dr \geqslant t\right). \tag{2.2.12}$$

Then,

$$h^{\varepsilon_1}_{\geqslant \ell}(u) = G\big(y(\ell - \xi'^0)\big). \tag{2.2.13}$$

Since $h^{\varepsilon_1}_{\geqslant \ell}$ is smooth, G is also smooth; it satisfies the following differential equation on the open half space $\big\{(\dot{\xi}, \xi)\,;\, \ell - \xi'^0 > 0\big\}$, traducing the $L^{h^{\varepsilon_1}}$ harmonicity of $h^{\varepsilon_1}_{\geqslant \ell}$. For our comfort, we shall write $\alpha(\xi) \equiv \ell - \xi'^0$.

$$\frac{\big(y\alpha(\xi)\big)^2}{2} G''\big(y\alpha(\xi)\big) + \left(\frac{d\, y\alpha(\xi)}{2} - 1\right) G'\big(y\alpha(\xi)\big) = 0;$$

that is

$$G''(r) + \left(\frac{d}{r} - \frac{2}{r^2}\right) G'(r) = 0. \tag{2.2.14}$$

We find

$$G'(r) = -C\, \frac{e^{-2/r}}{r^d} \mathbf{1}_{r>0}, \tag{2.2.15}$$

or $G(s) = C \int_s^{+\infty} \frac{e^{-2/r}}{r^d}\, dr$, for $s > 0$, where C is such that $G(0) = 1$, and $G \equiv 1$ on $\mathbb{R}^{\leqslant 0}$. The identification of this law is a result of Dufresne. The preceding simple proof is new.

Proposition **43 (Dufresne, [Duf90], [Yor92]).** *Let w be a Brownian motion . For $0 < a < b < \infty$,*

$$\mathbb{P}\left(\int_0^{\infty} e^{-w_s - \frac{d-1}{2}s} ds \in [a, b[\right) = C(d) \int_a^b \frac{e^{-2/u}}{u^d} du,$$

where $C(d)$ is a normalisation constant.

In our context, d is a dimension. The fact that it is integer has no importance in the preceding. So we have shown that for any $c > 0$,

$$\mathbb{P}\left(\int_0^{\infty} e^{-w_s - cs}\, ds \in [a, b[\right) = C(c) \int_a^b u^{-1-2c} e^{-2/u}\, du.$$

Yor and Matsumoto give an extensive study of the laws of exponential functionals of Brownian motion in [MY05a] and [MY05b], giving a completely different proof of this result.

Remark: We used corollary **32** to justify that $h^{\varepsilon_1}_{\geqslant\ell}$ is smooth. Hörmander's theorem is the heart of the proof of the equivalence stated in corollary **32**. This theorem is difficult. As concerns the smoothness of $h^{\varepsilon_1}_{\geqslant\ell}$, it seems irrelevant to use such a strong result to obtain such a simple fact. We give a simple proof of the smoothness of $h^{\varepsilon_1}_{\geqslant\ell}$ in the appendix **Back to Dufresne's integral**, in section **4.3**, p.96, based on an integration by parts formula.

b) Bounded $L^{h^{\varepsilon_1}}$-harmonic functions

The aim of this subsection is to show that bounded $L^{h^{\varepsilon_1}}$-harmonic functions depend only on y and ξ'^0. This goal is achieved in theorem **42**.

This property of bounded $L^{h^{\varepsilon_1}}$-harmonic functions will come from the following two results.

1. Let $u_0 = \big((Y_0, x), (Z_0, \xi'^{\geqslant 1})\big)$ and $\underline{u}_0 = \big((Y_0, \underline{x}), (Z_0, \underline{\xi}'^{\geqslant 1})\big)$ be two points in $\mathbb{H} \times \mathbb{R}^{1,d}$ with the same y and ξ'^0 coordinates.

   THEOREM **44 (Coupling theorem).** *Pick an $\varepsilon > 0$ We can couple two h^{ε_1}-processes, started from u_0 and $\underline{u}_0$, such that after the coupling time,*

   - $\dot{\underline{\xi}}_s = \dot{\xi}_s$ *and*
   - $\underline{\xi}_s = \xi_s + C_1\mathbf{g}'^1 + c_2\mathbf{g}'^2 + \cdots + c_d\mathbf{g}'^d$,

   where C_1 is a (random) constant and $c_2, ..., c_d$ (random) constants such that $|c_i| \leqslant \varepsilon$, $i = 2..d$.

2. Recall we can see the diffusion $\{u_s\}_{s\geqslant 0}$ on $\mathbb{H} \times \mathbb{R}^{1,d}$ as the projection of a diffusion $\big\{(g_s, \xi_s)\big\}_{s\geqslant 0}$ on the Poincaré group $\mathcal{G}$ of affine isometries of $\mathbb{R}^{1,d}$, with generator $\widetilde{L}$ described in equation (2.1.8).

   THEOREM **45 (Uniform continuity theorem).** *Bounded $\widetilde{L}^{h^{\varepsilon_1}}$-harmonic functions are right uniformly continuous.*

These two results will be used in the proof of theorem **42**.

***i)* The coupling** Fix $Y_0 \in \mathbb{R}^{>0}$, $Z_0 \in \mathbb{R}$ and $\varepsilon > 0$. Let $u_0 = \big((Y_0, x), (Z_0, \xi'^{\geqslant 1})\big)$ and $\underline{u}_0 = \big((Y_0, \underline{x}), (Z_0, \underline{\xi}'^{\geqslant 1})\big)$ be two points of $\big(\mathbb{R}^{>0} \times \mathbb{R}^{d-1}\big) \times \mathbb{R}^{1+d}$ qith the same y and ξ'^0 coordinate. We shall couple two trajectories started from u_0 and $\underline{u}_0$.

Consider the system

$$
\begin{aligned}
dy_s &= y_s dw^y_s + \frac{d}{2} y_s ds, & d\underline{y}_s &= \underline{y}_s dw^y_s + \frac{d}{2}\underline{y}_s ds,\\
dx_s &= y_s dw^x_s, & d\underline{x}_s &= \underline{y}_s d\underline{w}^x_s,\\
d\xi'^0_s &= \frac{ds}{y_s}, & d\underline{\xi}'^0_s &= \frac{ds}{\underline{y}_s},\\
d\xi'^1_s &= \frac{x_s^2 + y_s^2 + 1}{2y_s} ds, & d\underline{\xi}'^1_s &= \frac{\underline{x}_s^2 + \underline{y}_s^2 + 1}{2\underline{y}_s} ds,\\
d\xi'^{\geqslant 2}_s &= \frac{x_s}{y_s} ds, & d\underline{\xi}'^{\geqslant 2}_s &= \frac{\underline{x}_s}{\underline{y}_s} ds,
\end{aligned}
\tag{2.2.16}
$$

with initial conditions u_0 and $\underline{u}_0$, respectively, where w^y is a real Brownian motion, and $w^x, \underline{w}^x$ two $\mathbb{R}^{d-1}$ Brownian motions. The processes $w^y, w^x, \underline{w}^x$ and $\{u_s\}_{s\geqslant 0}$, $\{\underline{u}_s\}_{s\geqslant 0}$ are defined on some measurable space $(\Omega, \mathcal{F})$.

Remarks

1. Since u_0 and $\underline{u}_0$ have the same y-coordinate, and $\{y_s\}_{s\geqslant 0}$ and $\{\underline{y}_s\}_{s\geqslant 0}$ are driven by the same Brownian motion w^y, we have $\underline{y}_s = y_s$, for all $s \geqslant 0$.
2. As $\underline{\xi}'^0_0 = \xi'^0_0$, we also have $\underline{\xi}'^0_s = \xi'^0_s$, for all $s \geqslant 0$.

Define on $(\Omega, \mathcal{F})$ the filtration $\{\mathfrak{F}_s\}_{s\geqslant 0}$ generated by $\{u_s\}_{s\geqslant 0}$ and $\{\underline{u}_s\}_{s\geqslant 0}$: $\mathfrak{F}_s = \sigma\big((u_r, \underline{u}_r);\, r \leqslant s\big)$.

Given w^y and w^x, independent, we construct a Brownian motion $\underline{w}^x$ and an $\{\mathfrak{F}_s\}_{s\geqslant 0}$-stopping time T such that if one notes $\mathbf{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$ the law of the solution of the system (2.2.16), one has $\mathbf{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$-almost surely

- T is finite,
- for $s \geqslant T$, $\dot{\underline{\xi}}_s = \dot{\xi}_s$, and $\underline{\xi}_s = \xi_s + C_1 \mathbf{g}'^1 + c_2 \mathbf{g}'^2 + \cdots + c_d \mathbf{g}'^d$, where C_1 is a (random) constant and $c_2, ..., c_d$ (random) constants such that $|c_i| \leqslant \varepsilon$, $i = 2..d$.

i.a) To make things clearer, we first consider an analogous of system (2.2.16), where x is one dimensional, as well as $\xi'^{\geqslant 2}$. We note ξ'^2 instead of $\xi'^{\geqslant 2}$. The solutions $\{u_s\}_{s\geqslant 0}$, $\{\underline{u}_s\}_{s\geqslant 0}$ of the modified system live in $\big(\mathbb{R}^*_+ \times \mathbb{R}\big) \times (\mathbb{R} \times \mathbb{R}^2)$.

Note $\mathbb{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$ the law of the couple $\big\{(u_s, \underline{u}_s)\big\}_{s\geqslant 0}$ solution of the system (2.2.16), when $\underline{w}^x$ is independent of w^x and w^y.

Set

$$T = \inf\big\{s \geqslant 0\,;\, \underline{x}_s = x_s\ ,\ |\underline{\xi}'^2_s - \xi'^2_s| \leqslant \varepsilon\big\},$$

with the convention that $\inf \emptyset = +\infty$. The random time T is an $\{\mathfrak{F}_s\}_{s\geqslant 0}$-stopping time.

THEOREM **46.** *T is $\mathbb{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$-almost surely finite.*

Taking this theorem for granted, set, for $s \geqslant T$, $\underline{w}_s = w_s$. Then, the two stochastic differential equations of (2.2.16) are identical for $s \geqslant T$, and $\underline{x}_s = x_s$, and $\underline{\xi}'^2_s - \xi'^2_s$ is constant, with absolute value smaller than ε, for $s \geqslant T$.

Proof of theorem 46

Note ξ'^2 instead of $\xi'^{\geqslant 2}$. Write $z_s \equiv \frac{\underline{x}_s - x_s}{y_s}$. The objective is to show that the $\mathbb{R}^2$-valued process $\big\{(z_s, \underline{\xi}'^2 - \xi'^2 + \int_0^s z_u\, du)\big\}_{s\geqslant 0}$ $\mathbb{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$-almost surely reaches the set $\big\{(z, Z) \in \mathbb{R}^2\,;\, z = 0\,,\, |Z| \leqslant \varepsilon\big\}$ in a finite time.

LEMMA . *$\{z_s\}_{s\geqslant 0}$ is a diffusion, which is recurrent positive.*

$\lhd$ Use formulas (2.2.16) and Itô's formula to get

$$dz_s = (d\underline{w}^x_s - dw^x_s) - z_s dw^y_s - \frac{d}{2} z_s\, ds$$

The differential generator of $\{z_s\}_{s\geqslant 0}$ is

$$f \mapsto \frac{2 + z^2}{2} f'' - \frac{d}{2} z f'.$$

It has a unique invariant probability, which has a density with respect to Lebesgue's measure on $\mathbb{R}$, proportional to $(2 + z^2)^{-\frac{d}{2}-1}$([9]). $\rhd$

[9] We know from differential equations arguments that an invariant probability has a smooth density with respect to Lebesgue's measure dz on $\mathbb{R}$. Note it $m(z)$. It must satisfy the relation

$$\left(\frac{2 + z^2}{2} m(z)\right)'' = -\frac{d}{2}(z m(z))'.$$

If we note $u(z) = \frac{2+z^2}{2} m(z)$, u satisfies the equation $u''(z) = -\frac{d}{2}(\frac{2}{2+z^2} u(z))'$, *i.e.*, up to an additive constant $u'(z) = -\frac{d}{2}\frac{2}{2+z^2} u(z)$. So, $u(z) = (2 + z^2)^{-\frac{d}{2}}$, and $m(z)$ is proportional to $(2 + z^2)^{-\frac{d}{2}-1}$.

So we can look at its successive excursions outside $\{0\}$, of heigth $\geqslant 1$. Write

$$S_0 \equiv \inf\{s \geqslant 0;\ z_s = 0\}$$

and

$$S_n = \inf\{s \geqslant S_{n-1}\ ;\ \sup_{S_{n-1}\leqslant u\leqslant s} |z_u| \geqslant 1 \text{ and } z_s = 0\}.$$

Because of the Strong Markov Property, the excursions $\{z_s\}_{S_{n-1}\leqslant s\leqslant S_n}$, $n \geqslant 1$, are idependant and identical in law. So are the integrals $\{\int_{S_{n-1}}^{S_n} z_u\,du\}_{n\geqslant 1}$. So, using the following well known criterium[10] we are brought back to show that the random variable $\int_{S_0}^{S_1} z_u\,du$ is integrable and that the support of its law is non lattice.

Proposition. *Let μ be a probability on $\mathbb{R}$ having a moment of order 1. Let $\{X_n\}_{n\geqslant 1}$ be a sequence of independent random variables, of law μ. Then the random walk $\left\{\sum_{i=1}^{n} X_k\right\}_{n\geqslant 1}$ is recurrent in the closed subgroup generated by the support of μ if, and only if, $\int x\mu(dx) = 0$.*

We first show that the integrability condition is fulfilled.

Lemma 47. $\mathbb{E}^{\varepsilon_1}_{u_0,\underline{u}_0}\big[|\int_{S_0}^{S_1} z_u\,du|\big] < +\infty$.

$\lhd$ To evaluate this mean, we cut the integral in two parts: the first corresponding to the integral of z between S_0 and the first time H when $\{z_s\}_{s\geqslant 0}$ hits ± 1, and the second part $\int_H^{S_1} z_u\,du$. We show that

$$\mathbb{E}^{\varepsilon_1}_{u_0,\underline{u}_0}\Big[\Big|\int_{S_0}^{H} z_u\,du\Big|\Big] < +\infty, \text{ and } \mathbb{E}^{\varepsilon_1}_{u_0,\underline{u}_0}\Big[\Big|\int_{H}^{S_1} z_u\,du\Big|\Big] < +\infty.$$

The first one is easy to handle. For any $x \in \mathbb{R}$, note $\mathbb{P}_x$ the law of $\{z_s\}_{s\geqslant 0}$ started from x. Since $|z_s| \leqslant 1$ on $[S_0, H]$,

$$\mathbb{E}^{\varepsilon_1}_{u_0,\underline{u}_0}\Big[\Big|\int_{S_0}^{H} z_u\,du\Big|\Big] \leqslant \mathbb{E}_0[H] < +\infty(^{11}).$$

The second integral $\int_H^{S_1} z_u\,du$ is handled as follows.
Note $\tau_0 \equiv \inf\{s \geqslant 0\ ;\ z_s = 0\}$ the hitting time of $\{0\}$ by $\{z_s\}_{s\geqslant 0}$, and $g^0(x,y)$ the Green function of $\{z_s\}_{s\geqslant 0}$ *killed at time* τ_0.

$$\mathbb{E}^{\varepsilon_1}_{u_0,\underline{u}_0}\Big[\int_{H}^{S_1} z_u\,du\Big] = \mathbb{E}_1\Big[\int_0^{\tau} z_u\,du\Big] = \int_0^{+\infty} g^0(1,y)y\,dy.$$

We can find an explicit formula of the Green function $g^0(x,y)$.
Let φ be a smooth function on $\mathbb{R}^{>0}$, with compact support. Let $b > 0$ such that $\operatorname{supp}\varphi \subset (0,b)$. Note $\tau_{0,b} = \inf\{s > 0;\ z_s \in \{0,b\}\}$. Consider first the system

$$\begin{aligned} &\frac{2+z^2}{2} f''(z) - \frac{d}{2} z f'(z) = -\varphi(z), \\ &f(0^+) = 0,\ f(b^-) = 0, \end{aligned} \tag{2.2.17}$$

with unique solution the function $\int_0^\infty g^{0,b}(x,y)\varphi(y)\,dy$, where $g^{0,b}(x,y)$ is the Green function of $\{z_s\}$ *killed at time* $\tau_{0,b}$; then we let $b \to +\infty$. The dominated convergence theorem justifies the equalities $\stackrel{\bullet}{=}$.

[10] See [Chu01], Theorem 8.3.4, p.251, for instance.
[11] The diffusion $\{z_s\}_{s\leqslant H}$ is uniformly elliptic and has bounded drift. See [Bre92],Theorem 16.24, p.359.

$$\int_0^{+\infty} g^0(x,y)\varphi(y)\,dy = \mathbb{E}_x\Big[\int_0^{\tau_0} \varphi(z_u)\,du\Big] \overset{\bullet}{=} \lim_{b\to+\infty} \mathbb{E}_x\Big[\int_0^{\tau_{0,b}} \varphi(z_u)\,du\Big]$$
$$\overset{\bullet}{=} \lim_{b\to+\infty} \int_0^{+\infty} g^{0,b}(x,y)\varphi(y)\,dy = \int_0^{+\infty} \lim_{b\to+\infty} g^{0,b}(x,y)\,\varphi(y)\,dy.$$

System (2.2.17) can be solved explicitly, as it is a first order equation in f'. The set of solutions of the equation

$$\frac{2+z^2}{2}u'(z) - \frac{d}{2}zu(z) = -\varphi(z)$$

is found by the method of variation of the parameters. Any solution is of the form

$$u(z) = \lambda(2+z^2)^{\frac{d}{2}} + 2\,(2+z^2)^{\frac{d}{2}} \int_0^z \frac{-\varphi(r)\,dr}{(2+r^2)^{\frac{d+1}{2}}}, \tag{2.2.18}$$

where λ is a real constant.

Using the boundary conditions in (2.2.17), we find

$$\int_0^{+\infty} g^{0,b}(x,y)\varphi(y)\,dy = 2\frac{\int_0^b (2+u^2)^{\frac{d}{2}} \int_0^u \frac{\varphi(r)\,dr}{(2+r^2)^{\frac{d+1}{2}}}\,du}{\int_0^b (2+u^2)^{\frac{d}{2}}\,du} \int_0^x (2+u^2)^{\frac{d}{2}}\,du$$
$$- 2\int_0^x (2+u^2)^{\frac{d}{2}} \int_0^u \frac{\varphi(r)\,dr}{(2+r^2)^{\frac{d+1}{2}}}\,du. \tag{2.2.19}$$

We have

$$\lim_{b\to+\infty} \frac{\int_0^b (2+u^2)^{\frac{d}{2}} \int_0^u (2+r^2)^{-\frac{d+1}{2}}\varphi(r)\,dr\,du}{\int_0^b (2+u^2)^{\frac{d}{2}}\,du} = \int_0^{+\infty} (2+r^2)^{-\frac{d+1}{2}}\varphi(r)\,dr.$$

We deduce from equation (2.2.19) that $g^0(x,y) = 2(2+y^2)^{-\frac{d+1}{2}} \int_0^{x\wedge y}(2+u^2)^{\frac{d}{2}}\,du$. We see on this formula that

$$\mathbb{E}^{\varepsilon_1}_{u_0,\underline{u}_0}\Big[\int_H^{S_1} z_u\,du\Big] = \int_0^{+\infty} g^0(1,y)y\,dy < \infty,$$

as soon as $d \geqslant 2$ ([12]). ▷

As noted, it remains to prove that the closed group generated by the support of the law of $\int_{S_0}^{S_1} z_u\,du$ is non lattice to end the proof of the proposition **46** when $d=2$.

LEMMA **48.** *The closed group generated by the support of the law of $\int_{S_0}^{S_1} z_u\,du$ is non lattice.*

◁ We can suppose $S_0 = 0$.

Let $a > 0$. Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous function such that

- $\varphi(0) = 0$, $\varphi(1) = -\varepsilon < 0$,
- $\varphi(t) = 0$ only for $t = 0$ and some t_0 near 1,
- $\max_{s\in[0,1]} |\varphi(s)| \geqslant 1$,
- $\int_0^1 \varphi(r)\,dr \notin a\mathbb{Z}$.

[12] See the appendix **Coupling of a hypoelliptic diffusion in** $\mathbb{R}^2$, **4.4**, p.100 , for a generalisation of these calculations.

The support theorem **25** asserts that, for any $\eta > 0$, the event

$$\left\{ \sup_{r\in[0,1]} |z_r - \varphi(r)| \leqslant \eta \right\}$$

has positive probability. If one chooses $\eta > 0$ small enough,

- S_1 is close to t_0,
- $\max_{r\in[0,S_1]} |z_r| \geqslant 1$,
- $\int_0^{S_1} z_r\, dr$ is near $\int_0^1 \varphi(r)\, dr$.

Reducing $\eta > 0$ if necessary, we have $\left|\int_0^{S_1} z_r\, dr - \int_0^1 \varphi(r)\, dr\right| < \operatorname{dist}\left(\int_0^1 \varphi(r)\, dr,\, a\mathbb{Z}\right)$, and $\int_0^{S_1} z_r\, dr \notin a\mathbb{Z}$. So, $\int_0^{S_1} z_r\, dr$ cannot belong with probability 1 to some subgroup $a\mathbb{Z}$. ▷

***i.b)* Coupling in dimension $\geqslant 3$** – Note $x^1, ..., x^{d-1}$ the coordinates of $x \in \mathbb{R}^{d-1}$. Recall $\mathbb{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$ is the law of the diffusion solving system (2.2.16), where w^y, w^x, $\underline{w}^x$ are independent. We note $(w^x)^1$, ..., $(w^x)^{d-1}$ the $(d-1)$ coordinates of w^x (resp. $(\underline{w}^x)^i$ for $\underline{w}^x$).

First, use the coupling theorem **46** to find a $\mathbb{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$-almost surely finite $\{\mathfrak{F}_s\}_{s\geqslant 0}$-stopping time T_1 such that

$$\underline{x}^1_{T_1} = x^1_{T_1}, \text{ and } |\underline{\xi}'^{\,2}_{T_1} - \xi'^{\,2}_{T_1}| = \varepsilon.$$

From that time on, set $(\underline{w}^x)^1 = (w^x)^1$, $(\underline{w}^x)^{\geqslant 2}$ remaining independent of $(w^x)^{\geqslant 2}$ and w^y. For $s \geqslant T_1$, $\underline{x}^1_s = x^1_s$ and $\underline{\xi}'^{\,2}_s - \xi'^{\,2}_s$ is constant, with an absolute value less than or equal to ε.

We have $\underline{y}_{T_1} = y_{T_1}$. Then, using the strong Markov property, look at the processes $\left\{(\underline{y}_s, \underline{x}^2_s, \underline{\xi}^3_s)\right\}_{s\geqslant T_1}$ and $\left\{(y_s, x^2_s, \xi^3_s)\right\}_{s\geqslant T_1}$. We can use the coupling theorem **46** to couple the second coordinate $\underline{x}^2_s$ of $\underline{x}_s$ with x^2_s.

After that time T_2, set $(\underline{w}^x)^2 = (w^x)^2$. Then, for $s \geqslant T_2$, we have

- $\underline{x}^1_s = x^1_s$, and $\underline{x}^2_s = x^2_s$,
- $\underline{\xi}'^{\,3}_s - \xi'^{\,3}_s$ is constant, with an absolute value less than or equal to ε.

Repeat this procedure to couple each coordinate of $\underline{x}$ to the corresponding coordinates of x, using theorem **46**. After the coupling time T_{d-1}, $\underline{x}_s = x_s$ and for each $j = 2..d$, $\underline{\xi}'^j_s - \xi'^j_s$ is constant, with an absolute value less than or equal to ε. As $\underline{y}_s$ remained equal to y_s during the construction of the coupling, $\dot{\underline{\xi}}_s = \dot{\xi}_s$ from the time T_{d-1} on, and $\underline{\xi}'^1_s - \xi'^1_s$ remains constant.

Thus, we have shown the following theorem.

Theorem **49 (Coupling theorem).** *Let $\varepsilon > 0$ and $u_0 = (\dot{\xi}_0, \xi_0)$ and $\underline{u}_0 = (\dot{\underline{\xi}}_0, \underline{\xi}_0)$ be two points of $\mathbb{H} \times \mathbb{R}^{1,d}$ such that $\dot{\xi}_0$ and $\dot{\underline{\xi}}_0$ have the same y-coordinate, and $\xi'^0_0 = \underline{\xi}'^0_0$. We can find*

- *a probability filtered space $(\Omega, \mathcal{F}, \{\mathfrak{F}_s\}_{s\geqslant 0}, \mathbb{P}^{\varepsilon_1}_{u_0,\underline{u}_0})$,*
- *paths space adapted random variables $\{u_s\}_{s\geqslant 0}$ and $\{\underline{u}_s\}_{s\geqslant 0}$, on $(\Omega, \mathcal{F}, \{\mathfrak{F}_s\}_{s\geqslant 0})$,*
- *an $\{\mathfrak{F}_s\}_{s\geqslant 0}$-stopping time T,*

such that

1. *the law of* $\{u_s\}_{s\geqslant 0}$ *under* $\mathbb{P}^{\varepsilon_1}_{u_0,\underline{u}_0}$ *is* $\mathbb{P}^{\varepsilon_1}_{u_0}$, *and that of* $\{\underline{u}_s\}_{s\geqslant 0}$ *is* $\mathbb{P}^{\varepsilon_1}_{\underline{u}_0}$,
2. *after the coupling time* T, $\dot{\underline{\xi}}_s = \dot{\xi}_s$ *and* $\underline{\xi}_s = \xi_s + C_1\mathbf{g}'_1 + c_2\mathbf{g}'^2 + \cdots + c_d\mathbf{g}'^d$,

where C_1 *is a (random) constant and* $c_2, ..., c_d$ *(random) constants such that* $|c_i| \leqslant \varepsilon$, *for every* $i = 2..d$.

We need two ingredients in the receipe of the proof of theorem **55**. The first one is the preceding theorem, the second one is a uniform continuity property of $L^{h^{\varepsilon_1}}$-harmonic bounded functions.

To get this property, we go up to the group $\mathcal{G}$, where the diffusion $\big\{(g_s,\xi_s)\big\}_{s\geqslant 0}$ defined in **2.1.2** lives. Its generator was noted $\widetilde{L}$. A function f defined on $\mathbb{H}\times\mathbb{R}^{1,d}$ naturally extends to $\mathcal{G}$ by setting $f\big((g,\xi)\big) = f(g\varepsilon_0,\xi)$. For smooth functions f on $\mathcal{G}$, set

$$\widetilde{L}^{h^{\varepsilon_1}} f = \frac{\widetilde{L}(h^{\varepsilon_1} f)}{h^{\varepsilon_1}}.$$

We establish in this paragraph a uniform continuity property for $\widetilde{L}^{h^{\varepsilon_1}}$-harmonic bounded functions. Precisely, being on a non-commutative group, we must distinguish right and left uniform continuity. We show in theorem **53** that any bounded $\widetilde{L}^{h^{\varepsilon_1}}$-harmonic function h is right uniformly continuous: given $\varepsilon > 0$, there is a neighbourhood $\mathcal{V}_\varepsilon$ of Id in $\mathcal{G}$ such that

$$\forall\, \mathbf{e}\in\mathcal{G}, \forall\, \mathbf{e}'\in\mathcal{V}_\varepsilon, \quad |h(\mathbf{e}\mathbf{e}') - h(\mathbf{e})| \leqslant \varepsilon.$$

***ii)* Uniform continuity** **Notation** – We shall write $\mathbf{e} = (g,\xi)\in\mathcal{G}$. Note $\big\{\widetilde{P}^{\varepsilon_1}_s(\mathbf{e}, d\mathbf{e}')\big\}_{s\geqslant 0}$ the transition kernels of the $\widetilde{L}^{h^{\varepsilon_1}}$-diffusion.

Recall that we defined in **2.1.1** the action of an element $\mathbf{e} = (g,\xi)$ of $\mathcal{G}$ on $\mathbb{S}^{d-1}$:

$$(g,\xi).\sigma = \sigma' \text{ if } g\big(\mathbb{R}(\varepsilon_0+\sigma)\big) = \mathbb{R}(\varepsilon_0+\sigma').$$

LEMMA **50.** *Each element* $\mathbf{e}\in\mathcal{G}$ *can be written* $\mathbf{e} = \underline{\mathbf{e}}\widehat{\mathbf{e}}$, *with* $\underline{\mathbf{e}}$ *fixing* $\varepsilon_1\in\mathbb{S}^{d-1}$, *and* $\widehat{\mathbf{e}}$ *in a* compact *subset* $\widehat{\mathcal{G}}$ *of* $\mathcal{G}$.

◁ Let $\mathbf{e} = (g,\xi)\in\mathcal{G}$. Writting $\mathbf{e} = (\mathrm{Id},\xi)(g,0)$, we can drop (Id,ξ) apart. Use the halfspace model of $\mathbb{H}$ to describe the set of isometries of $\mathbb{H}$. It is well known that

- any isometry of $\mathbb{H}$ can be *uniquely* written as a product

$$g = t\,\lambda\,r,$$

  where t is a translation in a direction of the form $(0,\tau)\in\mathbb{R}^{>0}\times\mathbb{R}^{d-1}$, λ is the isometry $(y,x)\mapsto(\lambda y,\lambda x)$ and r is a hyperbolic rotation with center $(1,0)$,
- the set of hyperbolic rotations with center $(1,0)$ is isomorphic to the *compact* group $SO_d(\mathbb{R})$.

Each transform t,λ leaves ε_1 fixed, not r. Take $\underline{\mathbf{e}} = (t\lambda,\xi)$ and $\widehat{\mathbf{e}} = (r,0)$. ▷

Notation – For $\mathbf{e}\in\mathcal{G}$, $L_{\mathbf{e}}$ denotes the left translation on $\mathcal{G}$: $L_{\mathbf{e}}(\mathbf{e}') = \mathbf{e}\mathbf{e}'$.

LEMMA **51.** *The kernel* $\widetilde{P}^{\varepsilon_1}_s$ *is left invariant by any* $\underline{\mathbf{e}}$ *in* $\mathcal{G}$ *that fixes* ε_1: *for any such* $\underline{\mathbf{e}}$ *and any compactly supported smooth function* f,

$$\widetilde{P}^{\varepsilon_1}_t(f)\circ L_{\underline{\mathbf{e}}} = \widetilde{P}^{\varepsilon_1}_t(f\circ L_{\underline{\mathbf{e}}}).$$

◁ Write $\underline{\mathbf{e}} = (t\lambda,\xi)$, with t,λ as in the preceding lemma. Recall h^{ε_1} is a multiple of y^{d-1}. For any $\dot{\xi}\in\mathbb{H}$,

$$h^{\varepsilon_1}\circ L_{\underline{\mathbf{e}}}(\dot{\xi}) = h^{\varepsilon_1}\big(\mathbf{e}(\dot{\xi})\big) = h^{\varepsilon_1}\big(t(\lambda(\dot{\xi}))\big) = \lambda^{d-1}h^{\varepsilon_1}(\dot{\xi}). \tag{2.2.20}$$

The identity of the lemma is the integrated version of the identity

$$\left(\widetilde{L}^{h^{\varepsilon_1}} f\right) \circ L_{\underline{\mathbf{e}}} = \frac{\widetilde{L}(h^{\varepsilon_1} f)}{h^{\varepsilon_1}} \circ \widetilde{L}_{\underline{\mathbf{e}}} = \frac{\widetilde{L}\big((h^{\varepsilon_1} f) \circ \widetilde{L}_{\underline{\mathbf{e}}}\big)}{h^{\varepsilon} \circ \widetilde{L}_{\underline{\mathbf{e}}}} = \frac{\widetilde{L}\big(h^{\varepsilon_1} f \circ \widetilde{L}_{\underline{\mathbf{e}}}\big)}{h^{\varepsilon}} = \widetilde{L}^{h^{\varepsilon_1}}(f \circ L_{\underline{\mathbf{e}}}). \tag{2.2.21}$$

The first equality traduces the left invariance of L by *any* left translation, the second comes from identity (2.2.20). Here, f is any smooth function on $\mathcal{G}$ with compact support. We get from (2.2.21)

$$\widetilde{P}_t^{\varepsilon_1}(f) \circ L_{\underline{\mathbf{e}}} = \left(e^{-t\widetilde{L}^{h^{\varepsilon_1}}} f\right) \circ L_{\underline{\mathbf{e}}} = e^{-t\widetilde{L}^{h^{\varepsilon_1}}}\left(f \circ L_{\underline{\mathbf{e}}}\right) = \widetilde{P}_t^{\varepsilon_1}(f \circ L_{\underline{\mathbf{e}}}).$$

▷

Notation – Note $\widetilde{p}_1^{\varepsilon_1}(\mathbf{e}, \mathbf{e}')$ the density of the measure $\widetilde{P}_1^{\varepsilon_1}(\mathbf{e}, d\mathbf{e}')$ with respect to a Haar measure $Haar(d\mathbf{e}')$ on $\mathcal{G}$. It is a continuous function of $(\mathbf{e}, \mathbf{e}')$(13).

Because any bounded $\widetilde{L}^{h^{\varepsilon_1}}$-harmonic function satisfies

$$h(\mathbf{e}) = \int \widetilde{p}_1^{\varepsilon_1}(\mathbf{e}, \mathbf{e}')\, h(\mathbf{e}') Haar(d\mathbf{e}'),$$

we have for any $\widetilde{\mathbf{e}} \in \mathcal{G}$,

$$\big|h(\mathbf{e}) - h(\mathbf{e}\widetilde{\mathbf{e}})\big| \leqslant \|h\|_\infty \int_{\mathcal{G}} \big|\widetilde{p}_1^{\varepsilon_1}(\mathbf{e}, \mathbf{e}') - \widetilde{p}_1^{\varepsilon_1}(\mathbf{e}\widetilde{\mathbf{e}}, \mathbf{e}')\big| Haar(d\mathbf{e}'). \tag{2.2.22}$$

Let $\mathbf{e} = \underline{\mathbf{e}}\,\widehat{\mathbf{e}}$ be the decomposition of $\mathbf{e}$ given by lemma **50**, with $\underline{\mathbf{e}}$ and $\underline{\mathbf{e}}^{-1}$ fixing ε_1. Lemma **51** justifies the first equality

$$\begin{aligned}\int_{\mathcal{G}} \big|\widetilde{p}_1^{\varepsilon_1}(\mathbf{e}, \mathbf{e}') - \widetilde{p}_1^{\varepsilon_1}(\mathbf{e}\widetilde{\mathbf{e}}, \mathbf{e}')\big| Haar(d\mathbf{e}') &= \int_{\mathcal{G}} \big|\widetilde{p}_1^{\varepsilon_1}(\underline{\mathbf{e}}^{-1}\mathbf{e}, \underline{\mathbf{e}}^{-1}\mathbf{e}') - \widetilde{p}_1^{\varepsilon_1}(\underline{\mathbf{e}}^{-1}\mathbf{e}\,\widetilde{\mathbf{e}}, \underline{\mathbf{e}}^{-1}\mathbf{e}')\big| Haar(d\mathbf{e}') \\ &= \int_{\mathcal{G}} \big|\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})\big| Haar(d\mathbf{a}).\end{aligned} \tag{2.2.23}$$

LEMMA **52.** *The function* $\widetilde{\mathbf{e}} \in \mathcal{G} \mapsto \sup_{\widehat{\mathbf{e}} \in \widehat{\mathcal{G}}} \int_{\mathcal{G}} \big|\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})\big| Haar(d\mathbf{a})$ *converges to* 0 *as* $\widetilde{\mathbf{e}} \to \mathrm{Id}$.

◁ Let $\mathcal{U}$ be a compact neighbourhood of $\mathrm{Id} \in \mathcal{G}$, and $\varepsilon > 0$. The family of probabilities $\big\{\widetilde{P}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, .)\big\}_{\widehat{\mathbf{e}} \in \widehat{\mathcal{G}},\, \widetilde{\mathbf{e}} \in \mathcal{U}}$ is tight. Let $\underline{\mathcal{G}}$ be a compact subset of $\mathcal{G}$ such that for any $\widehat{\mathbf{e}} \in \widehat{\mathcal{G}}$, $\widetilde{\mathbf{e}} \in \mathcal{U}$,

$$\widetilde{P}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \underline{\mathcal{G}}) \geqslant 1 - \varepsilon.$$

Then,

$$\int_{\mathcal{G}} |\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})| Haar(d\mathbf{a}) \leqslant 2\,\varepsilon + \int_{\underline{\mathcal{G}}} |\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})| Haar(d\mathbf{a}) \tag{2.2.24}$$

As $\widehat{\mathbf{e}}$, $\widetilde{\mathbf{e}}$ and $\mathbf{a}$ are in compact subsets of $\mathcal{G}$ and the function $\widetilde{p}_1^{\varepsilon_1}(.,.)$ is continuous, $\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}, \mathbf{a})$ and $\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\widetilde{\mathbf{e}}, \mathbf{a})$ are bounded by a constant when $\widehat{\mathbf{e}} \in \widehat{\mathcal{G}}$, $\widetilde{\mathbf{e}} \in \mathcal{U}$ and $\mathbf{a} \in \underline{\mathcal{G}}$. So, the function

$$(\widehat{\mathbf{e}}, \widetilde{\mathbf{e}}) \mapsto \int_{\underline{\mathcal{G}}} |\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})| Haar(d\mathbf{a})$$

13 See section **2.2.1**.

is uniformly continuous. Since it is null on $\widehat{\mathcal{G}} \times \{0\}$, we get from (2.2.24)

$$\overline{\lim_{\widetilde{\mathbf{e}}\to \mathrm{Id}}} \int_{\mathcal{G}} |\widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}, \mathbf{a}) - \widetilde{p}_1^{\varepsilon_1}(\widehat{\mathbf{e}}\,\widetilde{\mathbf{e}}, \mathbf{a})| Haar(d\mathbf{a}) \leqslant 2\,\varepsilon.$$

As $\varepsilon > 0$ was chosen arbitrarily, the statement of the lemma follows. ▷

The inequality (2.2.22), together with lemma **52**, show that

THEOREM **53 (Uniform continuity theorem).** *Any bounded $\widetilde{L}^{h^{\varepsilon_1}}$-harmonic function is right uniformly continuous.*

The notation $\underline{\mathbf{e}}$ will no longer refer to the decomposition of lemma **50**.

c) Bounded $L^{h^{\varepsilon_1}}$-harmonic functions The usefulness of theorem **53** appears in the following lemma, which roughly says that a bounded $L^{h^{\varepsilon_1}}$-harmonic function $h(\dot{\zeta}, \zeta)$ does not vary much if $\dot{\zeta}$ remains near $\dot{\xi}$ and ζ moves in the pictured "ellipsoid".

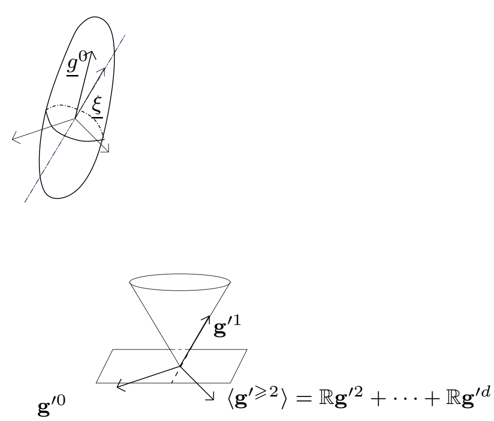


Figure 2.1: Trace on $\mathbb{R}^{1,d}$ of a neigbourhood $\mathbf{e}\mathcal{V}$ of $\mathbf{e} = (g, \xi) \in \mathcal{G}$

LEMMA **54.** *Let $\mathcal{V}$ be a neighbourhood of $Id \in \mathcal{G}$.*

1. *Let $\mathbf{e} = (g, \xi) \in \mathcal{G}$. Identify ξ to 0 on the straight line $\xi + \mathbb{R}\mathbf{g}'^1$. The intersection of $\xi + \mathbb{R}\mathbf{g}'^1$ with $\mathbf{e}\mathcal{V}$ contains an interval $(a(y), b(y))$ with the property that $a(y) \to -\infty$ and $b(y) \to +\infty$ as $y \to +\infty$.*
2. *For $\xi \in \mathbb{R}^{1,d}$, identify ξ to 0 in $\xi + \langle \mathbf{g}'^{\geqslant 2}\rangle$. There exists a constant $\varepsilon > 0$ depending on $\mathcal{V}$ only, such that for any $\mathbf{e} = (g, \xi) \in \mathcal{G}$, the intersection of $\xi + \langle \mathbf{g}'^{\geqslant 2}\rangle$ with $\mathbf{e}\mathcal{V}$ contains the Euclidian ball $B(0, \varepsilon)$.*

◁ **1)** For any two $\mathbf{e} = (g, \xi)$ and $\underline{\mathbf{e}} = (\underline{g}, \underline{\xi})$ in $\mathcal{G}$, $\mathbf{e}\underline{\mathbf{e}} = (g\underline{g}, \xi + g\underline{\xi})$.
$\mathcal{V}$ contains a product neighbourhood $\mathcal{V}_1 \times]-\varepsilon, \varepsilon[^{1+d} \subset SO_0(1,d) \times \mathbb{R}^{1,d}$, for some ε. We need to examine the set $g\,.\,]-\varepsilon, \varepsilon[^{1+d}$.

$$g\,\underline{\xi} = \underline{\xi}^0 g_0 + \underline{\xi}^1 g_1 + \ldots + \underline{\xi}^d g_d.$$

In the basis g', g^0 has coordinates $\left(\frac{1}{y}, \frac{|x|^2+y^2+1}{2y}, \frac{x}{y}\right)$. The set $g\,]-\varepsilon, \varepsilon[^{1+d} \cap \mathbb{R}g'^1$ contains the segment $\frac{|x|^2+y^2+1}{2y}\,]-\varepsilon, \varepsilon[$. We can take $a(y) = -\varepsilon\frac{y^2+1}{2y}$ and $b(y) = \varepsilon\frac{y^2+1}{2y}$.
2) Let V be a unit tangent vector to the halfspace hyperbolic space $\mathbb{R}^{>0} \times \mathbb{R}^{d-1}$, located at point (y, x). It has the form $y(u, v) \in \mathbb{R} \times \mathbb{R}^{d-1}$, where $u^2 + |v|_{\mathrm{Eucl}} = 1$. Note (y, x) the halfspace coordinates of g^0.

Since $\psi : \mathbb{R}^{>0} \times \mathbb{R}^{d-1} \to \mathbb{H}$ is an isometry, each $g^i, i = 1..d$, is of the form $D_{(y,x)}\psi(y(u,v))$ for some (u,v). So,

$$\left\{\underline{\xi}^1 g^1 + ... + \underline{\xi}^d g^d \; ; \; \underline{\xi} \in]-\varepsilon, \varepsilon[^d\right\} = D_{(y,x)}\psi\big(yB_{\mathbb{R}^d}(0,\varepsilon)\big),$$

where $B_{\mathbb{R}^d}(0,\varepsilon)$ is the open Euclidian ball of $\mathbb{R}^d$ with center 0 and radius ε. In particular, for $v \in \mathbb{R}^{d-1}$, $|v| \leqslant \varepsilon$, the vector $D_{(y,x)}\psi\big(y(0,v)\big)$ belongs to $\left\{\underline{\xi}^1 g^1 + ... + \underline{\xi}^d g^d \; ; \; \underline{\xi} \in]-\varepsilon, \varepsilon[^d\right\}$. In $\mathbf{g}'$-coordinates

$$D_{(y,x)}\psi\big(y(u,v)\big) = \left(\langle x, v\rangle + \left(1 - \frac{|x|^2 + y^2 + 1}{2y^2}\right) yu, \langle x, v\rangle + \left(1 - \frac{|x|^2 + y^2 - 1}{2y^2}\right) yu, v - u\frac{x}{y}\right),$$

so $D_{(y,x)}\psi\big(y(0,v)\big)$ has $\mathbf{g}'$-coordinates $(\langle x, v\rangle, \langle xv\rangle, v)$, which shows the second point of the lemma. ▷

The coupling of theorem 49, together with the uniform continuity theorem 53, are used with lemma 54 to prove the following result on $L^{h^{\varepsilon_1} h^{\varepsilon_1}_\ell}$-bounded harmonic functions.

Theorem 55. *Any $L^{h^{\varepsilon_1}}$-harmonic bounded function only depends on y and ξ'^0.*

◁ Let h be a bounded $L^{h^{\epsilon_1}}$-harmonic function, considered as a $\widetilde{L}^{h^{\epsilon_1}}$-harmonic function. Let $\eta > 0$ be given. Because of the right uniform continuity of h, there exists a neighbourhood $\mathcal{V}$ of $\mathrm{Id} \in \mathcal{G}$ such that for any $\widetilde{\mathbf{e}} \in \mathcal{V}$,

$$\forall\, \mathbf{e} \in \mathcal{G}, \; \big|h(\mathbf{e}) - h(\mathbf{e}\widetilde{\mathbf{e}})\big| \leqslant \eta.$$

Take two points of $\mathcal{G}$, $\mathbf{e}_0 = \big((g_0^0, ..., g_0^3), \xi_0\big)$ and $\underline{\mathbf{e}}_0 = \big((\underline{g}_0^0, ..., \underline{g}_0^3), \underline{\xi}_0\big)$, with g_0^0 and $\underline{g}_0^0$ $(\in \mathbb{H})$ having the same y-coordinate, and $\underline{\xi}_0'^0 = \xi_0'^0$.
Using the coupling time constructed in theorem **49**, and the stopping time theorem, we can write, for any $s \geqslant 0$,

$$h(\mathbf{e}_0) - h(\underline{\mathbf{e}}_0) = \mathbb{E}^{\epsilon_1}_{\mathbf{e}_0, \underline{\mathbf{e}}_0}\big[h(\mathbf{e}_{T+s}) - h(\underline{\mathbf{e}}_{T+s})\big].$$

The stopping time T was constructed to ensure that

$$\mathbf{e}_{T+s} = \Big(\big((y_{T+s}, x_{T+s}), g^1_{T+s}, g^2_{T+s}, g^3_{T+s}\big), \big(\xi'^0_{T+s}, \xi'^1_{T+s}, \xi'^2_{T+s}, \xi'^3_{T+s}\big)\Big)$$

and

$$\underline{\mathbf{e}}_{T+s} = \Big(\big((y_{T+s}, x_{T+s}), \underline{g}^1_{T+s}, \underline{g}^2_{T+s}, \underline{g}^d_{T+s}\big), \big(\xi'^0_{T+s}, \underline{\xi}'^1_{T+s}, \underline{\xi}'^2_{T+s}, \underline{\xi}'^3_{T+s}\big)\Big),$$

for some $(g^1_{T+s}, ..., g^d_{T+s})$, and $(\underline{g}^1_{T+s}, ..., \underline{g}^d_{T+s})$ that have no reason to be equals, but with $|\underline{\xi}'^2_{T+s} - \xi'^2_{T+s}|$ and $|\underline{\xi}'^3_{T+s} - \xi'^3_{T+s}|$ less than or equal to ε. Yet, there exists an isometry ρ of $\mathbb{R}^3 \subset \mathbb{R}^{1,3}$ such that

$$\underline{g}_{T+s}\, \rho = g_{T+s}.$$

Note $\underline{\mathbf{e}}_{T+s}\, \rho = (\underline{g}_{T+s}\, \rho, \underline{\xi}_{T+s})$. Since the function $h\Big(\big((y,x), g^1, g^2, g^3\big), \xi\Big)$ does not depend on g^1, g^2, g^3,

$$h(\underline{\mathbf{e}}_{T+s}\, \rho) = h(\underline{\mathbf{e}}_{T+s}).$$

All the more, we have

$$\underline{\mathbf{e}}_{T+s}\, \rho = \big(g_{T+s}, \xi_{T+s} + C_1\mathbf{g}'^1 + c_2\mathbf{g}'^2 + c_3\mathbf{g}'^3\big),$$

for some (random) constants C_1, and $|c_2|, |c_3| \leqslant \varepsilon$.
As $\{y_r\}_{r\geqslant 0}$ diverges $\mathbb{P}^{\epsilon_1}$-almost surely to $+\infty$, we know from lemma **54** that for s large enough, y_{T+s} is large enough to ensure that

$$\underline{\mathbf{e}}_{T+s}\rho \in \mathbf{e}_{T+s}\mathcal{V}.$$

Applying the bounded convergence theorem, it follows that

$$\big|h(\mathbf{e}_0) - h(\underline{\mathbf{e}}_0)\big| \leqslant \mathbb{E}^{\epsilon_1}_{\mathbf{e}_0, \underline{\mathbf{e}}_0}\Big[\lim_{s \to +\infty}\big|h(\mathbf{e}_{T+s}) - h(\underline{\mathbf{e}}_{T+s})\big|\Big] \leqslant \eta.$$

Since $\eta > 0$ is arbitrary, the result follows. ▷

d) Toward a second conditioning

Fix $\ell \in \mathbb{R}$. The function

$$h_\ell^{\varepsilon_1} : u_0 \mapsto -\partial_{\ell'} h^{\varepsilon_1}_{\geqslant \ell'}(u_0)_{| \ell'=\ell}$$

is $L^{h^{\varepsilon_1}}$-harmonic in the open halfspace $\{(\dot{\xi}_0, \xi_0) \in \mathbb{H} \times \mathbb{R}^{1,d}; \ell - \xi'^0 > 0\}$.

Notation – For $\xi \in \mathbb{R}^{1,d}$, define

$$\alpha(\xi) = \ell - \xi'^0.$$

$$h_\ell^{\varepsilon_1}(u_0) = y_0 \frac{e^{\frac{-2}{y_0\alpha(\xi_0)}}}{\big(y_0\alpha(\xi_0)\big)^d} \mathbf{1}_{y_0\alpha(\xi_0)>0} = -y_0 G'\big(y_0\alpha(\xi_0)\big) \; (^{14}). \tag{2.2.25}$$

We can see that bounded $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonic functions depend only on y and ξ'^0 from the following observation.

Notation – For $\ell' \leqslant \ell$, note $\tau_{\ell-\ell'}$ the translation of length $\ell - \ell'$ in the direction $\mathbf{g}'^0$.

LEMMA **56.** *If h is a bounded $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonic function , then $h \circ \tau_{\ell-\ell'}$ is a bounded $L^{h^{\varepsilon_1}h_{\ell'}^{\varepsilon_1}}$-harmonic function. This correspondence is a bijection.*

So, given $\ell' \leqslant \ell$, and h a bounded $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonic function,

$$L\big(h^{\varepsilon_1} h_{\ell'}^{\varepsilon_1} h \circ \tau_{\ell-\ell'}\big) = 0.$$

Thus, if h is a bounded $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonic function, the function

$$2^n \int_{\ell-2^{-n}}^{\ell} h_{\ell'}^{\varepsilon_1}(.)(h \circ \tau_{\ell-\ell'})(.) d\ell'$$

is a bounded $L^{h^{\varepsilon_1}}$-harmonic function[15]; so it depends only on y and ξ'^0. Its limit $h_\ell^{\varepsilon_1} h$, as $n \to +\infty$, also depends only on y and ξ'^0; as $h_\ell^{\varepsilon_1}$ only depends on y and ξ'^0, so does h.

COROLLARY **57.** *Let $\ell \in \mathbb{R}$. Bounded $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonic functions only depend on y and ξ'^0.*

Notation – Note $\mathbb{P}_{u_0}^{\varepsilon_1,\ell}$ the law of the $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-diffusion, started from u_0.

Sum up. We want to show that the function $h^{\varepsilon_1}h_\ell^{\varepsilon_1}$ is a minimal L-harmonic function. This amounts to show that the constants are the only bounded $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonic bounded functions. Since such functions only depend on y and ξ'^0, we shall obtain the result if we can couple these two coordinates for $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-diffusions started from different points.

The process $\{(y_s, \xi_s'^0)\}$ happens to be a diffusion, under $\mathbb{P}_{u_0}^{\varepsilon_1,\ell}$, whatever $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$. We shall show in propositions **63** and **64** that two independent trajectories of this process couple naturally, which will provide the conclusion.

The constant $\ell \in \mathbb{R}$ is fixed in the next subsection.

[14] G was defined in equation (2.2.12).

[15] It is $\leqslant 2^n \, \|h\|_\infty \mathbb{P}_.^{\varepsilon_1}\big(R_\infty^{\sigma_\infty} \in [\ell - 2^{-n}, \ell]\big) \leqslant 2^n \|h\|_\infty$.

2.2.3 A representation formula

a) Bounded $L^{h^{\varepsilon_1}h^{\varepsilon_1}_\ell}$-harmonic functions

First, see what the $h^{\varepsilon_1}_\ell$-transform $L^{h^{\varepsilon_1}h^{\varepsilon_1}_\ell}$ of $L^{h^{\varepsilon_1}}$ looks like. It is defined on the open halfspace

$$\{\alpha > 0\} = \{(\dot\xi, \xi) \in \mathbb{H} \times \mathbb{R}^{1,d}\,;\, \alpha(\xi) = \ell - \xi'^0 > 0\},$$

The transform adds a drift[16]

$$2\frac{y_0^2}{2}\frac{\partial_{y_0} h^\sigma_\ell}{h^\sigma_\ell}\partial_{y_0} = y_0^2\frac{G'(y_0\alpha) + y_0\alpha G''(y_0\alpha)}{y_0 G'(y_0\alpha)}\partial_{y_0} = y_0\left((1-d) + \frac{2}{y_0\alpha}\right)\partial_{y_0},$$

to $L^{h^{\varepsilon_1}}$. So, we have, for $f \in \mathcal{C}_0^\infty(\{\alpha > 0\})$,

$$L^{h^{\varepsilon_1}h^\sigma_\ell} f = \frac{y_0^2}{2}(\partial^2_{x_0} + \partial^2_{y_0})f + \left(\frac{2}{\alpha(\xi_0)} - \left(\frac{d}{2} - 1\right)y_0\right)\partial_{y_0} f + \partial_{\xi_0} f(\dot\xi_0).$$

The evolution of the $h^{\varepsilon_1}h^{\varepsilon_1}_\ell$-process is determined by the following stochastic differential system.

$$\begin{aligned}
dy_s &= y_s dw^y_s + y_s\left(\frac{2}{\alpha(\xi_s)y_s} - \left(\frac{d}{2} - 1\right)\right)ds,\\
dx_s &= y_s dw^x_s,\\
d\xi'^0_s &= \frac{ds}{y_s},\\
d\xi'^1_s &= \frac{|x_s|^2 + y_s^2 + 1}{2y_s}ds,\\
d\xi'^j_s &= \frac{x^{j-1}_s}{y_s}ds,\ j = 2..d.
\end{aligned} \tag{2.2.26}$$

It is defined until its explosion time S, defined as the infimum of its exit time from all compacta and $\inf\{s > 0; \alpha(\xi_s) = 0\}$.

Notation – We shall write $\alpha_s \equiv \alpha(\xi_s) = \ell - \xi'^0_s$.

As explained in the preceding paragraph, we are interested in the behaviour of the process (y, ξ'^0). We have

$$\xi'^0_s = \ell - \alpha_s;$$

so, we see on (2.2.26) that

FACT **58.** *The process $\{(y_s, \alpha_s)\}_{0\leqslant s<S}$ is a diffusion.*

Notice that since $d\alpha_s = -\frac{ds}{y_s} \leqslant 0$, α decreases.

***i)* Preliminary remarks**

LEMMA **59.** $d(y_s\alpha_s) = (y_s\alpha_s)dw^y_s + \left(1 - (\frac{d}{2} - 1)(y_s\alpha_s)\right)ds.$

◁ Use Itô's formula.

$$d(y_s\alpha_s) = y_s\frac{-ds}{y_s} + \alpha_s y_s dw^y_s + \alpha_s\left(\frac{2}{\alpha_s} - \left(\frac{d}{2} - 1\right)y_s\right)ds = (y_s\alpha_s)dw^y_s + \left(1 - \left(\frac{d}{2} - 1\right)y_s\alpha_s\right)ds.$$

▷

[16] We saw in the equation (2.2.14) that $\frac{u^2}{2}G''(u) + \left(\frac{d\,u}{2} - 1\right)G'(u) = 0$.

Corollary 60. • *The process $\{y_s\alpha_s\}_{0\leqslant s<S}$ is a positive recurrent diffusion on $\mathbb{R}^{>0}$.*

- *The $(h^{\varepsilon_1}h_\ell^{\varepsilon_1})$-process $\mathbb{P}_{u_0}^{\varepsilon_1,\ell}$-almost surely does not explode, for every starting point $u_0 \in \mathbb{H}\times\mathbb{R}^{1,d}$.*

The density[17] of the invariant probability of the diffusion $\{y_s\alpha_s\}_{s\geqslant 0}$ is proportional to $t^{-d}e^{-\frac{2}{t}}$ [18]. The equation governing the evolution of y_s is integrable. Use Itô 's formula to differentiate

$$y_s e^{-w_s^y - 2\int \frac{du}{y_u\alpha_u} + \frac{d-1}{2}s} \equiv y_s z_s.$$

With

$$dz_s = -z_s dw_s^y + \left(\frac{d}{2} - \frac{2}{y_s\alpha_s}\right) z_s ds$$

and

$$d\langle y_s, z_s\rangle = -y_s z_s\, ds,$$

one finds

$$d(y_s z_s) = 0,$$

that is

$$y_s = y_0 e^{w_s^y + 2\int_0^s \frac{du}{y_u\alpha_u} - \frac{d-1}{2}s}. \tag{2.2.27}$$

As we know the invariant probability of the positive recurrent diffusion $\{y_s\alpha_s\}_{s\geqslant 0}$, the ergodic theorem gives us an almost sure equivalent of the integral

$$\int_0^s \frac{du}{y_u\alpha_u} = \frac{\int_0^\infty t^{-(d+1)}e^{-\frac{2}{t}}}{\int_0^\infty t^{-d}e^{-\frac{2}{t}}} s + o(s)$$

An integration by parts gives $\frac{\int_0^\infty t^{-(d+1)}e^{-\frac{2}{t}}\,dt}{\int_0^\infty t^{-d}e^{-\frac{2}{t}}\,dt} = \frac{d-1}{2}$, so that the LIL and formula (2.2.27) provide a precise estimate of y_s.

Proposition 61. *We have almost surely:* $\log(y_s) = \frac{d-1}{2}s + o(s)$.

As awaited,

Lemma 62. *$\{\alpha_s\}_{s\geqslant 0}$ almost surely decreases to 0 as $s \to +\infty$.*

◁ As $\{\alpha_s\}_{s\geqslant 0}$ decreases,

$$\big\{\{\alpha_s\}_{s\geqslant 0} \text{ does not tends to } 0\big\} = \bigcup_{n\geqslant 1}\{\forall\, s\geqslant 0,\ \alpha_s \geqslant \frac{1}{n}\}.$$

On the event $\{\forall\, s\geqslant 0,\ \alpha_s\geqslant \frac{1}{n}\}$, $y_s\alpha_s \to +\infty$, since $y_s\to+\infty$. But we saw that the diffusion $\{y_s\alpha_s\}_{s\geqslant 0}$ is recurrent positive. It follows that the event $\{\forall\, s\geqslant 0,\ \alpha_s\geqslant\frac{1}{n}\}$ is of null probability, for any $n\geqslant 1$, and $\{\alpha_s\}_{s\geqslant 0}$ decreases to 0 almost surely. ▷

[17] With respect to Lebesgue's measure on $\mathbb{R}^{>0}$.

[18] This density $m(t)$ must satisfy the equation $\left(\frac{t^2}{2}m(t)\right)'' = \Big(\big(1-(\frac{d}{2}-1)t\big)m(t)\Big)'$. If we note $u(t) = \frac{t^2}{2}m(t)$, u must satisfy the equation $u''(t) = \big(\{\frac{2}{t^2} - \frac{d-2}{t}\}u(t)\big)'$, *i.e.*, up to an additive constant $u'(t) = \left\{\frac{2}{t^2} - \frac{d-2}{t}\right\}u(t)$. This gives $u(t) = t^{-(d-2)}e^{-\frac{2}{t}}$, and $m(t) = t^{-d}e^{-\frac{2}{t}}$. We left the constant apart.

To investigate the system, we will take $\left(\frac{1}{y_s\alpha_s}, \alpha_s\right)$ as coordinates rather than (y_s, α_s). Set $b_s \equiv \frac{1}{y_s\alpha_s}$. We have

$$\begin{aligned} d\left(\frac{1}{y_s\alpha_s}\right) &\equiv db_s = -b_s dw_s^y + b_s\left(\frac{d}{2} - b_s\right) ds, \\ d\alpha_s &= -\alpha_s b_s ds, \end{aligned} \tag{2.2.28}$$

or

$$\begin{aligned} db_s &= -b_s dw_s^y + b_s\left(\frac{d}{2} - b_s\right) ds \\ \alpha_s &= \alpha_0 e^{-\int_0^s b_u du}. \end{aligned} \tag{2.2.29}$$

The diffusion $\{b_s\}_{s\geqslant 0}$ is positive recurrent on $\mathbb{R}^{>0}$.

***ii)* An automatic coupling**

In this paragraph,

1) we show that two independent copies of (b, α), started from different points, meet with probability 1. So, two independent copies of (y, α), started from different points, meet with probability 1.

2) This implies that $L^{h^{\varepsilon_1} h_\ell^{\varepsilon_1}}$ has no non constant bounded harmonic functions only depending on y and α (or y and ξ'^0).

1) For $(b_0, \alpha_0) \in \mathbb{R}^{>0} \times \mathbb{R}^{>0}$, write $\mathbb{P}_{(b_0,\alpha_0)}$ the law of the diffusion (b, α), started from (b_0, α_0), defined on the canonical space $\Omega \equiv \mathcal{C}(\mathbb{R}^{\geqslant 0}, \mathbb{R}^2)$. For $\omega \in \Omega$ and $R > 0$, note $T_R(\omega) \equiv \inf\{s > 0; \log\alpha_s(\omega) \leqslant -R\}$. As $\log\alpha_s = \log\alpha_0 - \int_0^s b_r\, dr$, and b_r is continuous, > 0, T_R is (almost surely) a strictly increasing, continous function of R.

Proposition 63. *The process $\{b_{T_R}\}_{R\geqslant 0}$ is a recurrent diffusion.*

$\lhd$ On the one hand, $\{-\log(\alpha_s)\}_{s\geqslant 0}$ being an additive functional of $\{b_s\}_{s\geqslant 0}$, it is well known that the time transform of b by the inverse of $-\log(\alpha)$ remains a diffusion. On the other hand, since b is positive recurrent, one can invoke the ergodic theorem and find some constants $0 < c < C < +\infty$, such that one has almost surely $c\, s \leqslant -\log\alpha_s \leqslant C\, s$, from a random time $s_0(\omega)$ on. So, for R large enough ($\geqslant \frac{s_0}{c}$),

$$\frac{R}{C} \leqslant T_R \leqslant \frac{R}{c}.$$

Then, the recurrence of $\{b_s\}_{s\geqslant 0}$ implies that of $\{b_{T_R}\}_{R\geqslant 0}$. $\rhd$

It remains to use the following elementary result to prove point **1)**.

Proposition 64. *Let $\{\mathbb{P}_x\}_{x\in\mathbb{R}^{>0}}$ be the family of laws associated with a recurrent diffusion on $\mathbb{R}^{>0}$. Let $x_0 \neq x_1 \in \mathbb{R}^{>0}$ and $\mathbb{P}_{x_0,x_1}$ be the law of a couple $(\mathfrak{x}, \mathfrak{x}')$ of independent diffusions with laws $\{\mathbb{P}_x\}_{x\in\mathbb{R}^{>0}}$, started from x_0 and x_1, respectively. Then, $\mathbb{P}_{x_0,x_1}$-almost surely,* $\inf\{s > 0; \mathfrak{x}_s = \mathfrak{x}'_s\} < +\infty$.

$\lhd$ Map the state space $\mathbb{R}^{>0}$ on $\mathbb{R}$ by the scale function to get two independent recurrent diffusions $\underline{\mathfrak{x}}$ and $\underline{\mathfrak{x}}'$ on $\mathbb{R}$, in natural scale. These are continuous local martingales with brackets increasing to $+\infty$ as time goes to $+\infty$. So the local martingale $\underline{\mathfrak{x}} - \underline{\mathfrak{x}}'$, whose bracket

$$\langle\underline{\mathfrak{x}} - \underline{\mathfrak{x}}'\rangle = \langle\underline{\mathfrak{x}}\rangle + \langle\underline{\mathfrak{x}}'\rangle \to +\infty,$$

has the trajectories of a Brownian motion; in particular, it hits 0 in a finite time. Since the scale function is injective, $\underline{\mathfrak{x}}$ and $\underline{\mathfrak{x}}'$ coincide at that time. $\rhd$

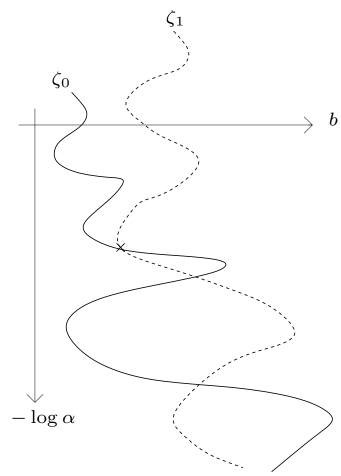


Figure 2.2: Coupling of two independent trajectories of (b, α)

The preceding two propositions show that *two independent copies of* (y, α)*, started from different points, almost surely meet at a positive time*[19].

2) Let us see why this fact implies that the differential generator of (b, α) has no non constant bounded harmonic functions.

Let $\zeta_0 \neq \zeta_1$ be two points of $\mathbb{R}^{>0} \times \mathbb{R}^{>0}$ and $\mathbb{P}_{\zeta_0}$, $\mathbb{P}_{\zeta_1}$, be the laws of the diffusion (b, α) started from ζ_0 and ζ_1 respectively. The system

$$\Big(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \big\{\sigma\big((\mathfrak{x}_s, \mathfrak{x}'_s); s \leqslant t\big)\big\}_{t \geqslant 0}, \mathbb{P}_{\zeta_0, \zeta_1} \equiv \mathbb{P}_{\zeta_0} \otimes \mathbb{P}_{\zeta_1}\Big)$$

describes the evolution of two independent copies of (b, α), started from ζ_0 and ζ_1. The applications $\mathfrak{x}$ and $\mathfrak{x}'$, $\Omega \times \Omega \to \Omega$, are the first and second projection, respectively.

Set

$$T_0(\omega, \omega') \equiv \inf\{s > 0; \mathfrak{x}_s(\omega, \omega') \in \mathfrak{x}'_{[0,+\infty[}(\omega, \omega')\}$$

and

$$T_1(\omega, \omega') \equiv \inf\{s > 0; \mathfrak{x}'_s(\omega, \omega') \in \mathfrak{x}_{[0,+\infty[}(\omega, \omega')\}.$$

These random times are not stopping times *with respect to the* σ*-algebra* $\big\{\sigma\big((\mathfrak{x}_s, \mathfrak{x}'_s); s \leqslant t\big)\big\}_{t \geqslant 0}$. Yet, the preceding tells us that they are finite $\mathbb{P}_{\zeta_0, \zeta_1}$-almost surely. As a consequence, the sets

$$\Omega_1 \equiv \{\omega' \in \Omega \ ; \ T_0(\omega, \omega') < +\infty, \ \mathbb{P}_{\zeta_0}(d\omega) - a.s.\}$$

and

$$\Omega_0 \equiv \{\omega \in \Omega \ ; \ T_1(\omega, \omega') < +\infty, \ \mathbb{P}_{\zeta_1}(d\omega') - a.s.\}$$

verify

$$\mathbb{P}_{\zeta_1}(\Omega_1) = \mathbb{P}_{\zeta_0}(\Omega_0) = 1.$$

Though $T_0(\omega, \omega')$ and $T_1(\omega, \omega')$ have no reasons to be equal, the monotonicity of $\{\log \alpha_s\}_{s \geqslant 0}$ implies that one has $\mathbb{P}_{\zeta_0, \zeta_1}$-almost surely

$$\mathfrak{x}_{T_0(\omega, \omega')}(\omega, \omega') = \mathfrak{x}'_{T_1(\omega, \omega')}(\omega, \omega').$$

[19]Actually, they meet at arbitrarily large times.

PROPOSITION **65.** *Let h be a bounded function, harmonic with respect to the differential generator of the diffusion (y,α). Then h is constant.*

$\lhd$ Let $\omega' \in \Omega_1$. The application $\omega \in \Omega \mapsto T_0(\omega,\omega')$ is a $\{\sigma(\mathfrak{x}_s; s \leqslant t)\}_{t\geqslant 0}$ stopping time, $\mathbb{P}_{x_0}(d\omega)$-almost surely finite, and $\big\{h(\mathfrak{x}_t(\omega,\omega'))\big\}_{t\geqslant 0}$ is a $\big\{\{\sigma(\mathfrak{x}_s\,;\, s\leqslant t)\}_{t\geqslant 0}, \mathbb{P}_{x_0}(d\omega)\big\}$ bounded martingale. The stopping time theorem applies.

$$h(\zeta_0) = \int h(\mathfrak{x}_{T_0(\omega,\omega')}(\omega,\omega'))\mathbb{P}_{\zeta_0}(d\omega);$$

integrating with respect to $\mathbb{P}_{\zeta_1}(d\omega')$, one gets

$$h(\zeta_0) = \int h(\mathfrak{x}_{T_0(\omega,\omega')}(\omega,\omega'))\mathbb{P}_{\zeta_0,\zeta_1}(d\omega,d\omega') \tag{2.2.30}$$

In the same way, one can show that

$$h(\zeta_1) = \int h(\mathfrak{x}_{T_1(\omega,\omega')}(\omega,\omega'))\mathbb{P}_{\zeta_0,\zeta_1}(d\omega,d\omega'). \tag{2.2.31}$$

As $\mathfrak{x}_{T_0(\omega,\omega')}(\omega,\omega') = \mathfrak{x}_{T_1(\omega,\omega')}(\omega,\omega')$, $\mathbb{P}_{\zeta_0,\zeta_1}(d\omega,d\omega')$-almost surely, $h(\zeta_0) = h(\zeta_1)$. $\rhd$

Since any bounded $L^{h^{\varepsilon_1}h_\ell^{\varepsilon_1}}$-harmonic function depends only on y and ξ'^0, that is on y and α, such function is an harmonic function with respect to the differential generator of the diffusion (y,α); so it is constant. So, the L-harmonic function $h^{\varepsilon_1}h_\ell^{\varepsilon_1}$ is minimal. We saw in **2.2.1** that this implies that any function $h^\sigma h_\ell^\sigma$, $\sigma \in \mathbb{S}^{d-1}, \ell \in \mathbb{R}$, is minimal.

THEOREM **66.** *The functions $h^\sigma h_\ell^\sigma$, $\sigma \in \mathbb{S}^{d-1}, \ell \in \mathbb{R}$, are L-harmonic minimal functions.*

b) A representation formula

From the probabilistic meaning of the functions h^σ and h_ℓ^σ and the facts on h-transform recalled in appendix **4.2**, this theorem gives us a decomposition of the constant function **1** as a mean of L-harmonic minimal functions:

$$\mathbf{1} = \int_{\mathbb{S}^{d-1}\times\mathbb{R}} h^\sigma h_\ell^\sigma \, d\sigma d\ell$$

Let $u_0 \in \mathbb{H}\times\mathbb{R}^{1,d}$ and $\sigma \in \mathbb{S}^{d-1}$. Choose a basis $g^\sigma \in SO_0(1,d)$ of $\mathbb{R}^{1,d}$ of the form $\{\varepsilon_0,\sigma,g_{\geqslant 2}\}$[20]. We saw in point **c** of **2.2.1** that if one uses g^σ-halfspace coordinates on $\mathbb{H}$ and $\{-g_1^\sigma, g_0^\sigma + g_1^\sigma, g_2^\sigma, ..., g_d^\sigma\}$-coordinates on $\mathbb{R}^{1,d}$, the law of R_∞^σ under $\mathbb{P}_{u_0}^\sigma$ has exactly the same expression as that of $R_\infty^{\varepsilon_1}$ under $\mathbb{P}_{u_0}^{\varepsilon_1}$, using the usual halfspace coordinates on $\mathbb{H}$ and $\mathbf{g}'$-linear coordinates on $\mathbb{R}^{1,d}$. Note (y,x) the g^σ-halfspace coordinates of a point $\dot{\xi} \in \mathbb{H}$. Remarking that $\frac{1}{y} = q\big(\dot{\xi},\varepsilon_0+\sigma\big)$, equation (2.2.13) and Dufresne's result (proposition 4.3) provide a coordinate free expression of the law of R_∞^σ under $\mathbb{P}_{u_0}^\sigma$. The following theorem give the unconditional version of this result.

PROPOSITION **67 (Unconditional law of $(\sigma_\infty, R_\infty^{\sigma_\infty})$).** *Let $u_0 \in \mathbb{H}\times\mathbb{R}^{1,d}$.*

1. *The limits*

$$\sigma_\infty = \lim_{s\to+\infty}\sigma_s, \text{ and } \quad R_\infty^{\sigma_\infty} = \lim_{s\to+\infty} q\big(\xi_s,\varepsilon_0+\sigma_\infty\big)$$

exist $\mathbb{P}_{u_0}$-almost surely.

[20]The notation $g_{\geqslant 2}$ is for a $(d-1)$-uple $\{g_2,...,g_d\}$.

2. Note $d\sigma$ the uniform probability on $\mathbb{S}^{d-1}$.

$$\begin{aligned}\mathbb{P}_{u_0}\big(\sigma_\infty \in d\sigma, R_\infty^{\sigma_\infty} \in \ell + d\ell\big) &= h^\sigma h_\ell^\sigma(u_0) d\sigma d\ell \\ &= \frac{h^\sigma(u_0)}{q\big(\dot{\xi}_0, \varepsilon_0+\sigma\big)} \left(\frac{q\big(\dot{\xi}_0, \varepsilon_0+\sigma\big)}{\ell - q\big(\dot{\xi}_0, \varepsilon_0+\sigma\big)}\right)^d \exp\left(-2\frac{q\big(\dot{\xi}_0, \varepsilon_0+\sigma\big)}{\ell - q\big(\dot{\xi}_0, \varepsilon_0+\sigma\big)}\right) d\sigma \, \mathbf{1}_{\ell > q\left(\dot{\xi}_0, \varepsilon_0+\sigma\right)} d\ell\end{aligned} \tag{2.2.32}$$

We shall use the classical method developped in the parabolic context to deduce from this identity a representation formula for the Poisson bounday of $(L, \mathbb{H}\times\mathbb{R}^{1,d})$ (see Doob, [Doo01], Chap.XIX, or Ancona, [Anc90], p.37, and the references given there). Before, we need to establish a compactness result in the set of L-harmonic bounded functions. Actually, since the good framework is the group $\mathcal{G}$, we first establish the result for $\widetilde{L}$-harmonic bounded functions.

2.2.4 Poisson boundary of $\big(L, \mathbb{H}\times\mathbb{R}^{1,d}\big)$

a) Compactness matters

In this paragraph, we shall use a result of Bony to obtain Harnack's compactness principle for $\widetilde{L}$-harmonic functions. Here is the result we would like to use.

THEOREM **68 (Bony's Harnack Inequality [Bon69]).** *Let $L' = \sum_{i=1}^r V_i^2 + V$ be a differential operator on a connected manifold $\mathbb{M}$ of dimension n such that the V_i's cannot be null all at the same time. Suppose the Lie algebra $\mathcal{L}(V_i)$ generated by the V_i's has rank n at every point. Then, for anycompact K contained in a chart $\{x^i\}$, every point $y_0 \in \mathbb{M}$ and every multi-index p, there exists a constant λ, depending on K, $\{x^i\}$ and p, such that every non negative L'-harmonic function h satisfies*

$$\sup_{x\in K} \Big|\frac{\partial^p h(x)}{\partial x^p}\Big| \leqslant \lambda\, h(y_0). \tag{2.2.33}$$

Before giving its proof, recall Bony demonstrated in [Bon69], Corollary 5.2, the following fact.

PROPOSITION **69.** *If the V_i's are not null all at the same time and rank$(V_1, ..., V_r, V) = n$, everywhere, then there exists a basis of the topology made up of open sets $\mathcal{U}$ for which the Dirichlet problem*

for $f \in \mathcal{C}(\partial\mathcal{U})$ find $u \in \mathcal{C}^2(\mathcal{U}) \cap \mathcal{C}(\overline{\mathcal{U}})$ such that $L'u = 0$ on $\mathcal{U}$ and $u_{|\partial\mathcal{U}} = f$

has a unique solution. These open sets are called ***elementary open sets****.*

Here is Bony's proof.

$\lhd$ Let x_0 be a point in K. On the one hand, one knows that given two relatively compact open neighbourhoods $\mathcal{O} \Subset \mathcal{O}'$ of x_0, small enough to be in a chart, and a multi-index p, there exists a constant c' such that any L'-harmonic function h satisfies[21]

$$\sup_{x\in\mathcal{O}} \Big|\frac{\partial^p h(x)}{\partial x^p}\Big| \leqslant c' \int_{\mathcal{O}'} h(x)dx.$$

On the other hand, one knows that when we are in an elementary open set $\mathcal{U}$, one has

$$h(y_0) \geqslant \beta \int_{\mathcal{U}} g_\beta^{\mathcal{U}}(y_0, x) h(x) dx,$$

where $g_\beta^{\mathcal{U}}$ is the Green function of the operator $L' - \beta$ on $\mathcal{U}$ with respect to the measure dx [22]. So, if

[21]This is a quantitative version of Hörmander's theorem on hypoellipticity.
[22]The existence of $g_\beta^{\mathcal{U}}$ is proved in Bony [Bon69].

we could suppose y_0, $\mathcal{O}$ and $\mathcal{O}'$ to be in an elementary open set and $g_\beta^{\mathcal{U}}(y_0, \mathcal{O}') \geqslant c'' > 0$ [23], we would have the Harnack inequality

$$\sup_{x \in \mathcal{O}} |\frac{\partial^p h(x)}{\partial x^p}| \leqslant \frac{c'}{c''\beta} h(y_0).$$

The hypothesis on the rank of the Lie algebra is made so as to give us the Strong Minimum Principle and, with it, the fact that $g_\beta^{\mathcal{U}}(., x_0)$ being > 0 somewhere must be > 0 everywhere; in particular $g_\beta^{\mathcal{U}}(y_0, x_0) > 0$. The continuity of $g_\beta^{\mathcal{U}}(y_0, .)$ gives us a neighbourhood $\mathcal{O}'$ of x_0 and a positive constant c'' such that $g_\beta^{\mathcal{U}}(y_0, \mathcal{O}') \geqslant c''$. c' is determined as soon as $\mathcal{O}'$, $\mathcal{O}$, p and the coordinates x are chosen. To obtain Harnak inequality not just for y_0, $\mathcal{O}$, $\mathcal{O}'$, in an elementary open set, use connectedness of $\mathbb{M}$ and compacity of K. ▷

This argument cannot be applied without any change to our situation, where the operator $\widetilde{L} = \frac{1}{2}\sum_{i=1}^d V_i^2 + V_0$ on $\mathcal{G}$ is such that $\mathrm{rank}\big(\mathcal{L}(V_1, ..., V_d)\big) < \dim \mathcal{G}$. Yet, the rank hypothesis is made to ensure the positivity of $g_\beta^{\mathcal{U}}(y_0, x_0)$. We get this positivity thanks to the support theorem **25**.

Given piecewise smooth, continuous controls, ϕ^i, $i = 1..d$, and $\mathbf{e}_0 \in \mathcal{G}$, the equation

$$d\big(\varphi(t)\big) = V_i\big(\varphi(t)\big)\phi_t^i dt + V_0\big(\varphi(t)\big)dt, \quad \varphi(0) = \mathbf{e_0}, \tag{2.2.34}$$

is the control equation associated with the ϕ^i's and $\mathbf{e}_0$. We note $\varphi(., \mathbf{e}_0)$ its unique solution.

Definition **70.** *Let $\mathcal{U}$ be an open set of $\mathcal{G}$ and let $\mathbf{e}_0 \in \mathcal{U}$. A point $\mathbf{z}$ is said* ***to be in the future of*** *$\mathbf{e}_0$* ***in*** *$\mathcal{U}$ if it is of the form $\varphi_{\mathbf{z}}(t; \mathbf{e}_0)$ for some $t > 0$, for some controls ϕ^i, and if $\varphi_{\mathbf{z}}([0, t]; \mathbf{e}_0) \subset \mathcal{U}$([24]). A set S is said to be in the future of $\mathbf{e}_0$ in $\mathcal{U}$ if each of its points is in the future of $\mathbf{e}_0$. The* ***future of*** *$\mathbf{e}_0$ is its future in $\mathcal{G}$.*

So, suppose $\mathbf{e_0}$ and $\mathcal{O}'$ are in an elementary open set $\mathcal{U}$ and $\mathcal{O}'$ is a subset of the future of $\mathbf{e_0}$ in $\mathcal{U}$. If one takes $\mathbf{z}$ in $\mathcal{O}'$, $\varphi_{\mathbf{z}}(.; \mathbf{e_0})$ spends a positive amount of time in $\mathcal{O}'$. Choose $\varepsilon > 0$ small enough for the ε-neighbourhood of $\varphi_{\mathbf{z}}([0, t]; \mathbf{e_0})$ to be included in $\mathcal{U}$, and the ball $B(\mathbf{z}, \varepsilon) \subset \mathcal{O}'$. Thus, by the support theorem 25, the diffusion $\{\mathbf{e}_s\}$, started from $\mathbf{e}_0$, spends a positive amount of time in $\mathcal{O}'$ with a positive probability, before leaving $\mathcal{U}$; that is

$$g_\beta^{\mathcal{U}}(\mathbf{e}_0, \mathcal{O}') > 0.$$

Without loss of generality, we can take $\mathcal{O}'$ smaller and suppose $g_\beta^{\mathcal{U}}(\mathbf{e}_0, \mathcal{O}') \geqslant c'' > 0$.

Now, suppose K is a compact in the inside of the future of $\mathbf{e_0}$ in $\mathcal{G}$. Let $\mathbf{z}_0 \in K$ and $\varphi_{\mathbf{z_0}}(.; \mathbf{e_0})$ a solution of (2.2.34) such that $\varphi_{\mathbf{z_0}}(T; \mathbf{e_0}) = \mathbf{z_0}$ for some $T > 0$. One can find a finite sequence $\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_s$ of points of $\varphi_{\mathbf{z_0}}([0, T]; \mathbf{e}_0)$, elementary open sets $\mathcal{U}_i$, $i = 0..s-1$, open sets $\mathcal{O}_i'$, $i = 1..s$, such that $\{\mathbf{e}_i\}$ and $\mathcal{O}_{i+1}'$ are included in $\mathcal{U}_i$, $\mathcal{O}_{i+1}'$ is in the future of $\mathbf{e}_i$ and $\mathbf{e}_{i+1} \in \mathcal{O}_{i+1}'$. The conclusion of Bony's Harnack inequality applies in each $\mathcal{U}_i$. If one takes $p = 0$ in (2.2.33) for $i = 1..s-1$ and use (2.2.33) with p for $i = s$, one has the result not for K but for a neighbourhood of $\mathbf{z}_0 \in K$. The compacity of K then yields the following version of Bony's result.

Theorem **71 (Harnack's Inequality).** *Let K be a compact of $\mathcal{G}$, located in the future of a point $\mathbf{e}_0$, small enough to be in a chart $\{\mathbf{e}^i\}$, and p be a mutli-index. There exists a constant λ, depending on K, $\{\mathbf{e}^i\}$ and p, such that every non negative $\widetilde{L}$-harmonic function h satisfies*

$$\sup_{\mathbf{e} \in K} |\frac{\partial^p h(\mathbf{e})}{\partial \mathbf{e}^p}| \leqslant lambda\, h(\mathbf{e}_0).$$

The smallness restriction is not a real one since any compact K can be covered by finitely many charts.

[23] $g_\beta^{\mathcal{U}}(y_0, \mathcal{O}') \geqslant c''$ means $\inf\{g_\beta^{\mathcal{U}}(y_0, x); x \in \mathcal{O}'\} \geqslant c''$.

[24] We note $\varphi_{\mathbf{z}}$ to recall this function is associated to $\mathbf{z}$.

Remarks

1. Notice that since we are on a group, and $\widetilde{L}$ is left invariant, the preceding construction gives the same constant λ if we replace $\mathbf{e}_0$ by $\mathbf{e}\mathbf{e}_0$ and K by $\mathbf{e}K$, whatever $\mathbf{e} \in \mathcal{G}$. It provides a proof that

   COROLLARY **72.** *Any bounded L-harmonic function is right uniformly continuous.*

   It also shows that any non negative L-harmonic function has a growth comparable to the growth of the volume of balls on $\mathcal{G}$.

2. This result is also true with $\widetilde{L}$ replaced by the heat operator $\widetilde{L} - \partial_t$.

We determined the future of $\mathbf{e}_0$ in proposition **26**; so, Harnack's inequality and Ascoli's theorem justify the

THEOREM **73 (Harnack's compactness principle).** *Let $\{h_n\}_{n\geqslant 0}$ be a sequence of $\widetilde{L}$-harmonic functions. If there is a point $\mathbf{e}_0 \in \mathcal{G}$ such that the sequence $\{h_n(\mathbf{e}_0)\}_{n\geqslant 0}$ is bounded, then $\{h_n\}_{n\geqslant 0}$ has a subsequence that uniformly locally $\mathcal{C}^\infty$ converges on $\mathcal{C}^{>0}(\mathbf{e}_0)$ [25]. The same result holds for L-harmonic functions with $\mathcal{C}^{\geqslant 0}(u_0)$.*

b) Intregral representation of L-harmonic bounded functions.

In this section, we use Choquet's representation theorem to obtain an explicit description of the set of bounded L-harmonic functions. The method is classical in the parabolic contect (see for instance Doob, [Doo01], Chapter XIX, or [Anc90], p.37, and the references therein). It provides a description of the invariant σ-algebra under any $\mathbb{P}_{u_0}$, $u_0 \in \mathbb{H} \times \mathbb{R}^{1,d}$.

The following lemma will enable us to define a compact, convex set, adapted to the situation.

LEMMA **74.** *For $n \in \mathbb{N}$, set $u_n \equiv (\varepsilon_0, -n\varepsilon_0) \in \mathbb{H} \times \mathbb{R}^{1,d}$. Then, for any $\sigma \in \mathbb{S}^{d-1}$ and $\ell \in \mathbb{R}$,*

$$\sum_{n\geqslant 0} h^\sigma h^\sigma_\ell(u_n) < +\infty.$$

$\triangleleft$ Since we found an explicit expression of $h^\sigma h^\sigma_\ell$ in equation (2.2.32), we read on the formula

$$h^\sigma h^\sigma_\ell(u_n) = (\ell + n)^{-d} \exp\left(-\frac{2}{\ell + n}\right)$$

the convergence of the series. $\triangleright$

Choose a sequence of positive numbers p_n such that $\sum_{n\geqslant 0} p_n = 1$ and set $\nu \equiv \sum_{n\geqslant 0} p_n \delta_{u_n}$. The set of $\mathbb{L}^1(\nu)$-integrable functions contains the set of bounded functions. We note $\langle f, \nu\rangle \equiv \int f d\nu$. Set

$$\mathbf{C}_\nu \equiv \{h \geqslant 0; h\ L-\text{harmonic}\ , \langle h, \nu\rangle\ < +\infty\}, \mathbf{K}_\nu \equiv \{h \in C_\nu; \langle h, \nu\rangle \leqslant 1\}, \text{and } \mathbf{K}^1_\nu \equiv \{h \in C_\nu; \langle h, \nu\rangle = 1\}. \tag{2.2.35}$$

$\mathbf{C}_\nu$ is a cone and $\mathbf{K}_\nu$ and $\mathbf{K}^1_\nu$ are convex subsets of $\mathbf{C}_\nu$. The functions $h^\sigma h^\sigma_\ell$ are in $\mathbf{C}_\nu$. Endow $\mathbf{C}_\nu$ with the topology of uniform convergence on compacta.
As a consequence of Harnack's inequality **71**, the only $h \in \mathbf{C}_\nu$ such that $\langle h, \nu\rangle = 0$ is the zero function.

[25] By "uniformly locally $\mathcal{C}^\infty$ converges", we mean that, for any $p \geqslant 0$, all the derivatives of h of order $\leqslant p$ uniformly converge on compacta.

PROPOSITION **75.** $\mathbf{K}_\nu$ *is compact.*

$\triangleleft$ The topology being metrisable, we check that any sequence of points of $\mathbf{K}_\nu$ has a convergent subsequence. Since $\langle h, \nu\rangle = \sum_{n\geqslant 0} p_n h(u_n) = 1$, for $h \in \mathbf{K}_\nu$, every $h \in \mathbf{K}_\nu$ satisfies $h(u_n) \leqslant \frac{1}{p_n}$. Fix n. The compacity principle **73** enables to extract from any sequence $\{h_p\}$ of points of $\mathbf{K}_\nu$ a subsequence that uniformly locally converges on $\mathcal{C}^{>0}(u_n)$. So, a diagonal extraction provides a subsequence uniformly locally converging on $\bigcup_{n\geqslant 0} \mathcal{C}^{>0}(u_n)$. The choice of the u_n's was made so as to ensure that $\mathcal{C}^{>0}(u_n)$ increases to $\mathbb{H}\times\mathbb{R}^{1,d}$. This limit belongs to $\mathbf{K}_\nu$ because $0 \leqslant \langle \underline{\lim}\, h_p, \nu\rangle \leqslant \underline{\lim}\,\langle h_p, \nu\rangle \leqslant 1$. $\triangleright$

PROPOSITION **76.** $\mathbf{C}_\nu$ *is a lattice with respect to its own order.*

$\triangleleft$ This is so because the set of non negative L-harmonic functions is itself a lattice with respect to its own order and if we have $0 \leqslant h' \leqslant h$, with $h \in \mathbf{C}_\nu$ then $h' \in \mathbf{C}_\nu$. $\triangleright$

In this context, *Choquet's representation theory*[26] applies.

THEOREM **77 (Choquet's representation theorem).** *1. Any point h of $\mathbf{K}^1_\nu$ can be uniquely written as*

$$h = \int \underline{h}\ \mu_h(d\underline{h}),$$

where μ_h is a Borel probability supported on the set of extremal points of $\mathbf{K}^1_\nu$.

2. Associate to each non null $h \in \mathbf{C}_\nu$, firstly, the point $\frac{h}{\langle h,\nu\rangle} \in \mathbf{K}^1_\nu$, then the probability $\mu_{\frac{h}{\langle h,\nu\rangle}}$ on the set of extremal points of $\mathbf{K}^1_\nu$. The application

$$h \in \mathbf{C}_\nu \mapsto \widetilde{\mu}_h \equiv \langle h, \nu\rangle \mu_{\frac{h}{\langle h,\nu\rangle}} \tag{2.2.36}$$

is a lattice isomorphism between $\mathbf{C}_\nu$ and the set of probability measures on $\mathbf{K}^1_\nu$([27]). In particular, if $h \leqslant h'$ then $\widetilde{\mu}_h \leqslant \widetilde{\mu}_{h'}$; so $\widetilde{\mu}_h = F\widetilde{\mu}_{h'}$, for some Borel function F, bounded $\widetilde{\mu}_{h'}$-almost everywhere.

As $0 < \langle h^\sigma h^\sigma_\ell, \nu\rangle < +\infty$, for all σ, ℓ, one can rewrite the integral representation (2.2.4) as

$$\mathbf{1} = \int_{\mathbb{S}^2\times\mathbb{R}} \frac{h^\sigma h^\sigma_\ell}{\langle h^\sigma h^\sigma_\ell, \nu\rangle}\ \langle h^\sigma h^\sigma_\ell, \nu\rangle\ d\sigma d\ell,$$

writing the function $\mathbf{1}$ as a mean of extremal points of $\mathbf{K}^1_\nu$.

The preceding decomposition is the unique Choquet's representation of $\mathbf{1}$ in $\mathbf{K}_\nu$:

$$\widetilde{\mu}_{\mathbf{1}} = \langle h^\sigma h^\sigma_\ell, \nu\rangle\ d\sigma d\ell.$$

As $\langle h^\sigma h^\sigma_\ell, \nu\rangle$ is a positive continuous function of σ and ℓ([28]), any $\widetilde{\mu}_{\mathbf{1}}$-almost sure equality is a $(d\sigma d\ell)$-almost sure equality.

Let $f \in C_\nu$ be any (non null) non negative L-harmonic bounded function. Since

- $f \leqslant \|f\|_\infty \mathbf{1}$, we have $\widetilde{\mu}_f = F\widetilde{\mu}_{\|f\|_\infty \mathbf{1}}$, for some function F on $\mathbb{S}^{d-1}\times\mathbb{R}$ such that one has $(d\sigma d\ell)$-almost everywhere $0 \leqslant F \leqslant 1$,

[26] As explained in Choquet, [Cho69a] and [Cho69b], Phelps, [Phe66], or Becker [Bec99].

[27] Endowed with its natural lattice structure: if $f = \frac{d\mu}{d(\mu+\mu')}$ and $g = \frac{d\mu'}{d(\mu+\mu')}$, $\frac{d(\mu\wedge\mu')}{d(\mu+\mu')} = f \wedge g$.

[28] It does not depend on σ.

- $\widetilde{\mu}_{\|f\|_\infty \mathbf{1}} = \|f\|_\infty \widetilde{\mu}_{\mathbf{1}} = \|f\|_\infty \langle h^\sigma h^\sigma_\ell, \nu\rangle \, d\sigma d\ell$,

we deduce from theorem **77** that

$$f = \langle f, \nu\rangle \int \frac{h^\sigma h^\sigma_\ell}{\langle h^\sigma h^\sigma_\ell, \nu\rangle} F(\sigma,\ell) \|f\|_\infty \langle h^\sigma h^\sigma_\ell, \nu\rangle \, d\sigma d\ell.$$

One sees one this expression that f is of the form

$$\int h^\sigma h^\sigma_\ell \mathbf{F}(\sigma,\ell) \, d\sigma d\ell$$

for some Borel function $\mathbf{F}$, bounded $(d\sigma d\ell)$-almost everywhere.

Conversely, for any ($(d\sigma d\ell)$-almost everywhere) bounded Borel function $\mathbf{F}$ on $\mathbb{S}^{d-1}\times\mathbb{R}$, the function

$$\int_{\mathbb{S}^{d-1}\times\mathbb{R}} \mathbf{F}(\sigma,\ell) h^\sigma h^\sigma_\ell \, d\sigma d\ell \geqslant 0$$

is bounded and L-harmonic. This provides a complete description of the Poisson boundary of L.

Theorem **78 (Poisson boundary of $(L, \mathbb{H}\times\mathbb{R}^{1,d})$).** *Any bounded L-harmonic function is of the form*

$$\int_{\mathbb{S}^{d-1}\times\mathbb{R}} \mathbf{F}(\sigma,\ell) h^\sigma(.) h^\sigma_\ell(.) \, d\sigma d\ell,$$

for some bounded (Borel) function $\mathbf{F}$ on $\mathbb{S}^{d-1}\times\mathbb{R}$. Reciprocally, such a formula defines a bounded L-harmonic function.

The description of the invariant σ-algebra of the diffusion $\{(\dot{\xi}_s, \xi_s)\}_{s\geqslant 0}$ given in theorem **11** follows from proposition **2.2.4**.

Indeed, given a bounded $Inv\big((\dot{\xi}, s, \xi_s)\big)$-measurable random variable X, the harmonic function (proposition **2.2.4**)

$$(\dot{\xi},\xi) \in \mathbb{H}\times\mathbb{R}^{1,d} \mapsto \mathbb{E}_{\dot{\xi},\xi}[X]$$

is of the form

$$\int_{\mathbb{S}^{d-1}\times\mathbb{R}} h^\sigma(\dot{\xi},\xi) h^\sigma_\ell(\dot{\xi},\xi) \mathbf{F}(\sigma,\ell) \, d\sigma d\ell,$$

for some bounded measurable function $\mathbf{F}$ on $\mathbb{S}^{d-1}\times\mathbb{R}$. That is,

$$\forall\, (\dot{\xi},\xi) \in \mathbb{H}\times\mathbb{R}^{1,d}, \quad \mathbb{E}_{\dot{\xi},\xi}[X] = \mathbb{E}_{\dot{\xi},\xi}\big[\mathbf{F}(\sigma_\infty, R^{\sigma_\infty}_\infty)\big].$$

Let $(\dot{\xi}_0,\xi_0) \in \mathbb{H}\times\mathbb{R}^{1,d}$ be given. Since the process $\big\{\mathbb{E}_{\dot{\xi}_s,\xi_s}[X]\big\}_{s\geqslant 0}$ is under $\mathbb{P}_{\dot{\xi}_0,\xi_0}$ a bounded martingale, it almost surely converges to X, and $\mathbf{F}(\sigma_\infty, R^{\sigma_\infty}_\infty)$; so, both quantities are $\mathbb{P}_{\dot{\xi}_0,\xi_0}$-almost surely equal.

Corollary **79 (Invariant σ-algebra of the relativistic diffusion).** *For any $(\dot{\xi}_0,\xi_0) \in \mathbb{H}\times\mathbb{R}^{1,d}$, the σ-algebras $Inv\big((\dot{\xi}, s, \xi_s)\big)$ and $\sigma\big(\sigma_\infty, R^{\sigma_\infty}_\infty\big)$ coincide up to $\mathbb{P}_{\dot{\xi}_0,\xi_0}$-null sets.*

Chapter 3

Geometric and algebraic representation of the Poisson boundary of L

3.1 A geometric description of the boundary

3.1.1 Introduction

a) Vocabulary — The following definitions are the natural extensions to manifolds of the concepts defined in $\mathbb{R}^{1,n-1}$.

- In an n-dimensional vector space identified with $\mathbb{R}^n$, any linear transform of the null cone of the quadratic form $(x^0)^2 - (x^1)^2 - ... - (x^{n-1})^2$, is called a ***Lorentzian cone***. The ***inside*** of this cone is $\{x \in \mathbb{R}^n;\ (x^0)^2 - (x^1)^2 - ... - (x^{n-1})^2 > 0\}$.
- A ***Lorentzian manifold*** $(\mathbb{M}, q)$ ***of dimension*** n is a n-dimensional manifold equiped with a $(0,2)$-tensor q with constant signature $(+, -, \cdots, -)$.
- By a 'path', we mean a $\mathcal{C}^1$ path.
- A ***causal path*** $\gamma : I \to \mathbb{M}$, with values in a Lorentzian manifold $(\mathbb{M}, q)$ is a path whose speed is always *non spacelike* : $\forall\ s \in I$, $q(\dot\gamma_s, \dot\gamma_s) \geqslant 0$, that is, γ_s is in the closure of the inside of the Lorentzian cone of $q_{|T_{\gamma_s}\mathbb{M}}$. If $q(\dot\gamma_s, \dot\gamma_s) > 0$ for all $s \in I$, the **path** is said to be **timelike**.

  The null cones of $q_{|T_{\gamma_s}\mathbb{M}}$ are made of two half cones. One of them is arbitrarily called the *future cone*, and the other, the *past cone*. We suppose we can make a smooth choice of future cones, *on the whole* $\mathbb{M}$; so, we can speak of a ***future-oriented causal curve*** γ if its speed is everywhere in the closure of the inside of the future cone. Such manifolds are said to be *causally orientable* – *cf.* O'Neill [O'N83], p.194.
- By ***past of a point*** $x \in \mathbb{M}$, we mean the set of points $y \in \mathbb{M}$ such that there exists a future-oriented *causal* path from y to x. It is traditionaly denoted $J^-(x)$. In $\mathbb{R}^{1,n-1}$, the past of a point x is the closure of the inside of the inferior lightcone with vertex x. If γ is a future causal path, the sets $J^-(\gamma_t)$, are increasing with t.
- For $A \subset \mathbb{M}$, the set $J^-(A) \equiv \bigcup_{x \in A} J^-(x)$ is called the ***past of*** A.

b) What happens to future-oriented causal paths leaving every compact of some Lorentzian manifold ? — The causal structure of a Lorentzian manifold $(\mathbb{M}, q)$ provides, under causality conditions (see [GKP72]), a equivalence relation on future-oriented causal paths, leaving every compact, that allows to give $\mathbb{M}$ a boundary.

Identify two future-oriented causal curves γ and γ', leaving every compact, if the *increasing* sets $J^-(\gamma_s)$ and $J^-(\gamma'_s)$ converge to the same set (*i.e.* $\bigcup_{s>0} J^-(\gamma_s) = \bigcup_{s>0} J^-(\gamma'_s)$). Yet, such a definition is meaningless if, for example, all points (or some) have the same past. So it seems reasonable to define such a relation just for Lorentzian manifolds enjoying the following property of "separation of points by their past".

(SP) *Two points are equal iff they have the same past*[1].

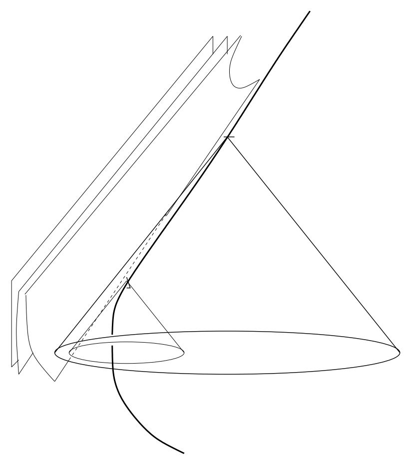


Figure 3.1: Past of a path

If a Lorentzian manifold $(\mathbb{M}, q)$ has this property, the set of all equivalence classes of future-oriented causal paths leaving every compact is called the ***causal boundary of*** $(\mathbb{M}, q)$ [2]. To be useful, this construction recquires a good understanding of the causal structure of the Lorentzian manifold.

There is another standard method to attack the question if one is only interested in lightlike hypersurfaces, null geodesics, causality, or other *conformal structures*.

In good situations, one can embed $(\mathbb{M}, q)$ conformally in a relatively compact subset of a Lorentzian manifold $(\widehat{\mathbb{M}}, \widehat{q})$.If the embedding is good enough, one can identify the causal boundary of $\mathbb{M} \subset \widehat{\mathbb{M}}$ as a subset of the set of points of $\overline{\mathbb{M}} \subset \widehat{\mathbb{M}}$, limits of future-oriented causal paths leaving every compact of $\mathbb{M}$.

Einstein universe $\mathbf{Ein}_n$, defined in the next section, will play the role of $\widehat{\mathbb{M}}$ for $\mathbb{R}^{1,n-1}$. We will note $\mathbf{C}$ the conformal boundary of $\mathbb{R}^{1,n-1} \subset \mathbf{Ein}_n$. We will see in theorem **88** that

- any point of $\mathbf{C}$, except one special point p, is the limit of a lightlike geodesic of $\mathbb{R}^{1,n-1}$,
- $\mathbf{C}$ can be identified with the causal boundary of $\mathbb{R}^{1,n-1}$,

[1] Globally hyperbolic Lorentzian manifolds have this property.

[2] We can define a similar object using the future $I^+(\gamma_s)$ of curves, past-directed, leaving every compact. Both boundaries can be used simultaneously to give $\mathbb{M}$ a larger boundary, as made in Geroch, Penrose, [GKP72], or in Hawking & Ellis [HE73], section 6.8, where much information on the subject can be found, among others, the definition of the topology of this larger boundary and the fact that, sometimes, it does not have any reasonable differentiable structure.

- a trajectory $\{\xi_s\}_{s\geqslant 0}$ almost surely converges to a point of $\mathbf{C}\backslash\{\mathbf{p}\}$.

In that sense, $\{\xi_s\}_{s\geqslant 0}$ eventually behaves as a lightlike geodesic.

In a first time, we construct Einstein universe with no reference $\mathbb{R}^{1,n-1}$. The link between the two spaces will be done in subsection **3.1.3**.

3.1.2 Einstein universe

a) Définition In the usual Riemannian conformal framework, the sphere $\mathbb{S}^n$ has the property that all the simply connected constant curvature Riemannian models conformally appear as open subsets of it: it is naturally endowed with its Riemannian metric of constant curvature $+1$, the complementary of a point has in its conformal class the flat metric on $\mathbb{R}^n$, thanks to the *stereographic projection*, and a half-sphere is conformally equivalent to the hyperbolic space, with constant curvature -1. Einstein universe can play the same role in the Lorentzian framework, the constant curvature models being Minkowski space, de Sitter space and Anti-de Sitter space, with curvature $0, 1, -1$, respectively. We will only deal with Minkowski's space, exhibiting a stereographic projection.

DEFINITION **80.** *Endow $\mathbb{R}^{2+n}$ with the quadratic form*

$$q^{2,n}(x) = -2x^0x^{n+1} + (x^1)^2 - (x^2)^2 - \cdots - (x^n)^2,$$

expressed in the canonical basis $(\widetilde{\varepsilon_0}, \widetilde{\varepsilon_1}, ..., \widetilde{\varepsilon_n}, \widetilde{\varepsilon_{n+1}})$ of $\mathbb{R}^{2+n}$. Endowed with this quadratic form, $\mathbb{R}^{2+n}$ will be denoted $\mathbb{R}^{2,n}$.

*A **vector** with $q^{2,n}$ null norm is said to be **null**.*

*Note $C^{2,n}$ the **lightcone** of $\mathbb{R}^{2,n}$, made up of all null vectors. $C^{2,n}\backslash\{0\}$ is a smooth manifold. Its tangent space at a point $v \in C^{2,n}\backslash\{0\}$ is denoted $T_vC^{2,n}$.*

DEFINITION **81.** *A plane E is said to be **totally degenerated** if the quadratic form $q^{2,n}$ is null on E.*

The following lemma is an elementary lemma on quadratic forms.

LEMMA . *$C^{2,n}$ has totally degenerated planes. They are all generated by vectors v and v', where $v \in C^{2,n}$ and $v' \in T_vC^{2,n}\backslash\{0\}$ is a null vector.*

These planes play a special role in the description of the lightlike geodesics of Einstein universe, described in proposition **85**.

Let $\pi : C^{2,n} \to \mathbb{RP}^{n+1}$ be the restriction to $C^{2,n}$ of the canonical projection of $\mathbb{R}^{2+n}\backslash\{0\}$ in $\mathbb{RP}^{n+1}$. Write

$$\mathbf{Ein}_n \equiv \pi(C^{2,n});$$

this is a hypersurface of $\mathbb{RP}^{n+1}$. Let $v \in C^{2,n}$ and set $p = \pi(v)$. The restriction of the quadratic form $q^{2,n}$ to $T_vC^{2,n}$ has signature $(0, +, -, \cdots, -)$, where 0 is for the v-direction. Since v is in the kernel of π, $T_p\mathbf{Ein}_n$ bears a Lorentzian cone, $\widetilde{C}(p)$, image of the lightcone $\underline{C}(v)$ of $q^{2,n}_{|T_vC^{2,n}}$, by $T_v\pi$. This cone $\widetilde{C}(p)$ is well defined:

$$T_{v'}\pi\big(\underline{C}(v')\big) = T_v\pi\big(\underline{C}(v)\big), \text{ if } \pi(v') = \pi(v).$$

It depends smoothly on p. This smooth distribution of Lorentzian cones gives $\mathbf{Ein}_n$ a natural conformal Lorentzian structure.

DEFINITION. ***Einstein universe** is this Lorentzian manifold together with its conformal structure:*

$$\big(\mathbf{Ein}_n, \{\widetilde{C}(p)\}_{p\in\mathbf{Ein}_n}\big).$$

Remarks

- The Anti-de Sitter space of dimension $n+1$ is defined as the projection in $\mathbb{RP}^{n+1}$ of the open set $\{x \in \mathbb{R}^{2+n}; q^{2,n}(x) > 0\}$. $\mathbf{Ein}_n$ is the *topological boundary* of this subspace of $\mathbb{RP}^{n+1}$.
- $\mathbf{Ein}_n$ is causally orientable[3].

The Einstein universe has a rich conformal group of transforms, inherited from the preceding construction. Actually, they are completely described by the following rigidity theorem [4].

THEOREM **82 (Liouville).**
- *$PO(2,n)$ acts transitively on $\mathbf{Ein}_n$ and respects its conformal structure.*
- *$PO(2,n)$ acts transitively on the bundle of lightlike directions over $\mathbf{Ein}_n$.*
- *Any local conformal transformation of $\mathbf{Ein}_n$ is the restriction of the action on $\mathbf{Ein}_n$ of a unique element of $PO(2,n)$.*

b) Lightlike geodesics of the Einstein universe — In the framework of conformal Lorentzian geometry, the notion of geodesic is not well defined generally; yet, that of *lightlike geodesic* is a well defined *geometric notion.*

PROPOSITION **83.** *Let g_0 be a Lorentzian metric on $\mathbf{Ein}_n$ with the $\widetilde{C}(p)$'s as null cones. Let $f : \mathbf{Ein}_n \to \mathbb{R}$ be a smooth map, and $e^{2f}g_0$ a conformal metric on $\mathbf{Ein}_n$. Any* lightlike g_0-geodesic $\gamma : I \to \mathbf{Ein}_n$ *can be reparametrised to give an $(e^{2f}g_0)$- lightlike geodesic.*

◁ Let ∇ be the Levi-Civita connection associated with g_0 and ∇' be the Levi-Civita connection associated with $e^{2f}g_0$. One has for every vector fields V, W on $\mathbf{Ein}_n$

$$\nabla'_V W = \nabla_V W + (V.f)W + (W.f)V - g_0(V,W)\nabla f,$$

where ∇f is the gradient of f with respect to the metric g_0. So, if $\gamma : I \to \mathbf{Ein}_n$ is a g_0-lightlike geodesic, $\nabla_{\dot\gamma}\dot\gamma = 0$, and

$$\nabla'_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + 2(\dot\gamma.f)\dot\gamma - g_0(\dot\gamma,\dot\gamma)\nabla f = 2(\dot\gamma.f)\dot\gamma,$$

since $\dot\gamma$ is null. We see on this equation that it suffices to make a time change (depending on f and γ) for the newly parametrised path to be a $(e^{2f}g_0)$-geodesic. Precisely, if $\underline{\gamma}(s) \equiv \gamma\big(t(s)\big)$,

$$\begin{aligned}\nabla'_{\dot{\underline{\gamma}}}\dot{\underline{\gamma}} &= t''(s)\dot\gamma\big(t(s)\big) + (t'(s))^2\nabla'_{\dot\gamma}\dot\gamma \\ &= t''(s)\dot\gamma\big(t(s)\big) + 2\big(t'(s)\big)^2(\dot\gamma.f)\dot\gamma\big(t(s)\big).\end{aligned} \tag{3.1.1}$$

So, take a solution of the equation $t''(s) + 2\big(t'(s)\big)^2(\dot\gamma.f)\big(t(s)\big) = 0.$ ▷

DEFINITION **84.** *A 1-dimensional manifold $\Gamma \subset \mathbf{Ein}_n$ is said to be a* ***lightlike geodesic*** *if any point ξ of Γ has a neighbourhood $\mathcal{U}$ such that $\Gamma \cap \mathcal{U}$ is of the form $\gamma(I)$ for some $\big(e^{2f}g_0\big)$-geodesic $\gamma : I \to \mathbf{Ein}_n$.*

Remark – The proof makes it clear why the notion of geodesic is not a conformal notion, except that of lightlike geodesic.

In our case, there is a simple geometric description of lightlike geodesics.

[3] One can find a non vanishing timelike vector field on $\mathbf{Ein}_n$, which implies the causal orientability (*cf.* O'neill [O'N83], p.194). Indeed, if one looks at $\mathbf{Ein}_n$ as the image of the quotient of $\mathbb{S}^1 \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n+1} \cap C^{2,n}$, by the antipodal map $(x,y) \in \mathbb{S}^1 \times \mathbb{S}^{n-1} \mapsto (-x,-y) \in \mathbb{S}^1 \times \mathbb{S}^{n-1}$, the image of ∂_x in $\mathbf{Ein}_n$ is a smooth timelike vector field.

[4] See [Fra02] for instance.

PROPOSITION **85.** *Any lightlike geodesic running through the point* $\pi(v)$ *is of the form* $\pi\big(\langle v, v'\rangle \cap\ C^{2,n}\big)$, *where* $v' \in T_vC^{2,n}$ *is a null vector. In short, the non parametrized lightlike geodesics are the projections in* Ein_n *of totally degenerated planes of* $\mathbb{R}^{2,n}$*; they are topological circles.*

$\lhd$ This proposition is proved in subsection **3.1.4**. $\rhd$

DEFINITION. *Let* p *be a point in* Ein_n*. The set of all lightlike geodesics running through* p *is called the* ***lightcone of*** p *and denoted* $\mathbf{C}(p)$*.*

From the preceding proposition **85**, one sees that if $p = \pi(v)$ and $v^{\perp}$ is the orthogonal of v with respect to $q^{2,n}$, then

$$\mathbf{C}(p) = \pi\big(C^{2,n} \cap v^{\perp}\big).$$

For example, if $p = \pi(\widetilde{\varepsilon}_0)$, $\mathbf{C}(p) = \pi\big(C^{2,n} \cap \{x_{n+1} = 0\}\big)$. The cone $\widetilde{C}(p)$ is the infinitesimal version of $\mathbf{C}(p)$.

From its simple geometric description, one sees that

PROPOSITION. $\mathbf{C}(p)\backslash\{p\}$ *is a hypersurface, diffeomorphic to* $\mathbb{R}\times\mathbb{S}^{n-2}$*.* $\mathbf{C}(p)$ *is compact and has a singularity at* p*.* $Ein_n\backslash\mathbf{C}(p)$ *is dense in* Ein_n*.*

Remarks

- $\mathbf{C}(p)$ does not separate $\mathbf{Ein}_n$.
- Notice that $\mathbf{C}(p)\backslash\{p\}$ is foliated by lightlike geodesics: any point of $\mathbf{C}(p)\backslash\{p\}$ belongs to a unique lightlike geodesic.

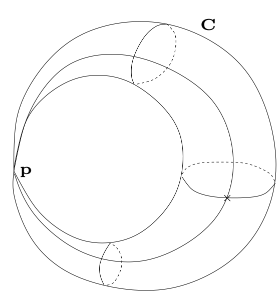


Figure 3.2: The lightcone of p

3.1.3 Where does Minkowski's space hide ?

The reason of the introduction of the Einstein universe, while speaking of the Minkowski space $\mathbb{R}^{1,n-1}$, is that one can see $\mathbb{R}^{1,n-1}$ as a dense subset of $\mathbf{Ein}_n$ having the same geometry as $\mathbb{R}^{1,n-1}$. In a (pseudo-) Riemannian framework, we would use the notion of isometry to give a precise meaning to this sentence. In a conformal framework, the notion of isometry is replaced by that of conformal equivalence.

DEFINITION **86.** *Let* $\big(E, \{C_E(p)\}_{p\in E}\big)$ *and* $\big(F, \{C_F(p)\}_{p\in F}\big)$ *be two manifolds with smooth distributions of Lorentzian cones on their tangent spaces. They are said to be* ***conformally equivalent*** *if one can find a smooth diffeomorphism* $\varphi : E \to F$ *which maps the tangent cones on* E *on the tangent cones on* F*:*

$$\forall\, p \in E\,,\ T_p\varphi\big(C_E(p)\big) = C_F\big(\varphi(p)\big).$$

Choose a point $p = \pi(v) \in \mathbf{Ein}_n$.

PROPOSITION **87.** *The dense open set* $\mathbf{Ein}_n \backslash \mathbf{C}(p)$ *is conformally equivalent to* $\mathbb{R}^{1,n-1}$*. In that sense,* $\mathbf{Ein}_n$ *provides a compactification of* $\mathbb{R}^{1,n-1}$ *in which* $\mathbf{C}(p)$ *is its boundary.*

◁ Because of the homogeneity of $\mathbf{Ein}_n$, we can suppose $p = \pi(\widetilde{\varepsilon}_0)$, so that

$$\mathbf{C}(p) = \pi\big(C^{2,n} \cap \{x_{n+1} = 0\}\big).$$

One defines a stereographic projection. We note q_0 the usual Lorentzian quadratic form on $\mathbb{R}^{1,n-1}$

$$q^{1,n-1}\big(\xi^0, \xi^1, ..., \xi^{n-1}\big) = (\xi^0)^2 - \big((\xi^1)^2 + ... + (\xi^{n-1})^2\big). \tag{3.1.2}$$

Let $i : (\mathbb{R}^{1,n-1}, q^{1,n-1}) \to (\mathbb{R}^{2,n}, q^{2,n})$ be the identification of $\mathbb{R}^n = \mathbb{R}^{1,n-1}$ to $\langle \varepsilon_1, ..., \varepsilon_n \rangle$. This is an isometry. The map

$$j : x \in \mathbb{R}^{1,n-1} \mapsto q^{1,n-1}(x,x)\widetilde{\varepsilon}_0 + 2i(x) + \widetilde{\varepsilon}_{n+1} \in C^{2,n} \tag{3.1.3}$$

is well defined and maps the lightcone of $\mathbb{R}^{1,n-1}$ to the lightcone of $T_{j(x)}C^{2,n}$, proving that the map $\pi \circ j$ is a conformal mapping [5]. The map $\pi \circ j$ is called a ***stereographic projection***. The only points of $\mathbf{Ein}_n$ that are not in the image of $\pi \circ j$ are the projection in $\mathbf{Ein}_n$ of the points of $C^{2,n}$ with $x_{n+1} = 0$, *i.e.* $\mathbf{C}(p)$. ▷

Remark that $q^{2,n}\big(j(x)\big) = 5q^{1,n-1}(x)$.

From now on, we identify $\mathbb{R}^{1,n-1}$ and $\pi \circ j(\mathbb{R}^{1,n-1}) \subset \mathbf{Ein}_n$, where $\pi \circ j$ is the stereographic projection constructed above. The point $\mathbf{p} = \pi(\widetilde{\varepsilon}_0)$ being fixed, we note $\mathbf{C}$ instead of $\mathbf{C}(\mathbf{p})$.

Write $(\varepsilon_0, ..., \varepsilon_{n-1})$ the canonical basis of $\mathbb{R}^{1,n-1}$.

The following proposition gives a more precise image of this embedding. Recall the geodesics of $\mathbb{R}^{1,n-1}$ are straight lines, run linearly.

THEOREM **88.** *1. For any non lightlike geodesic* $\xi + \mathbb{R}\, v$ *of* $\mathbb{R}^{1,n-1}$*,* $\xi + tv$ *converges to* $\mathbf{p}$ *as* $t \to +\infty$*.*

2. Any point in $\mathbf{C}\backslash\{\mathbf{p}\}$ *is the limit of a lightlike geodesic of* $\mathbb{R}^{1,n-1}$*. More precisely,*

- *Let* $v \in \mathbb{R}^{1,n-1}$ *be an isotrope vector. There is a lightlike geodesic* Δ_v *of* $\mathbf{C}$ *such that for every lightlike geodesic* $\xi + \mathbb{R}\, v$*, with direction* v*,* $\xi + tv$ *converges to a point of* $\Delta_v\backslash\{\mathbf{p}\}$ *as* $t \to +\infty$*. This lightlike geodesic of* $\mathbf{Ein}_n$ *depends only on the direction* $\mathbb{R}v$ *of* v*.*

Identify the set of null directions with $\mathbb{S}^{n-2}$*: if* $\mathbb{R}\, v = \mathbb{R}\,(\varepsilon_0 + \sigma)$*,* $\sigma \in \mathbb{S}^{n-2}$*, say that* v *has direction* σ*. We note* $\Delta_\sigma \equiv \Delta_v$*, if* v *has direction* $\sigma \in \mathbb{S}^{n-2}$*.*

Let $\sigma \in \mathbb{S}^{n-2}$ *and* $v = \varepsilon_0 + \sigma \in \mathbb{R}^{1,n-1}$*. This is a null vector with direction* σ*.*

- *Two* v*-directed geodesics* $\xi + \mathbb{R}\, v$ *and* $\xi' + \mathbb{R}\, v$ *converge to the same point iff* ξ' *is in the affine hyperplane* $\xi + v^\perp$*.*

The function $q(v,.)$ *is constant on* $\xi + v^\perp$*, note* $\ell \in \mathbb{R}$ *this constant. We note* $p_\sigma(\ell) \in \Delta_\sigma$ *the limit of* $\xi + \mathbb{R}\, v$*.*

- *Any point of* Δ_σ *is the limit of a lightlike geodesic* $\xi + \mathbb{R}\,(\varepsilon_0 + \sigma)$*, for some* $\xi \in \mathbb{R}^{1,n-1}$*.*

[5] The map π is injective on $j(\mathbb{R}^{1,n-1})$.

3. ***A natural parametrization of*** $\mathbf{C}\backslash\{\mathbf{p}\}$ — *The map*

$$(\sigma,\ell)\in\mathbb{S}^{n-2}\times\mathbb{R}\mapsto p_\sigma(\ell)\in\subset\mathbf{C}\backslash\{\mathbf{p}\}$$

is a diffeomorphism.

4. ***Asymptotic future of a point*** — *Let* $\xi\in\mathbb{R}^{1,n-1}$ *be given,*

$$\mathcal{C}^0(\xi)=\{\zeta\in\mathbb{R}^{1,n-1}\ ;\ q(\zeta-\xi)=0\}$$

its lightcone, and

$$\mathcal{C}^{\geqslant 0}(\xi)=\{\zeta\in\mathbb{R}^{1,n-1}\ ;\ q(\zeta-\xi)\geqslant 0\}$$

its future.

- $\overline{\mathcal{C}^0(\xi)}\cap\mathbf{C}=\{p_\sigma\big(q^{1,n-1}(\varepsilon_0+\sigma,\xi)\big);\ \sigma\in\mathbb{S}^{d-1}\}$ *is diffeomorphic to* $\mathbb{S}^{d-1}$.
- $\overline{\mathcal{C}^{\geqslant 0}(\xi)}\cap\big(\mathbf{C}\backslash\{p\}\big)=\{p_\sigma(\ell);\ \ell\geqslant q^{1,n-1}\big(\varepsilon_0+\sigma,\xi\big)\}$.

In both statements, $\overline{A}$ *is the closure of* $A\subset\mathbf{Ein}_n$ *in* $\mathbf{Ein}_n$.

5. *Let* $\sigma\in\mathbb{S}^{n-2}$, $\ell\in\mathbb{R}$.

   (a) *Note* $J^-\big(p_\sigma(\ell),\mathbb{R}^{1,n-1}\big)$ *the causal past of* $p_\sigma(\ell)$ *in* $\mathbb{R}^{1,n-1}$:

   $$J^-\big(p_\sigma(\ell),\mathbb{R}^{1,n-1}\big)=\{\zeta\in\mathbb{R}^{1,n-1};\exists\gamma:\mathbb{R}^{\geqslant 0}\to\mathbb{R}^{1,n-1},\ \textit{causal},\ \gamma_0=\zeta,\ \lim_{s\to+\infty}\gamma_s=p_\sigma(\ell)\}.$$

   Then, $J^-\big(p_\sigma(\ell),\mathbb{R}^{1,n-1}\big)=\{\zeta\in\mathbb{R}^{1,n-1};\ q^{1,n-1}(\zeta,\varepsilon_0+\sigma)\leqslant\ell\}$.

   (b) $\overline{J^-\big(p_\sigma(\ell),\mathbb{R}^{1,n-1}\big)}\cap\mathbf{C}=\{p_\sigma(r)\,;\ r\leqslant\ell\}$.

6. *Given a path* $\{\gamma_s\}_{s\geqslant 0}$ *in* $\mathbb{R}^{1,n-1}$, *write* $\gamma_s=(t_s,r_s\theta_s)\in\mathbb{R}^{\geqslant 0}\times\mathbb{R}^{\geqslant 0}\mathbb{S}^{d-2}$. *If* $r_s\to+\infty$, $\theta_s\to\theta$ *and* $t_s-r_s\to\ell\in\mathbb{R}$, *then* $\gamma_s\to p_\theta(\ell)\in\mathbf{C}\backslash\{\mathbf{p}\}$.

7. (a) *Every timelike path* $\{\gamma_s\}_{s\in I}$ *in* $\mathbb{R}^{1,n-1}$, *future-oriented and inextensible*[6], *converges toward a point of* $\mathbf{C}$

   (b) *Let* $\sigma\in\mathbb{S}^{n-2}$ *and* $\ell\in\mathbb{R}$.

   $$\gamma_s\underset{s\to+\infty}{\longrightarrow}p_\sigma(\ell)\ \textit{iff}\ q^{1,n-1}\big(\gamma_s,\varepsilon_0+\sigma\big)\underset{s\to+\infty}{\longrightarrow}\ell.$$

   If no such σ *and* ℓ *exist, then* $\gamma_s\underset{s\to+\infty}{\longrightarrow}\mathbf{p}$.

   (c) *If* $\gamma_s\to p_\sigma(\ell)$ *then* $J^-(\gamma_s)\underset{s\to+\infty}{\longrightarrow}\{\zeta\in\mathbb{R}^{1,n-1}\ ;\ q^{1,n-1}\big(\zeta,\varepsilon_0+\sigma\big)\leqslant\ell\}$; *if* $\gamma_s\to p$, *then* $J^-(\gamma_s)\underset{s\to+\infty}{\longrightarrow}\mathbb{R}^{1,n-1}$.

   So, we can identify $\mathbf{C}$ *with the* ***causal boundary of*** $\mathbb{R}^{1,n-1}$, *as defined in the introduction of this part.*

◁ The proofs are given in subsection **3.1.4**. ▷

The results of the preceding sections can be restated in this framework.

[6]That is, I is an interval of $\mathbb{R}$, the time component γ_t^0 of γ_t increases, and $\gamma:I\to\mathbb{R}^{1,n-1}$ has no proper $\mathcal{C}^1$ extension.

THEOREM **89.** *1. Let $u \in \mathbb{H} \times \mathbb{R}^n$. The process $\{\xi_s\}_{s\geqslant 0}$ $\mathbb{P}_u$-almost surely converges to a random point ξ_∞ of $\mathbf{C}\backslash\{\mathbf{p}\}$. In (σ, ℓ) coordinates, this random point has under $\mathbb{P}_u$ a law with density $h^\sigma h^\sigma_\ell(u)$ with respect to $d\sigma\, d\ell$, proportional to*

$$\frac{h^\sigma(u)}{q\big(\dot{\xi}, \varepsilon_0 + \sigma\big)} \left(\frac{q\big(\dot{\xi}, \varepsilon_0 + \sigma\big)}{\ell - q\big(\xi, \varepsilon_0 + \sigma\big)}\right)^d \exp\left(-2\frac{q\big(\dot{\xi}, \varepsilon_0 + \sigma\big)}{\ell - q\big(\xi, \varepsilon_0 + \sigma\big)}\right) \mathbf{1}_{\ell > q\left(\xi, \varepsilon_0 + \sigma\right)}.$$

2. For any $u \in \mathbb{H} \times \mathbb{R}^{1,d}$, the σ-algebras $Inv\big(\{u_s\}\big)$ and $\sigma\big(\xi_\infty\big)$ coincide up to $\mathbb{P}_u$-null sets.

3. Any bounded L-harmonic function is of the form $u \mapsto \mathbb{E}_u[F(\xi_\infty)]$, for a Borel bounded function F on $\mathbf{C}$.

3.1.4 Proofs of the geometric claims

Recall $(\widetilde{\varepsilon}_0, ..., \widetilde{\varepsilon}_{n+1})$ is the canonical basis of $\mathbb{R}^{2,n}$, $(\varepsilon_0, ..., \varepsilon_{n-1})$ that of $\mathbb{R}^{1,n-1}$, whose natural Lorentz quadratic form is denoted $q^{1,n-1}$. The map $i : \mathbb{R}^{1,n-1} \to \langle \widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_n\rangle$ is the linear identification of $\mathbb{R}^n$ with $\langle \widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_n\rangle$ sending ε_j on $\widetilde{\varepsilon}_{j+1}$.

First, we show that the lightlike geodesics of Einstein universe have the simple description given in proposition **85**.

PROPOSITION **90.** *The lightlike geodesics of $\mathbf{Ein}_n$ are the projections in $\mathbf{Ein}_n$ of totally isotrope planes of $\mathbb{R}^{2,n}$.*

The proof makes use of the conformal embedding of $\mathbb{R}^{1,n-1}$ in $\mathbf{Ein}_n$ as a dense subset, with boundary $\mathbf{C}$. We identify $\mathbb{R}^{1,n-1}$ with its stereographic projection.

$\lhd$ Using the transitive action of $PO(2, n)$ on the bundle of lightlike geodecis over $\mathbf{Ein}_n$, we are brought back to show that the lightlike geodesic $t \in \mathbb{R} \mapsto t(\varepsilon_0 + \varepsilon_1)$, (locally) ambeds into the projection of a totally degenerated plane of $\mathbb{R}^{2,n}$. We see on formula (3.1.3) defining j that $j\big(\gamma(\mathbb{R})\big) = \pi\big(\langle \widetilde{\varepsilon}_{n+1}, \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2\rangle\big)$. $\rhd$

Proof of theorem 88 — Let $v \in \mathbb{R}^{1,n-1}$. Set $\gamma_s = a + s\, v \in \mathbb{R}^{1,n-1}$, $s \in \mathbb{R}$. The curve γ is a geodesic of $\mathbb{R}^{1,n-1}$. The image of γ by j is the curve of $C^{2,n}$

$$j(\gamma_s) = s^2 q^{1,n-1}(v)\, \widetilde{\varepsilon}_0 + 2\, s\big(q^{1,n-1}(a, v)\, \varepsilon_0 + i(v)\big) + 2i(a) + q^{1,n-1}(a)\, \widetilde{\varepsilon}_0 + \widetilde{\varepsilon}_{n+1}. \tag{3.1.4}$$

1. If v is not null, $q^{1,n-1}(v) \neq 0$ and

$$\begin{aligned} \lim_{s\to+\infty} \pi\big(j(\gamma_s)\big) &= \lim_{s\to+\infty} \pi\Big(s^2 q^{1,n-1}(v)\, \widetilde{\varepsilon}_0 + 2\, s\big(q^{1,n-1}(a, v)\, \widetilde{\varepsilon}_0 + i(v)\big) + 2i(a) + q^{1,n-1}(a)\, \widetilde{\varepsilon}_0 + \widetilde{\varepsilon}_{n+1}\Big) \\ &= \lim_{s\to+\infty} \pi\big(\widetilde{\varepsilon}_0 + o(s^{-1})\big) = \pi(\widetilde{\varepsilon}_0) = \mathbf{p}. \end{aligned} \tag{3.1.5}$$

2. From now on, $v \in \mathbb{R}^{1,n-1}$ is a null vector. So, we have

$$\lim_{s\to+\infty} \pi\big(j(\gamma_s)\big) = \pi\big(q^{1,n-1}(a, v)\, \widetilde{\varepsilon}_0 + i(v)\big). \tag{3.1.6}$$

As noticed after proposition **3.1.2**, $\mathbf{C}\backslash\{\mathbf{p}\}$ is foliated by lightlike geodesics. The precise formulation of this fact is

$$\mathbf{C}\backslash\{\mathbf{p}\} = \coprod_{\sigma\in\mathbb{S}^{n-2}} \pi\Big(\mathbb{R}\widetilde{\varepsilon}_0 + i\big(\varepsilon_0 + \sigma\big)\Big) = \coprod_{\sigma\in\mathbb{S}^{n-2}} \Delta_\sigma\backslash\{\mathbf{p}\}, \tag{3.1.7}$$

which comes from the description of the lightlike geodesics of $\mathbf{Ein}_n$ and the definition of $\mathbf{C}$. We read on the formula (3.1.6) all the results stated in point *2*.

Remarks

- We see on the formula (3.1.6) that the map $(\sigma, \ell) \mapsto p_\sigma(\ell)$ is *injective*.
- *Proof of 6* – Any point $m \in \mathbb{R}^{1,n-1}$ can be written
$$m = q^{1,n-1}(m, \varepsilon_0 + \sigma_m)\varepsilon_0 + s_m(\varepsilon_0 + \sigma_m),$$
with $\sigma_m \in \mathbb{S}^{n-2}$ and $s_m \geqslant 0$. Using the formula (3.1.5), one sees that
$$\pi\big(j(m)\big) = \pi\Big(2s_m\{q^{1,n-1}(m, \varepsilon_0+\sigma_m)\varepsilon_0+i(\varepsilon_0+\sigma_m)\}+\{2q^{1,n-1}(m,\varepsilon_0+\sigma_m)\varepsilon_1+q^{1,n-1}(m,\varepsilon_0+\sigma_m)^2\varepsilon_0+\varepsilon_{n+1}\}\Big).$$
We read on this formula that if a sequence $\{m_p\}_{p\geqslant 0}$ of points of $\mathbb{R}^{1,n-1}$, satisfies $s_{m_p} \to +\infty$, $\sigma_{m_p} \underset{p\to+\infty}{\longrightarrow} \sigma$ and $q^{1,n-1}(m_p, (1, \sigma_{m_p}) \underset{p\to+\infty}{\longrightarrow} \ell$, then $\{m_p\}_{p\geqslant 0}$ converges toward $p_\sigma(\ell)$.

 Any sequence $\{m_p\}_{p\geqslant 0}$ leaving every compact, with $\{s_{m_p}\}_{p\geqslant 0}$ bounded, converges to $\mathbf{p}$.

3. Set
$$\varphi : \mathbb{S}^{n-2} \to \mathbf{C}\backslash\{\mathbf{p}\},\ \sigma \mapsto \pi\big(i(\varepsilon_0 + \sigma)\big)$$
the smooth map that associate to σ the point of $\mathbf{C}$, limit of the lightlike geodesic passing through 0, with direction σ. φ is a diffeomorphism on its image.

Set
$$\varphi(t, .) : \mathbb{R}^{1,n-1} \to \mathbb{R},\ \zeta \mapsto \zeta + t\varepsilon_0.$$
The element of $O(2, n)$ whose action on $\mathbf{Ein}_n$ induces the translation $\varphi(t, .)$ on $\mathbb{R}^{1,n-1} \subset \mathbf{Ein}_n$ has matrix
$$\begin{pmatrix} 1 & t & 0 & \cdots & 0 & t^2 \\ & 1 & 0 & \cdots & 0 & 2t \\ & & \ddots & & \vdots & 0 \\ & & & \ddots & 0 & \vdots \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$
We see on this expression that one can extend this smooth flow of translations to a smooth flow of transforms of $PO(2, n)$, $\phi(t, .)$, acting on $\mathbf{Ein}_n$.

LEMMA . *The flow $\{\phi(t, .)\}_{t\in\mathbb{R}}$ preserves each lightlike geodesic of $\mathbf{C}$, and is transitive on each $\Delta_\sigma\backslash\{\mathbf{p}\}$.*

◁ Since a point $p_\sigma(\ell)$ is the limit of a lightlike geodesic $\{\gamma_s\}_{s\geqslant 0} = \{a + s(\varepsilon_0 + \sigma)\}_{s\geqslant 0}$, with $\ell = q^{1,n-1}(a, \varepsilon_0 + \sigma)$
$$\phi\big(t, p_\sigma(\ell)\big) = \lim_{s\to+\infty} \phi(t, \gamma_s).$$
But $\phi(t, \gamma_s) = \pi \circ j(\gamma_s + t\varepsilon_0)$. So,
$$\begin{aligned} \phi\big(t, p_\sigma(\ell)\big) &= \lim_{s\to+\infty} \phi(t, \gamma_s) = \pi\Big(q^{1,n-1}\big(a + t\varepsilon_0, \varepsilon_0 + \sigma\big)\varepsilon_0 + i(\varepsilon_0 + \sigma)\Big) \\ &= \pi\Big((\ell + t)\varepsilon_0 + i(\varepsilon_0 + \sigma)\Big) = p_\sigma(\ell + t). \end{aligned} \tag{3.1.8}$$
So, the flow preserves each leaf of $\mathbf{C}$ and the transitiveness of the action comes from the formula (3.1.8) and the description (3.1.7). ▷

Since $\varphi(\sigma) = p_\sigma(0)$, formula (3.1.8) reads

$$\phi(\ell,\sigma) = p_\sigma(\ell),$$

and the assertion of point *3* comes from the flow property of ϕ.

4. The first point is a direct consequence of the preceding one. The second point comes from the fact that

$$\overline{\mathcal{C}^{\geqslant 0}(\xi)} = \overline{\bigcup_{t\geqslant 0} \mathcal{C}^0(\xi + t\varepsilon_0)}.$$

5. *5.(a)* Given $\zeta_0 \in \mathbb{R}^{1,n-1}$ with $q^{1,n-1}\big(\zeta_0, \varepsilon_0 + \sigma\big) \leqslant \ell$, find a timelike path from ζ_0 to the hyperplane $\big\{q^{1,n-1}\big(., \varepsilon_0 + \sigma\big) = \ell\big\}$. It hits the hyperplane at ζ_1. Then, follow the unique lightlike geodesic with direction σ, passing through ζ_1. The concatenation of the two paths is a path from ζ to $p_\sigma(\ell)$. So, $J^-\big(p_\sigma(\ell), \mathbb{R}^{1,n-1}\big) \supset \big\{q^{1,n-1}\big(., \varepsilon_0 + \sigma\big) \leqslant \ell\big\}$.

On the other side, if $q^{1,n-1}\big(\zeta_0, \varepsilon_0 + \sigma\big) > \ell$, $p_\sigma(\ell)$ cannot be in $\overline{\mathcal{C}^{\geqslant 0} \cap \big(\mathbf{C}\backslash\{\mathbf{p}\}\big)}$, since $\overline{\mathcal{C}^{\geqslant 0} \cap \big(\mathbf{C}\backslash\{\mathbf{p}\}\big)} = \big\{p_{\underline{\sigma}}(\underline{\ell})\,;\, \underline{\ell} \geqslant q^{1,n-1}\big(\varepsilon_0 + \underline{\sigma}, \zeta_0\big)\big\}$.

5.(b) is a direct consequence of *5.(a)*

7. The proof of the point *7.(a)* relies on an algebraic lemma.

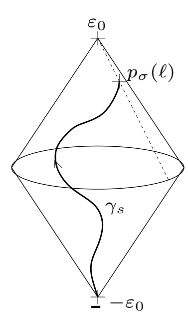
LEMMA **91.** *$PO(2,n)$ acts transitively on the pairs of points of $\mathbf{Ein}_n$ that are not on the same lightlike geodesic.*

> $\lhd$ Since any pair of points of $\mathbf{Ein}_n$ that are not on the same lightlike geodesic is of the form $\big(\pi(u), \pi(v)\big)$, with u and v null vectors, not orthogonal, we are brought back to show that $O(2,n)$ acts transitively on the set of such pairs (u,v), that is, $O(2,n)$ acts transitively on the set of planes of signature $(+,-)$. This is a well known (elementary) fact.
>
> $\rhd$

Let $\{\gamma_s\}_{s\geqslant 0}$ be a timelike path in $\mathbb{R}^{1,n-1}$, future-oriented and inextensible. The points γ_0 and $\mathbf{p}$ are not on the same lightlike geodesic of $\mathbf{Ein}_n$; from the lemma **91**, we can find an element T of $PO(2,n)$ that maps γ_0 and $\mathbf{p}$ to $-\varepsilon_0$ and ε_0 respectively. We show the statement *7.(a)* for the path $T\circ\gamma$, with $T(\mathbf{C})$ instead of $\mathbf{C}$. It implies the original statement.

We still note γ the path $T\circ\gamma$. Set $\mathcal{C}^{<0}\big(\varepsilon_0\big) = \big\{\xi \in \mathbb{R}^{1,n-1}\,;\, q^{1,n-1}(\xi - \varepsilon_0) < 0\big\}$.

Since γ is *timelike*, the problem now takes place in the diamond $\mathcal{D} \equiv \mathcal{C}^{>0}(-\varepsilon_0) \cap \mathcal{C}^{<0}(\varepsilon_0) \subset \mathbb{R}^{1,n-1}$.



One can easily show the following lemma.

LEMMA **92.** *The family $\{\mathcal{C}^{>0}(m - t\varepsilon_0) \cap \mathcal{C}^{<0}(m + t\varepsilon_0)\,;\, m \in \mathbb{R}^{1,n-1}, t > 0\}$ is a basis of the topology of $\mathbb{R}^{1,n-1}$.* [7]

[7] The statement is true in a much greater generality.

Definition **93.** *We call a neighbourhood of m of the form $\mathcal{C}^{\geqslant 0}(m - t\varepsilon_0) \cap \mathcal{C}^{\leqslant 0}(m + t\varepsilon_0)$ a* ***diamond neighbourhood of*** *m, noted $\mathcal{D}(m, t)$.*

Note $Cl(\gamma)$ the set of points of the past lightcone of ε_0 that are in the closure of γ, and suppose $Cl(\gamma)$ contains two distinct points p_1 and p_2. Let $\mathcal{D}(p_1, t_1)$ and $\mathcal{D}(p_2, t_2)$ two diamond neighbourhoods of p_1 and p_2, respectively, that do not intersect.

Since γ is a timelike path, and $p_1 \in Cl(\gamma)$, for any $s \geqslant 0$, the point $\gamma_s \in \mathcal{C}^{<0}(p_1)$. All the more, since $p_1 \in Cl(\gamma)$, the path γ enters $\mathcal{D}(p_1, t_1)$ at some time ; in particular, there is a time s_1 such that $\gamma_{s_1} \in \mathcal{C}^{>0}(p_1 - t_1\varepsilon_0)$. No timelike path started from a point of $\mathcal{C}^{>0}(p_1 - t_1\varepsilon_0)$ can exit from this set. So, γ remains in $\mathcal{C}^{>0}(p_1 - t_1\varepsilon_0) \cap \mathcal{C}^{\leqslant 0}(p_1) \subset D(p_1, t_1)$, from that time on. Thus, since $\mathcal{D}(p_1, t_1)$ and $\mathcal{D}(p_2, t_2)$ do not intersect, p_2 cannot belong to $Cl(\gamma)$.

We note γ_∞ the unique point of $\mathbf{C}$ in $Cl(\gamma)$.

7.(b) ***i)*** Let $\sigma \in \mathbb{S}^{n-2}$, $\ell \in \mathbb{R}$, be given, and $\{\gamma_s\}_{s\geqslant 0}$ be a timelike path such that $q^{1,n-1}(\gamma_s, \varepsilon_0+\sigma) \underset{s\to+\infty}{\longrightarrow} \ell$.

As $\{\gamma_s\}_{s\geqslant 0}$ is *timelike*, the function $s \in \mathbb{R}^{\geqslant 0} \to q^{1,n-1}\big(\gamma_s, \varepsilon_0 + \sigma\big)$ increases. So, from point *5.(a)*, the point γ_s belongs to the past of $p_\sigma(\ell)$ in $\mathbb{R}^{1,n-1}$, and from *5.(b)*, $\gamma_\infty \in \{p_\sigma(r)\,;\, r \leqslant \ell\}$.

Should γ_∞ be equal to some $p_\sigma(r)$, with $r < \ell$, then $q^{1,n-1}\big(\gamma_s, \varepsilon_0 + \sigma\big)$ would remain $\leqslant r$, from *5.(a)*, contradicting the hypothesis. This shows that $\gamma_\infty = p_\sigma(\ell)$.

ii) Reciprocally, suppose that $\gamma_\infty = p_\sigma(\ell)$, for some $\sigma \in \mathbb{S}^{n-2}$ and $\ell \in \mathbb{R}$. We know from 5.*(a)* that $q^{1,n-1}\big(\gamma_s, \varepsilon_0 + \sigma\big) \leqslant\leqslant$, for all $s \geqslant 0$. This function of s increases to a limit $r \leqslant \ell$. Point ***i)*** shows that $\gamma_\infty = p_\sigma(r)$, so $r = \ell$. Thus, given $\sigma \in \mathbb{S}^{n-2}$ and $\ell \in \mathbb{R}$,

$$\gamma_s \underset{s\to+\infty}{\longrightarrow} p_\sigma(\ell) \text{ iff } q^{1,n-1}(\gamma_s, \varepsilon_0 + \sigma) \underset{s\to+\infty}{\longrightarrow} \ell. \tag{3.1.9}$$

Since γ must converge to some point of $\mathbf{C}$ (*7.(a)*), we conclude from (3.1.9) that $\gamma_d \underset{s\to+\infty}{\longrightarrow} \mathbf{p}$ if we cannot find some $\sigma \in \mathbb{S}^{n-2}$ and $\ell \in \mathbb{R}$ such that (3.1.9) holds.

7.(c) Recall the definition of T given at the beginning of the proof of point *6.(a)*. If $\gamma \to \mathbf{p}$, then $T(\gamma_s) \to \varepsilon_0$ and the *past of* $T(\gamma_s)$ *in the diamond* $\mathcal{C}^{>0}(\widetilde{\varepsilon}_0) \cap \mathcal{C}^{<0}(\widetilde{\varepsilon}_0)$ converges to the whole diamond. In particular, any point of the image by T of the half line $\gamma_0 + \mathbb{R}^{\geqslant 0}\varepsilon_0$ eventually belongs to the past of $T(\gamma_s)$. So the whole half line $\gamma_0 + \mathbb{R}^{\geqslant 0}\varepsilon_0$ is included in $J^-(\gamma)$; but $J^-\big(\gamma_0 + \mathbb{R}^{\geqslant 0}\varepsilon_0\big) = \mathbb{R}^{1,n-1}$, so $J^-(\gamma) = \mathbb{R}^{1,n-1}$.

Now, let γ be a timelike path such that $\gamma_s \underset{s\to+\infty}{\longrightarrow} p_\sigma(\ell)$. We can find a timelike path $\{\rho_t\}_{t\geqslant 0}$, with $\rho_0 = \gamma_0$, which is eventually contained in the plane generated by $\widetilde{\varepsilon}_0$ and σ, and equal, in this plane, to

$$\rho_t := \big(\mathrm{ch}(t) - \ell\big)\sigma + \mathrm{sh}(t)\varepsilon_0.$$

Using point *6*, we see that the path $\{\rho_t\}_{t\geqslant 0}$ converges to $p_\sigma(\ell)$, and that

$$\mathcal{C}^{<0}(\rho_t) \underset{t\to+\infty}{\longrightarrow} \big\{\zeta \in \mathbb{R}^{1,n-1}\,;\, q^{1,n-1}\big(\zeta, \varepsilon_0 + \sigma\big) \leqslant \ell\big\}.$$

Any point $T(\rho_t)$ eventually belongs to $\mathcal{C}^{<0}\big(T(\gamma_s)\big) \cap \mathring{\mathcal{D}}$, since

$$\mathbf{1}_{\mathcal{C}^{<0}\left(T(\gamma_s)\right)\cap\mathring{\mathcal{D}}} \to \mathbf{1}_{\mathcal{C}^{<0}\left(T(p_\sigma(\ell))\right)\cap\mathring{\mathcal{D}}},$$

pointwise, and $T(\rho_t)$ belongs to $\mathcal{C}^{<0}\big(T(p_\sigma(\ell))\big) \cap \mathring{\mathcal{D}}$. So, $\rho\big(\mathbb{R}^{\geqslant 0}\big)$ belongs to $J^-(\gamma)$, and $J^-(\gamma_s)$ increases to $J^-\big(p_\sigma(\ell), \mathbb{R}^{1,n-1}\big) = \big\{\zeta \in \mathbb{R}^{1,n-1}\,;\, q^{1,n-1}\big(\zeta, \varepsilon_0 + \sigma\big)\big\}$.

3.2 An algebraic description of the boundary

The preceding section gives a geometric description of $Inv(\{u_s\})$. In this section, we provide an algebraic description of $Inv(\{u_s\})$. The way toward an algebraic treatment of the problem begins with the identification of the diffusion $\{u_s\}_{s\geqslant 0}$ as the projection of the diffusion $\{(g_s,\xi_s)\}_{s\geqslant 0}$ on the Poincaré group of affine isometries, with transition kernel $\{\widetilde{P}_s((g,\xi),.)$
$bigr\}_{s\geqslant 0}$.

We can make an algebraic study of the stochastic differential system driving $\{\mathbf{e}_s\}_{s\geqslant 0}$ by discretizing the time, looking at $\{\mathbf{e}_{n\delta}\}_{n\geqslant 0}$, for a fixed $\delta>0$. Because the vector fields generating the diffusion $\{\mathbf{e}_s\}_{s\geqslant 0}$ are left invariant, the sequence $\{\mathbf{e}_{n\delta}\}_{n\geqslant 0}$ has the law of a random walk on $\mathcal{G}$, with jump law $\widetilde{P}_\delta(\mathrm{Id},.)$. As concern the set of $\widetilde{L}$-harmonic bounded functions, we do not lose any information in that discretization.

PROPOSITION **94.** *Let $\delta>0$. A bounded function is $\widetilde{L}$-harmonic iff it is harmonic for the random walk on $\mathcal{G}$, with jump law $\widetilde{P}_\delta(Id,.)$.*

This proposition is proved in section **3.2.1**.

A lot of work was done to understand the long time behaviour of random walks on groups. Developping the work of Azencott [Aze70], Raugi succeeded in giving an algebraic description of the set of bounded harmonic functions associated with a random walk on a locally compact group, with denumerable basis. This part gives his description of the set of bounded $\widetilde{L}$-harmonic functions, through the study of the random walk.

We recall some definitions and basic facts about random walks on groups in section **3.2.1**. Section **3.2.2** describes the main algebraic tool we shall use: the Iwasawa decomposition of the semisimple group $SO_0(1,d)$. Using that decomposition, we examine the behaviour of several "coordinates" of the random walk in sections **3.2.3** and **3.2.4**, before giving an explicit description of the invariant σ-field of the random walk in section **3.2.5**. Then, we compare in section **3.2.6** the algebraic description of $Inv(\{\mathbf{e}_n\})$, given by Raugi's method, with the geometrical one obtained in the preceding section.

In the following, we note $\mathcal{G}=SO_0(1,d)\rtimes\mathbb{R}^{1,d}$.

3.2.1 Random walks on groups

a) Subadditive functions and moments of a measure on $\mathcal{G}$

DEFINITION. *A Borelian* ***function*** *$f:\mathcal{G}\to\mathbb{R}$ is said to be* ***subadditive*** *if*

$$\forall\, \mathbf{e},\mathbf{e}'\in\mathcal{G},\ \ f(\mathbf{e}\mathbf{e}')\leqslant f(\mathbf{e})+f(\mathbf{e}').$$

The function f is said to be a ***jauge*** *if there exists a constant $C\in\mathbb{R}$ such that*

$$\forall\, \mathbf{e},\mathbf{e}'\in\mathcal{G},\ \ f(\mathbf{e}\mathbf{e}')\leqslant f(\mathbf{e})+f(\mathbf{e}')+C.$$

The fundamental example of jauge is the jauge associated with a compact neighbourhood V of Id, generating $\mathcal{G}$([8]):

$$\forall \mathbf{e}\in\mathcal{G},\ f_V(\mathbf{e})=\inf\{n\geqslant 1\ ;\ \mathbf{e}\in V^n\}<\infty.$$

In our group $\mathcal{G}=SO_0(1,d)\times\mathbb{R}^{1,d}$, any compact neighbourhood V of Id generates $\mathcal{G}$. So, we drop the assumption "V generates $\mathcal{G}$" in the following.

The importance of this example comes from the following fact.

[8] $\mathcal{G}=\bigcup_{n\in\mathbb{N}}V^n$

Proposition **95 (Guivarch' [Gui76], p.**52)**.** *Let V be a compact neighbourhood of $Id \in \mathcal{G}$. For any jauge f, there exists a constant C, depending on V and f, such that*

$$f \leqslant C\, f_V.$$

As a consequence, all the jauges f_V are equivalent: if V' is another compact neighbourhood of Id, there exists constants $c(V,V')$ and $C(V,V')$ such that

$$c(V,V')f_V \leqslant f_{V'} \leqslant C(V,V')f_V.$$

As a result, if μ is a non negative measure on $\mathcal{G}$, and $p \in \mathbb{N}$, $\int f_V(\mathbf{e})^p \mu(d\mathbf{e})$ *is finite iff* $\int f_{V'}(\mathbf{e})^p \mu(d\mathbf{e})$ *is finite.*

Definition **96.** *Let $p \in \mathbb{N}$. A non negative Borel measure μ on $\mathcal{G}$ is said to have a* ***moment of order*** *p if*

$$\int f_V(\mathbf{e})^p \mu(d\mathbf{e}) < \infty.$$

This definition does not depend of the choice of the compact neighbourhood V of Id.

Remark – If $\|.\|$ is any algebra norm on $\mathcal{M}_{1+d}(\mathbb{R})$([9]), $\log \|.\|$ is subadditive. It follows from proposition **95** that if a measure μ on $\mathcal{G}$ has a first moment, $\log \|.\|$ is integrable with respect to μ. All norms on $\mathcal{M}_{1+d}(\mathbb{R})$ being equivalent, the result holds for any norm.

Proposition **97.** *The measure $\widetilde{P}_1(Id,.)$ has moments of any order.*

◁ We choose a particular V. To describe it, identify $SO_0(1,d)$ with the set of orthonormal frames on $\mathbb{H}$. Set

$$\widetilde{V} = \{(\dot{\xi},\mathcal{R}) \in \mathbb{OH}\ ;\ d(\dot{\xi},\varepsilon_0) \leqslant 1\},$$ [10]

and

$$V = \widetilde{V} \times B,$$

where B is the unit Euclidian ball of $\mathbb{R}^{1+d}$. We have

$$\widetilde{V}^n = \{(\dot{\xi},\mathcal{R}) \in \mathbb{OH}\ ;\ d(\dot{\xi},\varepsilon_0) \leqslant n\}.$$

We shall prove that one has

$$\sum_{n\geqslant 2} \mu\big((V^n)^c\big)n^p < +\infty,$$

for any $p \geqslant 0$, which implies that

$$\int f_V(\mathbf{e})^p \mu(d\mathbf{e}) = \mu(V) + \sum_{n\geqslant 2} \mu\left(V^n \backslash V^{n-1}\right) n^p < \infty.$$

The proof relies on the following elementary fact.

Lemma **98.** *Let $q \geqslant 1$, $\underline{g} \in \widetilde{V}^q$. The set $\{\underline{g}g\,;\, g \in \widetilde{V}^{2q}\}$ contains $\widetilde{V}^q$.*

[9] That is, we have $\|ab\| \leqslant \|a\|\|b\|$, for any matrices a, b.

[10] The notation $(\dot{\xi},\mathcal{R}) \in \mathbb{OH}$ means that $\dot{\xi} \in \mathbb{H}$ and $\mathcal{R}$ is an orthonormal basis of $T_{\dot{\xi}}\mathbb{H}$.

Notations – We note $\left[\frac{n}{9}\right]$ the integer part of $\frac{n}{9}$, and set $q_n = \left[\frac{n}{9}\right] - 1$.
Let $n \geqslant 18$. Given $\xi, \xi' \in B$, the point

$$(g_1, 0) \cdots (g_{q_n}, 0)(g_{q_n+1}, \xi)(g_{q_n+2}, 0) \cdots (g_{3q_n}, 0)(g_{3q_n+1}, \xi')(g_{3q_n+2}, 0) \cdots (g_{9q_n+3}, 0),$$

belongs to $\widetilde{V}^n$ and is equal to

$$(g_1 \cdots g_{9q_n+3}, g_1 \cdots g_{q_n}\xi + g_1 \cdots g_{q_n} g_{q_n+1} \cdots g_{3q_n}\xi').$$

We conclude from lemma **98** that

$$\widetilde{V}^{3q_n+1} \times \left(\widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B\right) \subset V^n.$$

The inclusion

$$\widetilde{V}^{q_n} \times \left(\widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B\right) \subset V^n,$$

will be sufficient to meet our purpose.
So,

$$\mu\big((V^n)^c\big) \leqslant \mu\Big(\Big(\widetilde{V}^{q_n} \times \left(\widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B\right)\Big)^c\Big) \leqslant \mu\left(\left(\widetilde{V}^{q_n}\right)^c \times \mathbb{R}^{1,d}\right) + \mu\left(\left\{(g,\xi) \in \mathcal{G} \; ; \; \xi \notin \left(\widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B\right)\right\}\right). \tag{3.2.1}$$

We majorize both terms of the right member of inequality (3.2.1) separately.
a) The first one is equal to $\widetilde{\mathbb{P}}_{\mathrm{Id}}\left(g_1 \notin \widetilde{V}^{q_n}\right)$. Noting (ρ_1, θ_1) the polar coordinates of the first vector of g_1 and $\mathbb{P}_{\varepsilon_0}$ the law of a Brownian motion on $\mathbb{H}$ started from ε_0,

$$\widetilde{\mathbb{P}}_{\mathrm{Id}}\left(g_1 \notin \widetilde{V}^{q_n}\right) \leqslant \mathbb{P}_{\varepsilon_0}\left(\rho_1 \geqslant q_n\right) \leqslant \mathbb{P}_{\varepsilon_0}\left(\sup_{s \in [0,1]} \rho_s \geqslant q_n\right)$$

We estimate this probability using the comparison theorem on the equation $d\rho_s = dw_s^\rho + \frac{d-1}{2}\coth(\rho_s)\, ds$, noting that $\coth(\rho) \leqslant 2$ as soon as $\rho \geqslant 2$. Precisely, one has

$$\sup_{s \in [0,1]} \rho_s \leqslant \sup_{s \in [0,1]} w_s^{(d-1),2},$$

where $w^{(d-1),2}$ is a Brownian motion with drift $(d-1)$, started from 2. Note $\mathbb{P}_2^{(d-1)}$ its law. Using formula 1.1.4 in [BS02], p.197, it follows that

$$\mathbb{P}_{(1,0)}\big(\sup_{s \in [0,1]} \rho_s \geqslant q_n\big) \leqslant \mathbb{P}_2^{(d-1)}\big(\sup_{s \in [0,1]} w_s^{(d-1),2} \geqslant q_n\big) \leqslant \frac{1}{2} f\left(\frac{q_n - d - 1}{\sqrt{2}}\right) + \frac{1}{2} e^{(d-1)(q_n-2)} f\left(\frac{q_n - 3 + d}{\sqrt{2}}\right), \tag{3.2.2}$$

with $f(x) = \frac{e^{-x^2}}{\sqrt{\pi}x}$. One concludes from this estimate that

$$\sum_{n \geqslant 2} \widetilde{\mathbb{P}}_{\mathrm{Id}}\left(g_1 \notin \widetilde{V}^{q_n}\right) n^p < \infty, \tag{3.2.3}$$

for any $p \geqslant 0$.
b) The second member is equal to $\widetilde{\mathbb{P}}_{\mathrm{Id}}\left(\xi_1 \notin \widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B\right)$.
To deal with it, we use the following lemma, whose proof is analogous to that of proposition **24**.
d denotes the hyperbolic distance in $\mathbb{H}$, and $B_{\mathbb{H}}(\varepsilon_0, R) = \{\zeta \in \mathbb{H} \, ; \; d(\varepsilon_0, \zeta) \leqslant R\}$.

LEMMA **99.** *Let $R > 0$. For any continuous path $\{\dot{\xi}_s\}_{s\in[0,1]}$ contained in $B_{\mathbb{H}}(\varepsilon_0, R)$, one has*

$$\int_0^1 \dot{\xi}_s \, ds \in ConvHull\big(B_{\mathbb{H}}(\varepsilon_0, R)\big) \subset \mathbb{R}^{1,d}.$$

LEMMA **100.** $ConvHull\big(B_{\mathbb{H}}(\varepsilon_0, q_n)\big) \subset \widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B$

○ In dimension $d = 1$, we see on figure 3.3 that the inclusion of the statement holds. In greater dimension, any point ξ of $ConvHull\big(B_{\mathbb{H}}(\varepsilon_0, q_n)\big)$ is in the plane $\langle \varepsilon_0, \xi \rangle$, where the 2-dimensional result applies.

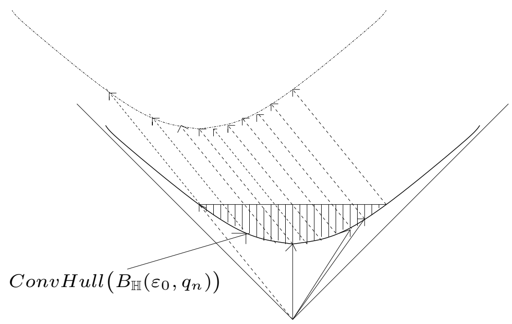


Figure 3.3: The dashed zone is in $\widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B$

○

So, we have

$$\mu\left(\{(g,\xi) \in \mathcal{G}\ ;\ \xi \notin \widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B\right) \leqslant \mathbb{P}_{\varepsilon_0}\left(\sup_{s\in[0,1]} \rho_s \geqslant q_n\right).$$

We use the same formula of [BS02] to obtain

$$\sum_{n\geqslant 2} \widetilde{\mathbb{P}}_{\mathrm{Id}}\left(\xi_1 \notin \widetilde{V}^{q_n}.B + \widetilde{V}^{q_n}.B\right) n^p < \infty,$$

for any $p \geqslant 0$. Together with (3.2.3) and (3.2.1), this inequality proves the result. ▷

b) Harmonic functions

Let μ be a probability on $\mathcal{G}$.

DEFINITION **101.** *Let $\mathbf{e} \in \mathcal{G}$. Any random sequence $\{\mathbf{e}_n\}_{n\geqslant 0}$ of points of $\mathcal{G}$ such that there exists a sequence $\{\underline{\mathbf{e}}_k\}_{k\geqslant 1}$ of iid $\mathcal{G}$-valued random variables, with law μ, satisfying the relation*

$$\mathbf{e}_n = \mathbf{e}\underline{\mathbf{e}}_1 ... \underline{\mathbf{e}}_n, \forall\, n \geqslant 1$$

is called a ***random walk on $\mathcal{G}$ with jump law μ, started from*** $\mathbf{e}$.

If we do not specify $\mathbf{e}$*, it means that we take* $\mathbf{e} = Id$.

Given $\mathbf{e}_0 \in \mathcal{G}$, note $\mathbb{P}^{\mu}_{\mathbf{e}_0}$ the image probability of $\mu^{\otimes \mathbb{N}^*}$ by the application

$$(\mathbf{e}_1, \mathbf{e}_2, ...) \mapsto (\mathbf{e}_0, \mathbf{e}_0\mathbf{e}_1, \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2, ...) \in \mathcal{G}^{\mathbb{N}}.$$

If $\mathbf{e}_n$ is the n^{th} coordinate on $\mathcal{G}^{\mathbb{N}}$, the sequence $\{\mathbf{e}_n\}_{n\geqslant 0}$ is under $\mathbb{P}^{\mu}_{\mathbf{e}_0}$ a right random walk on $\mathcal{G}$.

A μ-harmonic function is a function constant in mean along any trajectory of a right random walk with jump law μ. Precisely, DEFINITION **102.** *A Borel μ-integrable real* ***function*** *h on $\mathcal{G}$ is said to be* ***μ-harmonic*** *if for any $\mathbf{e} \in \mathcal{G}$,*

$$\int h(\mathbf{e}\mathbf{e}')\mu(d\mathbf{e}') = h(\mathbf{e}).$$

It is immediate to see that

LEMMA **103.** *The function h is harmonic iff $\{h(\mathbf{e}_n)\}_{n\geqslant 0}$ is a martingale with respect to the filtration generated by $\{\mathbf{e}_n\}_{n\geqslant 0}$.*

Dealing only with μ-(right) random walks, we shall talk of harmonic functions, with no more references to μ, without any risk of confusion. We shall note $\mathbb{P}_{\mathbf{e}_0}$ instead of $\mathbb{P}^{\mu}_{\mathbf{e}_0}$. We shall say "random walk" for "right random walk".

We now prove the proposition stated in the introduction of this part.

PROPOSITION **104.** *Let $\delta > 0$. A bounded function is $\widetilde{L}$-harmonic iff it is harmonic for the random walk on $\mathcal{G}$, with jump law $\widetilde{P}_\delta(Id,.)$.*

$\lhd$ • An $\widetilde{L}$-harmonic function obviously defines an harmonic function of the random walk.
• Let h be a bounded function, harmonic for the random walk. Setting $Y(\omega) = \lim h\big(\mathbf{e}_n(\omega)\big)$ when this limit exists, 0 elsewhere, we define an $Inv\big(\{\mathbf{e}_n\}\big)$-measurable random variable. As $Inv\big(\{\mathbf{e}_n\}\big) \subset \tau\big(\{\mathbf{e}_s\}\big)$ and $\tau\big(\{\mathbf{e}_s\}\big)$ and $Inv\big(\{\mathbf{e}_s\}\big)$ coincide under each $\mathbb{P}_{\mathbf{e}_0}$, up to $\mathbb{P}_{\mathbf{e}_0}$-null sets (Thereom **38**), proposition **32**, and the remark following it, shows that the function $\mathbf{e} \mapsto \mathbb{E}_{\mathbf{e}}[Y]$ is $\widetilde{L}$-harmonic. This function is equal to h. $\rhd$

PROPOSITION **105.** *1. Any bounded harmonic function h is right uniformly continuous.*

2. ***Right periods of a bounded harmonic function*** *– Let $\mathbf{e}_0 \in \mathcal{G}$, and h be a bounded harmonic function on $\mathcal{G}$. One has $\mathbb{P}_{\mathbf{e}_0}$-almost surely, for any $\mathbf{e} \in \mathrm{Supp}(\mu)$,*

$$\lim_{n\to+\infty} h(\mathbf{e}_n\mathbf{e}) = \lim_{n\to+\infty} h(\mathbf{e}_n).$$

Remark – The result of proposition **26** shows that $\mathrm{Supp}\,\mu$ is a semi-group.

$\lhd$ *1.* We already obtained this property as a corollary of Harnack's inequality (theorem **71**). We give here a simpler proof.
Let $\varepsilon > 0$. Let $\mathcal{V}$ be a compact neighbourhood of $\mathrm{Id} \in \mathcal{G}$.
On the one hand, $\widetilde{P}_1(.,.)$ inherits from the left invariance of the vectors fields generating $\{\widetilde{P}_s(.,.)\}_{s\geqslant 0}$ the invariance property:

$$\widetilde{p}_1(\mathbf{e},\mathbf{e}') = \widetilde{p}_1(\mathbf{e}^{-1}\mathbf{e}').$$

On the other hand, the family $\{\widetilde{P}_1(\widetilde{\mathbf{e}},.)\}_{\widetilde{\mathbf{e}}\in\mathcal{V}}$ is tight. Let K be compact subset of $\mathcal{G}$ such that

$$\forall\, \widetilde{\mathbf{e}} \in \mathcal{V} \quad \widetilde{P}_1(\widetilde{\mathbf{e}}, K^c) \leqslant \varepsilon.$$

Then, for any $\mathbf{e} \in \mathcal{G}$, $\widetilde{\mathbf{e}} \in \mathcal{V}$,

$$\begin{aligned}
|h(\mathbf{e}\widetilde{\mathbf{e}}) - h(\mathbf{e})| &= \left|\int \big\{\widetilde{p}_1\big((\mathbf{e}\widetilde{\mathbf{e}})^{-1}\mathbf{e}'\big) - \widetilde{p}_1\big(\mathbf{e}^{-1}\mathbf{e}'\big)\big\} h(\mathbf{e}')\, Haar(d\mathbf{e}')\right| \\
&= \left|\int \big\{\widetilde{p}_1\big(\widetilde{\mathbf{e}}^{-1}\mathbf{a}\big) - \widetilde{p}_1\big(\mathbf{a}\big)\big\} h(\mathbf{a})\, Haar(d\mathbf{a})\right| = \left|\int_{K^c} . + \int_K .\right| \\
&\leqslant 2\,\varepsilon \|h\|_\infty + \left|\int_K \big\{\widetilde{p}_1\big(\widetilde{\mathbf{e}}^{-1}\mathbf{a}\big) - \widetilde{p}_1\big(\mathbf{a}\big)\big\} h(\mathbf{a})\, Haar(d\mathbf{a})\right|
\end{aligned}$$

Since $\widetilde{p}_1$ is continuous[11], one can use dominated convergence theorem in the integral and obtain the existence of a neighbourhood $\widetilde{\mathcal{V}} \subset \mathcal{V}$ of Id$\in \mathcal{G}$ such that

$$\forall \mathbf{e} \in \mathcal{G}, \quad \sup_{\widetilde{\mathbf{e}} \in \widetilde{\mathcal{V}}} |h(\mathbf{e}\widetilde{\mathbf{e}}) - h(\mathbf{e})| \leqslant 2\, \|h\|_\infty \varepsilon + \varepsilon,$$

which shows the result.

2. Remark that

$$\mathbb{E}_{\mathbf{e}_0}\Big[\int \sum_{n\geqslant 0} (h(\mathbf{e}_n\mathbf{e}) - h(\mathbf{e}_n))^2 \mu(d\mathbf{e})\Big] = \sum_{n\geqslant 0} \big(\mathbb{E}_{\mathbf{e}_0}[h^2(\mathbf{e}_{n+1})] - \mathbb{E}_{\mathbf{e}_0}[h^2(\mathbf{e}_n)]\big) < +\infty.$$

It follows that for $\mu \otimes \mathbb{P}$-almost all $(\mathbf{e}, \omega) \in \mathcal{G} \times \Omega$,

$$\lim_{n\to+\infty} h(\mathbf{e}_n\mathbf{e}) = \lim_{n\to+\infty} h(\mathbf{e}_n).$$

Because of the right uniform continuity, we actually have a stronger result:

$$\mathbb{P}_{\mathbf{e}_0}\text{-almost surely for all } \mathbf{e} \in \mathrm{Supp}(\mu),\ \lim_{n\to+\infty} h(\mathbf{e}_n\mathbf{e}) = \lim_{n\to+\infty} h(\mathbf{e}_n).$$

▷

3.2.2 Iwasawa's decomposition of $SO_0(1,d)$

One can consult the book of Knapp [Kna02] for a general statement of Iwasawa's decomposition of semisimple Lie groups, as well as for basic definitions.

The Lie algebra of $SO_0(1,d)$ is

$$so(1,d) = \left\{ \begin{pmatrix} 0 & {}^t c \\ c & A \end{pmatrix} ;\ c \in \mathbb{R}^d, A \in so(d) \right\}.$$

Three subalgebras of $so(1,d)$ play an important role.

- $\mathfrak{a}$: the commutative algebra generated by the boost $\alpha \equiv \begin{pmatrix} 0 & 1 & (0) \\ 1 & 0 & (0) \\ (0) & (0) & \mathbf{0}_{d-1} \end{pmatrix}$ ([12]). $\mathfrak{a}$ is a maximal commutative subalgebra of $so(1,d)$.
- $\mathfrak{n} = \{X \in so(1,d)\ ;\ ad(\alpha)X = X\} = \left\{ \begin{pmatrix} 0 & 0 & {}^t h \\ 0 & 0 & {}^t h \\ h & -h & \mathbf{0}_{d-1} \end{pmatrix} ;\ h \in \mathbb{R}^{d-1} \right\}.$

FACT **106.** *1.* $\mathfrak{n}$ *is an Abelian subalgebra of* $so(1,d)$*, of dimension* $d-1$*:* $[\mathfrak{n},\mathfrak{n}] = 0$.

2. $\mathfrak{n}$ *is an ideal of* $\mathfrak{n} \oplus \mathfrak{a}$.

- $\mathfrak{k} = \left\{ \begin{pmatrix} 0 & \dots & 0 \\ \vdots & so(d) & \\ 0 & & \end{pmatrix} \right\}.$

[11] See the remarks following proposition **30**.
[12] We note $\mathbf{0}_{d-1}$ the $(d-1)\times(d-1)$ null matrix.

Set

- $\mathcal{A} = exp(\mathfrak{a}) = \left\{ \begin{pmatrix} \mathrm{ch}(t) & \mathrm{sh}(t) & (0) \\ \mathrm{sh}(t) & \mathrm{ch}(t) & (0) \\ (0) & (0) & \mathbf{1}_{d-1} \end{pmatrix} ;\ t \in \mathbb{R} \right\}$; it is a commutative subgroup of $SO_0(1,d)$ made up of hyperbolic rotations.
-

$$\mathcal{N} = exp(\mathfrak{n}) = \left\{ \begin{pmatrix} 1+\frac{|h|^2}{2} & -\frac{|h|^2}{2} & {}^t h \\ \frac{|h|^2}{2} & 1-\frac{|h|^2}{2} & {}^t h \\ h & -h & \mathrm{Id}_{d-1} \end{pmatrix} ;\ h \in \mathbb{R}^{d-1} \right\} (^{13}). \tag{3.2.4}$$

- $\mathcal{K} = exp(\mathfrak{k}) = \left\{ \begin{pmatrix} 1 & (0) \\ (0) & A \end{pmatrix} ;\ A \in SO_0(d) \right\}$.

PROPOSITION **107.** *1. $\mathcal{A}$ normalizes $\mathcal{N}$.*

2. For any $\zeta \in \mathbb{R}^{1,d}$ and any $N \in \mathcal{N}$, we have $\langle \varepsilon_0^* - \varepsilon_1^*, N\zeta \rangle = \langle \varepsilon_0^* - \varepsilon_1^*, \zeta \rangle$.

$\lhd$ *1. Let $X \in \mathfrak{n}$ and $A = e^{t\alpha} \in \mathcal{A}$.*

$$Ae^X A^{-1} = e^{t\alpha} eXe^{-t\alpha} = e^{Ad(e^{t\alpha})X} = e^{ad(e^{t\alpha})X} = e^{e^t X} \in \mathcal{N}.$$

2. For $n = \begin{pmatrix} 0 & 0 & {}^t h \\ 0 & 0 & {}^t h \\ h & -h & \mathbf{0}_{d-1} \end{pmatrix} \in \mathfrak{n}$, ${}^t n(\varepsilon_0 - \varepsilon_1) = 0$, so, $e^{{}^t n}(\varepsilon_0 - \varepsilon_1) = (\varepsilon_0 - \varepsilon_1)$. $\rhd$

The following decomposition is a special case of Iwasawa's decomposition of semisimple Lie groups. A proof can be found in [Kna02], p.374.

THEOREM **108 (Iwasawa).** *The multiplication $\mathcal{N} \times \mathcal{A} \times \mathcal{K} \to SO_0(1,d)$ given by $(N, A, K) \mapsto NAK$ is a diffeomorphism.*

Notations

- To distinguish the elements of $SO_0(1,d)$ from those of $so(1,d)$, we shall use bold letter or capital letters for the elements of the group and usual letters for those of the Lie algebra.
- It will be useful to define the projection $\pi_{\mathcal{N}} : SO_0(1,d) \to \mathcal{N}$ by $\pi_{\mathcal{N}}(\mathbf{g}) = N$, if $\mathbf{g} = NAK$ is Iwasawa's decomposition of $\mathbf{g}$. This map is smooth. We give analogous definitions of $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{K}}$.

Let $\{\mathbf{e}_n\}_{n\geqslant 0}$ be the random walk on $\mathcal{G}$, with jump law $\widetilde{P}_1(\mathrm{Id}, .)$. Note $\mathbf{e}_n = (\mathbf{g}_n, \xi_n) \in SO_0(1,d) \times \mathbb{R}^{1,d}$. We will use Iwasawa's decomposition of $SO_0(1,d)$ to understand the behaviour of $\mathbf{g}_n$. If we set

$$\mathbf{g}_n = N_n A_n K_n,$$

with $N_n \in \mathcal{N}$, $A_n = \begin{pmatrix} \mathrm{ch}(t_n) & \mathrm{sh}(t_n) & (0) \\ \mathrm{sh}(t_n) & \mathrm{ch}(t_n) & (0) \\ (0) & (0) & \mathbf{1}_{d-1} \end{pmatrix} \in \mathcal{A}$, $K_n \in \mathcal{K}$, we shall show that $\frac{t_n}{n} \underset{n\to+\infty}{\longrightarrow} -\frac{d-1}{2}$ and that $\{N_n\}_{n\geqslant 0}$ converge.

[13] $|h|$ is the Euclidian norm of $h \in \mathbb{R}^{d-1}$.

3.2.3 Behaviour of A_n and N_n

a) Behaviour of $\{A_n\}_{n\geqslant 0}$ – The hyperbolic rotation A_n is determined by t_n. One has a clear understanding of the behaviour of the sequence $\{t_n\}_{n\geqslant 0}$ if one looks at $\mathbf{g}_n$ as an element of $\mathbb{OH}$.

Let $\{\dot{\xi}_s\}_{s\geqslant 0}$ be a Brownian motion on $\mathbb{H}$, started from $\varepsilon_0 \in \mathbb{H}$, and $\{\dot{\xi}_s, \{\varepsilon_1(s), ..., \varepsilon_d(s)\}\}_{s\geqslant 0}$ be the path in $\mathbb{OH}$ obtained by parallel transport of the frame $\{\varepsilon_1, ..., \varepsilon_d\}$ along $\{\dot{\xi}_s\}_{s\geqslant 0}$. For any $s \geqslant 0$, the frame $\{\dot{\xi}_s, \varepsilon_1(s), ..., \varepsilon_d(s)\}$ of $\mathbb{R}^{1+d}$ belongs to $SO_0(1,d)$; in particular, g_n has the same law as $\{\dot{\xi}_n, \varepsilon_1(n), ..., \varepsilon_d(n)\}$. So, one can write

$$\dot{\xi}_n = N_n A_n K_n \varepsilon_0,$$

i.e.

$$\dot{\xi}_n = N_n A_n \varepsilon_0,$$

since $K_n\varepsilon_0 = \varepsilon_0$. Using the second point of proposition **107**, one obtains

$$\langle \varepsilon_0^* - \varepsilon_1^*, \dot{\xi}_n\rangle = \langle \varepsilon_0^* - \varepsilon_1^*, N_n A_n \varepsilon_0\rangle = \langle \varepsilon_0^* - \varepsilon_1^*, A_n \varepsilon_0\rangle = e^{-t_n}.$$

Write $\dot{\xi}_s = \mathrm{ch}(\rho_s) + \mathrm{sh}(\rho_s)\theta_s \in \mathbb{R}^{1,d}$, using the polar coordinates (ρ_s, θ_s) of $\dot{\xi}_s$. We saw in proposition **22** that

- $\rho_s = \frac{d-1}{2}s + o(s)$,
- $\langle \varepsilon_1^*, \theta_s\rangle = \theta_s^1$, almost surely converges toward a limit θ_∞^1 such that $|\theta_\infty^1| < 1$.

As a consequence,

$$\begin{aligned}\langle \varepsilon_0^* - \varepsilon_1^*, \dot{\xi}_n\rangle = \mathrm{ch}(\rho_n) - \mathrm{sh}(\rho_n)\theta_n^1 &= \frac{1-\theta_n^1}{2} e^{\rho_n} + o(1)\\ &= \frac{1-\theta_n^1}{2} e^{\frac{d-1}{2}n + o(n)} + o(1) = e^{-t_n}.\end{aligned}$$

PROPOSITION **109 (Behaviour of $\{t_n\}_{n\geqslant 0}$).** *One has the almost sure convergence* $\frac{t_n}{n} \underset{n\to+\infty}{\longrightarrow} -\frac{d-1}{2}$.

b) Behaviour of $\{N_n\}_{n\geqslant 0}$ Notations – $\{\mathbf{g}_n\}_{n\geqslant 0}$ is a random walk on $SO_0(1,d)$, with jump law $\nu(.) \equiv \widetilde{\mathbb{P}}_{\mathrm{Id}}(\mathbf{g}_1 \in .)$. Write

$$\mathbf{g}_n = Y_1 ... Y_n,$$

where the Y_i's are iid, with law ν. As a corollary of proposition **97**, we have the

COROLLARY **110.** *The probability ν has moments of any order.*

The identity

$$\mathbf{g}_{n+1} = \mathbf{g}_n Y_{n+1}$$

can be written as

$$\begin{aligned}N_{n+1}A_{n+1}K_{n+1} &= N_n A_n K_n Y_{n+1} = N_n A_n \pi_{\mathcal{N}}(K_n Y_{n+1}) \pi_{\mathcal{A}}(K_n Y_{n+1}) \pi_{\mathcal{K}}(K_n Y_{n+1})\\ &= \big(N_n A_n \pi_{\mathcal{N}}(K_n Y_{n+1}) A_n^{-1}\big)\big(A_n \pi_{\mathcal{A}}(K_n Y_{n+1})\big)\pi_{\mathcal{K}}(K_n Y_{n+1}).\end{aligned} \tag{3.2.5}$$

Since $\mathcal{A}$ normalizes $\mathcal{N}$, $A_n \pi_{\mathcal{N}}(K_n Y_{n+1}) A_n^{-1} \in \mathcal{N}$, and the right hand side of (3.2.5) is the unique Iwasawa decomposition of $\mathbf{g}_{n+1}$.

$$N_{n+1} = N_n A_n \pi_{\mathcal{N}}(K_n Y_{n+1}) A_n^{-1} \tag{3.2.6}$$

So, we will get the convergence of the sequence $\{N_n\}_{n\geqslant 0}$ if we can show that

$$\overline{\lim}\, \|\mathrm{Id} - A_n\pi_{\mathcal{N}}(K_nY_{n+1})A_n^{-1}\|^{1/n} < 1. \tag{3.2.7}$$

If one writes $\pi_{\mathcal{N}}(K_nY_{n+1}) = e^{h_n}$, one has

$$A_n\pi_{\mathcal{N}}(K_nY_{n+1})A_n^{-1} = e^{ad(e^{t_n\alpha}h_n)} = e^{e^{t_n}h_n},$$

because of the definition of $\mathcal{N}$. We see on the explicit formula (3.2.4) for $e^{e^{t_n}h_n}$ that since $\overline{\lim}\, e^{\frac{t_n}{n}} < 1$, the inequality

$$\overline{\lim}\, \|\mathrm{Id} - A_n\pi_{\mathcal{N}}(K_nY_{n+1})A_n^{-1}\|^{1/n} < 1$$

is equivalent to

$$\overline{\lim}\, \|h_n\|^{\frac{1}{n}} \leqslant 1.$$

We see on formula (3.2.4) that this last condition is equivalent to $\overline{\lim}\, \|e^{h_n}\|^{\frac{1}{n}} \leqslant 1$.

For each n,

$$\|e^{h_n}\| = \|\pi_{\mathcal{N}}(K_nY_{n+1})\| \leqslant \sup_{K\in\mathcal{K}} \|\pi_{\mathcal{N}}(KY_{n+1})\|.$$

Contrarily to the non iid sequence $\{e^{h_n}\}_{n\geqslant 0}$, the sequence $\big\{\sup_{K\in\mathcal{K}} \|\pi_{\mathcal{N}}(KY_{n+1})\|\big\}_{n\geqslant 0}$ is iid. An elementary lemma asserts that one has almost surely

$$\overline{\lim}\, \big\{\sup_{K\in\mathcal{K}} \|\pi_{\mathcal{N}}(KY_{n+1})\|\big\}^{\frac{1}{n}} \leqslant 1$$

if $\log\big(\sup_{K\in\mathcal{K}} \|\pi_{\mathcal{N}}(KY_1)\|\big)$ is an integrable random variable.

We know that $\log \|Y_1\|$ is integrable[14]. Or, since $\pi_{\mathcal{NA}}(KY_1) = KY_1\pi_{\mathcal{K}}(KY_1)^{-1}$, and $\mathcal{K}$ is compact, there exists constants $C_1, C_2 > 0$ (depending only on the norm we use) such that

$$\|\pi_{\mathcal{NA}}(KY_1)\| \leqslant C_1\|KY_1\| \leqslant C_2\|Y_1\|,$$

for all $K \in \mathcal{K}$. Now, if one chooses the norm $\|(a_{ij})\| = \max_{i,j}|a_{ij}|$, we see on the explicit formula

$$e^he^{t\alpha} = \begin{pmatrix} (\mathrm{ch}t)\left(1+\frac{|h|^2}{2}\right) - (\mathrm{sh}t)\frac{|h|^2}{2} & (\mathrm{sh}t)\left(1+\frac{|h|^2}{2}\right) - (\mathrm{ch}t)\frac{|h|^2}{2} & {}^th \\ (\mathrm{ch}t)\frac{|h|^2}{2} - (\mathrm{sh}t)\left(1-\frac{|h|^2}{2}\right) & (\mathrm{sh}t)\frac{|h|^2}{2} + (\mathrm{ch}t)\left(1-\frac{|h|^2}{2}\right) & {}^th \\ e^{-t}h & -e^{-t}h & \mathrm{Id}_{d-1} \end{pmatrix}$$

that $\|e^h\| \leqslant \|e^he^{t\alpha}\|$; so, for any $K \in \mathcal{K}$,

$$\|\pi_{\mathcal{N}}(KY_1)\| \leqslant \|\pi_{\mathcal{NA}}(KY_1)\| \leqslant C_2\|Y_1\|.$$

The conclusion comes from the integrability of $\log \|Y_1\|$.

Proposition 111. *The sequence $\{N_n\}_{n\geqslant 0}$ almost surely converges.*

[14] Remember the remark following the definition of the moment of a measure on $\mathcal{G}$.

3.2.4 Back to the group $\mathcal{G} = SO_0(1,d) \rtimes \mathbb{R}^{1+d}$

The convergence of $\{N_n\}_{n\geqslant 0}$ is analogous to the convergence of $\{\sigma_s\}_{s\geqslant 0}$, in the diffusion framework. We now find the equivalent of $R^{\sigma_\infty}_\infty$. To make things easier to handle, we make a change of basis.

Set $a = \mathrm{diag}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \mathrm{Id}_{d-1}\right)$.

LEMMA **112.** *The application $\varphi_a : (\mathbf{g},\xi) \in SO_0(1,d) \rtimes \mathbb{R}^{1+d} \mapsto (a\mathbf{g}a^{-1}, a\xi) \in aSO_0(1,d)a^{-1} \rtimes \mathbb{R}^{1+d}$ is a group isomorphism.*

$\lhd$

$$\begin{aligned}\big(\varphi_a.(\mathbf{g},\xi)\big)\big(\varphi_a(\mathbf{g}',\xi')\big) &= (a\mathbf{g}a^{-1}, a\xi)(a\mathbf{g}'a^{-1}, a\xi') = (a\mathbf{g}\mathbf{g}'a^{-1}, a\xi + a\mathbf{g}\xi')\\ &= \varphi_a\big((\mathbf{g},\xi)(\mathbf{g}',\xi')\big)\end{aligned}$$

$\rhd$

We shall note $\mathcal{G}^a = aSO(1,d)a^{-1} \rtimes \mathbb{R}^{1+d}$ and μ^a the image of μ by the isomorphism φ_a. The sequence $\{\varphi_a(\mathbf{e}_n)\}_{n\geqslant 0}$ is a random walk on $\mathcal{G}^a$, with jump law μ^a. We note $\mathbb{P}^a_{\mathbf{e}}$ its law, if it starts from $\mathbf{e} \in \mathcal{G}^a$. We shall note $\mathbf{y}_i$ its jumps, so that $\mathbf{e}_n = \mathbf{e}\mathbf{y}_1...\mathbf{y}_n$.

Decomposition of $\mathcal{G}^a$ – $\mathcal{G}^a$ is non longer a semi-simple Lie group, as $SO_0(1,d)$ is. Its semi-simple part $aSO_0(1,d)a^{-1}$ has a Lie algebra equal to

$$\left\{\begin{pmatrix} t & 0 & \sqrt{2}h^t\\ 0 & -t & \sqrt{2}k^t\\ \sqrt{2}k & \sqrt{2}h & (c)\end{pmatrix}\ ;\ t\in\mathbb{R},\ h,k\in\mathbb{R}^{d-1},\ (c)\in so(d-1)\right\}.$$

Note

$$aso(1,d)a^{-1} = \mathfrak{n}^a \oplus \mathfrak{a}^a \oplus \mathfrak{k}^a,$$

Iwasawa's decomposition of $aso(1,d)a^{-1}$.

- $\mathfrak{a}^a = \mathbb{R}\,\alpha^a = \mathbb{R}\,\mathrm{diag}(1,-1,\mathbf{0}_{d-1})$,
- $\mathfrak{n}^a = \{X \in aso(1,d)a^{-1}; \mathrm{ad}(\alpha^a)X = X\} = \left\{\begin{pmatrix} 0 & 0 & \sqrt{2}^t h\\ 0 & 0 & (0)\\ (0) & \sqrt{2}h & \mathbf{0}_{d-1}\end{pmatrix}\ ;\ h\in\mathbb{R}^{d-1}\right\}$,
- $\mathfrak{k}^a = a\mathfrak{k}a^{-1}$.

The corresponding groups are

- $\mathcal{A}^a = \{\mathrm{diag}(e^t, e^{-t}, 1, .., 1);\ t\in\mathbb{R}\}$,
- $\mathcal{N}^a = \left\{\begin{pmatrix} 1 & |h|^2 & \sqrt{2}^t h\\ 0 & 1 & (0)\\ (0) & \sqrt{2}h & \mathrm{Id}_{d-1}\end{pmatrix}\ ;\ h\in\mathbb{R}^{d-1}\right\}$,
- $\mathcal{K}^a = a\mathcal{K}a^{-1}$.

The Lie algebra ad operator is $ad\big((g,\xi)\big)(g',\xi') = \big([g,g'], g\xi' - [g,g']\xi\big)$.

The subvector space

$$\mathfrak{d}^a \equiv \mathbb{R}^{d+1} \oplus \mathfrak{n}^a \oplus \mathfrak{a}^a$$

of $\mathfrak{g}^a$ is a subalgebra of the Lie algebra $\mathfrak{g}^a$ of $\mathcal{G}^a$, and $\mathbb{R}^{d+1} \oplus \mathfrak{n}^a$ is an ideal of $\mathfrak{d}^a$. Define

- $\mathfrak{h}^a = \mathbb{R}^{1+d} \oplus \mathfrak{n}^a$,
- $\mathfrak{d}_-^a = \mathbb{R}\,\varepsilon_0 \oplus \mathfrak{n}^a$,
- $\mathfrak{d}_+^a = \langle \varepsilon_1, ..., \varepsilon_d \rangle \oplus \mathfrak{a}^a$,

so that $\mathfrak{d}^a = \mathfrak{d}_-^a \oplus \mathfrak{d}_+^a$.

$\mathfrak{d}_-^a$ and $\mathfrak{d}_+^a$ are Lie subalgebras of $\mathfrak{g}^a$, with associated groups $\mathcal{D}_-^a \equiv \mathcal{N}^a \times \mathbb{R}\varepsilon_0$ and $\mathcal{D}_+^a \equiv \mathcal{A}^a \times \langle \varepsilon_1, ..., \varepsilon_d \rangle$, respectively. $\mathfrak{d}^a$ is the Lie algebra of the group $\mathbb{R}^{1+d} \times \mathcal{N}^a \mathcal{A}^a$.

Since $\varepsilon_0 \in \ker(h), \forall\, h \in \mathfrak{n}^a$,

$$e^{r\varepsilon_0 \oplus h} = \left(e^h, er\varepsilon_0\right) \in \mathcal{D}_-^a.$$

So $(\mathcal{D}_-^a, .)$ is isomorphic to $(\mathfrak{d}_-^a, +)$.

The application

$$\mathcal{D}_-^a \times \mathcal{D}_+^a \times \mathcal{K}^a \to \mathcal{G}^a, \quad (\mathbf{d}_-, \mathbf{d}_+, \mathbf{k}) \mapsto \mathbf{d}_- \mathbf{d}_+ \mathbf{k} \tag{3.2.8}$$

is an isomorphism of analytical manifolds. Note p_-^a the projection on $\mathcal{D}_-^a$.

THEOREM **113 (Convergence theorem).** *The component $p_-^a(\mathbf{e}_n) = \left(e^{h_n}, r_n\varepsilon_0\right) \in \mathcal{D}_-^a$ of the random walk $\{\mathbf{e}_n\}_{n\geqslant 0}$ on $\mathcal{G}^a$, with jump law μ^a, converges $\mathbb{P}_{\mathbf{e}_0}^a$-almost surely, for every $\mathbf{e}_0 \in \mathcal{G}^a$.*

$\triangleleft$ ***i)*** The convergence of $\{e^{h_n}\}_{n\geqslant 0}$ was proved above (**3.2.3**).

ii) An elementary calculus shows the following lemma.

LEMMA **114.** *Let $\left(e^h, r\,\varepsilon_0\right) \in \mathcal{D}_-^a$ and $\mathbf{d} = \left(e^{\underline{h}}, \underline{r}\,\varepsilon_0\right)\left(e^{\underline{t}\alpha^a}, (0, \underline{\xi}^1, \underline{\xi}^{\geqslant 2})\right)$. We have*

$$p_-^a\left(\mathbf{d}\left(e^h, r\,\varepsilon_0\right)\right) = \left(e^{\underline{h} + e^{\underline{t}} h}, e^{\underline{t}} r + b(\mathbf{d})\right),$$

for some real $b(\mathbf{d})$ depending only on $\mathbf{d}$. So, $\mathbf{d}$ acts on the ε_0-part of $\mathcal{D}_-^a$ as an affine transform. Note it $f_{\mathbf{d}}(.)$.

Now,

$$\mathbf{e}_{n+1} = \mathbf{d}_{n+1}\mathbf{k}_{n+1} = \mathbf{e}_n\mathbf{y}_{n+1} = \mathbf{d}_n\mathbf{k}_n\mathbf{y}_{n+1} = \mathbf{d}_n\pi_{\mathbf{d}}(\mathbf{k}_n\mathbf{y}_{n+1})\pi_{\mathcal{K}}(\mathbf{k}_n\mathbf{y}_{n+1}),$$

so that

$$\mathbf{d}_{n+1} = \pi_{\mathcal{D}^a}(\mathbf{e}_0\mathbf{y}_1)\pi_{\mathcal{D}^a}(\mathbf{k}_1\mathbf{y}_2)...\pi_{\mathcal{D}^a}(\mathbf{k}_n\mathbf{y}_{n+1}). \tag{3.2.9}$$

Note $p_-^a(\mathbf{e}_n) = p_-^a(\mathbf{d}_n) = \left(e^{h_n}, r_n\varepsilon_0\right)$. Formula (3.2.9) tells us that

$$r_n = f_{\pi_{\mathcal{D}^a}(\mathbf{y}_1)} \circ f_{\pi_{\mathcal{D}^a}(\mathbf{k}_1\mathbf{y}_2)} \circ \cdots \circ f_{\pi_{\mathcal{D}^a}(\mathbf{k}_{n-1}\mathbf{y}_n)}(0).$$

Note $f_{\pi_{\mathcal{D}^a}(\mathbf{k}_{n-1}\mathbf{y}_n)}(r) = a_n r + b_n$ and write

$$M\left(\pi_{\mathcal{D}^a}(\mathbf{k}_{n-1}\mathbf{y}_n)\right) = \begin{pmatrix} a_n & b_n \\ 0 & 1 \end{pmatrix}$$

the matrix of $f_{\pi_{\mathcal{D}^a}(\mathbf{k}_{n-1}\mathbf{y}_n)}$.

Set, for $\mathbf{e} \in \mathcal{G}$,

$$\phi(\mathbf{e}) \equiv \sup_{\mathbf{k}\in\mathcal{K}^a} \log\left\|M\left(\pi_{\mathcal{D}^a}(\mathbf{k}\mathbf{e})\right)\right\|,$$

where $\|.\|$ is the endomorphism norm. $\mathcal{K}^a$ being compact, the supremum is finite.

LEMMA **115.** *For any $\mathbf{e}, \mathbf{e}' \in \mathcal{G}^a$, $f(\mathbf{e}\mathbf{e}') \leqslant f(\mathbf{e}) + f(\mathbf{e}')$.*

○ For any $\mathbf{k} \in \mathcal{K}^a$, $\mathbf{e}, \mathbf{e}' \in \mathcal{G}^a$,

$$\pi_{\mathcal{D}^a}(\mathbf{kee}') = \pi_{\mathcal{D}^a}\big(\pi_{\mathcal{D}^a}(\mathbf{ke})\pi_{\mathcal{K}^a}(\mathbf{ke})\mathbf{e}'\big) = \pi_{\mathcal{D}^a}(\mathbf{ke})\pi_{\mathcal{D}^a}\big(\pi_{\mathcal{K}^a}(\mathbf{ke})\mathbf{e}'\big).$$

The inequality of the lemma is a direct consequence of this identity. ○

The function ϕ being *subadditive*, is bounded by a multiple of a jauge (proposition **95**). Since μ^a has a first moment, and $\phi \geqslant 0$, one has

$$\int \phi(\mathbf{e})\mu(d\mathbf{e}) < \infty;$$

so, for any $c > 0$,

$$\sum_{p \geqslant 1} \mu(\phi \geqslant p\,c) < \infty.$$

As we have

$$\sum_{p \geqslant 2} \mathbb{P}\big(\log \|M\big(\pi_{\mathcal{D}^a}(\mathbf{k}_{p-1}\mathbf{y}_p)\big)\| \geqslant p\,c\big) \leqslant \sum_{p \geqslant 2} \mathbb{P}\big(\phi(\mathbf{y}_p) \geqslant p\,c\big) = \sum_{p \geqslant 2} \mu(\phi \geqslant p\,c) < \infty,$$

this implies that

$$\overline{\lim}\, \|M\big(\pi_{\mathcal{D}^a}(\mathbf{k}_{p-1}\mathbf{y}_p)\big)\|^{1/p} \leqslant e^c,$$

for any $c > 0$. Thus, one has $\mathbb{P}^a_{\mathbf{e}_0}$-almost surely

$$\overline{\lim}\, \|M\big(\pi_{\mathcal{D}^a}(\mathbf{k}_{p-1}\mathbf{y}_p)\big)\|^{1/p} \leqslant 1. \tag{3.2.10}$$

In particular,

$$\overline{\lim}\, |b_n|^{\frac{1}{n}} \leqslant 1.$$

Now, as

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & b_{n-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 \cdots a_n & b_1 + a_1 b_2 + a_1 a_2 b_3 + a_1 \cdots a_{n-1} b_n \\ 0 & 1 \end{pmatrix},$$

and $a_1 \cdots a_n = e^{t_n}$, we have $\overline{\lim}\, \|a_1 \cdots a_n\|^{\frac{1}{n}} = e^{-\frac{d-1}{2}} < 1$; so

$$r_n = b_1 + a_1 b_2 + a_1 a_2 b_3 + a_1 \cdots a_{n-1} b_n$$

converges as $n \to +\infty$. ▷

Notation – Note (N_∞, r_∞) the $\mathbb{P}^a_{\mathbf{e}_0}$-almost sure limit of $\{(N_n, r_n)\}_{n \geqslant 0}$.

We are now ready to determine the Poisson boundary of the random walk $\{\mathbf{e}_n\}_{n \geqslant 0}$ on $\mathcal{G}$.

3.2.5 Poisson boundary of the random walk

Noting that a bounded Borel function h on $\mathcal{G}$ is μ-harmonic iff the function $h \circ \varphi_a^{-1}$ is μ^a-harmonic, a description of the set of bounded μ^a-harmonic functions gives a description of the set of bounded μ-harmonic functions. So we consider the situation in $\mathcal{G}^a$ and note $\{\mathbf{e}_n\}_{n\geqslant 0}$ the μ^a-random walk started from Id; note $\mathbb{P}^a$ its law.

The sequence $\{\mathbf{e}\mathbf{e}_n\}_{n\geqslant 0}$ is under $\mathbb{P}^a$ a random walk started from $\mathbf{e}$.

Notations • Note $Y(\mathbf{e},\omega)$ the $\mathbb{P}^a$-almost sure limit of the sequence $\{p_-^a(\mathbf{e}\mathbf{e}_n(\omega))\}_{n\geqslant 0}$: $Y(\mathbf{e},\omega) = (N_\infty, r_\infty \varepsilon_0)$.

• Let h be a bounded μ^a-harmonic function on $\mathcal{G}^a$. We note $Z_h(\mathbf{e},\omega)$ the $\mathbb{P}^a$-almost sure limit: $\lim_{n\to+\infty} h(\mathbf{e}\mathbf{e}_n(\omega))$.

Theorem **116 (Poisson boundary of the random walk).** *For any $\mathbf{e} \in \mathcal{G}^a$, the σ-algebras $Inv(\{\mathbf{e}_n\})$ and $\sigma(r_\infty, h_\infty)$ coincide up to $\mathbb{P}^a_{\mathbf{e}}$-null sets. With different words, the Poisson boundary of the random walk $\{\mathbf{e}_n\}_{n\geqslant 0}$ on $\mathcal{G}^a$ (or $\mathcal{G}$) can be identified with $\mathcal{G}^a/\mathcal{D}^a_+\mathcal{A}^a\mathcal{K}^a$.*

Lemma **117 (Fundamental lemma).** *It suffices to show that any bounded $\mathcal{D}^a_-$-left invariant μ^a-harmonic function is constant.*

$\lhd$ Since $Inv(\{\mathbf{e}_n\})$ is generated under $\mathbb{P}^a_{\mathbf{e}}$ by the random variables of the form $Z_h(\mathbf{e},.)$, we show that, under the hypothesis of the statement, any such random variable coincide with a measurable function of $Y(\mathbf{e},.)$, up to $\mathbb{P}^a_{\mathbf{e}}$-null sets.
As h is right uniformly continuous (Corollary **72**), there is a measurable set $\Omega_1 \subset \Omega$, of $\mathbb{P}^a$-probability 1 such that $\forall\,\omega \in \Omega_1$, $\forall\,\mathbf{e} \in \mathcal{G}^a$, the limit $Z_h(\mathbf{e},\omega)$ exists.
So, given $\mathbf{e} \in \mathcal{G}^a$, we define a random variable setting

$$\phi_{\mathbf{e}}(\mathbf{e}',\omega) = Z\big(\mathbf{e}Y(\mathbf{e}',\omega)^{-1}\mathbf{e}',\omega\big).$$

This random variable is shift invariant as a function of ω, and $\mathcal{D}^a_-$-left invariant as a function of $\mathbf{e}'$. We conclude from our hypothesis that it must be $\mathbb{P}^a$-almost surely a constant function of $\mathbf{e}'$. Note $c(\mathbf{e})$ this constant.
It is elementary to check that the application $\mathbf{e} \in \mathcal{G}^a \to c(\mathbf{e})$ is measurable. For $\omega \in \Omega_1$, for any $\mathbf{e},\mathbf{e}' \in \mathcal{G}^a$,

$$Z\big(\mathbf{e}Y(\mathbf{e}',\omega)^{-1}\mathbf{e}',\omega\big) = c(\mathbf{e}).$$

Taking $\mathbf{e} = Y(\mathbf{e}',\omega)$ yields $Z(\mathbf{e}',\omega) = c\big(Y(\mathbf{e}',\omega)\big)$. $\rhd$

It remains to show the main point.

Proposition **118.** *Any bounded $\mathcal{D}^a_-$-left invariant μ^a-harmonic function is constant.*

$\lhd$ ***1)*** The proof relies on a
Lemma **119.** *Let $\mathbf{e} \in \mathcal{G}^a$. Suppose $\mathbf{x} \in \mathcal{G}^a$ is such that one can write $\mathbf{x}\mathbf{e}\mathbf{e}_n = \mathbf{e}\mathbf{e}_n\mathbf{x}'_n$, where the sequence $\{\mathbf{x}'_n\}_{n\geqslant 0}$ has $\mathbb{P}^a$-almost surely a converging subsequence. Then, for any μ^a-harmonic function h, one has*

$$h(\mathbf{x}\mathbf{e}) = h(\mathbf{e}),$$

for any $\mathbf{e} \in \mathcal{G}^a$.

○ **a)** We first show that property *2* of proposition **105** holds with the set

$$T^a_{\geqslant 0} \equiv \{\mathbf{e} = (g,\xi) \in \mathcal{G}^a\,;\, q(a^{-1}\xi) \geqslant 0\}(^{15})$$

instead of the support of μ^a (which is a semi-group).

[15] This set is equal to the image by φ_a of the support of the diffusion $\{\mathbf{e}_s\}_{s\geqslant}$ on $\mathcal{G}$, started from $(\mathrm{Id}, 0)$.

μ^a is the time 1 of a family $\{\widetilde{P}_s^a(\mathbf{e},.)\}_{s\geqslant 0,\mathbf{e}\in\mathcal{G}^a}$ of (smooth) transition kernels, associated with a left invariant stochastic differential equation on $\mathcal{G}^a$.[16]

$$\mu^a = \widetilde{P}_1^a(\mathrm{Id},.).$$

We know from proposition **104** that for any $p \geqslant 1$, h is $\widetilde{P}_{\frac{1}{p}}^a(\mathrm{Id},.)$-harmonic. Note $\mathbb{P}_{\mathbf{e}_0}^{a,\frac{1}{p}}$ the law of the $\widetilde{P}_{\frac{1}{p}}^a(\mathrm{Id},.)$-random walk started from $\mathbf{e}_0 \in \mathcal{G}^a$, and $T_{\frac{1}{p}}^a$ the support of $\widetilde{P}_{\frac{1}{p}}^a(\mathrm{Id},.)$, which is a semi-group. Applying proposition **105**, one gets

$$\mathbb{P}_{\mathbf{e}_0}^{a,\frac{1}{p}}\text{-almost surely}, \forall\, \mathbf{e} \in T_{\frac{1}{p}}^a, \quad \lim h(\mathbf{e}_n\mathbf{e}) = \lim h(\mathbf{e}_n);$$

this implies that

$$\mathbb{P}_{\mathbf{e}_0}^{a}\text{-almost surely}, \forall\, \mathbf{e} \in T_{\frac{1}{p}}^a, \quad \lim h(\mathbf{e}_n\mathbf{e}) = \lim h(\mathbf{e}_n).$$

Then, use the right uniform continuity of h to say that

$$\mathbb{P}_{\mathbf{e}_0}^{a}\text{-almost surely}, \forall\, \mathbf{e} \in \overline{\bigcup_{p\geqslant 1} T_{\frac{1}{p}}^a}, \quad \lim h(\mathbf{e}_n\mathbf{e}) = \lim h(\mathbf{e}_n).$$

The set $\overline{\bigcup_{p\geqslant 1} T_{\frac{1}{p}}^a}$ is exactly the set $T_{\geqslant 0}^a$. It enjoys the property that $\left(T_{\geqslant 0}^a\right)\left(T_{\geqslant 0}^a\right)^{-1} = \mathcal{G}^a$.

b) Now, let $\mathbf{x} \in \mathcal{G}^a$ as in the statement. We can write

$$\lim \mathbf{x}'_{n_p} = \mathbf{s}\mathbf{t}^{-1},$$

with $\mathbf{s},\mathbf{t} \in T_{\geqslant 0}^a$, for some increasing subsequence $\{n_p\}_{p\geqslant 0}$. Then, for any $n \geqslant 0$,

$$h(\mathbf{x}\mathbf{e}\mathbf{e}_n) = h(\mathbf{x}\mathbf{e}\mathbf{e}_n\mathbf{t}) = h(\mathbf{e}\mathbf{e}_n\mathbf{x}'_n\mathbf{t}).$$

Using the right uniform continuity of h, one gets the $\mathbb{P}^a$-almost sure equalities

$$\begin{aligned}\lim_{n\to+\infty} h(\mathbf{x}\mathbf{e}\mathbf{e}_n) &= \lim_{p\to+\infty} h(\mathbf{x}\mathbf{e}\mathbf{e}_{n_p}) = \lim_{p\to+\infty} (\mathbf{e}\mathbf{e}_{n_p}\mathbf{x}'_{n_p}\mathbf{t}) \\ &= \lim_{p\to+\infty} h(\mathbf{e}\mathbf{e}_{n_p}\mathbf{s}) = \lim_{n\to+\infty} h(\mathbf{e}\mathbf{e}_n\mathbf{s}) = \lim_{n\to+\infty} h(\mathbf{e}\mathbf{e}_n),\end{aligned} \tag{3.2.11}$$

and taking mean

$$h(\mathbf{x}\mathbf{e}) = h(\mathbf{e}).$$

○

2) For $\mathbf{d} = \left(e^h, \xi\right) \in \mathcal{D}^a$ and $\mathbf{x} = (\mathrm{Id}, \underline{\xi})$, with $\underline{\xi} \in \langle \varepsilon_1, ..., \varepsilon_d\rangle$,

$$\begin{aligned}\mathbf{d}^{-1}\mathbf{x}\mathbf{d} &= \left(e^{-h}, -e^{-h}\xi\right)\left(\mathrm{Id}, \underline{\xi}\right)\left(e^h, \xi\right) = \left(\mathrm{Id}, e^{-h}\underline{\xi}\right) \\ &= \left(\mathrm{Id}, er\varepsilon_0\right)\left(\mathrm{Id}, \underline{\xi}'\right),\end{aligned} \tag{3.2.12}$$

for some unique $r \in \mathbb{R}$ and $\underline{\xi}' \in \langle \varepsilon_1, ..., \varepsilon_d\rangle$. Note $\mathbf{x}' = \left(\mathrm{Id}, \underline{\xi}'\right)$. So

$$\mathbf{x}\mathbf{d} = \mathbf{d}\,(\mathrm{Id}, er\varepsilon_0)\,\mathbf{x}' = (\mathrm{Id}, er\varepsilon_0)\,\mathbf{d}\mathbf{x}',$$

since $\mathcal{N}^a$ leaves ε_0 stable. Note that $(\mathrm{Id}, er\varepsilon_0) \in \mathcal{D}_-^a$.

[16] $\{\widetilde{P}_s^a(\mathbf{e},.)\}_{s\geqslant 0,\mathbf{e}\in\mathcal{G}^a}$ is the image by φ_a of the family $\{\widetilde{P}_s(.,.)\}$ of transition kernels of the diffusion $\{\mathbf{e}_s\}_{s\geqslant 0}$ on $\mathcal{G}$.

Now, let h be a $\mathcal{D}^a_-$-left invariant μ^a-harmonic function. Write $\mathbf{e} = \left(e^h, \xi\right) e^{t\alpha^a}\mathbf{k} \equiv \mathbf{dak}$, with $h \in \mathfrak{n}^a, \xi \in \mathbb{R}^{1+d}$ and $\mathbf{k} \in \mathcal{K}^a$. For $x = (\mathrm{Id}, \underline{\xi})$, with $\underline{\xi} \in \langle \varepsilon_1, ..., \varepsilon_d\rangle$, the left $\mathcal{D}^a_-$-invariance of h justifies the equality $\overset{\bullet}{=}$ below.

$$h(\mathbf{xe}) = h(\mathbf{xdak}) = h\big((\mathrm{Id}, er\varepsilon_0)\mathbf{dx'ak}\big) \overset{\bullet}{=} h\left(\mathbf{dx'ak}\right)$$
$$= h\left(\mathbf{ek}^{-1}\mathbf{a}^{-1}\mathbf{x'ak}\right).$$

Applied to $\mathbf{ee}_n$ and its decomposition $\mathbf{ee}_n = \left(e^h_n, \xi_n\right) e^{t_n\alpha^a}\mathbf{k}_n \equiv \mathbf{d}_n\mathbf{a}_n\mathbf{k}_n$, $h_n \in \mathfrak{n}^a, \xi_n \in \mathbb{R}^{1+d}$ and $\mathbf{k}_n \in \mathcal{K}^a$, it gives

$$h(\mathbf{xee}_n) = h(\mathbf{ee}_n\mathbf{k}_n^{-1}\mathbf{a}_n^{-1}\mathbf{x}'_n\mathbf{a}_n\mathbf{k}_n),$$

with $\mathbf{x}'_n = \big(\mathrm{Id}, (0, \underline{\xi}^1, \sqrt{2}\underline{\xi}^1 h_n^{\geqslant 2} + \underline{\xi}^{\geqslant 2})\big)$.
The $\mathbb{R}^{1,d}$-part of $\mathbf{a}_n^{-1}\mathbf{x}'_n\mathbf{a}_n\mathbf{k}_n$ is equal to $(0, e^{t_n}\underline{\xi}^1, \sqrt{2}\underline{\xi}^1 h_n^{\geqslant 2} + \underline{\xi}^{\geqslant 2})$. As $\{\mathbf{k}_n\}_{n\geqslant 0}$ moves in a compact set, it has a converging subsequence $\{\mathbf{k}_{n_p}\}_{\geqslant 0}$. Thus, the $\mathbb{R}^{1,d}$-part of $\mathbf{k}_{n_p}^{-1}\mathbf{a}_{n_p}^{-1}\mathbf{x}'_{n_p}\mathbf{a}_n\mathbf{k}_{n_p}$ converges. Its $SO_0(1,d)$-part is equal to Id for any $p \geqslant 0$. So, lemma **119** applies, and gives

$$\forall\, \mathbf{e} \in \mathcal{G}^a, \quad h(\mathbf{xe}) = h(\mathbf{e}).$$

As any element of $\mathcal{D}^a$ can be written $\mathbf{d}_-\left(\mathrm{Id}, \underline{\xi}\right)$, for some $\mathbf{d} \in \mathcal{D}^a_-$ and $\underline{\xi} \in \langle \varepsilon_1, ..., \varepsilon_d\rangle$, h happens to be $\mathcal{D}^a$-left invariant.

3) Now, set $\mathbf{x} = \left(e^{t\alpha^a}, 0\right) \in \mathcal{A}^a$. For $\mathbf{d} = \left(e^h, \xi\right) \in \mathcal{D}^a$,

$$\mathbf{xd} = \left(e^{t\alpha^a}, 0\right)\left(e^h, \xi\right) = \left(e^{e^t h}, e^{t\alpha^a}\xi\right)\left(e^{t\alpha^a}, 0\right) = \mathbf{d'x},$$

for $\mathbf{d}' \in \mathcal{D}^a$.
So, if one writes an element $\mathbf{e} \in \mathcal{G}^a$ as $\mathbf{e} = \left(e^h, \xi\right)\left(e^{t\alpha^a}, 0\right)\mathbf{k} \equiv \mathbf{dak}$, $h \in \mathfrak{n}^a, \mathbf{k} \in \mathcal{K}^a$, the $\mathcal{D}^a$-left invariance of h enables to write

$$h(\mathbf{xe}) = h(\mathbf{xdak}) = h(\mathbf{d'xak}) = h(\mathbf{xak}) = h\left(\mathbf{dakk}^{-1}\mathbf{a}^{-1}\mathbf{xak}\right)$$
$$= h\left(\mathbf{ek}^{-1}\mathbf{a}^{-1}\mathbf{xak}\right).$$

As $\mathcal{A}^a$ is commutative, $\mathbf{a}^{-1}\mathbf{xa} = \mathbf{x}$, and

$$h(\mathbf{xe}) = h\left(\mathbf{ek}^{-1}\mathbf{xk}\right).$$

Applied to the decomposition of $\mathbf{ee}_n = \left(e^h_n, \xi_n\right)\left(e^{t_n\alpha^a}, 0\right)\mathbf{k}_n$, it gives

$$h(\mathbf{xee}_n) = h(\mathbf{ee}_n\mathbf{k}_n^{-1}\mathbf{xk}_n).$$

Since the sequence $\{\mathbf{k}_n^{-1}\mathbf{xk}_n\}_{n\geqslant 0}$ has for $\mathbb{P}^a$-almost all ω a converging subsequence ($\mathcal{K}^a$ is compact), lemma **119** gives

$$h(\mathbf{xe}) = h(\mathbf{e}).$$

So h is $\mathcal{D}^a\mathcal{A}^a$-left invariant.

4) Now, a $\mathcal{D}^a\mathcal{A}^a$-left invariant function h defines a function $\underline{h}$ on $\mathcal{D}^a\backslash\mathcal{G} \simeq \mathcal{N}^a\mathcal{A}^a\backslash aSO_0(1,d)a^{-1}$. Define the projection ν^a of μ^a on $aSO_0(1,d)a^{-1}$ by setting $\nu^a(B) = \mathbb{P}^a_{(\mathrm{Id},0)}(g_1 \in B)$, for a Borel set B of $aSO_0(1,d)a^{-1}$. The μ^a-harmonicity of h becomes for $\underline{h}$ the property

$$\forall\, x \in \mathcal{N}^a\mathcal{A}^a\backslash aSO_0(1,d)a^{-1}, \quad \int_{aSO_0(1,d)a^{-1}} \underline{h}(x.g)\nu^a(dg) = \underline{h}(x).$$

Using the same method as in propositions **24** and **26**, it is easy to show that ν^a charges any open set of $aSO_0(1,d)a^{-1}$. Since $\underline{h}$ is continuous and $\mathcal{D}^a\backslash\mathcal{G} \simeq \mathcal{N}^a\mathcal{A}^a\backslash aSO_0(1,d)a^{-1}$ is compact, $\underline{h}$ is constant (maximum principle). ▷

3.2.6 Correspondence between the algebraic and the geometric representations

To establish this correspondance, we come back to the group $\mathcal{G}$, where we come back to the notation $\mathbf{e}_n = (g_n, \xi_n)$, since it will no longer be question of elements of the Lie algebra $\mathfrak{g}$.

Let $h_\infty \in \mathbb{R}^{d-1}$ such that N_∞ is constructed from h_∞ using formula (3.2.4).

THEOREM **120 (Correspondence theorem).** *1. $h_\infty \in \mathbb{R}^{d-1}$ is the stereographic projection of $\sigma_\infty \in \mathbb{S}^{d-1}$.*

2. $r_\infty = \frac{1+|h_\infty|^2}{\sqrt{2}} R_\infty^{\sigma_\infty}$.

Theorem **116** asserts that any bounded function is of the form $\mathbf{e} \in \mathcal{G} \mapsto \mathbb{E}_\mathbf{e}[H(h_\infty, r_\infty)]$, for some bounded Borel function H on $\mathbb{R}^{d-1} \times \mathbb{R}$. Since any $\widetilde{L}$-harmonic function h on $\mathcal{G}$ is a μ-harmonic function , it is of the preceding form. Theorem **120** tells us that we can actually write $h(\mathbf{e}) = \mathbb{E}_\mathbf{e}[F(\sigma_\infty, R_\infty^{\sigma_\infty})]$, for some bounded Borel function F on $\mathbb{S}^{d-1} \times \mathbb{R}$. So, theorem **116** implies that

COROLLARY **121 (Poisson boundary of $\widetilde{L}$).** *Any bounded $\widetilde{L}$-harmonic function h on $\mathcal{G}$ is of the form*

$$h(\mathbf{e}) = \int_{\mathbb{S}^{d-1}\times\mathbb{R}} h^\sigma h^\sigma_\ell(\mathbf{e}) F(\sigma, \ell) d\sigma d\ell,$$

for some bounded Borel function F on $\mathbb{S}^{d-1} \times \mathbb{R}$.

$\lhd$ *1.* The expression $\dot{\xi}_n = g_n \varepsilon_0 = N_n A_n K_n \varepsilon_0 = N_n A_n \varepsilon_0 = N_n \begin{pmatrix} \mathrm{ch} t_n \\ \mathrm{sh} t_n \\ (0) \end{pmatrix}$ gives the coordinates of $\dot{\xi}_n$ in the canonical basis.

$$\dot{\xi}_n = \begin{pmatrix} \frac{1}{2}(|h_n|^2+1)e^{-t_n} + \frac{e^{t_n}}{2} \\ \frac{|h_n|^2-1}{2} e^{-t_n} + \frac{e^{t_n}}{2} \\ e^{-t_n} h_n \end{pmatrix}$$

Since it is also equal to $\left(\mathrm{ch}\rho_n, (\mathrm{sh}\rho_n)\sigma_n\right)$, we get $\sigma_\infty = \lim_{n\to+\infty} \frac{\dot{\xi}_n^{\geqslant 1}}{\dot{\xi}_n^0}$, that is

$$\sigma_\infty = \lim_{n\to+\infty} \begin{pmatrix} \frac{|h_n|^2-1+e^{2t_n}}{|h_n|^2+1+e^{2t_n}} \\ \frac{2h_n}{|h_n|^2+1+e^{2t_n}} \end{pmatrix} = \begin{pmatrix} \frac{|h_\infty|^2-1}{|h_\infty|^2+1} \\ \frac{2h_\infty}{|h_\infty|^2+1} \end{pmatrix}.$$

This is the analytical expression of the stereographic projection of $h_\infty \in \mathbb{R}^{d-1}$ on $\mathbb{S}^{d-1}$.

2. Not to confuse the two random walks on $\mathcal{G}$ and $\mathcal{G}^a$, we note $\{(g_n, \xi_n)\}_{n\geqslant 0}$ the random walk on $\mathcal{G}$ and $\{(\overline{g}_n, \overline{\xi}_n)\}_{n\geqslant 0}$ its image by φ_a on $\mathcal{G}^a$.

$$\overline{g}_n = a g_n a^{-1}, \quad \overline{\xi}_n = a\xi_n.$$

a) Theorem **113** asserts that we can write

$$(\overline{g}_n, \overline{\xi}_n) = \left(\overline{N}_n, (\overline{\xi}^0_n, 0, ...)\right)\left(\overline{A}_n \overline{K}_n, \left(0, \overline{\xi}'^{\geqslant 1}_n\right)\right),$$

with $\left(\overline{N}_n, (\overline{\xi}^0_n, 0, ...)\right)$ converging and $\overline{\xi}_n = \overline{\xi}^0_n \varepsilon_0 + \overline{N}_n{}^t (0, \overline{\xi}'^{\geqslant 1}_n)$. Since $\overline{N}_n \varepsilon_0 = \varepsilon_0$,

$$\overline{\xi}_n = \overline{N}_n{}^t \left(\overline{\xi}^0_n, \overline{\xi}'^{\geqslant 1}_n\right).$$

So the convergence of $\overline{\xi}_n^0$ reads

$$\left\langle \varepsilon_0^*, \overline{N}_n^{-1}\overline{\xi}_n \right\rangle \text{ converges,}$$

i.e.

$$\langle {}^t\overline{N}_n^{-1} \varepsilon_0^*, \overline{\xi}_n \rangle \text{ converges.}$$

As ${}^t\overline{N}_n^{-1}\varepsilon_0^*$ has coordinates $\begin{pmatrix} 1 \\ |h_n|^2 \\ \sqrt{2}h_n \end{pmatrix}$ in the basis $(\varepsilon_0^*, ..., \varepsilon_d^*)$ of $\left(\mathbb{R}^{1+d}\right)^*$, and $\overline{\xi}_n = a\xi_n$,

$$\left\langle {}^t a \begin{pmatrix} 1 \\ |h_n|^2 \\ \sqrt{2}h_n \end{pmatrix}, \xi_n \right\rangle \text{ converges.}$$

This quantity is equal to

$$\begin{aligned}
\left\langle \frac{1+|h_n|^2}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{1-|h_n|^2}{1+|h_n|^2} \\ \frac{2h_n}{1+|h_n|^2} \end{pmatrix}, \xi_n \right\rangle &= \frac{1+|h_n|^2}{\sqrt{2}} q\left(\begin{pmatrix} 1 \\ \frac{1-|h_n|^2}{1+|h_n|^2} \\ \frac{2h_n}{1+|h_n|^2} \end{pmatrix}, \xi_n \right) \\
&= \frac{1+|h_n|^2}{\sqrt{2}} q\big((1,\sigma_n), \xi_n\big).
\end{aligned}$$

b) So it remains to prove that $q\big((1,\sigma_n),\xi_n\big)$ and $q\big((1,\sigma_\infty),\xi_n\big)$ have the same limit.
For $\sigma \in \mathbb{S}^{d-1}$,

$$\begin{aligned}
q\big((1,\sigma),\xi_n\big) &= q\big((1,\sigma),\xi_0\big) + \int_0^n \mathrm{ch}\rho_s \, ds - \int_0^n (\mathrm{sh}\rho_s)(\sigma_s,\sigma)_{\mathrm{Eucl}} \, ds \\
&= q\big((1,\sigma),\xi_0\big) + \int_0^n (\mathrm{ch}\rho_s - \mathrm{sh}\rho_s) \, ds + \int_0^n (\mathrm{sh}\rho_s)\big(1-(\sigma_s,\sigma)_{\mathrm{Eucl}}\big) \, ds.
\end{aligned}$$

So,

$$\begin{aligned}
q\big((1,\sigma_n),\xi_n\big) - q\big((1,\sigma_\infty),\xi_n\big) &= q\big((1,\sigma_n),\xi_0\big) - q\big((1,\sigma_\infty),\xi_0\big) + \\
&\quad \int_0^n (\mathrm{sh}\rho_s)\big\{\big(1-(\sigma_s,\sigma_n)_{\mathrm{Eucl}}\big) - \big(1-(\sigma_s,\sigma_\infty)_{\mathrm{Eucl}}\big)\big\} \, ds \\
&= o(1) + \int_0^\infty (\mathrm{sh}\rho_s)\big\{\big(1-(\sigma_s,\sigma_n)_{\mathrm{Eucl}}\big) - \big(1-(\sigma_s,\sigma_\infty)_{\mathrm{Eucl}}\big)\big\}\mathbf{1}_{s\leqslant n} \, ds.
\end{aligned}$$

We apply the dominated convergence theorem to show that the integral tends to 0. For it, we need to majorize $\big\{\big(1-(\sigma_s,\sigma_n)_{\mathrm{Eucl}}\big) - \big(1-(\sigma_s,\sigma_\infty)_{\mathrm{Eucl}}\big)\big\}\mathbf{1}_{s\leqslant n}$ by some function $f(s)$, independant of n, such that $\int_0^\infty (\mathrm{sh}\rho_s) f(s) \, ds < \infty$.
Note $d_{s,n}$ the spherical distance between σ_s and σ_n, and $d_{s,\infty}$ the spherical distance between σ_s and σ_∞. Since

$$\begin{aligned}
1-(\sigma_s,\sigma_n)_{\mathrm{Eucl}} &= \frac{d_{s,n}^2}{2} + o(d_{s,n}^2), \\
1-(\sigma_s,\sigma_n)_{\mathrm{Eucl}} &= \frac{d_{s,\infty}^2}{2} + o(_{s,\infty}^2),
\end{aligned}$$

we estimate $d_{s,n}$ and $d_{s,\infty}$.
As $\sigma_s = \Sigma\,(T_s)$, where Σ is a Brownian motion on $\mathbb{S}^{d-1}$ and T_s a converging random time change, independant of Σ, we can use the estimate on the continuity modulus of Σ to majorize $d_{s,n}$ and $d_{s,\infty}$.
In their article [BCM87], P.Baldi and M.Chaleyat-Maurel showed that Lévy's estimate on the continuity modulus of the real Brownian motion has an analogous for elliptic diffusions, provided one replaces the Euclidian geometry of $\mathbb{R}$ by the Riemannian geometry associated with the diffusion. For Brownian motion on $\mathbb{S}^{d-1}$, it is the usual geometry induced by the ambient Euclidian space. So, if $\mathrm{osc}(\Sigma; u, v)$ denotes the oscillation of Σ on the time interval $[u, v]$, one has almost surely

$$\mathrm{osc}(\Sigma; u, v) \leqslant \sqrt{3(v-u)\log\frac{1}{v-u}},$$

provided $v - u$ is small enough (and $[u, v]$ is in a fixed interval). Since

$$d_{s,n}, d_{s,\infty} \leqslant \mathrm{osc}(\Sigma; T_s, T_\infty)$$

we have

$$d_{d,n}^2 + d_{s,\infty}^2 \leqslant 2\,\mathrm{osc}(\Sigma; T_s, T_\infty)^2 \leqslant 6(T_\infty - T_s)\log\frac{1}{T_\infty - T_s}.$$

for s large enough.
As $T_\infty - T_s = \int_s^\infty \frac{dr}{\mathrm{sh}^2\rho_r} \leqslant \int_s^\infty \frac{4\,dr}{(e^{\rho_r}-1)^2}$, and the function $\varepsilon \mapsto \varepsilon \log\frac{1}{\varepsilon}$ increases on $]0, \frac{1}{e}[$,

$$d_{d,n}^2 + d_{s,\infty}^2 \leqslant 24\left(\int_s^\infty \frac{dr}{(e^{\rho_r}-1)^2}\right)\log\left(\int_s^\infty \frac{dr}{(e^{\rho_r}-1)^2}\right)^{-1},$$

for s large enough.
Let $\varepsilon > 0$. We saw in section **2.1.2** that the inequalities $\frac{d-1}{2+\varepsilon}r \leqslant \rho_r \leqslant \frac{d-1}{2-\varepsilon}r$, hold for r large enough; so, we have a majoration of the form

$$d_{d,n}^2 + d_{s,\infty}^2 \leqslant Ce^{-2\frac{d-1}{2+\frac{\varepsilon}{2}}s},$$

for s large enough. The same kind of majoration holds for

$$\left\{\left(1 - (\sigma_s, \sigma_n)_{\mathrm{Eucl}}\right) - \left(1 - (\sigma_s, \sigma_\infty)_{\mathrm{Eucl}}\right)\right\}\mathbf{1}_{s\leqslant n},$$

independantly of n, for s large. As we eventually have $\mathrm{sh}\rho_s \leqslant e^{\frac{d-1}{2-\varepsilon}s}$,

$$(\mathrm{sh}\rho_s)\left\{\left(1 - (\sigma_s, \sigma_n)_{\mathrm{Eucl}}\right) - \left(1 - (\sigma_s, \sigma_\infty)_{\mathrm{Eucl}}\right)\right\}\mathbf{1}_{s\leqslant n} \leqslant C\exp\left(-(d-1)\left(\frac{2}{2+\frac{\varepsilon}{2}} - \frac{1}{2-\varepsilon}\right)s\right).$$

The majorant is integrable for $\varepsilon > 0$ small enough. ▷

Chapter 4

Appendices

4.1 Dudley's results

We begin this appendix by giving a dynamical description of the class of relativistic processes $\{(\dot{\xi}_s, \xi_s)\}_{s\geqslant 0}$ discovered by Dudley and classified by theorem **7**, of Karpelevich, Schur and Tutubalin. This description gives a clearer image of the pathwise behaviour of the process. It uses results on Lévy processes on Lie groups. Consult the first pages of the book *Lévy processes in Lie groups*, of Liao, [Lia04], for basic definitions and to get a picture of the state of art on the subject.

Section **4.1.2** contains the proofs of Dudley's results about the asymptotic behaviour of any non exploding Markov process on $\mathbb{H}$ with a $SO_0(1,d)$-invariant transition kernel, stated in theorem **8**, in the Introduction. We note $\{\dot{\xi}_s\}_{s\geqslant 0}$ such a process on $\mathbb{H}$ and $\mathbb{P}_{\dot{\xi}_0}$ its law, when started from $\dot{\xi}_0 \in \mathbb{H}$.

4.1.1 Dynamic description of the general relativistic process in $\mathbb{H}\times\mathbb{R}^{1,d}$

Since

$$\xi_s = \xi_0 + \int_0^s \dot{\xi}_r \, dr, \tag{4.1.1}$$

is determined by $\{\dot{\xi}_s\}_{s\geqslant 0}$, we concentrate on the $\mathbb{H}$-valued Markov process $\{\dot{\xi}_s\}_{s\geqslant 0}$. The family $\{\mathbb{P}_{\dot{\xi}}\}_{\dot{\xi}\in\mathbb{H}}$ of its laws satisfy the two conditions

1. $\forall\,\dot{\xi}\in\mathbb{H},\ \forall\, t\geqslant 0,\ \mathbb{P}_{\dot{\xi}}(\dot{\xi}_t\in\mathbb{H}) = 1$,
2. $\forall\,\dot{\xi}\in\mathbb{H},\ \forall\, t\geqslant 0,\ \mathbb{P}_{g(\dot{\xi})}(\dot{\xi}_t\in g(A)) = \mathbb{P}_{\dot{\xi}}(\dot{\xi}_t\in A)$, for any isometry g of $\mathbb{H}$ and any Borelian subset A of $\mathbb{H}$.

Notation – Write $\mathcal{L}$ the generator of $\{\dot{\xi}_s\}_{s\geqslant 0}$.

Hunt showed in [Hun56] (in a more general setting) that considering $\mathbb{H}$ as a homogeneous space of $SO_0(1,d)$ provides the key to the understanding of the operator $\mathcal{L}$. The following version of his theorem is taken from [Lia04], theorem 1.1, p11, and theorem 2.1, p.42, and stated in our framework.

Note $V_1, ..., V_d$ the left invariant vector fiels on $SO_0(1,d)$ whose values in Id are $E_1 = \varepsilon_1\otimes\varepsilon_0^* + \varepsilon_0\otimes\varepsilon_1^*$, ..., $E_d = \varepsilon_d\otimes\varepsilon_0^* + \varepsilon_0\otimes\varepsilon_d^*$, respectively. Order the set $\{E_{ij},\ 1\leqslant i<j\leqslant \frac{d(d-1)}{2}\} \equiv \{\mathbf{E}_k; 1\leqslant k\leqslant \frac{d(d-1)}{2}\}$ of spatial infinitesimal rotations[1], using the lexicographical order, and define the left invariant vector fields on $SO_0(1,d)$

$$V_{d+k}(g) = g\mathbf{E}_k.$$

[1] $\exp(tE_{ij})$ is the matrix of a rotation of angle t in the plane $\langle\varepsilon_i,\varepsilon_j\rangle$.

Notation – $n = \frac{d(d+1)}{2}$ is the dimension of $SO_0(1,d)$.

THEOREM **122 (Hunt, [Hun56]).** *We identify* $\mathbb{H}$ *and* $SO_0(1,d)/SO(d)$. *Let* $\widetilde{\mathcal{L}}$ *be the generator of a Lévy process* $\{g_t\}_{t\geqslant 0}$ *on* $SO_0(1,d)$.

1.
   - *The domain of* $\widetilde{\mathcal{L}}$ *contains the set of smooth functions on* $SO_0(1,d)$, *with compact support.*
   - *We can find smooth functions* $x_1,...,x_n$ *on* $SO_0(1,d)$, *which form a coordinate system on* $SO_0(1,d)$, *near the identity, such that for any smooth function* f *on* $SO_0(1,d)$, *with compact support, and* $g\in SO_0(1,d)$,

$$\widetilde{\mathcal{L}}f(g) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}V_jV_kf(g) + \sum_{i=1}^{n} c_iV_if(g) + \int_{SO_0(1,d)} \big(f(gh) - f(h) - \sum_{i=1}^{n} x_i(h)V_if(g)\big)\,\mathbf{n}(dh),$$

   where a_{ij}, c_i *are constants,* $\{a_{ij}\}$ *is a nonnegative definite symmetric matrix and* $\mathbf{n}$ *is a measure on* $SO_0(1,d)$ *such that*

$$\mathbf{n}(\{Id\}) = 0,\ \mathbf{n}\left(\sum_{i=1..n} x_i^2\right) < \infty,\ \mathbf{n}(U^c) < \infty$$

   for any neighbourhood U *of Id.*

2.
   - *The domain of* $\mathcal{L}$ *contains the set of smooth functions on* $\mathbb{H}$, *with compact support.*
   - *We can find smooth functions* $y_1,...,y_d$ *on* $\mathbb{H}$, *which form a coordinate system on* $\mathbb{H}$, *near* ε_0, *such that for any smooth function* f *on* $\mathbb{H}$, *with compact support,*

$$\mathcal{L}f(\varepsilon_0) = c\,\triangle^{\mathbb{H}} f(\varepsilon_0) + \int_{\mathbb{H}} \big(f(\dot{\xi}) - f(\varepsilon_0) - \sum_{i=1}^{d} y_i(\dot{\xi})\partial_{y_i} f(\varepsilon_0)\big)\, n(d\dot{\xi}),$$

   where $c \geqslant 0$ *is a constant, and* n *is an* $SO_0(1,d)$*-invariant measure on* $\mathbb{H}$ *satisfying*

$$n(\varepsilon_0) = 0,\ n\left(\sum_{i=1..d} y_i^2\right) < \infty,\ n(U^c) < \infty$$

   for any neighbourhood U *of* ε_0.

The introduction of Lévy processes on $SO_0(1,d)$ is justified by the following

COROLLARY **123 ([Lia04], Theorem** 2.2, **p.**43**).** *There is a right* $SO(d)$*-invariant left Lévy process* $\{g_t\}_{t\geqslant 0}$ *in* $SO_0(1,d)$ *such that its canonical projection on* $\mathbb{H}$ *has the same law as* $\{\dot{\xi}_s\}_{s\geqslant 0}$.

This corollary brings us back to a dynamical description of a (left) Lévy process on $SO_0(1,d)$. Applebaum and Kunita gave in [AK93] a characterization of Lévy processes in Lie groups as solutions of integral equations involving stochastics integrals with respect to a Brownian motion and a Poisson random measure. We state it in our framework.

Define the random measure N on $\mathbb{R}_+ \times SO_0(1,d)$ by

$$N\big([0,t]\times B\big) = \big|\{s\in]0,t];\ g_{s^-}^{-1}g_s \neq \mathrm{Id},\ \text{and}\ g_{s^-}^{-1}g_s \in B\}\big|,$$

for $t \geqslant 0$ and B a Borelian subset of $SO_0(1,d)$. As $\{g_t\}_{t\geqslant 0}$ is a Lévy process, N is a Poisson random measure; it counts the number of jumps of $\{g_t\}_{t\geqslant 0}$. For a fixed B, the process $\{N\big([0,t]\times B\big)\}_{t\geqslant 0}$ is $\{\mathcal{F}_t^g\}_{t\geqslant 0}$-adapted. Note $\overline{N}$ the compensated random measure of N.

THEOREM **124 ([AK93]).** *There exists an n-dimensional $\{\mathcal{F}_t^g\}_{t\geqslant 0}$-adapted Brownian motion $\{w_s\}_{s\geqslant 0}$ with covariance matrix $\{a_{ij}\}$, independent of N, such that for any compactly supported $\mathcal{C}^2$ function f on $SO_0(1,d)$,*

$$f(g_t) = f(g_0) + \sum_{i=1}^{n} \int_0^t V_i f(g_{s^-}) \circ dw_s^i + \sum_{i=1}^{n} c_i \int_0^t V_i f(g_{s^-})\, ds + \int_0^t \int_{SO_0(1,d)} \big(f(g_{s^-}h) - f(g_{s^-})\big)\overline{N}(ds\, dh)$$
$$+ \int_0^t \int_{SO_0(1,d)} \Big(f(g_{s^-}h) - f(g_{s^-}) - \sum_{i=1}^{n} x_i(h) V_i f(g_{s^-})\Big)\, ds\, \mathbf{n}(dh)$$

Together with theorem **122** and equation (4.1.1), this theorem provides a *dynamical description* of the most general relativistic process in $\mathbb{H} \times \mathbb{R}^{1,d}$.

Applebaum gave in [App00], theorem 3, p.396, a *pathwise description* of a Lévy process in a Lie group G. Defining "by analogy with the Euclidian case a compound Poisson process as the composition of a random number of iid G-valued random variables where the number of terms taken depends on the value of a Poisson process", he employes this compound Poisson process to describe the paths of a Lévy process in G "as the almost sure limit of a sequence of Brownian motions interlaced with random jumps". This gives an intuitive picture of the paths of a Lévy process.

The results of Applebaum and al. came 30 years after the results of Karpelevich & al., and Hunt. Fortunately, they are not needed to get a good idea of the long time behaviour of $\{\dot{\xi}_s\}_{s\geqslant 0}$. The following section gives a proof of Dudley's results about it.

4.1.2 Dudley's results

THEOREM **125.** *1. The process $\{\dot{\xi}_s\}_{s\geqslant 0}$ on $\mathbb{H}$ is $\mathbb{P}_{\dot{\xi}_0}$-almost surely transient, whatever $\dot{\xi}_0 \in \mathbb{H}$.*

2. Note (ρ_s, σ_s) the polar coordinates of $\dot{\xi}_s$. The direction σ_s of $\dot{\xi}_s$ converges $\mathbb{P}_{\dot{\xi}_0}$-almost surely when $s \to +\infty$.

Recall the definition of the Fourier transform of a radial probability, given in the Introduction, in equation (1.0.3). The radial probability μ is the law of ρ_1.

$\lhd$ *1.* The proof is based on a

LEMMA **126.** *For every $R > 0$, there exists a constant $C(R)$ such that we have*

$$\mathbb{P}_{\varepsilon_0}\big(\rho_n \leqslant R\big) \leqslant C(R)\widehat{\mu}(0)^n,$$

for any $n \geqslant 1$. In particular, $\mathbb{P}_{\varepsilon_0}(\rho_n \leqslant R,$ i.o.$) = 0$.

$\bigcirc$ Note $\mu^{\star n} = \mu \star \cdots \star \mu$, n times. As in the Euclidian framework,

$$\widehat{\mu \star \nu} = \widehat{\mu}\,\widehat{\nu}.$$

The function $\rho \in \mathbb{R}_+ \to \frac{\rho}{\text{sh}\rho}$ decreases. So, given $R > 0$,

$$\mu^{\star n}(\rho \leqslant R) = \int_{\rho\leqslant R} \mu^{\star n}(d\dot{\xi}) \leqslant \frac{\text{sh}R}{R} \int_{\rho\leqslant R} \frac{\rho}{\text{sh}\rho}\mu^{\star n}(d\dot{\xi})$$
$$\leqslant \frac{\text{sh}R}{R} \int \frac{\rho}{\text{sh}\rho}\mu^{\star n}(d\dot{\xi}) = \frac{\text{sh}R}{R}\widehat{\mu^{\star n}}(0) = \frac{\text{sh}R}{R}\widehat{\mu}(0)^n.$$

Since μ is not concentrated on ε_0 and $\frac{\rho}{\text{sh}\rho} < 1$, if $\rho > 0$, $\widehat{\mu}(0) < 1$. Applying Borel-Cantelli lemma gives $\mathbb{P}_{\varepsilon_0}(\rho_n \leqslant R,\ i.o.) = 0$. $\bigcirc$

Were the process $\{\dot{\xi}_s\}_{s\geqslant 0}$ recurrent, it would come back in a ball $\{\rho \leqslant r'\}$ at arbitrarily large times. Note

$$S_1 = \inf\{s \geqslant 0\,;\, \rho_s \geqslant 2r'\}, \tau_1 = \{s \geqslant S_1\,;\, \rho_s \leqslant r'\},$$

and, recursively,

$$S_n = \inf\{s \geqslant \tau_{n-1}\,;\, \rho_s \geqslant 2r'\}, \tau_n = \{s \geqslant S_{n-1}\,;\, \rho_s \leqslant r'\}.$$

These stopping times are all $\mathbb{P}_{\dot{\xi}_0}$-almost surely finite.

The process $\{\dot{\xi}_s\}_{s\geqslant 0}$ enjoying the strong Markov property, and its transition kernels being radial, the excursions $\{\rho_s\}_{\tau_k\leqslant s<\tau_{k+1}}$, $k \geqslant 1$, of $\{\rho_s\}_{s\geqslant 0}$ outside $[0, r']$, of height $\geqslant 2r'$, would be independent and identicaly distributed. The lifetime of $\{\dot{\xi}_s\}_{s\geqslant 0}$ being $\mathbb{P}_{\dot{\xi}_0}$-almost surely infinite, one could choose a constant $r > 0$ sufficiently large to have

$$\forall\, \dot{\xi}_0 \in \mathbb{H}, \quad \mathbb{P}_{\dot{\xi}_0}\left(\sup_{0\leqslant r\leqslant 1} d(\dot{\xi}_s, \dot{\xi}_0) \leqslant r\right) \geqslant \frac{1}{2}.$$

One would deduce from Borel-Cantelli's lemma that we have $\mathbb{P}_{\dot{\xi}_0}$-almost surely

$$\overline{\lim_{k\to+\infty}}\, \mathbf{1}_{\dot{\xi}_{[\tau_k,\tau_k+1[}\subset B(\varepsilon_0, r+r')} = 1.$$

The interval $[\tau_k, \tau_k + 1[$ containing an integer, we would have

$$\overline{\lim_{n\to+\infty}}\, \mathbf{1}_{\dot{\xi}_n\subset B(\varepsilon_0, r+r')} = 1,$$

contradicting lemma **126**.

2. The point $\dot{\xi}_0 \in \mathbb{H}$ is fixed in this proof. If $\dot{\xi}$ has polar coordinates $(\rho_, \sigma)$, we sometimes write $\mathbb{P}_{\rho,\sigma}$ for $\mathbb{P}_{\dot{\xi}_0}$.

The heart of the proof is in the 2-dimensional situation, and we do the proof in that case.

We need a

LEMMA **127.** *There exists a positive function $f(n) > 0$, increasing to ∞, with the following property. Whatever $n \geqslant 1$, there exists $\rho_n \geqslant f(n)$ such that $\forall\, \rho \geqslant \rho_n$, $\forall \sigma_0 \in \mathbb{S}^{d-1}$, $\forall\, s \geqslant 0$,*

$$\mathbb{P}_{\rho,\sigma_0}\left(\rho_s \leqslant f(n) \text{ or } d_{\mathbb{S}^{d-1}}(\sigma_s, \sigma_0) \geqslant \frac{1}{n}\right) \leqslant \frac{1}{n^2}.(^2)$$

○ Use halfspace coordinates on $\mathbb{H}$.

Using an isometry, we can suppose that the geodesic with polar coordinates $\{(\rho, \sigma_0)\}_{\rho\geqslant 0}$ is the line $\{(y, 0)\}_{y=1..0}$ in halfspace coordinates.

The quantities $r(\alpha, y)$ and r_n are defined on figure 4.1. Elementary calculations give

$$r(\alpha, y) = y\frac{\cos\alpha + 1}{|\sin\alpha|};$$

this is a decreasing function of $\alpha \in]0, \pi[$.

Set $y_n = \frac{\sin\frac{1}{2n^2}}{1+\cos\frac{1}{2n^2}} r_n$. For $0 < y \leqslant y_n$ and $|\alpha| \in]\frac{1}{2n^2}, \pi[$,

$$r(\alpha, y) \leqslant r_n.$$

[2] $d_{\mathbb{S}^{d-1}}$ is the Euclidian spherical distance on $\mathbb{S}^{d-1}$.

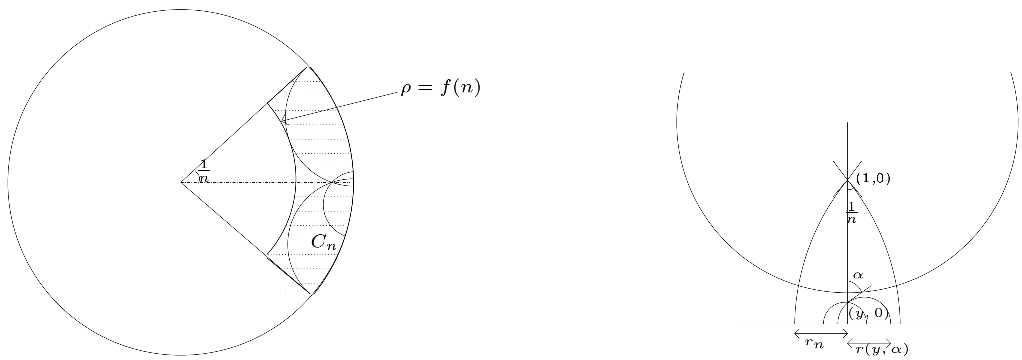


Figure 4.1:

All the more, another elementary calculation shows that the hyperbolic ball with center $(1,0)$ and radius $-\log \frac{r_n}{1+\cos\frac{1}{2n^2}}$ does not intersect any geodesic started from y, with direction α, if $y \leqslant y_n$ and $|\alpha| \in]\frac{1}{2n^2}, \pi[$.
Set

$$f(n) := -\log \frac{r_n}{1+\cos\frac{1}{2n^2}}.$$

Now, given, $s \geqslant 0$, the distribution of $\dot{\xi}_s$ under $\mathbb{P}_{(y,0)}$ is of the form $h(\rho^y)d\rho^y d\sigma^y$, for some $h \geqslant 0$, if one uses polar coordinates (ρ^y, σ^y) associated with the point $(y,0)$. Note (ρ_s^y, α_s^y), with $\alpha_s^y \in [-\pi, \pi[$, these coordinates. The polar coordinates of $\dot{\xi}_s$ (associated with the point $(1,0)$) are still noted (ρ_s, σ_s).
Take $y \leqslant y_n$. With probability $1 - \frac{1}{n^2}$, $|\alpha_s^y| \in]\frac{1}{2n^2}, \pi]$. We see from the preceding that, on this event, the point $\dot{\xi}_s$ satisfies $\rho_s \geqslant f(n)$ and $d_{\mathbb{S}^{d-1}}(\sigma_s, \sigma_0) \leqslant \frac{1}{n}$. The result follows, with

$$\rho_n = -\log y_n = -\log\left(\frac{\sin\frac{1}{2n^2}}{1+\cos\frac{1}{2n^2}} r_n\right).$$

○

Set

$$C_n(\dot{\xi}_0) \equiv \left\{\dot{\xi} \in \mathbb{H}\,;\ \rho \geqslant f(n),\ \text{and}\ d_{\mathbb{S}^{d-1}}(\sigma_s, \sigma_0) \leqslant \frac{1}{n}\right\}.$$

With this notation, lemma **127** reads

$$\forall\, \rho \geqslant \rho_n,\ \forall\, s \geqslant 0, \quad \mathbb{P}_{\rho,\sigma_0}\big(\dot{\xi}_s \notin C_n(\dot{\xi}_0)\big) \leqslant \frac{1}{n^2}. \tag{4.1.2}$$

Now, set for each integer $n \geqslant 1$,

$$\tau_n \equiv \inf\{s \geqslant 0; \rho_s = \rho_n\}.$$

We know from point 1 that this hitting time is $\mathbb{P}_{\dot{\xi}_0}$-almost surely finite. Let $s > 0$.
Using figure 4.2, one sees as in the proof of lemma **127** that there exists a constant $0 < \varepsilon < \frac{1}{2}$ such that for any $n \geqslant 1$ and any $\dot{\xi} \in C_n(\dot{\xi}_0)$ with $\rho = f(n)$, one has

$$\forall\, s \geqslant 0\ ,\ \mathbb{P}_{\dot{\xi}}\big(\dot{\xi}_s \notin C_n(\dot{\xi}_0)\big) \geqslant \varepsilon.(^3) \tag{4.1.3}$$

[3] $\dot{\xi}_s \notin C_n(\dot{\xi}_0)$ if its direction is not in the angular shaded region. As the law of this direction is the uniform probability on $\mathbb{S}^{d-1}$, the result follows.

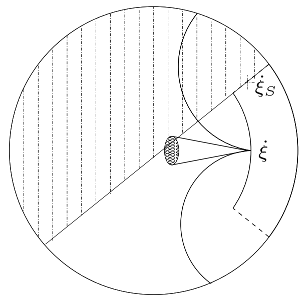


Figure 4.2:

Then, one has

$$\mathbb{P}_{\dot\xi_0}\big(\dot\xi_{\tau_n+s} \notin C_n(\dot\xi_{\tau_n})\big) \geqslant \varepsilon\, \mathbb{P}_{\dot\xi_0}\big(\exists\, r \in [\tau_n, \tau_n + s[\,;\ \dot\xi_r \notin C_n(\dot\xi_{\tau_n})\big). \tag{4.1.4}$$

To obtain this inequality, we applied the strong Markov property to the first exit time S from $C_n(\dot\xi_{\tau_n})$:

- if $d_{\mathbb{S}^{d-1}}(\sigma_S, \sigma_{\tau_n}) = \frac{1}{n}$, one knows that

$$\mathbb{P}_{\dot\xi_S}\big(\dot\xi_{s-S} \notin C_n(\dot\xi_{\tau_n})\big) \geqslant \mathbb{P}_{\dot\xi_S}\big(\dot\xi_{s-S} \text{ not in the half-space shaded in figure 4.2}\big) = \frac{1}{2},$$

- if $\rho(\dot\xi_S) = f(n)$, apply inequality (4.1.3).

One deduces from inequalities (4.1.2) and (4.1.4) that

$$\mathbb{P}_{\dot\xi_0}\big(\exists\, r \geqslant \tau_n\,;\ \dot\xi_r \notin C_n(\dot\xi_{\tau_n})\big) \leqslant \frac{1}{\varepsilon\, n^2}.$$

Borel-Cantelli lemma applies, and shows that one has $\mathbb{P}_{\dot\xi_0}$-almost surely $\dot\xi_{[\tau_n,+\infty[} \subset C_n(\dot\xi_{\tau_n})$, evnetually. Thus, the oscillation of $\{\sigma_s\}_{s\geqslant 0}$ on the interval $[\tau_n, +\infty[$ is less than or equal to $\frac{2}{n}$; as $\tau_n \to +\infty$, this implies the $\mathbb{P}_{\dot\xi_0}$-almost sure convergence of $\{\sigma_s\}_{s\geqslant 0}$. ▷

4.2 h-transform

In this appendix, we recall the link existing between h-transform and conditional processes.

a) h**-transform** — Recall we note $p_s(u,v)$ the transition density of the diffusion $\{u_s\}_{s\geqslant 0}$.

Let h be a positive L-harmonic function. Following for instance Kaï Laï Chung, [CZ95], Chap.5, we can define for all $s>0$, $u,v\in\mathbb{H}\times\mathbb{R}^{1,d}$,

$$p_s^h(u,v)=\frac{p_s(u,v)\,h(v)}{h(u)}.$$

p^h determines a Markov process on the state space $\left(\mathbb{H}\times\mathbb{R}^{1,d}\right)\cup\{\partial\}$, where ∂ is an extra point needed in the definition of the transition probabilities. This process is called the **h-transform diffusion**, and its law is denoted $\mathbb{P}^h$. Its **life time** S is its exit time from all compacta, and its differential generator

$$L^h(f)=\frac{L(hf)}{h}.$$

The following absolute continuity relation is fundamental. Its proof can be found in the book [Dyn02] of Dynkin, p.103. We use his notations.

THEOREM **128 (Absolute continuity relation).** *For every stopping time τ and every $\mathcal{F}_{\leqslant\tau}$ non negative random variable Y, one has*

$$\mathbb{E}_u^h[Y\mathbf{1}_{\tau<S}]=\mathbb{E}_u\left[Y\frac{h(u_\tau)}{h(u)}\mathbf{1}_{\tau<S}\right].$$

COROLLARY .

1. *For each $u_0\in\mathbb{H}\times\mathbb{R}^{1,d}$, $\mathbb{P}_{u_0}^h$ -almost every path of the h-transform diffusion is continuous.*
2. *The h-transform diffusion enjoys the strong Markov property.*

Proofs of these statements can be found in Kaï Laï Chung [CZ95], pp.133 – 134, and Durrett [Dur84], p.102.

b) h**-transform and conditional diffusion**

THEOREM **129.** *Let $u_0\in\mathbb{H}\times\mathbb{R}^{1,3}$. For any $\Lambda\in\mathcal{F}$, $\mathbb{P}_{u_0}^{h^{\sigma_\infty(\omega)}}(\Lambda)$ is a version of $\mathbb{E}_{u_0}[\Lambda|\sigma(\sigma_\infty(\omega))]$.*

$\triangleleft$ We have to see that for any bounded Borel function $f:\mathbb{S}^{d-1}\to\mathbb{R}$, $u_0\in\mathbb{H}\times\mathbb{R}^{1,d-1}$ and $\Lambda\in\mathcal{F}$,

$$\mathbb{E}_{u_0}[f(\sigma_\infty)\mathbf{1}_\Lambda]=\mathbb{E}_{u_0}[f(\sigma_\infty))\mathbb{P}_{u_0}^{h^{\sigma(\infty)}}(\Lambda)]. \tag{4.2.1}$$

Fix f and u_0. The set of Λ's which satisfies the preceding equation is a *monotone class.* Let $\{K_n\}_{n\geqslant 0}$ be an exhaustion of $\mathbb{H}\times\mathbb{R}^{1,d}$ by an increasing sequence of compacta ; write $S_n=\inf\{s>0;u_s\notin K_n\}$ the exit time from K_n. The algebra $\bigcup_{n\geqslant 0}\mathcal{F}_{\leqslant S_n}$ generates $\mathcal{F}$:

$$\sigma(\bigcup_{n\geqslant 0}\mathcal{F}_{\leqslant S_n})=\mathcal{F}.$$

So, it suffices to show the relation (4.2.1) with $\Lambda\in\mathcal{F}_{\leqslant S_n}$. For such a Λ, the absolute continuity relation **128** justifies the crucial equality $\stackrel{\bullet}{=}$ bellow.

$$\begin{aligned}\mathbb{E}_{u_0}[f(\sigma_\infty)\mathbf{1}_\Lambda] &= \mathbb{E}_{u_0}[\mathbf{1}_\Lambda \mathbb{E}_{u_0}[f(\sigma_\infty)|\mathcal{F}_{\leqslant S_n}]] = \mathbb{E}_{u_0}[\mathbf{1}_\Lambda \mathbb{E}_{u_{S_n}}[f(\sigma_\infty)]] = \mathbb{E}_{u_0}[\mathbf{1}_\Lambda \int h^\sigma(u_{S_n}) f(\sigma)\, d\sigma] \\ &= \int f(\sigma) h^\sigma(u_0) \mathbb{E}_{u_0}[\mathbf{1}_\Lambda \frac{h^\sigma(u_{S_n})}{h^\sigma(u_0)}]\, d\sigma \overset{\bullet}{=} \int f(\sigma) h^\sigma(u_0) \mathbb{E}^{h^\sigma}_{u_0}[\mathbf{1}_\Lambda]\, d\sigma \\ &= \mathbb{E}_{u_0}[f(\sigma_\infty(\omega)) \mathbb{P}^{h^{\sigma_\infty(\omega)}}_{u_0}(\Lambda)].\end{aligned}$$

▷

4.3 Back to Dufresne's integral

Let $u \in \mathbb{H} \times \mathbb{R}^{1,d}$. We showed in section **2.2.2** that the $Inv(\{u_s\})$-measurable random variable

$$R_\infty^{\varepsilon_1} = \lim_{s \to +\infty} q(\xi_s, \varepsilon_0 + \varepsilon_1)$$

has, under $\mathbb{P}_u$, a law related to that Dufresne's integral $\int_0^{+\infty} e^{-w_s - \frac{d-1}{2}s}\, ds$. Precisely,

$$h_{\geqslant \ell}^{\varepsilon_1}(u) = \mathbb{P}_u\left(R_\infty^{\varepsilon_1} \geqslant \ell\right) = \mathbb{P}\left(\int_0^{+\infty} e^{-w_s - \frac{d-1}{2}s}\, ds \geqslant y(\ell - \xi'^0)\right),$$

where $u = (\dot{\xi}, \xi)$, $\dot{\xi}$ having halfspace coordinates (y, x), and ξ having $\mathbf{g}'$-coordinates $(\xi'^0, ..., \xi'^d)$.

We used in section **2.2.2** the a priori smoothness of the function $h_{\geqslant \ell}^{\sigma}$ given by proposition **32** to say it is L^{h^σ}-harmonic. We deduced from this fact that the random variable $\int_0^{+\infty} e^{-w_s - \frac{d-1}{2}s}\, ds$ has a smooth density with respect to Lebesgue's measure on $\mathbb{R}$. The equation $L^{h^\sigma} h_{\geqslant \ell}^{\sigma} = 0$, enabled its identification:

$$\mathbb{P}\left(\int_0^{\infty} e^{-w_s - \frac{d-1}{2}s} ds \in r + dr\right) = \mathbf{1}_{r>0} e^{-2/r} r^{-d} dr$$

Proposition **32** is a consequence of Hörmander's hypoellipticity theorem ; one can find its use a heavy price to pay.

In this paragraph, we give a direct proof of the smoothness of the law of $\int_0^{\infty} e^{-w_s - \frac{d-1}{2}s} ds$, using an integration by part formula. Inequality (4.3.8) will enable us to apply the following well known fact.

Proposition **130.** *Let ν be a Borel probability on $\mathbb{R}$ and an integer $k \geqslant 2$. Suppose there exists a constant $C_k > 0$ such that for any smooth function ϕ with bounded derivatives,*

$$\left| \int \phi^{(k)}(x) \nu(dx) \right| \leqslant C_k \|\phi\|_\infty.$$

Then ν has a density with respect to Lebesgue measure, of class $\mathcal{C}^{k-2}$.

◁ Applied with $\phi(t) = e^{i\lambda t}$, the hypothesis tells us that the continuous fonction $\widehat{\nu}(\lambda)$ decreases at least as λ^{-k}:

$$|\widehat{\nu}(\lambda)| \leqslant C_k \lambda^{-k} \tag{4.3.1}$$

$\widehat{\nu}$ being integrable, the inversion formula tells us that

$$\nu(t) = \int e^{it\lambda} \hat{\nu}(\lambda) d\lambda.$$

Because of (4.3.1), the integral is a $\mathcal{C}^{k-2}$ function of t. ▷

We shall take advantage of the following writing

$$\begin{aligned} \int_0^\infty e^{w_u - c\,u}\, du &= \int_0^1 e^{(w_u - u\,w_1) + (w_1 - c)u}\, du + e^{-c} e^{w_1} \int_1^\infty e^{(w_u - w_1) - c(u-1)}\, du \\ &= \int_0^1 e^{\mathfrak{p}_u + (w_1 - c)u}\, du + e^{-c} e^{w_1} \int_1^\infty e^{\widetilde{w}_u - c(u-1)}\, du, \end{aligned} \tag{4.3.2}$$

in which one $\int_0^\infty e^{w_u - c\,u}\, du$ appears as a functional of the Brownian bridge $\{\mathfrak{p}_u\}_{0 \leqslant u \leqslant 1} = \{w_u - u\,w_1\}_{0 \leqslant u \leqslant 1}$, of w_1, and of the Brownian motion $\{\widetilde{w}_u\}_{u \geqslant 0} = \{w_{u+1} - w_1\}_{u \geqslant 0}$. These three processes are independant under $\mathbb{P}$. From now on, we note

$$\Omega = \mathcal{C}\big([0,1], \mathbb{R}\big) \times \mathbb{R} \times \mathcal{C}(\mathbb{R}^+, \mathbb{R}),$$

endowed with the product σ-algebra of the Borelian σ-algebras of each factor. We put on this measurable space the probability

$$\mathbb{Q} = \mathbf{P}^{[0,1]} \otimes \mathbb{W} \otimes \widetilde{\mathbb{P}},$$

where $\mathbf{P}^{[0,1]}$ is the law of the Brownian bridge, $\mathbb{W}$ is a standard Gaussian law, and $\widetilde{\mathbb{P}}$ the Wiener measure on $\mathcal{C}(\mathbb{R}^+,\mathbb{R})$. We shall note $\omega = (\mathfrak{p}, w_1, \widetilde{w})$ an element of Ω, and w the Brownian motion constructed from $\mathfrak{p}$, w_1, $\widetilde{w}$.

Not to complicate the writing, we shall take $c = 1$. We shall show that the random variable $\int_0^\infty e^{w_u - u}\, du$ has a $\mathcal{C}^2$ density with respect to Lebesgue's measure on $\mathbb{R}$.

4.3.1 The actors and their qualities

We shall pay attention to the following quantities.

- $X(\omega) = \int_0^\infty e^{w_s - s} ds$,
- For $k \geqslant 1$, $X^{(k)}(\omega) = \int_0^1 u^k\, e^{\mathfrak{p}_u + (w_1 - 1)u}\, du + e^{w_1 - 1} \int_0^\infty e^{\widetilde{w}_u - u}\, du$
- $F(\omega) = \frac{1}{X^{(1)}(\omega)}$.

ϕ will be a smooth bounded function with bounded derivatives of all order.

In the following, we shall write $\mathbb{L}^p$ for $\mathbb{L}^p(\mathbb{Q}(dw))$, and will note $\mathbb{E}$ the mean operator under $\mathbb{Q}$.

For $\widetilde{\mathbb{P}}$-almost all $\widetilde{w}$ and every $\mathfrak{p}$, $X(\mathfrak{p}, ., \widetilde{w})$ and $F(\mathfrak{p}, ., \widetilde{w})$ are $\mathcal{C}^\infty$ functions of w_1. The derivatives of $X(\mathfrak{p}, ., \widetilde{w})$ are equal to $X^{(k)}(\mathfrak{p}, ., \widetilde{w})$ and that of $F(\mathfrak{p}, ., \widetilde{w})$ are explicit:

$$F^{(1)}(\mathfrak{p}, ., \widetilde{w}) = -\big(X^{(2)} F^2\big)(\mathfrak{p}, ., \widetilde{w}),$$
$$F^{(2)}(\mathfrak{p}, ., \widetilde{w}) = -\big(2F^{(1)} X^{(2)} F + F^2 X^{(3)}\big)(\mathfrak{p}, ., \widetilde{w}),$$

and so on. We shall write

$$\partial_{w_1}\big(\phi(X)F\big) = X^{(1)}\phi'(X)F + \phi(X)F^{(1)} = \phi'(X) + \phi(X)F^{(1)}.$$

Proposition 131. *1. $X \in \mathbb{L}^p$, for all $1 \leqslant p < 2$. As $0 \leqslant X^{(k)} \leqslant X$, $X^{(k)} \in \mathbb{L}^p$, for all $1 \leqslant p < 2$.*

2. $F \in \mathbb{L}^p$, whatever $p \geqslant 1$. All its derivatives with respect to w_1 are also in the spaces $\mathbb{L}^p$, $p \geqslant 1$.

$\lhd$ *1.* Let $0 < a < 1$ be a constant. The function $e^{w_r - r}$ is integrable over $(0, +\infty($ on an event of probability 1. One can apply on this event Jensen's inequality to the function $\frac{1}{a} e^{w_r - (1-a)r}$ and the probability $a e^{-ar} \mathbf{1}_{r>0}$; one obtains

$$\left(\int_0^\infty e^{w_r - r}\, dr\right)^p = \left(\int_0^\infty \frac{e^{w_r - (1-a)r}}{a} a e^{-ar}\, dr\right)^p \leqslant a^{1-p} \int_0^\infty e^{p w_r - (1-a)pr} e^{-ar}\, dr.$$

So we have

$$\mathbb{E}\left[\left(\int_0^\infty e^{w_r - r}\, dr\right)^p\right] \leqslant a^{1-p} \int_0^\infty e^{\frac{p^2 r}{2} - \big((1-a)p + a\big) r}\, dr < \infty,$$

if

$$\frac{p^2}{2} < (1-a)p + a. \tag{4.3.3}$$

Since one can find a constant $0 < a < 1$, small enough to satisfy condition (4.3.3), for any $p < 2$, the result follows.

2. Estimate $\mathbb{Q}(F(w) > r)$.

$$\begin{aligned}
&\mathbb{Q}(F(w) > r) = \mathbb{Q}\left(\int_0^1 e^{\mathfrak{p}_u+(w_1-1)u}\,du + e^{w_1-1}\int_0^\infty e^{\widetilde{w}_u-u}\,du < \frac{1}{r}\right)\\
&\leqslant \mathbb{Q}\left(e^{w_1}\int_0^\infty e^{\widetilde{w}_u-u}\,du < \frac{e}{r}\right) \leqslant \mathbb{Q}\left(e^{w_1}\int_0^1 e^{\widetilde{w}_u-u}\,du < \frac{e}{r}\right)\\
&\leqslant \mathbb{Q}\left(e^{w_1}\int_0^1 e^{\widetilde{w}_u-u}\,du < \frac{e}{r}\ ;\ \inf_{u\in[0,1]}\widetilde{w}_u \geqslant -\frac{\ln(r)}{2}\right) + \mathbb{Q}\left(e^{w_1}\int_0^1 e^{\widetilde{w}_u-u}\,du < \frac{e}{r}\ ;\ \inf_{u\in[0,1]}\widetilde{w}_u < -\frac{\ln(r)}{2}\right)\\
&\leqslant \mathbb{Q}\left(e^{w_1}\int_0^1 e^{\frac{-\ln(r)}{2}-u}\,du < \frac{e}{r}\ ;\ \inf_{u\in[0,1]}\widetilde{w}_u \geqslant -\frac{\ln(r)}{2}\right) + \mathbb{Q}\left(\inf_{u\in[0,1]}\widetilde{w}_u < -\frac{\ln(r)}{2}\right)\\
&\leqslant \mathbb{Q}\left(w_1 \leqslant -\frac{\ln(r)}{2} + \mathrm{Cte}\right) + 2\,\mathbb{Q}\left(\widetilde{w}_1 \geqslant \frac{\ln(r)}{2}\right).
\end{aligned} \tag{4.3.4}$$

Each of these terms is of the same order: $\frac{1}{r^{\frac{\ln(r)}{8}}\ln(r)}$.
This function of r decreases enough to ensure that we have

$$\mathbb{E}[F^p] = \int_0^\infty r^{p-1}\mathbb{P}(F > r)dr < \infty.$$

for all $p \geqslant 1$.
As concern the derivatives of F, we treat the first two ones $F^{(1)}$ and $F^{(2)}$, higher derivatives being handled in the same way.
From the almost sure inequality

$$0 \leqslant X^{(2)}F = \frac{X^{(2)}}{X^{(1)}} \leqslant 1,$$

one gets

$$|X^{(2)}F^2| \leqslant F.$$

It follows that $F^{(1)} = -X^{(2)}(\omega)F(\omega)^2$ belongs to all the spaces $\mathbb{L}^p$, $p \geqslant 1$.
In the same way, as $X^{(3)}(\omega)F(\omega) = \frac{X^{(3)}(\omega)}{X(\omega)} \leqslant 1$, the random variable $F^2X^{(3)}$ happens to be in all the spaces $\mathbb{L}^p$. Besides, one knows that

$$0 \leqslant X^{(2)}\,F \leqslant 1,$$

and

$$F^{(1)} \in \mathbb{L}^p,\ \forall p \geqslant 1.$$

Thus, $F^{(2)} = -2F^{(1)}X^{(2)}F - F^2X^{(3)}$ is in all the $\mathbb{L}^p$'s, $p \geqslant 1$. ▷

4.3.2 Integration by parts

THEOREM **132 (Integration by parts formula).** *Let ϕ be any $\mathcal{C}^1$ bounded function with bounded derivative. Each member of the equality is well defined, and one has*

$$\mathbb{E}\big[w_1\,\phi(X)F\big] = \mathbb{E}\big[\partial_{w_1}\big(\phi(X)F\big)\big]. \tag{4.3.5}$$

◁ w_1 and F belonging to the spaces $\mathbb{L}^p$, $w_1 F$ is integrable. $\phi(X)$ is bounded. *Fubini's theorem* and the integration formula for a standard Gaussian measure on $\mathbb{R}$, $\mathcal{N}(dw_1)$, justify the following

$$\begin{aligned}\mathbb{E}[w_1\,\phi(X)F] &= \mathbb{P}^{[0,1]}\otimes\widetilde{\mathbb{P}}\Big(\int w_1\phi(X(\mathfrak{p},w_1,\widetilde{w}))F(X(\mathfrak{p},w_1,\widetilde{w}))\,\mathcal{N}(dw_1)\Big)\\ &= \mathbb{P}^{[0,1]}\otimes\widetilde{\mathbb{P}}\Big(\int \partial_{w_1}\big(\phi(X(\mathfrak{p},w_1,\widetilde{w}))F(X(\mathfrak{p},w_1,\widetilde{w}))\big)\,\mathcal{N}(dw_1)\Big)\\ &= \mathbb{E}\big[\partial_{w_1}\big(\phi(X)F\big)\big]\end{aligned}$$

▷

As $\partial_{w_1}\big(\phi(X)F\big) = X^{(1)}\phi(X)F + \phi(X)F^{(1)} = \phi'(X) + \phi(X)F^{(1)}$, the following corollary holds.

COROLLARY **133.** *There exists a constant $C_1 > 0$ such that the following inequality holds for every bounded function $\phi \in \mathcal{C}^1$, with bounded derivatives of all order.*

$$\big|\mathbb{E}[\phi'(X)]\big| \leqslant C_1\|\phi\|_\infty.$$

To get the same kind of estimates with $\phi^{(2)}$ instead of ϕ', apply the integration by part fomrula to ϕ' to get

$$\mathbb{E}\big[\phi^{(2)}(X)\big] = \mathbb{E}\big[\phi^{(1)}(X)\big\{Fw_1 - F^{(1)}\big\}\big]. \tag{4.3.6}$$

Write $K = Fw_1 - F^{(1)}$. As F and w_1 belong to all the spaces $\mathbb{L}^p$, Fw_1 is also in all the spaces $\mathbb{L}^p$, as well as K. The same holds for KF, which is for $\widetilde{\mathbb{P}}$-almost all $\widetilde{w}$ and every $\mathfrak{p}$ a smooth function of w_1, with derivative

$$\partial_{w_1}(KF) = K^{(1)}F + KF^{(1)}.$$

To estimate $\mathbb{E}[\phi'(X)K]$, one applies the integration by part formula not to $\phi(X)F$, but to $\phi(X)KF$. The use of the formula is justified as in the proof of the theorem. It gives

$$\mathbb{E}\big[\phi'(X)K\big] = \mathbb{E}\big[\phi(X)\big\{KF\,w_1 - \partial_{w_1}(KF)\big\}\big]. \tag{4.3.7}$$

So, we have

$$\mathbb{E}[\phi'(X)K] \leqslant \|\phi\|_\infty\,\|KF\,w_1 - \partial_{w_1}(KF)\|_{\mathbb{L}^1}.$$

We deduce from (4.3.6) that we can find a constant C_2 such that

$$\big|\mathbb{E}\big[\phi^{(2)}(X)\big]\big| \leqslant C_2\|\phi\|_\infty.$$

To get the same estimates with $\phi^{(3)}(X)$, one applies the integration by part formula to the functional $\phi(X)w_1F^2K$; then to get the following theorem, to the functional $\phi(X)(w_1F)^2FK$.

THEOREM **134.** *There exists a constant C_4 such that for every bounded function ϕ, with bounded derivatives of all order,*

$$\big|\mathbb{E}\big[\phi^{(4)}(X)\big]\big| \leqslant C_4\|\phi\|_\infty. \tag{4.3.8}$$

As was pointed out in proposition **130**, this inequality implies that the law of $X(w)$ has a $\mathcal{C}^2$ density with respect to Lebesgue's measure. The equation (2.2.14)([4]) that enabled its identification shows that this density is actually $\mathcal{C}^\infty$. We could get this fact by iterating indefinitely the integration by part formula (4.3.5).

[4]Where one takes $c = 1$.

4.4 Coupling of a hypoelliptic diffusion on $\mathbb{R}^2$

Let σ and b be two continuous functions on $\mathbb{R}$. Write $a = \frac{\sigma^2}{2}$. We suppose that

1. $a > 0$,
2. $\int_{-\infty}^{+\infty} a^{-1}(z)e^{\int_0^z \frac{b(r)}{a(r)}\,dr}\,dz < +\infty$,
3. the stochastic differential equation

$$dz_s = \sigma(z_s)dw_s + b(z_s)ds,$$

has the pathwise uniqueness property and that its solutions enjoy the strong Markov property.

Condition 2 ensures that the diffusion has an invariant probability; so it does not explode. This invariant probability is given by the formula

$$\mu(dz) = c\, a^{-1}(z)\, e^{\int_0^z \frac{b(r)}{a(r)}\,dr}\,dz,$$

where $c = \left(\int_{-\infty}^{\infty} a^{-1}(z)\, e^{\int_0^z \frac{b(r)}{a(r)}\,dr}\,dz\right)^{-1}$ ([5]). We set

$$m(z) = a^{-1}(z)\, e^{\int_0^z \frac{b(r)}{a(r)}\,dr}.$$

We make the hypotheses that

$$\begin{aligned} &\int_0^{+\infty} zm(z)\,dz < \infty, \quad \int_{-\infty}^0 zm(z)\,dz < \infty \\ &\int_0^{+\infty} e^{-\int_0^s \frac{b(r)}{a(r)}\,dr}\,ds = +\infty, \quad \int_{-\infty}^0 e^{-\int_0^s \frac{b(r)}{a(r)}\,dr}\,ds = +\infty. \end{aligned} \tag{4.4.1}$$

It is interesting to notice that under these hypotheses the scheme used in the proof of theorem **46** can be used to provide a "successful coupling" of two trajectories of the diffusion $\{(z_s, \int_0^s z_r\,dr)bigr\}_{s\geqslant 0}$ on $\mathbb{R}^2$, started from different points.

Let $(z_0, Z_0) \in \mathbb{R}^2$ and $(z'_0, Z'_0) \in \mathbb{R}^2$. Note $\{z_s\}_{s\geqslant 0}$ the solution of the stochastic differential equation

$$dz_s = \sigma(z_s)dw_s + b(z_s)ds,$$

with initial condition z_0, and $\{z'_s\}_{s\geqslant 0}$ the solution of the equation

$$dz_s = \sigma(z_s)dw'_s + b(z_s)ds,$$

with initial condition z_0, where w' is a Brownian motion independent of w. The processes z and z' are independent. Both can be defined on the measurable space $\Omega = \mathcal{C}(\mathbb{R}^{\geqslant 0}, \mathbb{R})$, equiped with its Borelian σ-algebra $\mathcal{F}$.

We set, for $s > 0$

$$Z_s = Z_0 + \int_0^s z_u\,du \text{ and } Z'_s = Z'_0 + \int_0^s z'_u\,du.$$

The dynamic of $\{(z_s, Z_s)\}_{s\geqslant 0}$ is illustrated in figure 5.1. The process Z_s increases if $z_s > 0$, decreases if $z_s < 0$.

As $\sigma > 0$, we can use the Support Theorem,**25**, to see that

[5]The condition $\int_{-\infty}^{+\infty} \frac{e^{\int_0^z \frac{b(r)}{a(r)}\,dr}}{a(z)}\,dz < \infty$ is a necessary and sufficient requirement to ensure the positive recurrence of $\{z_s\}_{s\geqslant 0}$.

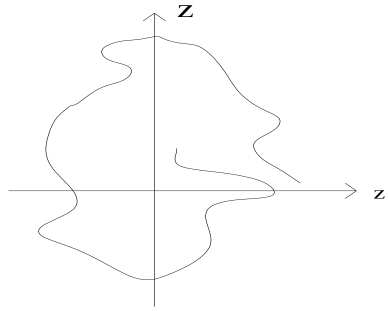


Figure 4.3: Dynamics of the diffusion

LEMMA **135.** *For any $\varepsilon > 0$ and any continuous function $s \in [0,1] \mapsto (\gamma_s, \Gamma_s) \in \mathbb{R}^2$, such that*

- $|\gamma_0 - z_0| \leqslant \varepsilon$, $|\Gamma_0 - Z_0| \leqslant \varepsilon$,
- Γ_s *increases if* $\gamma_s > 0$, *decreases if* $\gamma_s < 0$

one has

$$\mathbb{P}\Big(\sup_{s\in[0,1]} |z_s - \gamma_s| \leqslant \varepsilon \text{ and } \sup_{s\in[0,1]} |Z_s - \Gamma_s| \leqslant \varepsilon\Big) > 0$$

Note $\{\mathcal{F}_s^{(1)}\}_{s\geqslant 0}$ the filtration generated by $\{z_s\}_{s\geqslant 0}$: $\mathcal{F}_s^{(1)} = \sigma\big(z_r;\, r \leqslant s\big)$; and $\{\mathcal{F}_s^{(2)}\}_{s\geqslant 0}$ the filtration generated by $\{z'_s\}_{s\geqslant 0}$: $\mathcal{F}_s^{(2)} = \sigma\big(z'_r;\, r \leqslant s\big)$. The two diffusions $\{(z_s, Z_s)\}_{s\geqslant 0}$ and $\{(z'_s, Z'_s)\}_{s\geqslant 0}$ on $\mathbb{R}^2$ are independent. Note $\mathbb{P}$ the probability on $(\Omega\times\Omega, \mathcal{F}\otimes\mathcal{F})$, product of the probability laws of $\{(z_s, Z_s)\}_{s\geqslant 0}$ and $\{(z'_s, Z'_s)\}_{s\geqslant 0}$. This probability implicitely depends on z_0, z'_0, Z_0, Z'_0, which are fixed quantities in the sequel.

THEOREM **136.** *Note (z, Z) the coordinates on $\mathbb{R}^2$. The hypoelliptic differential operator*

$$\frac{\sigma(z)^2}{2}\partial_z^2 + b(z)\partial_z + z\,\partial Z$$

on $\mathbb{R}^2$, has no non constant bounded harmonic functions.

$\triangleleft$ Set

$$R_0 = \inf\{s \geqslant 0\,;\ z_s = 1\} \quad S_0 = \inf\{s \geqslant R_0\,;\ z_s = 0\} \text{ and } T_0 = \inf\{s \geqslant S_0\,;\ z_s = -1\}.$$

Define recursively, for $n \geqslant 1$,

$$R_n = \inf\{s \geqslant T_{n-1}\,;\ z_s = 1\} \quad S_n = \inf\{s \geqslant R_n\,;\ z_s = 0\} \text{ and } T_n = \inf\{s \geqslant S_n\,;\ z_s = -1\}.$$

These random times are $\mathcal{F}^{(1)}$-stopping times. We define similar random variables R'_n, S'_n, T'_n, with z'. Pay attention to the random variables $I_n = Z_{R_n}$ and $I'_n = Z'_{R'_n}$. Using hypotheses (4.4.1), we can make a similar proof to that of lemma **47** and show that for any $n \geqslant 1$

$$\mathbb{E}[|I_n - I_{n-1}|] < +\infty(^6). \tag{4.4.2}$$

$I'_n - I'_{n-1}$ has the same law as $I_n - I_{n-1}$.

But we know from the Strong Markov property that the random variables $\{I_n - I_{n-1}\}_{n\geqslant 1}$ are independent, identicaly distributed. So, the sequence $\{I_n\}_{n\geqslant 0}$ is a real random walk . The same holds for

[6] The details are writen in the proof of the proposition closing this section.

$\{I'_n\}_{n\geqslant 0}$; these two random walks are independent, with the same jump law. Using inequality (4.4.2), we see that $\{I_n - I'_n\}_{n\geqslant 0}$ is a *positive recurrent real random walk.*
Using this fact together with lemma **135** and the Strong Markov property, the Borel-Cantelli lemma implies that the set

$$\mathcal{N} \equiv \Big\{ n \in \mathbb{N}\,;\ \frac{1}{2} \leqslant I_n - I'_n \leqslant 1,\ \{(z_s, Z_s)\}_{s\in[R_n,S_n]} \cap \{(z'_s, Z'_s)\}_{s\in[R'_n,S'_n]} \neq \emptyset \Big\}$$

is $\mathbb{P}$-almost surely infinite.

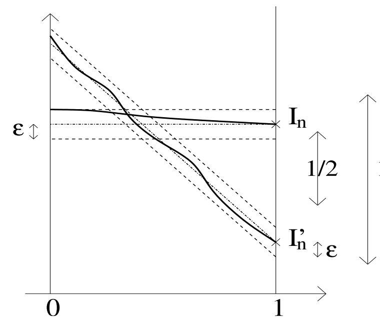


Figure 4.4: This configuration has a positive probability

The end of the proof follows the same road as that of the second point in the paragraph **An automatic coupling**, in section **2.2.3**.
Note $\mathbb{P}_{z_0,Z_0}$ the (marginal) law of $bigl\{(z_s, Z_s)bigr\}_{s\geqslant 0}$, $\mathbb{P}_{z'_0,Z'_0}$ that of $bigl\{(z'_s, Z'_s)bigr\}_{s\geqslant 0}$. A generic element in $\Omega\times\Omega$ is denoted (ω,ω').
Define

$$T(\omega,\omega') = \inf\{s \geqslant 0\,;\, \exists\, n \geqslant 0,\ R_n(\omega) \leqslant s \leqslant S_n(\omega),\ \frac{1}{2} \leqslant I_n(\omega) - I'_n(\omega') \leqslant 1,$$
$$(z_s(\omega), Z_s(\omega)) \in \{(z'_r(\omega'), Z'_r(\omega'))\}_{r\in[R'_n(\omega'),S'_n(\omega')]}\big\}$$

and

$$T'(\omega,\omega') = \inf\{s \geqslant 0\,;\, \exists\, n \geqslant 0,\ R'_n(\omega') \leqslant s \leqslant S'_n(\omega'),\ \frac{1}{2} \leqslant I_n(\omega) - I'_n(\omega') \leqslant 1,$$
$$(z'_s(\omega'), Z'_s(\omega')) \in \{(z_r(\omega), Z_r(\omega))\}_{r\in[R_n(\omega),S_n(\omega)]}\big\}.$$

The measurable subsets of Ω

$$\Omega_{z'} \equiv \{\omega' \in \Omega\,;\ T(\omega,\omega') < \infty,\, \mathbb{P}_{z_0,Z_0}(d\omega) - a.s.\}$$

and

$$\Omega_{z} \equiv \{\omega \in \Omega\,;\ T'(\omega,\omega') < \infty,\, \mathbb{P}_{z'_0,Z'_0}(d\omega') - a.s.\}$$

verify

$$\mathbb{P}_{z_0,Z_0}(\Omega_z) = \mathbb{P}_{z'_0,Z'_0}(\Omega_{z'}) = 1.$$

Because of the *monotonicity of* Z *(resp.* Z'*) on* $[R_n,S_n]$ *(resp.* $[R'_n,S'_n]$), one has $\mathbb{P}$-almost surely

$$(z,Z)_{T(\omega,\omega')} = (z',Z')_{T'(\omega,\omega')}.$$

The proof of proposition **65** can be used to conclude. ▷

One can find in the works of Ben Arous & al. [BACK95] and Kendall & Price [KP04] more information on couplings of hypoelliptic diffusions.

We end this section with the proof of the following generalisation of the calculation made during the proof of lemma **47**.

PROPOSITION **137.** *Consider the stochastic differential equation*

$$dz_s = \sigma(z_s)dw_s + b(z_s)ds,$$

where $\sigma > 0$ *is a continuous function, as well as* b[7]*. The Green function* $g^0(t,r)$ *of the diffusion* z *killed when hitting* 0 *is given by the formula*

$$g^0(t,r) = \left(\int_0^{t\wedge r} e^{-\int_0^s \frac{b(u)}{a(u)}\,du}\,ds\right)\frac{e^{\int_0^r \frac{b(u)}{a(u)}\,du}}{a(r)},\ \ 0 \leqslant t,r.$$

◁ We proceed exactly as in the proof of lemma **47**, by first finding the Green function $g^{0,v}(t,r)$ of the diffusion z *killed when it exits* $(0,v)$, and then letting $v \to +\infty$. We find $g^{0,v}(t,r)$ by solving explicitly the equation

$$a(t)f''(t) + b(t)f'(t) = -\varphi(t), \quad f(0^+) = 0,\ f(v^-) = 0, \tag{4.4.3}$$

where φ is a smooth function with compact support included in $(0,v)$. The solution of (4.4.3) is also

$$\int_0^{+\infty} g^{0,v}(t,r)\varphi(r)\,dr.$$

Note it $f_v(t)$ for the moment. Equation (4.4.3) is a first order equation in f'_v. We find its solutions using the method of variation of the parameters.

$$f'_v(t) = ce^{-\int_0^t \frac{b(r)}{a(r)}\,dr} - e^{-\int_0^t \frac{b(r)}{a(r)}\,dr}\int_0^t \frac{e^{\int_0^r \frac{b}{a}}}{a(r)}\varphi(r)\,dr,$$

where c is a constant. Remark that the density $m(r)$ of the unique invariant probability of the diffusion appears in that formula.

$$f'_v(t) = ce^{-\int_0^t \frac{b(r)}{a(r)}\,dr} - e^{-\int_0^t \frac{b(r)}{a(r)}\,dr}\int_0^t m(r)\varphi(r)\,dr.$$

One can choose the constant so as to have $f_v(0^+) = f_v(v^-) = 0$. It depends on the parameter v:

$$c_v = \frac{\int_0^v e^{-\int_0^s \frac{b}{a}}\int_0^s m(r)\varphi(r)\,drds}{\int_0^v e^{-\int_0^s \frac{b}{a}}\,ds},$$

With that choice, we obtain

$$\int_0^{+\infty} g^{0,v}(t,r)\varphi(r)\,dr = f_v(t) = \frac{\int_0^v e^{-\int_0^s \frac{b}{a}}\int_0^s m(r)\varphi(r)\,drds}{\int_0^v e^{-\int_0^s \frac{b}{a}}\,ds}\int_0^t e^{-\int_0^s \frac{b(r)}{a(r)}\,dr}\,ds - \int_0^t e^{-\int_0^s \frac{b(r)}{a(r)}\,dr}\int_0^s m(r)\varphi(r)\,dr\,ds.$$

Since $\int_0^\infty e^{-\int_0^s \frac{b}{a}}\,ds = +\infty$ and the function φ is compactly supported,

$$c_v \underset{v\to+\infty}{\longrightarrow} \int_0^\infty m(r)\varphi(r)\,dr,$$

[7]These continuity hypotheses are not necessary, but harmless.

and

$$\int_0^{+\infty} g^0(t,r)\varphi(r)\,dr = f_{+\infty}(t) = \left(\int_0^{\infty} m(r)\varphi(r)\,dr\right)\int_0^t e^{-\int_0^s \frac{b}{a}}\,ds - \int_0^t e^{-\int_0^s \frac{b}{a}}\left(\int_0^s m(r)\varphi(r)\,dr\right)\,ds.$$

Elementary manipulations yield

$$\int_0^{+\infty} g^0(t,r)\varphi(r)\,dr = \int_0^{\infty}\Big(\int_0^{t\wedge r} e^{-\int_0^s \frac{b(u)}{a(u)}\,du}\,ds\Big)m(r)\varphi(r)\,dr.$$

▷

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