Higher order paracontrolled calculus and applications for 3D PAM and Burgers system with multiplicative noise

I. BAILLEUL, F. BERNICOT, and D. FREY

Abstract. We sharpen in this work the tools of paracontrolled calculus in order to provide a complete analysis of the parabolic Anderson model equation in a 3-dimensional setting, in either bounded or unbounded domains equipped with a sub-Laplacian structure. We develop for that purpose a higher order paracontrolled calculus via semigroups methods. The technical core of this machinery is the introduction of a pair of intertwined space-time paraproducts on parabolic Hölder spaces, with good continuity properties, as well as some continuity properties of iterated commutators and correctors built from paraproducts and resonant operators. Given the scope of our semigroup methods in terms of operators and geometry of the ambient space, the application of our tools to the study of the 3-dimensional parabolic Anderson model equation provides results that go beyond the case of $\mathbb{R}^3$ with its Laplace operator very recently studied by Hairer and Labbé with the tools of regularity structures.

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Introduction

It is probably understated to say that the work [23] of M. Hairer has opened a new era in the study of stochastic singular parabolic partial differential equations. It provides a setting where one can make sense of a product of a distribution with parabolic non-positive Hölder regularity index, say \(a\), with a function with non-negative regularity index, say \(b\), even in the case where \(a + b\) is non-positive, and where one can make sense of and solve a large class of parabolic stochastic singular partial differential equations by fixed point methods. The parabolic Anderson model equation (PAM)

\[(\partial_t + L)u = u\zeta,\]

studied in Section 4 in a 3-dimensional unbounded background, is an example of such an equation. It makes sense in that setting to work with a distribution \(\zeta\) of Hölder exponent \(\alpha - 2\), for some \(\alpha < \frac{1}{2}\), while one expects the solution \(u\) to the equation to be of parabolic Hölder regularity \(\alpha\), making the product \(u\zeta\) undefined.

We will also be interested in the corresponding 3D Burgers system

\[(\partial_t + L)u + (u, \nabla)(u) = u\zeta,\]

where \(u\) is now valued in 3D and \(\zeta\) a 3D valued white noise.

This 3D Burgers system plays a very important role in the theory of PDEs coming from fluid mechanic (initially) and later from condensed matter physics, statistical physics, ... . It has been proposed by Burgers (1930s) as a simple model of the dynamics for Navier-Stokes equations. A change of variables, namely the Hopf-Cole transformation, allows us to reduce the deterministic quasilinear parabolic equation to the heat equation, thus allowing the derivation of exact solutions in the closed form. Despite of this simple transformation, the study of Burgers system is still very fashionable because of a benchmark model that can be used to understand the basic features of the interaction between nonlinearity and dissipation. Motivated by the intention to reinstate Burgers equation as a model for turbulence, stochastic variants has been the topic of numerous recent works, by adding a random forcing term (see for example [7, 13] for the 1D case with space-time white noise, [14, 22] for the 1D extension to the multiplicative noise and more recently [9] for a multi-dimensional result with an additive noise). The Hopf-Cole transformation allows us to link the stochastic Burgers system with additive noise to the heat equation with multiplicative noise (PAM Model). However, it is not clear how to study / solve the stochastic Burgers system with multiplicative noise, which we are going to provide a precise study through the paracontrolled calculus approach.

Also, this has to be seen as a first step to understand the multiplicative 3D stochastic incompressible Navier-Stokes equation, where the incompressibility brings the difficulty to deal with the Leray projector to keep the vanishing divergence property (as recently initiated with an additive noise in [21]).

The way out of this quandary found by M. Hairer has its roots in T. Lyons’ theory of rough paths, which already faced the same problem by addressing the question of making sense of and solving controlled differential equations

\[dz_t = V_i(z_t) dX^i_t\]

in \(\mathbb{R}^d\) say, driven by an \(\mathbb{R}^\ell\)-valued \(\frac{1}{p}\)-Hölder control \(X = (X^1, \ldots, X^\ell)\), with \(p \geq 2\), and where \(V_i\) are sufficiently regular vector fields on \(\mathbb{R}^d\). One expects a solution path to be \(\frac{1}{p}\)-Hölder continuous as well, in which case the product \(V_i(z_t) dX^i_t\), or the integral \(\int_0^t V_i(z_s) dX^i_s\), cannot be given an intrinsic meaning as \(\frac{2}{p} - 1 \leq 0\), in our case. Typical realizations of a Brownian path have that regularity. Lyons’ deep insight was to realize that one can make sense of and solve equation (1.2) if one assumes that there is given an enriched version of the driving signal \(X\) that formally consists of \(X\) together with its non-existing iterated integrals. The theory of regularity structures rests on the same
philosophy and the idea that these additional objects should be used as a basis to give a local description of the unknown $u$, in the same way as polynomials are used to describe locally $C^k$ functions.

At the very same time that M. Hairer built his theory, Gubinelli, Imkeller and Perkowski proposed in [19] another implementation of that philosophy building on a different notion of local description of a distribution, using paraproducts on the torus. The machinery of paracontrolled distributions introduced in [19] rests on a first order Taylor expansion of a distribution that happened to be sufficient to deal with the stochastic parabolic Anderson equation (1.1) on the 2-dimensional torus, the stochastic Burgers equation in one space dimension [19] and the $\Phi^4_3$ equation on the 3-dimensional torus. The KPZ equation can also be dealt with using this setting [20].

Following Bony’s approach [8], the paraproduct used in [19] is defined in terms of Fourier analysis and does not allow for the treatment of equations outside the flat background of the torus or, if one is ready to work with weighted functional spaces, the Euclidean space. Their machinery allows them to work in a certain range of irregularity for the singular distribution in the equation under study, which prevents them for instance from considering the 3-dimensional parabolic Anderson model equation. The geometric restriction on the background was greatly relaxed in our previous work [3] by building paraproducts from the heat semigroup associated with the operator $L$ in the semilinear equation. A theory of paracontrolled distributions can then be considered in doubling metric measure spaces where one has small time Gaussian estimates on the heat kernel and its ‘gradient’ – see [3]. This setting already offers situations where the theory of regularity structures is not known to be working. The stochastic parabolic Anderson model equation in a 2-dimensional doubling manifold was considered as an example.

We extend in the present work the scope of the paracontrolled approach by introducing higher order paracontrolled calculus, which can be understood as higher order ‘Taylor’ expansions in a paracontrolled setting. This will allow us for instance to study the PAM equation in a 3-dimensional unbounded setting. From a technical point of view, this requires the introduction of a number of commutators and correctors built from our para-product. Working in unbounded spaces with weighted functional spaces also requires a careful treatment which was not done so far.

Hairer and Labbè have very recently studied the same equation in $\mathbb{R}^3$ from the point of view of regularity structures [25]. Our results give an alternative approach, and provide a non-trivial extension of their result to a non-flat setting, with a possibly wider range of operators $L$ than can be treated presently in the theory of regularity structures.

The geometric and functional setting in which we lay down our study is described in Section 2. In short, we work on a doubling metric measure space $(M, d, \mu)$, equipped with a sub-Laplacian operator $L$ given by the finite sum of squares of ”vector fields”, which is assumed to generate a heat semigroup whose kernel and its iterated derivatives satisfy Gaussian pointwise bounds; precise conditions are given in the item Conditions in the beginning of Section 2.1. Such a setting covers a number of interesting cases.

One can use the semigroup to construct in an intrinsic way the scale of spatial Hölder space $C^0(M)$ on $M$ and a scale of parabolic Hölder spaces $C^\alpha([0,T] \times M)$, and prove some Schauder-type regularity estimate for the heat semigroup. One of our main contributions is the introduction of a pair of paraproducts built from the heat semigroup, intertwined via the resolution map that provides the solution to the equation $(\partial_t + L)v = f$, with zero initial condition, as a function of the right hand side $f$; see Section 3. This allows us to use exact formulas where commutators were used previously, with no hope to get
a fine description for it. These two paraproducts have the same algebraic structure and the same analytic properties, most importantly a cancellation property that is introduced in Section 2.2. The technical core of the paracontrolled calculus introduced in [19] is a continuity estimate for a corrector that allows to make sense of an a priori undefined term by compensating it by another potentially undefined term with a simpler structure. We prove in Section 3.2 more continuity results for iterated correctors and some modified commutators that appear useful in developing our higher order paracontrolled calculus in the example of the parabolic Anderson model equation in Section 4.

**Theorem 1.** Let \((M, d, \mu)\) be a doubling metric measure space that is Ahlfors regular of dimension 3, and let \(L\) be a second order differential operator on \(M\) that satisfies the assumptions Conditions given below in Section 2.1. Let \(\xi\) stand for a time-independent white noise on \(M\), and let \(\xi^\varepsilon := (e^{-\varepsilon L})\xi\) be its regularization via the heat semigroup. Given \(\alpha \in (\frac{2}{5}, \frac{3}{2})\), there exists a sequence \((\varepsilon^\alpha)_{0<\varepsilon\leq 1}\) of time-independent and deterministic functions such that the following holds. For every finite positive time horizon \(T\) and every initial data \(u_0 \in C^{4\alpha}(M)\), the solution \(u^\varepsilon\) of the renormalized equation

\[
\partial_t u^\varepsilon + Lu^\varepsilon = u^\varepsilon (\xi^\varepsilon - c^\varepsilon), \quad u^\varepsilon(0) = u_0
\]

converges in probability to the solution \(u \in C^\alpha([0, T] \times M)\) of the parabolic Anderson model equation on \(M\). The result holds with \(T = \infty\) if \(\mu(M)\) is finite.

The notion of solution to the (PAM) equation will be explained in Section 4. The constraint \(\alpha > \frac{3}{5}\) on the regularity index is somewhat irrelevant and can be weakened to \(\alpha > 0\). Hairer and Labbé [25] are able to work in the range \(-\frac{1}{2} < \alpha \leq 0\), in the setting of regularity structures; we do not know how to deal with such a situation in our setting.

Note on the other hand that we described in the Appendix of [3] how to extend paracontrolled calculus to a Sobolev setting. Together with the present work, this allows to prove in Section 3.2 more continuity results for iterated correctors and some modified commutators that appear useful in developing our higher order paracontrolled calculus in the example of the parabolic Anderson model equation in Section 4.

**Notation.** Let us fix some notation that will be used throughout the work. Given a metric measure space \((M, d, \mu)\), we shall denote its parabolic version by \((\mathcal{M}, \rho, \nu)\), where \(\mathcal{M} := M \times \mathbb{R}\) is equipped with the parabolic metric

\[
\rho((x, \tau), (y, \sigma)) = d(x, y) + \sqrt{|\tau - \sigma|}
\]

and where, for every \((x, \tau) \in \mathcal{M}\) and positive radius \(r\), the parabolic ball \(B_{\mathcal{M}}((x, \tau), r)\) has the volume

\[
\nu(B_{\mathcal{M}}((x, \tau), r)) = r^2 \mu(B(x, r)).
\]

Given an unbounded linear operator \(L\) on \(L^2(M)\), we denote by \(\mathcal{D}_2(L)\) its domain. We give here the definition of a distribution, as it is understood in the present work. The definition will always be associated with the operator \(L\) described in Subsection 2.1 below.

**Definition.** We fix a point \(o \in M\) and then define a Fréchet space of test functions on \(\mathcal{M}\) setting

\[
\mathcal{S}_o := \left\{ f \in \bigcap_{n \geq 0} C^\infty_n(\mathcal{D}_2(L^n)); \forall a_1, a_2, a_3 \in \mathbb{N}, \left\| (1 + d(o, \cdot))^a_1 \partial^a_3 L^a_2 f \right\|_{2, d\nu} < \infty \right\},
\]
equipped with the metric
\[ \|f\| := \sup_{a_1,a_2,a_3 \in \mathbb{N}} 1 \land \left\| \left( 1 + d(o, \cdot) \right)^{a_1} \partial^a f \right\|_{2,d\nu}. \]

A distribution is a continuous linear functional on \( S_o \); we write \( S'_o \) for the set of all distributions.

(Let us point out that the arbitrary choice of the point \( o \in M \) is only relevant in the case of an unbounded ambient space \( M \); even in that case, the space \( S_o \) does not depend on \( o \), for \( o \) ranging inside a bounded subset of \( M \).)

As a last bit of notation, we shall always denote by \( K_Q \) the kernel of an operator \( Q \), and write \( \lesssim_T \) for an inequality that holds up to a positive multiplicative constant that depends only on \( T \).

2 Geometric and functional settings

We describe in this section the geometric and functional setting in which we shall construct our space-time paraproducts in Section 3. These paraproducts will be used in Section 4 to solve the parabolic Anderson model equation \((1.1)\) in a 3-dimensional unbounded setting, and give a kind of high order Taylor expansion of its solution. We shall work in a sub-Laplacian setting under fairly general conditions. Parabolic Hölder spaces can be defined purely in terms of the associated semigroup, see Section 2.3. In Section 2.4 we prove Schauder estimates, which will be one of our main tools in the study of the 3-dimensional parabolic Anderson model equation in Section 4. The cancellation properties put forward in Section 2.2 are fundamental for proving some continuity results in Section 3.

2.1 Sub-Laplacian framework

Our basic setting in this work will be a volume doubling metric measure space \((M,d,\mu)\), with \( M \) a manifold or a discrete set; all kernels mentioned in the sequel are with respect to the measure \( \mu \). We are going to introduce in the sequel a number of tools to analyze singular partial differential equations involving a parabolic operator on \( \mathbb{R}_+ \times M \)
\[ \mathcal{L} := \partial_t + L, \]
with \( L \) built from first order differential/difference operators \((V_i)_{i=1..\ell_0}\) on \( M \), satisfying the Leibniz rule
\[ V_i(fg) = fV_i(g) + gV_i(f) \]
for all functions \( f,g \in \mathcal{D}(V_i) \) with \( fg \in \mathcal{D}(V_i) \). Denoting by \( \mathcal{D}(V_i) \) the domain in \( L^2(M) \) of \( V_i \), we assume that
\[ \mathcal{D} := \bigcap_{i=1}^{\ell_0} \mathcal{D}(V_i) \]
is dense in \( L^2(M) \); we shall work in the sequel with
\[ L = -\sum_{i=1}^{\ell_0} V_i^2, \]
and identify \( L \) with its smallest closed extension in \( L^2(M) \). Given a tuple \( I = (i_1,\ldots,i_k) \) in \( \{1,\ldots,\ell_0\}^k \), we shall set \(|I| := k\) and
\[ V_I := V_{i_k} \cdots V_{i_1}. \]
**Conditions.** We shall assume throughout that

- the operator $L$ is injective, has a bounded $H^\infty$-calculus on $L^2(M)$, and $-L$ generates a holomorphic semigroup $(e^{-tL})_{t>0}$ on $L^2(M)$,
- the semigroup has regularity estimates at any order, which means that for every tuple $I$, the operators $t^{|I|/2}V_I e^{-tL}$ and $e^{-tL}t^{|I|/2}V_I$ have kernels $K(x,y)$ satisfying the Gaussian estimate
  \[ |K(x,y)| \lesssim \frac{1}{\mu(B(x,\sqrt{t}))} e^{-\frac{d(x,y)^2}{c}} \tag{2.1} \]
  and the following regularity estimate: for $d(x,z) \leq \sqrt{t}$
  \[ |K(x,y) - K(z,y)| \lesssim \frac{d(y,z)}{\sqrt{t}} V(x,\sqrt{t}) e^{-\frac{d(x,y)^2}{c}} \tag{2.2} \]
  for some constants which may depend on $|I|$,
- the heat semigroup is conservative $(e^{-tL})(1_M) = 1_M$ for every $t > 0$, where $1_M$ stands for the constant function on $M$ – or in a weak sense that $L(1_M) = 0$.

We first point out that regularity property \[2.2\] for $|I| = k$ can (usually) be obtained through \[2.1\] for $k + 1$ as soon as we have a kind of "finite-increment" formula
\[ |K(x,y) - K(z,y)| \lesssim d(x,z) \sup_{j \in \{x,z\}} \sup_{w \in (x,z)} |X_j K(w,y)| \]
where $(x,y)$ stands for a geodesic joining $x$ to $z$ and of length $d(x,z)$.

As a matter of fact, it suffices for the present work to assume that the semigroup has regularity estimates of large enough order. Here are a few examples of such a geometrical setting.

(a) **Euclidean domains.** In the particular case of the Euclidean space, all of the current work can be reformulated in terms of Fourier transform rather than in terms of heat semigroup; which may make some reasoning a bit more usual but does not really simplifies anything. The case of a bounded domain with its Laplacian associated with Neumann boundary conditions fits our framework if the boundary is sufficiently regular.

(b) **Riemannian manifolds.** Assume $M$ is a parallelizable $d$-dimensional manifold with a smooth global frame field $V = (V_1,\ldots,V_d)$. One endows $M$ with a Riemannian structure by turning $V$ into orthonormal frames. The above assumption on the heat kernel holds true if $M$ has bounded geometry, that is if

(i) the curvature tensor and all its covariant derivatives are bounded in the frame field $V$,
(ii) Ricci curvature is bounded from below,
(iii) and $M$ has a positive injectivity radius;

see for instance [12] or [30]. One can actually include the Laplace operator in this setting by working with its canonical lift to the orthonormal frame bundle, given by $\frac{1}{2} \sum_{i=1}^d H_i^2 + \frac{1}{2} \sum_{1 \leq j < k \leq d} V_{jk}$, where the $H_i$ are the canonical horizontal vector fields of the Levi-Civita connection, and the $V_{jk}$ are the canonical vertical vector fields on the orthonormal frame bundle, inherited from its $SO(\mathbb{R}^d)$-principal bundle structure. The bundle $OM$ is parallelizable and satisfies the assumptions **Conditions** if the Riemannian base manifold $M$ satisfies the above three conditions (i–iii).
(c) **Carnot-Caratheodory spaces and Lie groups.** Let $M$ be a $d$-dimensional manifold and $(V_1, \ldots, V_l)$ be a family of smooth vector fields on $M$. Recall they are said to satisfy Hörmander’s condition if the Lie algebra generated by the $V_i$ has dimension $d$ at all points of $M$. One can then equip $M$ with a metric, named after Carnot and Caratheodory, as follows. Let $A_V$ be the family of absolutely continuous curves $\zeta : [a,b] \rightarrow M$, such that there exist $\ell$ measurable functions $c_j : [a,b] \rightarrow [0,1]$ such that one has
\[
\sum_{j=1}^{\ell} c_j(t)^2 \leq 1 \quad \text{and} \quad \zeta'(t) = \sum_{j=1}^{\ell} c_j(t)V_j(\zeta(t))
\]
at almost all times $t \in [a,b]$. The Carnot-Caratheodory distance between two points $x$ and $y$ of $M$ is then defined as
\[
d_{CC}(x,y) := \inf \left\{ T > 0; \text{ there exists } \zeta \in A_V \text{ with } \zeta(0) = x \text{ and } \zeta(T) = y \right\}.
\]
- **Lie groups** provide interesting classes of examples. Assume we are given a family $(V_1, \ldots, V_n)$ of left-invariant vector fields on a Lie group $G$, satisfying Hörmander’s condition. The associated Carnot-Caratheodory distance function is left-invariant, so that balls have a volume that only depends on their radius, and the metric space $(G, d_{CC})$ is of homogeneous type [21, 28]. Two cases can happen when the group $G$ is unimodular. It is either doubling or the volume of balls has an exponential growth [21]; nilpotent Lie groups are for instance doubling [16]. We refer the reader to [29, Thm 5.14] and [12, Section 3, Appendix 1] for a detailed study of the heat semigroup in such a setting; the heat kernel and its high-order derivatives satisfy Gaussian upper bounds, so the above condition holds true.
- **Carnot groups** are important examples in so far as they provide universal local models for sub-Riemannian diffusions. Recall a nilpotent Lie group is called a Carnot group if it admits a stratification, that is one can split its Lie algebra $\mathfrak{g}$ into
\[
\mathfrak{g} = V_1 \oplus \ldots \oplus V_r
\]
where the linear subspaces $V_i$ satisfy $[V_i, V_j] = V_{i+j}$ for $i = 1, \ldots, r-1$ and $[V_1, V_r] = 0$. (By $[V_i, V_j]$, we denote the subspace of $\mathfrak{g}$ generated by the elements $[X,Y]$ where $X \in V_i$ and $Y \in V_j$.) Write $n_i$ for the dimension of $V_i$ and $d = n_1 + \cdots + n_r$ for the dimension of $G$. One defines a family $(\delta_\lambda)_{\lambda > 0}$ of dilations on $\mathfrak{g}$ by the formula
\[
\delta_\lambda(x) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}), \quad x^{(i)} \in V_i.
\]
The couple $G = (G, \delta_\lambda)$ is called a homogeneous Carnot group of step $r$ and $n_1$ generators if $\delta_\lambda$ is an automorphism of $\mathfrak{g}$ for every $\lambda > 0$ and if the first $n_1$ elements of the Jacobian basis of $\mathfrak{g}$ span a Lie subalgebra of maximal dimension $d$. The number $Q := \sum_{i=1}^{r} i n_i$ is called the homogeneous dimension of $G$.

The Heisenberg group $H^d$ is for instance a Carnot group of dimension $Q = 2d+2$. We refer the reader to [18] for an introduction to pseudo-differential operators in this setting using a kind of Fourier transform involving irreducible representations and to [2] for a complete work about pseudo-differential calculus on Heisenberg groups.

(d) **Discrete setting.** We underline here that our setting is flexible enough to handle problems set in a discrete setting. Consider as an example the following network given for a large enough integer $N$,
\[
\mathbb{T}_N^d := \frac{1}{N}(\mathbb{Z}/(N\mathbb{Z}))^d,
\]
which can be seen as a discrete approximation of the torus $\mathbb{T}_N^d = (\mathbb{R}/\mathbb{Z})^d$, for $N \to \infty$. On $\mathbb{Z}_N^d := (\mathbb{Z}/N\mathbb{Z})^d$, or $\mathbb{Z}^d$, the discrete positive Laplacian $L_{\mathbb{Z}_N^d}$ is defined by the formula

$$L_N(f)(x) := -\frac{1}{4} \sum_{i=1}^{d} \left( f(x + 2e_i) + f(x - 2e_i) \right) + \frac{d}{2} f(x),$$

where $(e_i)_{i=1}^d$ is the canonical basis of $\mathbb{Z}_N^d$. Such a Laplacian fits our sub-Laplacian framework, in the sense that we have

$$L_N(f) = -\sum_{i=1}^{d} V_i^2(f),$$

with the difference operators

$$(V_i f)(x) := \frac{1}{2} \left( f(x + e_i) - f(x - e_i) \right).$$

Even though the operators $V_i$ do not satisfy an exact Leibniz rule, we have

$$V_i(fg)(x) = V_i(f(x)) g(x + e_i) + f(x - e_i) V_i(g)(x);$$

translation by a unit vector of the canonical basis preserves any Hölder space. This discrete Laplacian generates a heat semigroup $(e^{-tL_{\mathbb{Z}_N^d}})$, whose heat kernel, and its iterated discrete gradients, satisfy the pointwise Gaussian bounds

$$K_{e^{-tL_{\mathbb{Z}_N^d}}}(x,y) \lesssim t^{-\frac{d}{2}} e^{-c \frac{d(x,y)^2}{t}},$$

under the restriction $t \leq d(x,y)$. We refer the reader to [13] and references therein for characterization of such pointwise estimates for infinite graphs and to [26] for the particular case of discrete groups.

Estimates for the heat kernel of the Laplacian of $\mathbb{T}_N^d$ can be deduced from the latter. Indeed, by using the canonical change of variables $x \mapsto Nx$ which realizes a bijection between $\mathbb{T}_N^d$ and $\mathbb{Z}_N^d$, we deduce that for every $x, y \in \mathbb{T}_N^d$

$$K_{e^{-tL_{\mathbb{T}_N^d}}}(x,y) = N^d K_{e^{-tL_{\mathbb{Z}_N^d}}}(Nx,Ny).$$

So for $t \geq N^{-1}$ and every $x, y \in \mathbb{T}_N^d$, one has $d(x,y) \lesssim 1 \lesssim tN$, and so

$$K_{e^{-tL_{\mathbb{T}_N^d}}}(x,y) \lesssim t^{-\frac{d}{2}} e^{-c \frac{d(x,y)^2}{t}}.$$

Therefore these Gaussian pointwise estimates, and similar ones for the iterated gradient of the semigroup, are satisfied on $\mathbb{T}_N^d$, but only at the scales with $t > 1/N$.

Anticipating over the developments of Section 4 on the parabolic Anderson model equation (PAM), the above restricted heat kernel estimates on $\mathbb{T}_N^d$ tell us that one can only solve the “projection” of (PAM) equation through the projection operator $P_{1/N} := e^{-tL_{\mathbb{T}_N^d}/N}$, that its it only make sense to solve the equation

$$(2.3) \quad P_{1/N} \left( \partial_t u + L_N u \right) = P_{1/N} \left( P_{1/N}(u).P_{1/N}(\xi) \right).$$

We refer the reader to a forthcoming work [11] by K. Chouck, J. Gairing and N. Perkowski, where the authors studied the 2-dimensional discrete version of the (PAM) equation in the Fourier analysis language.

Uniform bounds for the solutions $u_N$ of equation (2.3) can be obtained in $C^\alpha(\mathbb{T}_N^d)$, as $N$ varies. Since $C^\alpha(\mathbb{T}_N^d)$ can be canonically embedded into $C^\alpha(\mathbb{T}_N^d)$,
by expanding in Fourier series as done in [11], this gives a sequence of functions $\bar{u}_N$ uniformly bounded in $C^\alpha(T^d)$ which may be shown by compactness to converge to the unique solution of the continuous (PAM) equation on the torus $T^d$. This procedure will be detailed in [11] in the 2-dimensional setting; combining the details of [11] with the present extension of the paracontrolled calculus, it is possible to prove in a 3-dimensional setting an invariance principle similar to that proved in [11] in a 2-dimensional setting.

2.2. Approximation operators and cancellation

We introduce in this section a fundamental notion of approximation operators. Some of them possess cancellation effects as a kind of orthogonality property quantified by condition (2.9) below. Note that we shall be working in a parabolic setting with mixed cancellation effects in time and space.

All computations below make sense for a choice of large enough integers $b, \ell_1$ to be fixed later. The following parabolic Gaussian-like kernels $(G_t)_{0 < t \leq 1}$ will be used as reference kernels in this work. For $0 < t \leq 1$ and $\sigma \leq \tau$, if $d(x, y) \leq 1$, set

$$G_t((x, \tau), (y, \sigma)) := \nu(B_M((x, \tau), \sqrt{t}))^{-1} \left(1 + \frac{\rho((x, \tau), (y, \sigma))}{t}\right)^{-\ell_1},$$

otherwise we set

$$G_t((x, \tau), (y, \sigma)) := \mu(B(x, 1))^{-1} e^{-cd(x,y)^2t} \left(1 + \frac{|\tau - \sigma|}{t}\right)^{-\ell_1} \left(1 + \frac{d(x, y)^2}{t}\right)^{-\ell_1}$$

for $d(x, y) \geq 1$; set $G_t \equiv 0$ if $\tau \leq \sigma$. We do not emphasize the dependence of $G$ on the positive constant $c$ in the above definition, and we shall allow ourselves to abuse notations and write $G_t$ for two functions corresponding to two different values of that constant. So we have for instance, for $s, t \in (0, 1)$, the estimate

$$\int_M G_t((x, \tau), (y, \sigma)) G_s((y, \sigma), (z, \lambda)) \nu(dy d\sigma) \lesssim G_{t+s}((x, \tau), (z, \lambda)).$$

Indeed, in $G_t$ kernel the spatial variables can be separated from the time-variable. Then in both space or time variables, the previous inequality only comes from classical estimates for convolution with fast decay at infinity for the functions, see [3, Lemma A.5] for example.

This somewhat unnatural definition of a “Gaussian” kernel is justified by the fact that we shall mainly be interested in local regularity matters; the definition of $G$ in the domain $\{d(x, y) \geq 1\}$ is only technical and will allow us to obtain global estimates with weights. Presently, note that a large enough choice of constant $\ell_1$ ensures that we have

$$\sup_{t \in (0, 1]} \sup_{(x, \tau) \in M} \int_M G_t((x, \tau), (y, \sigma)) \nu(dy d\sigma) < \infty,$$

so any linear operator on $M$, with a kernel pointwisely bounded by some $G_t$ is bounded in $L^p(\nu)$ for every $p \in [1, \infty]$.

A last bit of notation is needed before we introduce the cancellation property for a family of operators in a parabolic setting. Given a real-valued integrable function $\phi$ on $\mathbb{R}$, set

$$\phi_t(\cdot) := \frac{1}{t} \phi\left(\frac{\cdot}{t}\right);$$
the family \( (\phi_t)_{0 < t \leq 1} \) is uniformly bounded in \( L^1(\mathbb{R}) \). We also define the “convolution” operator \( \phi^* \) associated with \( \phi \) via the formula
\[
\phi^*(f)(\tau) := \int_0^\infty \phi(\tau - \sigma) f(\sigma) d\sigma.
\]

Note that if \( \phi \) has support in \( \mathbb{R}_+ \), then the operator \( \phi^* \) has a kernel supported on the same set \( \{ (\sigma, \tau) : \sigma \leq \tau \} \) as our Gaussian-like kernel. Moreover, we let the reader to check that if \( \phi_1, \phi_2 \) are two \( L^1 \)-functions with \( \phi_2 \) supported on \([0, \infty)\) then an easy computation yields that
\[
(\phi_1 \ast \phi_2)^* = \phi_1^* \circ \phi_2^*
\]
where \( \phi_1 \ast \phi_2 \) is the usual convolution.

Given an integer \( b \geq 1 \), we define a special family of operators on \( L^2(M) \) setting
\[
Q_t^{(b)} := \gamma_b^{-1}(tL)^b e^{-tL} \quad \text{and} \quad -t\partial_t P_t^{(b)} = Q_t^{(b)},
\]
with \( \gamma_b := (b-1)! \); so \( P_t^{(b)} \) is an operator of the form \( p_b(tL)e^{-tL} \), for some polynomial \( p_b \) of degree \( b-1 \), with value 1 in 0. Under the above Conditions assumptions, the operators \( P_t^{(b)} \) and \( Q_t^{(b)} \) both satisfy the Gaussian regularity estimates (2.1) at any order
\[
(2.6) \quad \left| K_{\frac{|j|}{2}} V_{\frac{|j|}{2}} (x,y) \right| \vee \left| K_{\frac{|j|}{2}} V_{\frac{|j|}{2}} (x,y) \right| \leq \frac{1}{\mu(B(x, \sqrt{t}))} e^{-\frac{d(x,y)^2}{c}},
\]
with \( R \) standing here for \( P_t^{(b)} \) or \( Q_t^{(b)} \).

The parameters \( b \) and \( \ell_1 \) will be chosen large enough, and fixed throughout the paper. See Proposition 13 and the remark after Proposition 14 for the precise choice of \( b \) and \( \ell_1 \).

**Definition.** Let an integer \( a \in [0, 2b] \) be given. The following collection of families of operators is called the standard collection of operators with cancellation of order \( a \), denoted by \( SO^a \). It is made up of all the space-time operators
\[
\left( (t \frac{|j|}{2}) V_j \right) L^{\frac{a-|j|-2k}{2}} P_t^{(c)} \otimes m_t^* \right)_{0 < t \leq 1}
\]
where \( k \) is an integer with \( 2k + |J| \leq a \), and \( c \in [1, b] \), and \( m \) is any smooth function supported on \([\frac{1}{2}, 2]\) such that
\[
(2.7) \quad \int \tau^i m(\tau) d\tau = 0,
\]
for all \( 0 \leq i \leq k-1 \), with the first \( b \) derivatives bounded by 1. These operators are uniformly bounded in \( L^p \) for every \( p \in [1, \infty] \). So a standard collection of operators \( \mathcal{Q} \) can be seen as a map \( \mathcal{Q} : t \to Q_t \) from \([0, 1]\) to \( B(L^p(M)) \), the set of bounded linear operators on \( L^p(M) \). Moreover, such a function is bounded. We also set \( SO := \bigcup_{0 \leq a \leq 2b} SO^a \).

The cancellation effect is quantified in Proposition 2 below; note here that it makes sense at an intuitive level to say that \( L^{\frac{a-|j|-2k}{2}} \) encodes cancellation in the space-variable of order \( a - |J| - 2k \), that \( V_j \) encodes a cancellation in space of order \( |J| \) and that the moment condition (2.7) encodes a cancellation property in the time-variable of order \( k \) for the convolution operator \( m_t^* \). Since we are in the parabolic scaling, a cancellation of order \( k \) in time corresponds to a cancellation of order \( 2k \) in space, so that \( V_j L^{\frac{a-|j|-2k}{2}} P_t^{(c)} \otimes m_t^* \) has a space-time cancellation property of order \( a \). We invite the reader to check that each operator \( (t \frac{|j|}{2}) V_j (tL)^{\frac{a-|j|-2k}{2}} P_t^{(c)} \otimes m_t^* \) has a kernel pointwisely bounded above by some \( G_t \). We give one more definition before stating the cancellation property.
Proof – Given an operator \( Q_t := \left( t^{\frac{|j|}{2}} V_I \right) \phi(tL) \), with \(|I| \geq 1\), defined by functional calculus from some appropriate function \( \phi \), we write \( Q_t^* \) for the formal dual operator
\[
Q_t^* := t^{\frac{|j|}{2}} \phi(tL) V_I
\]
For \( I = \emptyset \), and \( Q_t = \phi(tL) \), we set \( Q_t^* := Q_t \). For an operator \( Q_t \) as above we set
\[
(Q_t \otimes m_t^*)^* = Q_t^* \otimes m_t^*.
\]
Note that the above definition is not related to any classical notion of duality and let emphasize that we do not assume that \( L \) is self-adjoint in \( L^2(\mu) \). This notation is only used to indicate that a \( Q_t \), resp. \( Q_t^* \), operator can be composed on the right, resp. on the left, by another operator \( \psi(L) \), for a suitable function \( \psi \), due to the functional calculus on \( L \).

Proposition 2. Consider \( Q^1 \in SO^{a_1} \) and \( Q^2 \in SO^{a_2} \) two collections encoding cancellation, and set \( a := \min(a_1, a_2) \). Then for every \( s,t \in (0,1] \), the composition \( Q_s^1 Q_t^2 \) has a kernel pointwisely bounded by
\[
(2.8)  \quad |K_{Q_s^1 Q_t^2}((x, \tau), (y, \sigma))| \lesssim \left( \frac{ts}{(s+t)^2} \right)^{\frac{a}{2}} \mathcal{G}_{t+s}((x, \tau), (y, \sigma)).
\]

The above mentioned orthogonality property of standard operators with cancellation is encoded in the factor \( \left( \frac{ts}{(s+t)^2} \right)^{\frac{a}{2}} \) that appears in the above estimate. This factor is small as soon as \( s \) or \( t \) is small compared to the other.

Proof – Given \( Q^1_s = s^{\frac{n_1}{2}} V_{J_1}(sL)^{-a_1-j_1-2k_1} P_{s(c_1)}^{(1)} \otimes m_s^{(1)*} \) and \( Q^2_t = (tL)^{-a_2-j_2-2k_2} P_{t(c_2)}^{(2)} t^{\frac{d_2}{2}} V_{J_2} \otimes m_t^{(2)*} \) a standard operator and the dual of another, we have
\[
Q^1_s Q^2_t = s^{\frac{n_1}{2}} t^{\frac{n_2}{2}} e^{2k_1} e^{2k_2} V_{J_1} L^{-a_1-j_1-2k_1+a_2-j_2-2k_2} P_{s(c_1)}^{(1)} P_{t(c_2)}^{(2)} V_{J_2} \otimes \left(m_s^{(1)*} \ast m_t^{(2)} \right).
\]
Assume, without loss of generality, that \( 0 < s \leq t \). Then the kernel of the time-convolution operator \( m_s^{(1)*} \ast m_t^{(2)} \) is given by
\[
K_{m_s^{(1)*} \ast m_t^{(2)}}(\tau - \sigma) = \int m_s^{(1)} \left( \frac{\tau - \lambda}{s} \right) m_t^{(2)} \left( \frac{\lambda - \sigma}{t} \right) d\lambda_{st},
\]
Since \( m_s^{(1)} \) has vanishing \( k_1 \) first moments, we can perform \( k_1 \) integration by parts and obtain that
\[
|K_{m_s^{(1)*} \ast m_t^{(2)}}(\tau, \sigma)| \lesssim \left( \frac{s}{t} \right)^{k_1} \int \partial^{-k_1} m_s^{(1)} \left( \frac{\tau - \lambda}{s} \right) \partial^{k_1} m_t^{(2)} \left( \frac{\lambda - \sigma}{t} \right) \frac{d\lambda}{st},
\]
where we slightly abuse notations and write \( \partial^{-k_1} m_s^{(1)} \) for the \( k_1 \)th primitive of \( m_s^{(1)} \) null at 0. Then we get
\[
|K_{m_s^{(1)*} m_t^{(2)}}(\tau, \sigma)| \lesssim \left( \frac{s}{t} \right)^{k_1} \int \left( 1 + \frac{|\tau - \lambda|}{s} \right)^{-\ell_1 + 2} \left( 1 + \frac{\lambda - \sigma}{t} \right)^{-\ell_1 + 2} \frac{d\lambda}{st}
\]
\[
\lesssim \left( \frac{s}{t} \right)^{k_1} \left( 1 + \frac{|\tau - \sigma|}{s + t} \right)^{-\ell_1} (s + t)^{-1}.
\]
In the space variable, the kernel of \( V_{J_1} L^{-a_1-j_1-2k_1+a_2-j_2-2k_2} P_{s(c_1)}^{(1)} P_{t(c_2)}^{(2)} V_{J_2} \) is bounded above by
\[
(s + t)^{-a_1-j_1-2k_1+a_2-j_2-2k_2} \mu(B(x, \sqrt{s + t}))^{-1} e^{-\frac{d(x,y)^2}{s + t}},
\]
as a consequence of the property \(2.6\). Altogether, this gives

\[
\left| K_{Q^1 Q^2^*} \left( (x, \tau), (y, \sigma) \right) \right| \lesssim \left( \frac{s}{t} \right)^{k_1} s^{\alpha_1 - 2k_1} t^{\alpha_2 - 2k_2} (s + t)^{-\alpha_1 + 2k_1 - \alpha_2 + 2k_2} \mathcal{G}_{t+s} \left( (x, \tau), (y, \sigma) \right)
\]

\[
\lesssim \left( \frac{s}{t} \right)^{k_2} \mathcal{G}_{t+s} \left( (x, \tau), (y, \sigma) \right)
\]

\[
\lesssim \left( \frac{s}{t} \right)^{k} \mathcal{G}_{t+s} \left( (x, \tau), (y, \sigma) \right),
\]

where we used that \( s \leq t \) and \( a \leq a_1 \).

\[ \square \]

**Definition.**

- Define \( \mathcal{O} \) as the set of families \( (P_t)_{0 \leq t \leq 1} \) of linear operators on \( \mathcal{M} \) with kernels pointwisely bounded by

\[
\left| K_{P_t} \left( (x, \tau), (y, \sigma) \right) \right| \lesssim \mathcal{G}_t \left( (x, \tau), (y, \sigma) \right).
\]

- Let \( 0 \leq a \leq 2b \) be an integer. We define the subset \( \mathcal{O}^a \) of \( \mathcal{O} \) of families of operators with the cancellation property of order \( a \) as the set of elements \( Q \) of \( \mathcal{O} \) with the following cancellation property. For every \( 0 < s, t \leq 1 \) and every standard family \( Q^s \in \mathcal{SO}^{a'} \), with \( a' \in [a, 2b] \), the operator \( Q_t Q^s \) has a kernel pointwisely bounded by

\[
\left| K_{Q_t Q^s} \left( (x, \tau), (y, \sigma) \right) \right| \lesssim \left( \frac{st}{(s + t)^2} \right)^{\frac{a}{2}} \mathcal{G}_{t+s} \left( (x, \tau), (y, \sigma) \right).
\] (2.9)

Here are a few examples. Consider a smooth function \( \chi \) on \([2^{-1}, 2]\), and an integer \( c \geq 1 \) and a tuple \( I \) of indices.

- The families \( \left( Q^{(\frac{2}{1})}_t \otimes \chi^i_t \right)_{0 \leq t \leq 1} \) and \( \left( t^{|I|} V_I P_t^{(c)} \otimes \chi^i_t \right)_{0 \leq t \leq 1} \) belong to \( \mathcal{O}^a \) if \( |I| \geq a \);

- If \( \int \tau^k \chi(\tau) \, d\tau = 0 \) for all integer \( k = 0, \ldots, a - 1 \), then we can see by integration by parts along the time-variable that \( \left( P_t^{(c)} \otimes \chi^i_t \right)_{0 \leq t \leq 1} \in \mathcal{O}^a \).

- If \( \int \tau^k \chi(\tau) \, d\tau = 0 \) for all integer \( k = 0, \ldots, a_2 \) with \( a_1 + a_2 = a \), then the families \( \left( Q^{(\frac{2}{1})}_t \otimes \chi^i_t \right)_{0 \leq t \leq 1} \) and \( \left( t^{|I|} V_I P_t^{(c)} \otimes \chi^i_t \right)_{0 \leq t \leq 1} \), where \( |I| \geq a_1 \), both belong to \( \mathcal{O}^a \).

We see on these examples that cancellation in the parabolic setting can encode some cancellations in the space variable, the time-variable or both at a time.

We introduced above the operators \( Q_t^{(b)} \) and \( P_t^{(b)} \) acting on \( \mathcal{M} \). We end this section by introducing their parabolic counterpart. Choose arbitrarily a smooth real-valued function \( \varphi \) on \( \mathbb{R} \), with support in \([1^{-1}, 2]\), unit integral and such that for every integer \( k = 1, \ldots, b \)

\[
\int \tau^k \varphi(\tau) \, d\tau = 0.
\]

Set

\[
P_t^{(b)} := P_t^{(b)} \otimes (\varphi \ast \varphi)^*, \quad \text{and} \quad Q_t^{(b)} := -t \partial_t P_t^{(b)},
\]

where \( \varphi \ast \varphi \) is the convolution. Denote by \( M_\tau \) the multiplication operator in \( \mathbb{R} \) by \( \tau \). An easy computation yields that

\[
Q_t^{(b)} = Q_t^{(b)} \otimes (\varphi \ast \varphi)^* + P_t^{(b)} \otimes \psi_t
\]
where \( \psi = (\varphi \ast \varphi) + 2(M \ast \varphi) \ast \varphi' \) is defined as
\[
\psi(t) = \partial_t \left[ \tau.(\varphi \ast \varphi)(t) \right]
= (\varphi \ast \varphi)(t) + 2(M \ast \varphi) \ast \varphi'(t) = (\varphi \ast \varphi)(t) + 2 \int_{\mathbb{R}} \sigma \varphi(\sigma) \varphi'(t - \sigma) d\sigma.
\]

Note that, from its very definition, a parabolic operator \( Q_t^{(b)} \) belongs at least to \( \mathcal{O}^2 \). Note also that due to the normalization of \( \varphi \), then for every \( f \in L^p(\mathbb{R}) \) supported on \([0, \infty)\) then
\[
\varphi_t^a(f) \xrightarrow{t \to 0} f \quad \text{in } L^p.
\]

So, the operators \( P_t \) weakly tend to the identity on \( L^p_0(\mathcal{M}) \) (the set of functions \( f \in L^p(\mathcal{M}) \) with time-support included in \([0, \infty)\) ) \( p \in [1, \infty) \), and \( C^0_0(\mathcal{M}) \) (the set of functions \( f \in C^0(\mathcal{M}) \) with time-support included in \([0, \infty)\) ) as \( t \) goes to \( 0 \); so we have the following \textbf{Calderón reproducing formula}: for every continuous function \( f \in L^\infty(\mathcal{M}) \) with time-support in \([0, \infty)\), then
\[
(2.10) \quad f = \int_0^1 Q_t^{(b)}(f) \frac{dt}{t} + P_1^{(b)}(f).
\]

This formula will play a fundamental role for us. Noting that the measure \( \frac{dt}{t} \) gives unit mass to intervals of the form \([2^{-i-1}, 2^{-i}]\), and considering the operator \( Q_t^{(b)} \) as a kind of multiplier roughly localized at frequencies of size \( t^{-\frac{1}{2}} \), Calderón’s formula appears as nothing else than a continuous time analogue of the Paley-Littlewood decomposition of \( f \), with \( \frac{dt}{t} \) in the role of the counting measure.

\[\text{2.3. Parabolic Hölder spaces}\]

We define in this section space and space-time weighted Hölder spaces, with possibly negative regularity index, and give a few basic facts about them. The setting of weighted function spaces is needed for the applications to the parabolic Anderson model equation on unbounded domains studied in Section 4.

Let us start recalling the following well-known facts about Hölder space on \( M \), and single out a good class of weights on \( M \). A function \( w : M \to [1, \infty) \) will be called a \textbf{spatial weight} if one can associate to any positive constant \( c_1 \) a positive constant \( c_2 \) such that
\[
(2.11) \quad \sup_{x,y \in M, \quad d(x,y) \leq 1} \frac{w(x)}{w(y)} \leq c_2, \quad \text{and} \quad w(x) e^{-c_1 d(x,y)} \leq c_2 w(y), \quad \forall x, y \in M.
\]

Given \( 0 < \alpha \leq 1 \), the classical metric Hölder space \( H^\alpha_w \) is defined as the set of real-valued functions \( f \) on \( M \) with finite \( H^\alpha_w \)-norm, defined by the formula
\[
\|f\|_{H^\alpha_w} := \|w^{-1} f\|_{L^\infty(M)} + \sup_{0 < d(x,y) \leq 1} \left| \frac{f(x) - f(y)}{w(x) d(x,y)^{\alpha}} \right| < \infty.
\]

**Definition.** For \( \alpha \in (-3, 3) \) and \( w \) a spatial weight, define \( C^\alpha_w := C^\alpha_w(M) \) as the closure of \( C^0 \) (the set of bounded and continuous functions) for \( C^\alpha_w \)-norm, defined by the formula
\[
\|f\|_{C^\alpha_w} := \|w^{-1} e^{t f}\|_{L^\infty(M)} + \sup_{0 < t \leq 1} t^{-\frac{\alpha}{2}} \left\| w^{-1} Q_t^{(a)} f \right\|_{L^\infty(M)};
\]
this norm does not depend on the integer \( a > \frac{|\alpha|}{2} \), and one can prove (see [3]) that the two spaces \( H^\alpha_w \) and \( C^\alpha_w \) coincide and have equivalent norms when \( 0 < \alpha < 1 \).
These notions have parabolic counterparts which we now introduce. A \textbf{space-time weight} is a function \( \omega : \mathcal{M} \to [1, \infty) \) with \( \omega(x, \cdot) \) non-decreasing for every \( x \in \mathcal{M} \), and such that there exists two constants \( c_1 \) and \( c_2 \) with

\[
\sup_{d(x, y) \leq 1; \tau \geq 0} \frac{\omega(x, \tau)}{\omega(y, \tau)} \leq c_2, \quad \omega(x, \tau) e^{-c_1 d(x, y)} \leq c_2 \omega(y, \tau), \quad \forall (x, \tau), (y, \tau) \in \mathcal{M}.
\]

The function \( w_\tau := \omega(\cdot, \tau) \) is a spatial weight for every time \( \tau \). For \( 0 < \alpha \leq 1 \) and a space-time weight \( \omega \), the metric parabolic Hölder space \( H_\omega^\alpha = H_\omega^\alpha(\mathcal{M}) \) is defined as the set of all functions on \( \mathcal{M} \) with finite \( H_\omega^\alpha \)-norm, defined by the formula

\[
\| f \|_{H_\omega^\alpha} := \| \omega^{-1} f \|_{L^\infty(\mathcal{M})} + \sup_{0 < \rho((x, \tau), (y, \sigma)) \leq 1; \tau \geq \sigma} \frac{|f(x, \tau) - f(y, \sigma)|}{\omega(x, \tau) \rho((x, \tau), (y, \sigma))^{\alpha}}.
\]

As in the above space setting one can recast this definition in a more functional setting, using the parabolic standard operators. This requires the use of the following elementary result.

\textbf{Lemma 3.} Let \( T \) be a linear operator on \( \mathcal{M} \) with a kernel \( K_T \) pointwisely bounded by a Gaussian kernel \( G_t \), for some \( t \in (0, 1] \). Then for every space-time weight \( \omega \), we have

\[
\| \omega^{-1} T f \|_{L^\infty(\mathcal{M})} \lesssim \| \omega^{-1} f \|_{L^\infty(\mathcal{M})}.
\]

\textbf{Proof –} Indeed, for every \( (x, \tau) \in \mathcal{M} \) we have

\[
\omega(x, \tau)^{-1} |T f(x, \tau)| \leq \int_{\mathcal{M}} G_t((x, \tau), (y, \sigma)) \frac{\omega(y, \sigma)}{\omega(x, \tau)} \omega(y, \sigma)^{-1} |f(y, \sigma)| \nu(dy d\sigma)
\]

\[
\lesssim \int_{\mathcal{M}} G_t((x, \tau), (y, \sigma)) \frac{\omega(y, \sigma)}{\omega(x, \sigma)} \omega(y, \sigma)^{-1} |f(y, \sigma)| \nu(dy d\sigma)
\]

\[
\lesssim \int_{\mathcal{M}} G_t((x, \tau), (y, \sigma)) \omega(y, \sigma)^{-1} |f(y, \sigma)| \nu(dy d\sigma)
\]

\[
\lesssim \| \omega^{-1} f \|_{L^\infty},
\]

where the implicit constants in \( G_t \) may change from one line to another and where we used (2.5) in the last line.

\( \square \)

\textbf{Definition.} For \( \alpha \in (-3, 3) \) and a space-time weight \( \omega \), we define the parabolic Hölder space \( C_\omega^\alpha := C_\omega^\alpha(\mathcal{M}) \) as the closure (in the set of distributions) of the set of bounded and continuous functions on \( \mathcal{M} \) for the \( C_\omega^\alpha \)-norm, defined by

\[
\| f \|_{C_\omega^\alpha} := \sup_{Q \in SO_k} \| \omega^{-1} Q_1(f) \|_{L^\infty(\mathcal{M})} + \sup_{Q \in SO_k} \sup_{0 \leq t \leq 1} t^{-\frac{\alpha}{2}} \| \omega^{-1} Q_t(f) \|_{L^\infty(\mathcal{M})}.
\]

Building on Calderón’s formula (2.10), one can prove as in [3] that the two spaces \( H_\omega^\alpha \) and \( C_\omega^\alpha \) coincide and have equivalent norms when \( 0 < \alpha < 1 \).

\textbf{Proposition 4.} For \( \alpha \in (0, 1) \) and every space-time weight \( \omega \), the two spaces \( H_\omega^\alpha \) and \( C_\omega^\alpha \) coincide and have equivalent norms.

\textbf{Proof –} We first check that \( H_\omega^\alpha \) is continuously embedded into \( C_\omega^\alpha \). So fix a function \( f \in H_\omega^\alpha \), then by Lemma 3 we easily deduce that

\[
\sup_{Q \in SO_k} \| \omega^{-1} Q_1(f) \|_{L^\infty(\mathcal{M})} \lesssim \| \omega^{-1} f \|_{L^\infty(\mathcal{M})}.
\]
For the high frequency part, we consider $t \in (0,1]$ and for $Q \in SO^k$ with $\alpha < k \leq 2b$ then $Q_t$ has at least a cancellation of order 1, hence
\[
Q_t[f](x,\tau) = Q_t[f - f(x,\tau)](x,\tau) = \int K_{Q_t}(x,\tau), (y,\sigma) [f(y,\sigma) - f(x,\tau)] \nu(dyd\sigma).
\]
Due to the kernel support of $Q_t$, the integrated quantity is non-vanishing (and so relevant) only for $\tau \geq 0$. If $\rho((x,\tau), (y,\sigma)) \leq 1$, then by definition
\[
|f(y,\sigma) - f(x,\tau)| \leq \omega(x,\tau) \rho((x,\tau), (y,\sigma))^\alpha \|f\|_{H^L}
\]
and if $\rho((x,\tau), (y,\sigma)) \geq 1$, then by the property of the weight we have
\[
|f(y,\sigma) - f(x,\tau)| \leq (\omega(x,\tau) + \omega(y,\tau)) \|\omega^{-1}f\|_{L^\infty(M)}.
\]
Hence
\[
|Q_t[f](x,\tau)| \lesssim \omega(x,\tau) \left\{ \int_{\rho \leq 1} G_t((x,\tau), (y,\sigma)) \rho((x,\tau), (y,\sigma))^\alpha \nu(dyd\sigma) + \int_{\rho \geq 1} G_t((x,\tau), (y,\sigma)) \left( 1 + \frac{\omega(y,\tau)}{\omega(x,\tau)} \right) \nu(dyd\sigma) \right\} \|f\|_{H^L} \lesssim \omega(x,\tau) t^{\alpha/2} \|f\|_{H^L},
\]
uniformly in $(x,\tau) \in M$ and $t \in (0,1)$; which concludes the proof of $H^L \hookrightarrow C^\alpha$. Then we now prove the reverse embedding : $C^\alpha \hookrightarrow H^L$. So fix a function $f \in C^\alpha$. For the low-frequency part, we easily have due to the Calderón reproducing formula
\[
\|\omega^{-1}f\|_{L^\infty(M)} \lesssim \|\omega^{-1}P_1^{(1)}f\|_{L^\infty(M)} + \int_0^1 \|\omega^{-1}Q_t^{(1)}f\|_{L^\infty(M)} \frac{dt}{t} \lesssim \|f\|_{H^L},
\]
since $\alpha > 0$. Then fix $(x,\tau)$ and $(y,\sigma)$ in $M$ with $\rho := \rho((x,\tau), (y,\sigma)) \leq 1$ and $\tau \geq \sigma$. We still decompose
\[
f = P_1^{(1)}f + \int_0^t Q_t^{(1)}f \frac{dt}{t}.
\]
For $t < \rho^2$, we have
\[
|Q_t^{(1)}f(x,\tau)| \lesssim t^{\alpha/2} \omega(x,\tau) \|f\|_{C^\alpha}
\]
and
\[
|Q_t^{(1)}f(y,\sigma)| \lesssim t^{\alpha/2} \omega(y,\sigma) \|f\|_{C^\alpha} \lesssim t^{\alpha/2} \omega(x,\tau) \|f\|_{C^\alpha}
\]
where we used that the weight is increasing in time and then that $d(x,y) \leq \rho \leq 1$ with the property of the weight. So we may integrate over $t < \rho^2$ and we have
\[
\int_0^{\rho^2} |Q_t^{(1)}f(x,\tau) - Q_t^{(1)}f(y,\sigma)| \frac{dt}{t} \lesssim \left( \int_0^{\rho^2} t^{\alpha/2} \frac{dt}{t} \right) \omega(x,\tau) \|f\|_{C^\alpha} \lesssim \rho^\alpha \omega(x,\tau) \|f\|_{C^\alpha}.
\]
For the low frequencies part, $Q_t^{(1)}$ with $\rho^2 \leq t \leq 1$ or $P_1^{(1)}$, we use that
\[
|Q_t^{(1)}f(x,\tau) - Q_t^{(1)}f(x,\sigma)| \lesssim |\tau - \sigma|^{1/2} \left( \sup_{\varsigma \in (\sigma,\tau)} \left| \partial_\tau Q_t^{(1)}f(x,\varsigma) \right| \right) \left( \sup_{\varsigma \in (\sigma,\tau)} \left| Q_t^{(1)}f(x,\varsigma) \right| \right) \lesssim \rho \omega(x,\tau) t^{(\alpha/2)} \|f\|_{C^\alpha},
\]
where we used that \( \rho \leq 1 \) with the fact that both \((t\partial_x Q_t^{(1)})_t\) and \((Q_t^{(1)})_t\) are collections of type \( \mathcal{SO}^1 \) (with a cancellation of order at least 1) and that the weight is non-decreasing in time. Similarly we can estimate the variation in space with the assumed finite-increment representation \([2,2]\),

\[
\left| Q_t^{(1)} f(x,\sigma) - Q_t^{(1)} f(y,\sigma) \right| \lesssim \|X_j Q_t^{(1)} f(z,\varsigma)\| \\
\lesssim \rho \omega(x,\tau,t)^{(\alpha-1)/2} \|f\|_{C^\alpha_\omega}.
\]

So we get

\[
\int_{\rho^2}^1 \left| Q_t^{(1)} f(x,\tau) - Q_t^{(1)} f(y,\sigma) \right| \frac{dt}{t} \lesssim \rho \left( \int_{\rho^2}^1 t^{(\alpha-1)/2} \frac{dt}{t} \right) \omega(x,\tau,t) \|f\|_{C^\alpha_\omega} \\
\lesssim \rho^\alpha \omega(x,\tau,t) \|f\|_{C^\alpha_\omega},
\]

because of \( \alpha < 1 \). We also obtain a similar estimate for \( P_t^{(1)} \) instead of \( Q_t^{(1)} \), which concludes the proof of \( C^\alpha_\omega \hookrightarrow \mathcal{H}^\alpha_\omega \).

\[\Box\]

The restriction \( \alpha \in (-3,3) \) is irrelevant and will be sufficient for our purpose in this work; taking \( b \) large enough we can allow regularity of as large an order as we want.

The next proposition introduces an intermediate space whose unweighted version was first introduced in the setting of paracontrolled calculus in \([19]\), and used in \([3]\). To fix notations, and given a space-time weight \( \omega \), we denote by \( \left( C^2_{\tau} L^\infty_x \right)(\omega) = \left( L^\infty_x C^2_{\tau} \right)(\omega) \) the set of parabolic distributions such that

\[
\sup_{x \in \mathbb{M}} \|f(x,)\|_{C^2_{\omega(x,)}(\mathbb{R}^+)} < \infty.
\]

Also \( \left( L^\infty_x C^\alpha_x \right)(\omega) \) stands for the set of parabolic distributions such that

\[
\sup_{\tau} \|f(,\tau)\|_{C^\alpha_\omega(M)} < \infty.
\]

**Proposition 5.** Given \( \alpha \in (0,2) \) and a space-time weight \( \omega \), set

\[
\mathcal{E}^\alpha_\omega := \left( C_{\tau}^{\alpha/2} L^\infty_x \right)(\omega) \cap \left( L^\infty_x C^\alpha_x \right)(\omega).
\]

Then \( \mathcal{E}^\alpha_\omega \) is continuously embedded into \( C^\alpha_\omega \). Furthermore, if \( \alpha \in (0,1) \), the spaces \( \mathcal{E}^\alpha_\omega, C^\alpha_\omega \) and \( \mathcal{H}^\alpha_\omega \) are equal, with equivalent norms.

**Proof** – We first check that \( \mathcal{E}^\alpha_\omega \) is continuously embedded into \( C^\alpha_\omega \), and fix for that purpose a function \( f \in \mathcal{E}^\alpha_\omega \). As done in \([3, \text{ Proposition 2.12}]\), we know that for all integers \( k, j \) with \( k + \frac{j}{2} > \frac{\alpha}{2} \) and every space function \( g \in C^0(M) \), we have

\[
\left\| t^j \overline{V}_j (tL)^k e^{-tL} g \right\|_{L^\infty(M)} \lesssim t^\frac{j}{2} \|g\|_{C^0(M)}
\]

for any subset of indices \( J \) with \( |J| = j \). So consider a generic standard family \( \left( t^{j/2} \overline{V}_j (tL)^a - j/2 - k P_t^{(c)} \otimes m_t^* \right)_{0 < t \leq 1} \) in \( \mathcal{SO}^a \), with \( 3 \leq a \leq b \), and a smooth function \( m \) with vanishing first \( k \) moments. If \( k = 0 \) we have seen that we have

\[
\left\| \omega^{-1} t^\frac{j}{2} \overline{V}_j (tL)^{a/2 - j/2} P_t^{(c)} f(,\tau) \right\|_{L^\infty(M)} \lesssim t^\frac{j}{2} \|f(\tau)\|_{C^\alpha_\omega},
\]

for every \( \tau \), so

\[
\left\| \omega^{-1} t^\frac{j}{2} \overline{V}_j (tL)^{a/2 - j/2} P_t^{(c)} \otimes m_t^* (f) \right\|_{L^\infty(M)} \lesssim t^\frac{j}{2} \|f\|_{L^\infty_t C^\alpha_\omega(\omega)}
\]
since $m_t^*$ is a $L^\infty(\mathbb{R})$-bounded operator as a convolution with an $L^1$-normalized function.

If $k = 1$ (or $k \geq 1$), the same reasoning shows that we have
\[
\|\omega(x, \cdot)^{-1} m_t^*(f)(x, \cdot)\|_{L^\infty(\mathbb{R}_+)} \lesssim t^{\frac{\omega}{2}} \|f(x, \cdot)\|_{C^0_{\omega(x, \cdot)}(\mathbb{R}_+)}
\]
for every $x \in M$, since $\frac{\omega}{2} \in (0, 1)$, and $m$ encodes a cancellation at order 1 in time as it has a vanishing first moment. Hence
\[
\left\|\omega^{-1} t^{\frac{\omega}{2}} V_j(tL)^{\alpha/2-j/2} P_t^{(e)} \otimes m_t^*(f)\right\|_{L^\infty(M)} \lesssim t^{\frac{\omega}{2}} \|f\|_{C^0_{\omega} L^\infty_x(\omega)}^rac{\omega}{2},
\]
which concludes the proof of the embedding $\mathcal{E}^\omega_0 \hookrightarrow C_0^\omega$. The remainder of the statement is elementary since $C_0^\omega = H_\omega^0$ is embedded in $\mathcal{E}^\omega_0$.

Before turning to the definition of an intertwined pair of parabolic paraproducts we close this section with two other useful continuity properties involving the H"older spaces $C^\omega_\alpha$.

**Proposition 6.** Given $\alpha \in (0, 1)$, a space-time weight $\omega$, some integer $a \geq 0$ and a standard family $\mathcal{P} \in \mathcal{SO}^a$, there exists a constant $c$ depending only on the weight $\omega$, such that
\[
\omega(x, \tau)^{-1} \left| (\mathcal{P}_t f)(x, \tau) - (\mathcal{P}_s f)(y, \sigma) \right| \lesssim \left( s + t + \rho((x, \tau), (y, \sigma)) \right)^2 e^{cd(x,y)} \|f\|_{C^\omega_\alpha},
\]
uniformly in $s, t \in (0, 1]$ and $(x, \tau), (y, \sigma) \in M$, with $\tau \geq \sigma$.

**Proof —** We explain in detail the most difficult case corresponding to $\mathcal{P} \in \mathcal{SO}^0$, so $\mathcal{P}$ encodes no cancellation. Then $\mathcal{P}_t$ takes the form
\[
\mathcal{P}_t = P_t^{(e)} \otimes m_t^*
\]
for some integer $c \geq 1$ and some smooth function $m$. There is no loss of generality in assuming that $\int m(\tau) \, d\tau$ is equal to 1, as $\mathcal{P}$ is actually an element of $\mathcal{SO}^1$ if $m$ has zero mean – this case is treated at the end of the proof.

In this setting, since $f$ is bounded and continuous, we have the pointwise identity
\[
f = \lim_{t \to 0} \mathcal{P}_t(f).
\]

**i)** Consider first the case where $\rho((x, \tau), (y, \sigma)) \leq 1$, so it follows from condition (2.12) that
\[
\omega(x, \tau)^{-1} \leq \omega(x, \sigma)^{-1} \lesssim \omega(y, \sigma)^{-1}.
\]

We decompose
\[
\omega(x, \tau)^{-1} \left| (\mathcal{P}_t f)(x, \tau) - (\mathcal{P}_s f)(y, \sigma) \right|
\leq \omega(x, \tau)^{-1} \left| f(x, \tau) - f(y, \sigma) \right| + \omega(x, \tau)^{-1} \left| (\mathcal{P}_t f)(x, \tau) - f(x, \tau) \right|
\]
\[
+ \omega(x, \tau)^{-1} \left| (\mathcal{P}_t f)(y, \sigma) - f(y, \sigma) \right|
\lesssim \omega(x, \tau)^{-1} \left| f(x, \tau) - f(y, \sigma) \right| + \omega^{-1} \left( \mathcal{P}_t f - f \right) \|_{L^\infty(M)} + \omega^{-1} \left( \mathcal{P}_s f - f \right) \|_{L^\infty(M)}.
\]

We have
\[
\omega(x, \tau)^{-1} \left| f(x, \tau) - f(y, \sigma) \right| \leq \rho((x, \tau), (y, \sigma)) a \|f\|_{H_0^\omega} \lesssim \rho((x, \tau), (y, \sigma)) a \|f\|_{C_0^\omega}.
\]

For the two other terms, we use that
\[
\left\|\omega^{-1} (\mathcal{P}_t f - f)\right\|_{L^\infty(M)} \leq \int_0^t \omega^{-1} u \partial_u \mathcal{P}_u f \left| \frac{du}{u} \right|,
\]
and note that
\[ u\partial_u P_u = Q_u^{(c)} \otimes m_u + P_u^{(c)} \otimes k_u \]
with \( k(\tau) = \partial_\tau [\tau m(\tau)] \), is actually the sum of two terms in \( SO^{\geq 1} \) since it is clear for the first one and the function \( k \) has a vanishing first moment. It follows by definition of the Hölder spaces with \( \alpha < 1 \), that we have
\[ \| \omega^{-1} (P_t f - f) \|_{L^\infty(M)} \lesssim \left( \int_0^t u^{\frac{\alpha}{2}} \frac{du}{u} \right) \| f \|_{C^\alpha \omega} \lesssim t^{\frac{\alpha}{2}} \| f \|_{C^\alpha \omega}. \]
A similar estimate holds by replacing \( t \) by \( s \), which then concludes the proof in this case.

ii) In the case where \( \rho((x, \tau), (y, \sigma)) \geq 1 \), we do not use the difference and use condition (2.12) on the weight \( \omega \) to write directly
\[ \omega(x, \tau)^{-1} \left| (P_t f)(x, \tau) - (P_s f)(y, \sigma) \right| \leq \omega(x, \tau)^{-1} \left| (P_t f)(x, \tau) \right| + \omega(x, \tau)^{-1} \left| (P_s f)(y, \sigma) \right| \]
\[ \leq \| \omega^{-1} P_t f \|_{L^\infty(M)} + \omega(x, \sigma)^{-1} \left| (P_t f)(y, \sigma) \right| \]
\[ \lesssim \| \omega^{-1} P_t f \|_{L^\infty(M)} + e^{cd(x,y)} \| \omega^{-1} P_s f \|_{L^\infty(M)}, \]
for some positive constant \( c \). Since we know by Lemma 2 that \( P_t \) and \( P_s \) are bounded in \( L^\infty(\omega) \), we deduce that
\[ \omega(x, \tau)^{-1} \left| (P_t f)(x, \tau) - (P_s f)(y, \sigma) \right| \lesssim e^{cd(x,y)} \| f \|_{L^\infty(M)} \]
\[ \lesssim e^{cd(x,y)} \| f \|_{C^\alpha \omega}, \]
since \( C^\alpha \omega \subset L^\infty(\omega) \), given that \( \alpha > 0 \). The awaited estimate follows from that point.

• In the easier situation where \( P \in SO^a \) for some integer \( a \geq 1 \), we can perform the same reasoning and use in addition the fact that
\[ \lim_{t \to 0} P_t (f) = 0, \]
so it is easier since we do not have to deal with the first term \( f(x, \tau) - f(y, \sigma) \).

With an analogous reasoning (indeed simpler) we may prove the following.

**Proposition 7.** Given \( \alpha \in (-3, 0) \), a space-time weight \( \omega \) and a standard family \( P \in SO^0 \), one has
\[ \| P_t f \|_{L^\infty(M)} \lesssim t^{\alpha/2} \| f \|_{C^\alpha \omega}, \]
uniformly in \( t \in (0, 1] \).

**Proof** – The proof follows the same idea as the one for Proposition 6. Indeed, we use that since \( P \) is a standard family then
\[ P_t f = \int_t^1 (-s \partial_s P_s) f \frac{ds}{s} + P_t f. \]
The key point is that \( (-s \partial_s P_s) f \) can be split into a finite sum of families of \( SO^{\geq 1} \), which allows us to conclude as previously.
2.4. Schauder estimates We provide in this subsection a Schauder estimate for the heat semigroup in the scale of weighted parabolic Hölder spaces. This quantitative regularization effect of the heat semigroup will be instrumental in the proof of the well-posedness of the parabolic Anderson model (PAM) equation studied in Section 4. Define here formally the linear resolution operator for the heat equation by the formula

\begin{equation}
\mathcal{R}(v)_T := \int_0^T e^{-((\tau - \sigma)L)\nu_\sigma} \, d\sigma.
\end{equation}

We fix in this section a finite positive time horizon \( T \) and consider the space

\[
\mathcal{M}_T := M \times [0,T],
\]
equipped with its parabolic structure. Denote by \( L_{\infty}^\infty \) the corresponding function space over \([0,T]\).

We first state a Schauder estimates, whose proof can be found in [3, 19].

**Proposition 8.** Given \( \beta \in \mathbb{R} \) and a space-time weight \( \omega \), we have

\[
\| \mathcal{R}(v) \|_{C^{\beta+2}_{L_{\infty}T}(\omega)} \lesssim \|v\|_{(L_{\infty}^\infty C^\beta_\omega)(\omega)}.
\]

We shall actually prove a refinement of this continuity estimate in the specific case where \( \omega \) has a special structure motivated by the study of the parabolic Anderson model equation to be done in Section 4. This weight was first introduced by Hairer and Labbé in the study of the PAM equation in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), in the setting of regularity structures [24, 25]. Let \( o = o_{\text{ref}} \) the reference point in \( M \) (fixed in the definition of \( \mathcal{S}_{\text{ref}} \) at the beginning), and set

\[
p_a(x) := (1 + d(o_{\text{ref}}, x))^{-a}, \quad \varpi(x, \tau) := e^{\kappa \tau} e^{(1+\kappa)(1+d(o_{\text{ref}}, x))},
\]

for \( 0 < a < 1 \) and a positive constant \( \kappa \). (The introduction of an extra exponential factor \( e^{\kappa \tau} \) in our space-time weight \( \varpi \) will allow us to get around an iterative step in the forthcoming application of the fixed point theorem used to solve PAM equation, as done in [24, 25].) The space-time weight \( \varpi \) satisfies condition (2.12) on \([0, T] \times M\), uniformly with respect to \( \kappa > 0 \). The above special weights satisfy in addition the following crucial property, already used in [24, 25]. We have

\begin{equation}
p_a(x) \varpi(x, \sigma) \lesssim \kappa^{-\theta} (\tau - \sigma)^{-a-\theta} \varpi(x, \tau),
\end{equation}

for every \( \theta \) small enough, uniformly with respect to \( x \in M, \kappa > 0 \) and \( 0 < \sigma < \tau \leq T \). The next improved Schauder-type continuity estimate shows how one can use the above inequality for the specific weights to compensate a gain on the weight by a loss of regularity.

**Proposition 9.** Given \( \beta \in \mathbb{R}, a \in (0, 1) \) and \( \theta \in (0, 1) \) small enough such that \( a + \theta < 1 \), we have the continuity estimate

\[
\| \mathcal{R}(v) \|_{C^{\beta+2+2(a-\theta)}_{L_{\infty}T}(\varpi)} \lesssim \kappa^{-\theta} \|v\|_{(L_{\infty}^\infty C^\beta_\varpi)(\varpi p_a)}.
\]

Moreover if \(-2 + 2(a + \theta) < \beta < 0\), then

\[
\| \mathcal{R}(v) \|_{C^{\beta+2-2(a-\theta)}_{L_{\infty}T}(\varpi p_a)} \lesssim \kappa^{-\theta} \|v\|_{(L_{\infty}^\infty C^\beta_\varpi)(\varpi p_a)}.
\]

**Proof** — Let us first check the regularity in space. So consider an integer \( c \geq |\beta|/2 + 1 \) and a parameter \( r \in (0, 1] \). Then for every fixed time \( \tau \in [0, T] \) we have

\[
Q_{\tau}^{(c)}(\mathcal{R}(v)) = \int_0^T Q_{\tau}^{(c)} e^{-(\tau - \sigma)L} \nu_\sigma \, d\sigma.
\]
By using the specific property \((2.14)\) of the weights \(p_a\) and \(\varpi\), one has
\[
\left\| \varpi_\tau^{-1} Q^1_{\tau} e^{-\tau \sigma} L v_\sigma \right\|_{L^\infty(M)} \lesssim \left( \frac{r}{r + \tau - \sigma} \right)^c \left\| \varpi_\tau^{-1} Q^1_{\tau - \sigma} v_\sigma \right\|_{L^\infty(M)} \\
\lesssim \kappa^{-\theta} \left( \frac{r}{r + \tau - \sigma} \right)^c (r + \tau - \sigma)^{\frac{\beta}{2}} (\tau - \sigma)^{-a - \theta} \left\| v_\sigma \right\|_{C^{\beta}_{p_0 = \sigma}}.
\]

So by integrating and using that \(c\) is taken large enough, we see that
\[
\left\| \varpi_\tau^{-1} Q^1_{\tau} (\mathcal{R}(v)_\tau) \right\|_{L^\infty(M)} \lesssim \kappa^{-\theta} \left\{ \int_0^\tau \left( \frac{r}{r + \tau - \sigma} \right)^c (r + \tau - \sigma)^{\frac{\beta}{2}} (\tau - \sigma)^{-a - \theta} d\sigma \right\} \left\| \mathcal{R}(v)_\sigma \right\|_{L^\infty(M)} (p_0 \varpi) \\
\lesssim \kappa^{-\theta} \tau^{-1-a-\theta} \left\| \mathcal{R}(v)_\sigma \right\|_{L^\infty(M)} (p_0 \varpi).
\]

This holds uniformly in \(r \in (0, 1]\) and \(\tau \in [0, T]\) and so one concludes the proof of the first statement with the global inequality
\[
\left\| \varpi_\tau^{-1} \mathcal{R}(v)_\tau \right\|_{L^\infty(M)} \lesssim \kappa^{-\theta} \left\{ \int_0^\tau (\tau - \sigma)^{-a-\theta} d\sigma \right\} \left\| \mathcal{R}(v)_\sigma \right\|_{L^\infty(M)} (p_0 \varpi) \lesssim \kappa^{-\theta} \tau^{1-a-\theta} \left\| \mathcal{R}(v)_\sigma \right\|_{L^\infty(M)} (p_0 \varpi).
\]

For the second statement, we note that for \(0 \leq \sigma < \tau \leq T\) we have
\[
\mathcal{R}(v)_\tau - \mathcal{R}(v)_\sigma = \left( e^{-(\tau - \sigma)L} - \text{Id} \right) \mathcal{R}(v)_\sigma + \int_\sigma^\tau e^{-(\tau - r)L} v_r \ dr
\]
\[
= \int_0^{\tau - \sigma} Q^1_{\tau} (\mathcal{R}(v)_\sigma) \frac{dr}{r} + \int_\sigma^\tau e^{-(\tau - r)L} v_r \ dr.
\]

We have by the previous estimate
\[
\left\| \varpi_\tau^{-1} \int_0^{\tau - \sigma} Q^1_{\tau} (\mathcal{R}(v)_\sigma) \frac{dr}{r} \right\|_{L^\infty(M)} \lesssim \kappa^{-\theta} \left\{ \int_0^{\tau - \sigma} \frac{dr}{r} \right\} \left\| \mathcal{R}(v)_\sigma \right\|_{C^{\beta}_{p_0 = \sigma}} \lesssim \kappa^{-\theta} (\tau - \sigma)^{\frac{\beta}{2} + 1 - a - \theta} \left\| \mathcal{R}(v)_\sigma \right\|_{L^\infty(M)} (p_0 \varpi),
\]

where we used that \(\varpi_\tau \geq \varpi_\sigma\) for \(\sigma \leq \tau\). Moreover, since \(\beta\) is negative, we also have
\[
\left\| \varpi_\tau^{-1} \int_\sigma^\tau e^{-(\tau - r)L} v_r \ dr \right\|_{L^\infty(M)} \lesssim \kappa^{-\theta} \int_\sigma^\tau \int_{\tau - r}^1 (\tau - r)^{-a-\theta} \left\| p_a \varpi_\tau^{-1} Q^1_s v_r \right\|_{L^\infty(M)} \frac{ds}{s} + \left\| \varpi_\tau^{-1} e^{-L} (v_r) \right\|_{L^\infty(M)} \ dr
\]
\[
\lesssim \kappa^{-\theta} \int_\tau^\tau \left\| v_r \right\|_{C^{\beta}_{p_0 = \sigma}} (\tau - \sigma)^{-a-\theta} \int_{\tau - r}^1 \frac{ds}{s} \left\| e^{-L} (v_r) \right\|_{C^{\beta}_{p_0 = \sigma}} \ dr
\]
\[
\lesssim \kappa^{-\theta} (\tau - \sigma)^{\frac{\beta}{2} + 1 - a - \theta} \left\| v \right\|_{L^\infty(M)} (p_0 \varpi),
\]

where we used \((2.14)\) and \(\frac{\beta}{2} + 1 - a - \theta > 0\).

The following result comes as a consequence of the proof (combined with Lemma 3); we single it out here for future reference.
Lemma 10. Let $T$ be a linear operator on $\mathcal{M}$ with a kernel pointwisely bounded by $G_t$ for some $t \in (0, 1]$. Then for every $a + \theta \in (0, 1)$, we have
\[ \|T\|_{L^\infty_{\text{wp}}(\mathcal{M}) \rightarrow L^\infty_{\text{wp}}(\mathcal{M})} \lesssim \kappa^{-\theta} t^{-a-\theta}. \]

3 Time-space paraproducts

As mentioned in the introduction, we shall use paraproducts as a means of comparing a given parabolic distribution with a model one, in the Hölder scale. Quickly said, and without making sense of the objects yet, we shall consider a formula like
\[ u = \Pi_v(W) + (\cdots) \]
as a kind of first order Taylor expansion of $u$ in terms of $W$, where $\Pi$ has to be thought of as a paraproduct. With an eye on the parabolic Anderson model equation (1.1), we expect to solve the latter by finding a fixed point of the map $u \mapsto \mathcal{R}(u \zeta)$, where $\mathcal{R}$ is the resolution operator of the parabolic operator $(\partial_t + L)$, introduced in (2.13), provided one can make sense of the ill-defined product $u \zeta$. We shall indeed be able to do that in a precise setting for which we shall need the continuity of a number of iterated commutators and correctors, studied in section 3.2; this will provide us with a formula for $u \zeta$ of the form
\[ u \zeta = \Pi_v(W) + (\cdots). \]

Given that we want to have
\[ u = \mathcal{R}(u \zeta) = \mathcal{R}(\Pi_v(W)) + (\cdots), \]
it is very tempting here to write
\[ \mathcal{R}(\Pi_v(W)) = \Pi_v(\mathcal{R}(W)) + [\mathcal{R}, \Pi_v](W), \]
and work with the commutator $[\mathcal{R}, \Pi_v]$. This is what was done in [19, 3] to study the 2-dimensional (PAM) equation on the torus and more general settings. Unfortunately, this commutator does not have the regularity properties needed to push the analysis of the (PAM) equation far enough in a 3-dimensional setting. As a way out of this problem, we introduce another paraproduct $\tilde{\Pi}_v(\cdot)$, taylor-made to deal with that problem, and intertwined to $\Pi_v(\cdot)$ via $\mathcal{R}$, that is
\[ \mathcal{R} \circ \tilde{\Pi}_v = \Pi_v \circ \mathcal{R}. \]

We show in section 3.1 that $\Pi$ and $\tilde{\Pi}$ have the same analytic properties. The introduction of semigroup methods for the definition and study of paraproducts is relatively new; we refer the reader to different recent works where such paraproducts have been used and studied [4, 6, 1, 5].

3.1. Intertwined paraproducts We introduce in this section a pair of intertwined paraproducts that will be used to analyze the a priori ill-posed terms in the right hand side of the parabolic Anderson model equation in the next section. We follow here for that purpose the semigroup approach developed in [3] based on the pointwise Calderón’s reproducing formula
\[ f = \int_0^1 Q_t^{(b)} f \frac{dt}{t} + P_1^{(b)} f, \]
for every bounded and continuous function \( f \). This formula says nothing else here than the fact that

\[
\lim_{t \to 0} \mathcal{P}^{(b)}_t = \text{Id}
\]

in a weak sense. (This is a direct consequence of the fact that the operator \((\varphi \star \varphi)^*_t\) tends to the identity operator, since \( \varphi \) has unit integral.) We can thus write formally for two continuous and bounded functions \( f, g \)

\[
(3.1) \quad fg = \lim_{t \to 0} \mathcal{P}^{(b)}_t \left( \mathcal{P}^{(b)}_t f \cdot \mathcal{P}^{(b)}_t g \right) = - \int_0^1 t \partial_t \left\{ \mathcal{P}^{(b)}_t \left( \mathcal{P}^{(b)}_t f \cdot \mathcal{P}^{(b)}_t g \right) \right\} \frac{dt}{t} + \Delta_{-1}(f, g)
\]

\[
= \int_0^\infty \left\{ \mathcal{P}^{(b)}_t \left( \mathcal{Q}^{(b)}_t f \cdot \mathcal{P}^{(b)}_t g \right) + \mathcal{P}^{(b)}_t \left( \mathcal{P}^{(b)}_t f \cdot \mathcal{Q}^{(b)}_t g \right) + \mathcal{Q}^{(b)}_t \left( \mathcal{P}^{(b)}_t f \cdot \mathcal{P}^{(b)}_t g \right) \right\} \frac{dt}{t} + \Delta_{-1}(f, g)
\]

where

\[
\Delta_{-1}(f, g) := \mathcal{P}^{(b)}_1 \left( \mathcal{P}^{(b)}_1 f \cdot \mathcal{P}^{(b)}_1 g \right)
\]

stands for the “low-frequency part” of the product of \( f \) and \( g \), and where we implicitly make the necessary assumptions on \( f \) and \( g \) for the above formula to make sense. This decomposition corresponds to an extension of Bony’s well-known paraproduct decomposition \( \mathbb{S} \) to our setting given by a semigroup.

The integral exponent \( b \) has not been chosen so far. Choose it here even and no smaller than \( 6 \). Using iteratively the Leibniz rule for the differentiation operators \( V_t \) or \( \partial_t \), generically denoted \( D \),

\[
D(\phi_1)\phi_2 = D(\phi_1 \cdot \phi_2) - \phi_1 \cdot D(\phi_2),
\]

we see that \( \mathcal{P}^{(b)}_t \left( \mathcal{Q}^{(b)}_t f \cdot \mathcal{P}^{(b)}_t g \right) \) can be decomposed as a finite sum of terms taking the form

\[
\mathcal{A}^{I,J}_{k,\ell}(f, g) := \mathcal{P}^{(b)}_t \left( t^{|I|+k} V_I \partial^k_\tau \right) \left( \mathcal{S}^{(b/2)}_t f \cdot (t^{|J|+\ell} V_J \partial^\ell_\tau) \mathcal{P}^{(b)}_t g \right)
\]

where \( \mathcal{S}^{(b/2)} \in \mathcal{O}^{b/2} \) and the tuples \( I, J \) and integers \( k, \ell \) satisfy the constraint

\[
\frac{|I| + |J|}{2} + k + \ell = b.
\]

Denote by \( \mathcal{I}_b \) the set of all such \((I, J, k, \ell)\). We then have the identity

\[
\int_0^1 \mathcal{P}^{(b)}_t \left( \mathcal{Q}^{(b)}_t f \cdot \mathcal{P}^{(b)}_t g \right) \frac{dt}{t} = \sum_{\mathcal{I}_b} a^{I,J}_{k,\ell} \int_0^1 \mathcal{A}^{I,J}_{k,\ell}(f, g) \frac{dt}{t},
\]

for some coefficients \( a^{I,J}_{k,\ell} \). Similarly, we have

\[
\int_0^1 \mathcal{Q}^{(b)}_t \left( \mathcal{P}^{(b)}_t f \cdot \mathcal{P}^{(b)}_t g \right) \frac{dt}{t} = \sum_{\mathcal{I}_b} b^{I,J}_{k,\ell} \int_0^1 \mathcal{B}^{I,J}_{k,\ell}(f, g) \frac{dt}{t},
\]

with \( \mathcal{B}^{I,J}_{k,\ell}(f, g) \) of the form

\[
\mathcal{B}^{I,J}_{k,\ell}(f, g) := \mathcal{S}^{(b/2)}_t \left( \left\{ (t^{|I|+k} V_I \partial^k_\tau) \mathcal{P}^{(b)}_t f \right\} \cdot \left\{ (t^{|J|+\ell} V_J \partial^\ell_\tau) \mathcal{P}^{(b)}_t g \right\} \right),
\]

for some coefficients \( b^{I,J}_{k,\ell} \). So we have at the end the decomposition

\[
f \cdot g = \sum_{\mathcal{I}_b} a^{I,J}_{k,\ell} \int_0^1 \left( \mathcal{A}^{I,J}_{k,\ell}(f, g) + \mathcal{A}^{I,J}_{k,\ell}(g, f) \right) \frac{dt}{t} + \sum_{\mathcal{I}_b} b^{I,J}_{k,\ell} \int_0^1 \mathcal{B}^{I,J}_{k,\ell}(f, g) \frac{dt}{t},
\]

which leads us to the following definition.
Definition. Given $f \in \bigcup_{\alpha \in (0,1)} C^\alpha$ and $g \in L^\infty(\mathcal{M})$, we define the paraproduct $\Pi^{(b)}_g(f)$ by the formula
\[
\Pi^{(b)}_g(f) := \int_0^1 \left\{ \sum_{I_b, \frac{dI}{4} + k > \frac{b}{4}} a^{I,J}_{k,\ell} A^{I,J}_{k,\ell}(f,g) + \sum_{I_b, \frac{dI}{4} + k > \frac{b}{4}} b^{I,J}_{k,\ell} B^{I,J}_{k,\ell}(f,g) \right\} \frac{dt}{t},
\]
and the resonant term $\Pi^{(b)}(f,g)$ by the formula
\[
\Pi^{(b)}(f,g) := \int_0^1 \left\{ \sum_{I_b, \frac{dI}{4} + k \leq \frac{b}{4}} a^{I,J}_{k,\ell} \left( A^{I,J}_{k,\ell}(f,g) + A^{I,J}_{k,\ell}(g,f) \right) + \sum_{I_b, \frac{dI}{4} + k = \frac{dI}{4} + \ell = \frac{b}{4}} b^{I,J}_{k,\ell} B^{I,J}_{k,\ell}(f,g) \right\} \frac{dt}{t}.
\]

With these notations, Calderón’s formula becomes
\[
fg = \Pi^{(b)}_g(f) + \Pi^{(b)}_f(g) + \Pi^{(b)}(f,g) + \Delta_{-1}(f,g)
\]
with the “low-frequency part”
\[
\Delta_{-1}(f,g) := \mathcal{P}^{(b)}_1 \left( \mathcal{P}^{(b)}_1 f \cdot \mathcal{P}^{(b)}_1 g \right).
\]

If $b$ is chosen large enough, then all of the operators involved in paraproducts and resonant term have a kernel pointwisely bounded by a kernel $\mathcal{G}_i$ at the right scaling. Moreover,

(a) the paraproduct term $\Pi^{(b)}_g(f)$ is a finite linear combination of operators of the form
\[
\int_0^1 \mathcal{Q}^{1*}_t \left( \mathcal{Q}^{2}_t f \cdot \mathcal{P}^{1}_t g \right) \frac{dt}{t}
\]
with $\mathcal{Q}^{1}, \mathcal{Q}^{2} \in SO^\frac{b}{4}$, and $\mathcal{P}^{1} \in SO$.

(b) the resonant term $\Pi^{(b)}(f,g)$ is a finite linear combination of operators of the form
\[
\int_0^1 \mathcal{P}^{1}_t \left( \mathcal{Q}^{1}_t f \cdot \mathcal{Q}^{2}_t g \right) \frac{dt}{t}
\]
with $\mathcal{Q}^{1}, \mathcal{Q}^{2} \in SO^\frac{b}{4}$ and $\mathcal{P}^{1} \in SO$.

Note that since the operators $\mathcal{Q}^{1}_t$ and $\mathcal{P}^{1}_t$ are of the type $\mathcal{Q}^{(c)}_t$, $\mathcal{P}^{(c)}_t$ or a $\mathcal{P}^{(c)}_t V_t$, they can easily be composed on the left with another operator $\mathcal{Q}^{(d)}_t$; this will simplify the analysis of the paraproduct and resonant terms in the parabolic Hölder spaces. Note also that $\Pi^{(b)}_f(1) = \Pi^{(b)}(f,1) = 0$, and that we have the identity
\[
\Pi^{(b)}_1(f) = f - \mathcal{P}^{(b)}_1 \mathcal{P}^{(b)}_1 f,
\]
as a consequence of our choice of the renormalizing constant. Therefore the paraproduct with the constant function $1$ is equal to the identity operator, up to the strongly regularizing operator $\mathcal{P}^{(b)}_1 \mathcal{P}^{(b)}_1$.

One can prove the following continuity estimates in exactly the same way as in [3]. Note first that if $\omega_1, \omega_2$ are two space-time weights, then $\omega := \omega_1 \omega_2$ is also a space-time weight.

Proposition 11. Let $\omega_1, \omega_2$ be two space-time weights, and set $\omega := \omega_1 \omega_2$.

(a) For every $\alpha, \beta \in \mathbb{R}$ and every positive regularity exponent $\gamma$, we have
\[
\left\| \Delta_{-1}(f,g) \right\|_{\mathcal{C}^\gamma} \lesssim \left\| f \right\|_{\mathcal{C}^\alpha_1} \left\| g \right\|_{\mathcal{C}^\beta_2}
\]
for every $f \in \mathcal{C}^\alpha_{\omega_1}$ and $g \in \mathcal{C}^\beta_{\omega_2}$.
(b) For every $\alpha \in (-3, 3)$ and $f \in C_{\omega_1}^\alpha$, we have
\[
\left\| \Pi_{(b)}^g(f) \right\|_{C_{\omega_0}^\alpha} \lesssim \|\omega_1^{-1}g\|_{C_{\omega_1}^\alpha} \|f\|_{C_{\omega_1}^\alpha}
\]
for every $g \in L^\infty(\omega_2^{-1})$, and
\[
\left\| \Pi_{(b)}^g(f) \right\|_{C_{\omega_2}^{\alpha+\beta}} \lesssim \|g\|_{C_{\omega_2}^\beta} \|f\|_{C_{\omega_1}^\alpha}
\]
for every $g \in C_{\omega_2}^\beta$ with $\beta < 0$ and $\alpha + \beta \in (-3, 3)$.

(c) For every $\alpha, \beta \in (-\infty, 3)$ with $\alpha + \beta > 0$, we have the continuity estimate
\[
\left\| \Pi_{(b)}(f, g) \right\|_{C_{\omega_2}^{\alpha+\beta}} \lesssim \|f\|_{C_{\omega_1}^\alpha} \|g\|_{C_{\omega_2}^\beta}.
\]

(The range $(-3, 3)$ for $\alpha$ (or $\alpha + \beta$) is due to the fact that all the operators involving a cancellation used in this estimate satisfy a cancellation of order at least $\nu + 10 > 3$. We simply write 3 in the above statement, which will be sufficient for our purpose.)

These regularity estimates can be refined if one uses the specific weights $\varpi$ and $p_\alpha\varpi$ introduced in Subsection 2.4.

**Proposition 12.** For every $\alpha \in (-3, 3)$ and $a, \theta \in (0, 1)$ with $\alpha - a - \theta \in (-3, 3)$ and $f \in C_{p_a}^\alpha$, we have

- for every $g \in L^\infty(\varpi)$
  \[
  \left\| \Pi_{(b)}^g(f) \right\|_{C_{\varpi}^{\alpha-a-\theta}} \lesssim \kappa^{-\theta}\|\varpi^{-1}g\|_{\infty} \|f\|_{C_{p_a}^\alpha};
  \]

- for every $g \in C_{\varpi}^\beta$ with $\beta < 0$ and $\alpha + \beta - a \in (-3, 3)$
  \[
  \left\| \Pi_{(b)}^g(f) \right\|_{C_{\varpi}^{\alpha+\beta-2(a+\theta)}} \lesssim \kappa^{-\theta}\|g\|_{C_{\varpi}^\beta} \|f\|_{C_{p_a}^\alpha}.
  \]

The proof of this result is done along exactly the same lines as the proof of Proposition 11 using as an additional ingredient the elementary Lemma 10.

We shall use the above paraproduct in our study of the parabolic Anderson model equation to give sense to the a priori undefined product $u\zeta$ of a $C^\alpha$ function $u$ on $M$ with a $C^{\alpha-2}$ distribution $\zeta$ on $\mathcal{M}$, while $2\alpha - 2 \leq 0$. We develop for that purpose a multicontrolled setting. Roughly speaking, we shall solve the PAM equation
\[
(\partial_t + L)u = u\zeta
\]
by finding a fixed point to the map $u \mapsto v$, where $v := \mathcal{R}(u\zeta)$. We would like to set for that purpose a setting where the product $u\zeta$ can be decomposed as a sum of the form
\[
u = \sum_{i=1}^{3} \Pi_{u_i}^{(b)}(Y_i) + (> 0),
\]
where $(> 0)$ stands for a function in $C^\eta$, for some positive regularity exponent $\eta$. (More motivation for this ansatz is given in section 4.) We would then have
\[
v = \sum_{i=1}^{3} \mathcal{R}\left(\Pi_{u_i}^{(b)}(Y_i)\right) + \mathcal{R}(> 0),
\]
which we would like to write under the form
\[ v = \sum_{i=1}^{3} \Pi_{u_i}^{(b)}(\mathcal{R}(Y_i)) + (\cdots), \]
commuting the resolution operator \( \mathcal{R} \) with the paraproduct. The commutation is not perfect though and only holds up to a correction term involving the regularizing commutator operator \([\mathcal{R}, \Pi_g(\cdot)]\), whose regularizing effect happens to be too limited for our purposes. This motivates us to introduce the following operator.

**Definition.** We define a modified paraproduct \( \widetilde{\Pi}^{(b)}_g \) setting
\[ \widetilde{\Pi}^{(b)}_g(f) := \mathcal{R} \left( \Pi^{(b)}_g(Lf) \right). \]

The next proposition shows that if one chooses the parameters \( \ell_1 \) that appears in the reference kernels \( \mathcal{G}_t \), and the exponent \( b \) in the definition of the paraproduct large enough, then the modified paraproduct \( \widetilde{\Pi}^{(b)}_g(\cdot) \) has the same algebraic/analytic properties as \( \Pi^{(b)}_g(\cdot) \).

**Proposition 13.** If the ambient space \( M \) is bounded, then for a large enough choice of constants \( \ell_1 \) and \( b \), the modified paraproduct \( \widetilde{\Pi}^{(b)}_g(f) \) is a finite linear combination of operators of the form
\[ \int_0^1 Q_1^{1*} \left( Q_2^2 f \cdot P_1^1 g \right) \frac{dt}{t} \]
with \( Q^1 \in \mathcal{O}_-^{\ell_1 - 2} \), \( Q^2 \in \mathcal{S} \mathcal{O}_+^{\ell_2} \) and \( P^1 \in \mathcal{S} \mathcal{O} \).

If the space \( M \) is unbounded, then the result still holds on the parabolic space \([0, T] \times M\) for every \( T > 0 \) (with implicit constants depending on \( T \)).

(The \( Q_1^{1*} \) in the decomposition of \( \Pi^{(b)}_g(f) \) was in \( \mathcal{S} \mathcal{O} \) rather than just in \( \mathcal{O} \); this is the only difference.)

**Proof** – Given the structure of \( \Pi^{(b)}_g(f) \) as a sum of terms of the form
\[ \int_0^1 Q_1^{1*} \left[ Q_2^2(\cdot), P_1^1 g \right] \frac{dt}{t} \]
with \( P^1 \in \mathcal{S} \mathcal{O} \) and \( Q^1, Q^2 \in \mathcal{S} \mathcal{O}_+^{\ell_2} \), it suffices to look at
\[ \int_0^1 \left( t^{-1} \mathcal{R} \right) Q_1^{1*} \left( Q_2^2(tL)(\cdot), P_1^1 g \right) \frac{dt}{t}. \]

We have \( P^1 \in \mathcal{S} \mathcal{O} \), and it is easy to check that \( \left( Q_2^2(tL) \right)_{0 < t \leq 1} \) also belongs to \( \mathcal{S} \mathcal{O}_+^{\ell_2 - 2} \subset \mathcal{S} \mathcal{O}_+^{\ell_2} \). In so far as
\[ \mathcal{R} Q_1^{1*} = (Q_1^{1} \mathcal{R})^*, \]
it remains to prove that the family \( \widetilde{Q}^1 := \left( Q_1^1 t^{-1} \mathcal{R} \right)_{0 < t \leq 1} \) belongs to \( \mathcal{O}_+^{\ell_2 - 2} \), whereas \( Q^1 \) is essentially given here by
\[ Q_1^1 = \left( t^{|I| + k} V_j \partial_r^k \right) P_1^{(b)} \]
with \( |I| + k > \frac{\ell_2}{4} \). Note in particular that we have either \( |I| \geq \frac{\ell_2}{4} \) or \( k \geq \frac{\ell_2}{8} \). We check in the first two steps of the proof that \( \widetilde{Q} \in \mathcal{O} \) in both cases provided \( b \) is chosen big enough. The third step is dedicated to proving that \( \widetilde{Q} \in \mathcal{O}_+^{\ell_2 - 1} \).

**Step 1.** Assume here that \( |I| \geq \frac{\ell_2}{4} \). The kernel \( K \) of \( Q_1^1 \circ (t^{-1} \mathcal{R}) \) is given by
\[ K((x, \tau), (y, \sigma)) = \int_{-\infty}^{\infty} K_{t^{\frac{|I|}{2}} V_j e^{-(\lambda - \sigma) \xi}}(x, y) \frac{(\varphi * \varphi)(\tau - \lambda)}{t^{\frac{|I|}{2}}} \frac{d\lambda}{t^{\frac{|I|}{2}}}. \]
So by the Gaussian estimates of the operator \( t^{\frac{|I|}{2}} V_I P_t^{(h)} e^{-(\lambda-\sigma)L} \) at scale \( \max(t, \lambda - \sigma)^{1/2} \), and since \( |I| \geq \frac{b}{2} \), we deduce that

\[
\begin{align*}
\left| K^{(\frac{|I|}{2})} V_I P_t^{(h)} e^{-(\lambda-\sigma)L} (x, y) \right| & \lesssim \left( \frac{t}{t + \lambda - \sigma} \right)^{\frac{b}{2} - \frac{\sigma}{2}} G_{t + \lambda - \sigma}(x, y) \\
& \lesssim \left( \frac{t}{t + \lambda - \sigma} \right)^{\frac{b}{2} - \frac{\sigma}{2} - \ell_1} \mu(B(x, \sqrt{t}))^{-1} \left( 1 + \frac{d(x, y)^2}{t + \lambda - \sigma} \right)^{-\ell_1} \mu(B(x, \sqrt{t}))^{-1} \left( 1 + \frac{d(x, y)^2}{t} \right)^{-\ell_1}
\end{align*}
\]

if \( b \) is chosen large enough for \( \frac{b}{2} - \frac{\sigma}{2} - \ell_1 \) to be non-negative. Using the smoothness of \( \varphi \) we then deduce that \( |K((x, \tau), (y, \sigma))| \) is bounded above by

\[
\begin{align*}
\mu(B(x, \sqrt{t}))^{-1} & \left( 1 + \frac{d(x, y)^2}{t} \right)^{-\ell_1} \int_{\sigma}^{\infty} \left( \frac{t}{t + \lambda - \sigma} \right)^{\frac{b}{2} - \frac{\sigma}{2} - \ell_1} \left( 1 + \frac{\tau - \lambda}{t} \right)^{-\ell_1} \frac{d\lambda}{t^2} \\
& \lesssim \frac{1}{t \mu(B(x, \sqrt{t}))} \left( 1 + \frac{d(x, y)^2}{t} \right)^{-\ell_1} \left( 1 + \frac{|\tau - \sigma|}{t} \right)^{-\ell_1} .
\end{align*}
\]

So we get the upper bound

(3.3) \[ |K((x, \tau), (y, \sigma))| \lesssim \nu(B_M((x, \tau), \sqrt{t}))^{-1} \left( 1 + \frac{d(x, y)^2 + |\tau - \sigma|}{t} \right)^{-\ell_1} . \]

If \( d(x, y) \leq 1 \), this is exactly the desired estimate. If \( d(x, y) \geq 1 \) and one works on a finite time interval \([0, T]\) then we keep the information that \(|\lambda - \sigma| \leq T\) and so the exponentially decreasing term in the Gaussian kernel on the spatial variable allows us to keep in all the previous computations an extra coefficient of the form

\[ \mu(B_M(x, 1))^{-1} e^{-\frac{d(x, y)^2}{t+\lambda}} \]

which is exactly the decay required in the definition of the class \( \mathcal{O} \).

**Step 2.** Assume now that \( k \geq b/8 \). We work with the above formula for the kernel \( K \) and use the cancellation effect in the time variable by integrating by parts in \( \lambda \) for transporting the cancellation from time to space variable. So starting from formula (3.2), the “boundary term” in the integration by parts

\[ K^{(\frac{|I|}{2})} V_I P_t^{(h)} e^{-(\lambda-\sigma)L} (x, y) (t\partial_\tau)^{k-1}(\varphi \ast \varphi)_t(\tau - \lambda) \]

is vanishing for \( \lambda \to \infty \), and equal to

\[ K^{(\frac{|I|}{2})} V_I P_t^{(h)} (x, y) (t\partial_\tau)^{k-1}(\varphi \ast \varphi)_t(\tau - \sigma) \]

for \( \lambda = \sigma \). The latter term satisfies estimate (3.3). So up to a term denoted by \((\checkmark)\), bounded as desired, we see that \( K((x, \tau), (y, \sigma)) \) is equal to

\[ (\checkmark) + \int_{\sigma}^{\infty} K^{(\frac{|I|}{2})} V_I P_t^{(h)} L^{-(\lambda-\sigma)L} (x, y) (-t\partial_\lambda)^{k-1}(\varphi \ast \varphi)_t(\tau - \lambda) \frac{d\lambda}{t^2} , \]

where we used that by analyticity of \( L \) in \( L^1(M) \)

\[ \partial_\lambda e^{-(\lambda-\sigma)L} = -Le^{-(\lambda-\sigma)L} . \]

Doing \( k \) integration by parts provides an identity of the form

\[ K((x, \tau), (y, \sigma)) = (\checkmark) + \int_{\sigma}^{\infty} K^{(\frac{|I|}{2} + k)} V_I P_t^{(h)} L^{-(\lambda-\sigma)L} (x, y)(\varphi \ast \varphi)_t(\tau - \lambda) \frac{d\lambda}{t^2} , \]
where (✓) stands for a term with (3.3) as an upper bound. This procedure leaves us with a kernel which has an order of cancellation at least $b/8$ in space; we can then repeat the analysis of Step 1 to conclude.

**Step 3.** The proof that $\tilde{Q}^1$ actually belongs to $O^{b/8 - 2}$ is very similar, with details largely left to the reader. The above two steps make it clear that the study of $\tilde{Q}^1$ reduces to the study of operators with a form similar to that of the elements of $SO$. We have provided all the details in Proposition 2 of how one can estimate the composition between such operators and obtain an extra factor encoding the cancellation property. The cancellation result on $\tilde{Q}^1$ comes by combining the arguments of Proposition 2 with the two last steps.

Let us give some details for the particular case where the family $Q$ belongs to $SO^a$ for some $a \geq b/8 - 1$ and commutes with $R$; this covers in particular the case where $Q$ is built in space only with the operator $L$ with no extra $V_i$ involved. Let us then take $s, t \in (0, 1)$ and consider the kernel of the operator $\tilde{Q}^1_t Q^*_s$. Note first that

$$\tilde{Q}^1_t Q^*_s = \left( Q^1_t Q^*_s \right) \circ (t^{-1} R)$$

$$= \frac{t + s}{t} \left( Q^1_t Q^*_s \right) (t + s)^{-1} R.$$

Since $Q^1 \in O^{\frac{b}{2}}$, we know that $Q^1_t Q^*_s$ is an operator with a kernel with decay at scale $(t + s)^{\frac{b}{2}}$ with an extra factor $\left( \frac{st}{(t+s)^2} \right)^{\frac{b}{2}}$. We may also consider that

$$Q^1_t Q^*_s = \left( \frac{st}{(t+s)^2} \right)^{\frac{b}{n}} \tilde{Q}^2_{t+s} (t+s)^{-1} R$$

for some operator $\tilde{Q}^2_{t+s}$ having $b/8$-order of cancellation and a kernel with decay at scale $(s + t)^{1/2}$. So by what we did in the two first steps we also obtain that $\tilde{Q}^2_{t+s}(t + s)^{-1} R$ has a kernel with decay at scale $(t + s)^{\frac{1}{2}}$, for a large enough choice of $b$. (Indeed, note that $Q^2$ is very similar to the operators studied in the two first steps: easily analyzed as a function of the space-variable, while, as far as the time-variable is concerned, the composition of convolution preserves the main properties needed on the functions – vanishing moments.) At the end, we conclude that

$$\tilde{Q}^1_t Q^*_s = \left( \frac{st}{(t+s)^2} \right)^{\frac{b}{n} - 1} \tilde{Q}^2_{t+s}$$

with $\tilde{Q}^2_{t+s}$ having fast decreasing kernel at scale $(s + t)^{1/2}$. That concludes the fact that $\tilde{Q}^1 \in O^{b/8 - 2}$.

The following continuity estimate is then a direct consequence of Proposition 13 (since the latter implies that we can reproduce the same argument as for the standard paraproduct in Proposition 12).

**Proposition 14.** For every $\alpha \in (-3, 3)$ and $\alpha, \theta \in (0, 1)$ with $\alpha - \alpha - \theta \in (-3, 3)$ and $f \in C^\alpha_{p_\alpha}$, we have

$$\left\| \tilde{\Pi}^{(b)}_g (f) \right\|_{C^\alpha_{p_\alpha - \theta}} \lesssim \kappa^{-\theta} \|w^{-1} g\|_\infty \|f\|_{C^\alpha_{p_\alpha}}$$

for every $g \in L^\infty_w$. 

> The following continuity estimate is then a direct consequence of Proposition 13 (since the latter implies that we can reproduce the same argument as for the standard paraproduct in Proposition 12).
Last, note the normalization identity
\[ \tilde{\Pi}_1(f) = f - \mathcal{R} P_1^{(b)} P_1^{(b)} (\mathcal{L} f) \]
for every distribution in \( f \in S'_o \); it reduces to
\[ \tilde{\Pi}_1(f) = f - P_1^{(b)} P_1^{(b)} (f) \]
if \( f|_{\tau=0} = 0 \). (Use here the support condition on \( \varphi \) in the definition of \( P \).) Let us also point out here the strongly regularizing effect of the two operators \( P_1^{(b)} P_1^{(b)} \) and \( \mathcal{R} P_1^{(b)} P_1^{(b)} \mathcal{L} \), which satisfy the continuity estimate
\[ \|T\|_{C^\alpha \to C^\beta} \lesssim 1 \]
for any \( \alpha, \beta \in (-3, 3) \) and any space-time weight \( \omega \).

We shall fix from now on the parameters \( b \) and \( \ell_1 \), large enough for the above result to hold true.

### 3.2. Iterated commutators and correctors

We state and prove in this section a number of continuity estimates for some correctors and commutators that will be useful in our study of the 3-dimensional parabolic Anderson model equation in Section 4.

**Definition 15.** Let us introduce the following a priori unbounded multilinear operators on \( S'_o \).

- **A modified commutator on paraproducts**
  \[ R^1(f, g, u) := \Pi_u^{(b)} \left( \tilde{\Pi}_g^{(b)}(f) \right) - \Pi_g^{(b)} \left( \tilde{\Pi}_u^{(b)}(f) \right), \]
  and its iterated versions
  \[ R^2(f, g, u, v) := R^1 \left( \tilde{\Pi}_g^{(b)}(f), u, v \right) - \Pi_g^{(b)} \left( R^1(f, u, v) \right) \]
  and
  \[ R^3(f, g, u, v, w) := R^2 \left( \tilde{\Pi}_g^{(b)}(f), u, v, w \right) - \Pi_g^{(b)} \left( R^2(f, u, v, w) \right). \]

- **A corrector**
  \[ C^1(f, g, u) := \Pi_g^{(b)} \left( \tilde{\Pi}_u^{(b)}(f) \right) - g \Pi_u^{(b)}(f), \]
  and its iterated versions
  \[ C^2(f, g, u, v) := C^1 \left( \tilde{\Pi}_u^{(b)}(f), u, v \right) - g C^1(f, u, v) \]
  and
  \[ C^3(f, g, u, v, w) := C^2 \left( \tilde{\Pi}_u^{(b)}(f), u, v, w \right) - g C^2(f, u, v, w). \]

**Proposition 16.** Given some space-time weights \( \omega_1, \omega_2, \omega_3 \), set \( \omega := \omega_1 \omega_2 \omega_3 \). Let \( \alpha, \beta, \gamma \) be Hölder regularity exponents with \( \alpha \in (-3, 3), \beta \in (0, 1) \) and \( \gamma \in (-\infty, 0) \). Then if
\[ \alpha + \beta < 3, \quad \text{and} \quad \delta := \alpha + \beta + \gamma \in (-3, 3), \]
we have
\[ \| R^1(f, g, u) \|_{C^\delta \omega} \lesssim \| f \|_{C^{\alpha}_{\omega_1}} \| g \|_{C^{\beta}_{\omega_2}} \| u \|_{C^{\gamma}_{\omega_3}}, \]
for every \( f \in C^{\alpha}_{\omega_1} \), \( g \in C^{\beta}_{\omega_2} \) and \( u \in C^{\gamma}_{\omega_3} \), so the modified commutator defines a trilinear continuous map from \( C^{\alpha}_{\omega_1} \times C^{\beta}_{\omega_2} \times C^{\gamma}_{\omega_3} \) to \( C^\delta \omega \).
Proof – Recall that $\Pi_1^{(b)}$, resp. $\tilde{\Pi}_1^{(b)}$, is given by a finite sum of operators of the form
\[
A_1^b(\cdot) := \int_0^1 Q_t^1 \left( Q_t^2(\cdot) P_t^1(g) \right) \frac{dt}{t},
\]
resp.
\[
\tilde{A}_1^b(\cdot) := \int_0^1 \tilde{Q}_t^1 \left( \tilde{Q}_t^2(\cdot) P_t^1(g) \right) \frac{dt}{t},
\]
where $Q^1, Q^2, \tilde{Q}^2$ belong at least to $SO^3$ and $\tilde{Q}^1$ is an element of $O^3$. We describe similarly $\Pi_2^{(b)}$ as a finite sum of operators of the form
\[
A_2^b(\cdot) := \int_0^1 Q_t^2 \left( Q_t^4(\cdot) P_t^2(u) \right) \frac{dt}{t}.
\]
Thus, we need to study a generic modified commutator
\[
A_2^b \left( \tilde{A}_2^b(f) \right) - A_2^b \left( A_2^b(f) \right),
\]
and introduce for that purpose the intermediate quantity
\[
\mathcal{E}(f,g,u) := \int_0^1 Q_s^3 \left( Q_s^4(f) \cdot P_s^1(g) \cdot P_s^2(u) \right) \frac{ds}{s}.
\]
Note here that due to the normalization $\Pi_1 \simeq I_d$, up to some strongly regularizing operator, there is no loss of generality in assuming that
\[
(3.5) \quad \int_0^1 \tilde{Q}_t^1 \tilde{Q}_t^2 \frac{dt}{t} = \int_0^1 Q_t^1 Q_t^2 \frac{dt}{t} = \int_0^1 Q_t^3 Q_t^1 \frac{dt}{t} = I_d.
\]

**Step 1. Study of $A_2^b \left( \tilde{A}_2^b(f) \right) - \mathcal{E}(f,g,u)$.** We shall use a family $Q$ in $SO^a$, for some $a > |\delta|$, to control the Hölder norm of that quantity. By definition, and using the normalization (3.5), the quantity $Q_s \left( A_2^b \left( \tilde{A}_2^b(f) \right) - \mathcal{E}(f,g,u) \right)$ is, for every $r \in (0,1)$, equal to
\[
\int_0^1 \int_0^1 Q_s \tilde{Q}_s^3 \left\{ Q_s \tilde{Q}_s^4 \left( \tilde{Q}_s^2(f) P_s^1(g) \cdot P_s^2(u) \right) \right\} ds \frac{dt}{st} - \int_0^1 Q_s \tilde{Q}_s^3 \left( Q_s^4(f) \cdot P_s^1(g) \cdot P_s^2(u) \right) \frac{ds}{s} dt,
\]
where in the last line the variable of $P_s^1(g)$ is the one of $Q_3^s$, and so it is frozen through the action of $Q_4^s \tilde{Q}_4^1$. Then using that $g \in C^\beta$ with $\beta \in (0,1)$, we know by Proposition 3 that we have, for $\tau \geq \sigma$,
\[
\omega_2(x,\tau)^{-1} \left| (P_s^1 y)(x,\tau) - (P_s^1 y)(y,\sigma) \right| \lesssim (s + t + \rho((x,\tau),(y,\sigma))^2)^{\frac{\beta}{2}} e^{c d(x,y)} \|g\|_{C^\beta_2}.
\]
Note that it follows from equation (2.4) that the kernel of $Q_4^s \tilde{Q}_4^1$ is pointwise bounded by $G_{t+s}$, and allowing different constants in the definition of $G$, we have
\[
(3.6) \quad G_{t+s}((x,\tau),(y,\sigma)) \left( s + t + d(x,y)^2 \right)^{\frac{\beta}{2}} e^{c d(x,y)} \lesssim (s + t)^{\frac{\beta}{2}} G_{t+s}((x,\tau),(y,\sigma)).
\]
So using Lemma 3 and the cancellation property of the operators $Q$ at an order no less than $a$ (resp. 3) for $Q$ (resp. the other collections $Q^1, \tilde{Q}^1$), we deduce that
\[
\left\| \omega^{-1} Q_s \left( A_2^b \left( \tilde{A}_2^b(f) \right) - \mathcal{E}(f,g,u) \right) \right\|_{\infty} \lesssim \|f\|_{C^a_k} \|g\|_{C^\beta_2} \|u\|_{C^\gamma_2} \int_0^1 \int_0^1 \left( \frac{sr}{(s+r)^2} \right)^{\frac{\beta}{2}} \left( \frac{st}{(s+t)^2} \right)^{\frac{\beta}{2}} t^{\frac{\beta}{2}} (s+t)^{\frac{\beta}{2}} \frac{ds \, dt}{st}.
\]
where we used that $\gamma$ is negative to control $P^2_u(u)$. The integral over $t \in (0,1)$ can be computed since $\alpha > -3$ and $\alpha + \beta < 3$, and we have

\[
\left\| \omega^{-1} Q_r \left( A^2_u \left( \tilde{A}^1_g(f) \right) - \mathcal{E}(f, g, u) \right) \right\|_{\infty} \\
\lesssim \|f\|_{C^\alpha_{\omega_2}} \|g\|_{C^\beta_{\omega_3}} \|u\|_{C^\gamma_{\omega_4}} \int_0^1 \int_0^1 \left( \frac{sr}{(s + r)^2} \right)^{\frac{3}{2}} \frac{ds}{s} \frac{dr}{r},
\]

uniformly in $r \in (0,1)$ because $|a| > \delta$. That concludes the estimate for the high frequency part. We repeat the same reasoning for the low-frequency part by replacing $Q_r$ with $Q_1$ and conclude that

\[
\left\| A^2_u \left( \tilde{A}^1_g(f) \right) - \mathcal{E}(f, g, u) \right\|_{C^\alpha_{\omega_2}} \lesssim \|f\|_{C^\alpha_{\omega_1}} \|g\|_{C^\beta_{\omega_2}} \|u\|_{C^\gamma_{\omega_3}}.
\]

**Step 2.** **Study of $A^1_g \left( A^2_u(f) \right) - \mathcal{E}(f, g, u).** This term is almost the same as that of Step 1 and can be treated in exactly the same way. Note that $Q_r \left( A^1_g \left( A^2_u(f) \right) - \mathcal{E}(f, g, u) \right)$ is equal, for every $r \in (0,1)$, to

\[
\int_0^1 \int_0^1 Q_r Q^1_t \left( Q^2_s \left( \mathcal{Q}^4_s g \right) P^2_s(u) \right) \cdot P^1_t(g) \frac{ds}{st} - \int_0^1 \int_0^1 Q_r Q^3_s \left( \mathcal{Q}^4_s g \right) \cdot P^1_s(g) \cdot P^2_s(u) \frac{ds}{s} = \int_0^1 \int_0^1 Q_r Q^1_t \left( Q^2_s \left( \mathcal{Q}^4_s g \right) \cdot P^1_t(g) \right) - \int_0^1 \int_0^1 Q_r Q^3_s \left( \mathcal{Q}^4_s g \right) \cdot P^1_s(g) \cdot P^2_s(u) \right) \frac{ds}{st},
\]

where in the last line the variable of $P^1_t(g)$ is the one of $Q^1_s$ (and so it is frozen through the action of $Q^3_s$). The same proof as in Step 1 can be repeated. \(\square\)

**Remark 17.** The above proof actually shows the following property of the operator

\[
\overline{R}^1 := f \mapsto R^1(f, g, u),
\]

where $g \in C^\beta_{\omega_2}$ and $u \in C^\gamma_{\omega_4}$ are fixed. For all families $Q^1, Q^2 \in \mathcal{O}^a$ for some $a > 0$, the linear operator $Q^1_t \overline{R}^1 Q^2_s$ has a kernel pointwisely bounded by

\[
(t + s)^{-\frac{a+3}{2}} \omega_2(x, \tau) \omega_3(x, \tau) \left( \frac{st}{(s + t)^2} \right)^{\frac{a}{2}} G_{t+s}(x, \tau, (y, \sigma)) \|g\|_{C^\beta_{\omega_2}} \|u\|_{C^\gamma_{\omega_4}}.
\]

**Proposition 18.** Given space-time weights $\omega_1, \omega_2, \omega_3, \omega_4$, set $\omega := \omega_1 \omega_2 \omega_3 \omega_4$. Let $\alpha, \beta, \gamma, \delta$ be Hölder regularity exponents with $\alpha \in (-3,3)$, $\beta, \gamma \in (0,1)$ and $\delta \in (-\infty, 0)$. Then if $\alpha + \beta < 3$, and $\sigma := \alpha + \beta + \gamma + \delta \in (-3,3)$

the iterated commutator $R^2$ is a continuous 4-linear map from $C^a_{\omega_1} \times C^\beta_{\omega_2} \times C^\gamma_{\omega_3} \times C^\delta_{\omega_4}$ to $C^\sigma_{\omega_5}$.

**Proof —** Fix functions $u \in C^\gamma_{\omega_4}$ and $v \in C^\delta_{\omega_5}$, and define $\overline{R}^1 : f \mapsto R^1(f, u, v)$, to write

\[
R^2(f, g, u, v) := \Pi^{(b)}_g \overline{R}^1(f) - R^1 \Pi^{(b)}_g(f).
\]

With the same notations as in the proof of Proposition [10] and working with the relations (3.3), we write

\[
\Pi^{(b)}_g \overline{R}^1(f) = \int_0^1 Q^1_t \left( Q^2_t \overline{R}^1 f \cdot P^1_t g \right) \frac{dt}{t} = \int_0^1 Q^1_t \left( Q^2_t \overline{R}^1 \tilde{Q}^*_t g \cdot P^1_t g \right) \frac{ds}{s} \frac{dt}{t}.
\]
Expanding $R^3 \tilde{\Pi}^b_i(f)$ correspondingly, we get

\begin{equation}
R^2(f, g) = \int_0^1 \int_0^1 Q^1 f \tilde{Q}^1_i \tilde{\Pi}^b_i(f, g - g_i) \, ds \, dt,
\end{equation}

where the variable of $P_i g$ is that of $Q^1_i$. Since $g \in C^\beta_{\omega_2}$ with $\beta \in (0, 1)$, we know from Proposition $\square$ that for $(x, \tau), (y, \sigma) \in M$

\[
\omega(x, \tau)^{-1} \left| (P^1_i g)(x, \tau) - (P^1_i g)(y, \sigma) \right| \lesssim (t + s + \rho((x, \tau), (y, \sigma))^2)^\frac{\beta}{2} e^{\rho(x, y)} \|g\|_{C^\beta_{\omega_2}}.
\]

As above, fix a collection $Q$ of $SO^\alpha$, for some $\alpha > |\sigma|$, to control H"older norms. We need to estimate

\[
\left\| \omega^{-1} Q_i R^2(f, g) \right\|_{L^\infty(M)}.
\]

Using decomposition (3.7) and Lemma $\mathbb{3}$ we have

\begin{equation}
\left\| \omega^{-1} Q_i R^2(f, g) \right\|_{L^\infty(M)} \lesssim \int_0^1 \int_0^1 \left( \frac{r t}{(r + t)^2} \right) \frac{1}{s} I_{s, t} \, ds \, dt,
\end{equation}

where

\[I_{s, t} := \sup_{(x, \tau) \in M} \frac{1}{\omega(x, \tau)} \left[ Q^1_i \tilde{Q}^1_i \left( \tilde{Q}^2_i f \cdot (P^1_i g(x, \tau) - P^1_i g) \right) \right](x, \tau).
\]

Due to Remark $\mathbb{17}$ we have a pointwise estimate of the kernel of $Q^2_i \tilde{Q}^1_i$, so with the pointwise regularity estimate on $g$ and (3.6) we deduce that

\[
I_{s, t} \lesssim (s + t)^{\frac{\alpha + \beta + \gamma + \delta + \theta}{2}} \||g||_{C^\alpha_{\omega_1}} ||u||_{C^\beta_{\omega_2}} ||v||_{C^\gamma_{\omega_3}} ||w||_{C^\delta_{\omega_4}}.
\]

We deduce from that estimate and the fact that $|\sigma| < \alpha$, that

\[
\left\| \omega^{-1} Q_i R^2(f, g) \right\|_{L^\infty(M)} \lesssim r^{\frac{\alpha + \beta}{2}} ||g||_{C^\alpha_{\omega_1}} ||u||_{C^\beta_{\omega_2}} ||v||_{C^\gamma_{\omega_3}} ||w||_{C^\delta_{\omega_4}},
\]

uniformly in $r \in (0, 1)$. A similar analysis of the low frequency of $R^2(f, g)$ can be done.

One can give as in Remark $\mathbb{17}$ a pointwise estimate for the kernel of this iterated commutator, and prove the following continuity result along the same line of reasoning as above; we leave the details to the reader.

**Proposition 19.** Given space-time weights $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$, set $\omega := \prod_{i=1}^5 \omega_i$. Let $\alpha, \beta, \gamma, \delta, \theta$ be H"older regularity exponents with $\alpha \in (-3, 3)$, $\beta, \gamma, \theta \in (0, 1)$ and $\theta \in (-\infty, 0)$. Then if

\[\alpha + \beta < 3, \quad \text{and} \quad \sigma := \alpha + \beta + \gamma + \delta + \theta \in (-3, 3),\]

the iterated modified commutator $R^3$ is a continuous linear map from $C^\alpha_{\omega_1} \times C^\beta_{\omega_2} \times C^\gamma_{\omega_3} \times C^\delta_{\omega_4} \times C^\theta_{\omega_5}$ to $C^\sigma_{\omega}$.

We now turn to the study of the continuity properties of the corrector $C^1(f, g, u) = \Pi^b_i(f, u) - g \Pi^b_i(f, u)$, and its iterated versions $C^2$ and $C^3$. The next result was proved in an unweighted setting in [3, Proposition 3.6]; elementary changes in the proof give the following weighted counterpart.
Proposition 20. Given space-time weights $\omega_1, \omega_2, \omega_3$, set $\omega := \omega_1 \omega_2 \omega_3$. Let $\alpha, \beta, \gamma$ be Hölder regularity exponents with $\alpha \in (-3, 3)$, $\beta \in (0, 1)$ and $\gamma \in (-\infty, 3]$. Set $\delta := (\alpha + \beta) \wedge 3 + \gamma$. If
\[
0 < \alpha + \beta + \gamma < 1 \quad \text{and} \quad \alpha + \gamma < 0
\]
then the corrector $C^1$ is a continuous trilinear map from $C^\alpha_{\omega_1} \times C^\beta_{\omega_2} \times C^\gamma_{\omega_3}$ to $C^\delta_{\omega}$.

Proposition 21. Given space-time weights $\omega_1, \omega_2, \omega_3, \omega_4$, set $\omega := \prod_{i=1}^4 \omega_i$. Let $\alpha, \beta, \gamma, \delta$ be Hölder regularity exponents with $\alpha \in (-3, 3)$, $\beta, \gamma \in (0, 1)$ and $\delta \in (-\infty, 0)$. Then, if $\alpha + \beta + \gamma < 3$ and
\[
\sigma := \alpha + \beta + \gamma + \delta \in (0, 3), \quad \alpha + \beta + \delta < 0 \quad \text{and} \quad \alpha + \gamma + \delta < 0,
\]
the iterated corrector $C^2$ defines a continuous linear map from $C^\alpha_{\omega_1} \times C^\beta_{\omega_2} \times C^\gamma_{\omega_3} \times C^\delta_{\omega_4}$ to $C^\sigma_{\omega}$.

Proof – Fix some functions $u \in C^\alpha_{\omega_1}$ and $v \in C^\beta_{\omega_4}$ and define the operators $C^1 : f \mapsto C^1(f, u, v)$ and
\[
C^2 : (f, g) \mapsto C^1(\tilde{\Pi}^b_g(f) - g \cdot C^1(f)),
\]
so that
\[
C^2(f, g, u, v) = C^2(f, g).
\]
Using the same notation as in the proof of Proposition 16, omitting for convenience the indices on the different collections $\mathcal{Q}$ and $\mathcal{P}$, and working with the relations (3.5), we write
\[
\bar{C}^1(\tilde{\Pi}^b_g(f)) = \int_0^1 \bar{C}^1(\tilde{\Pi}^b_g(\tilde{Q}_s f \cdot \mathcal{P}_s g)) \frac{ds}{s},
\]
\[
g \cdot \bar{C}^1(f) = g \cdot \bar{C}^1(\tilde{\Pi}^b_1(f)) = g \cdot \int_0^1 \bar{C}^1(\tilde{Q}_s f \cdot \mathcal{P}_s 1) \frac{ds}{s}.
\]
Note that (due to the conservation property of the heat semigroup associated with $L$) $\mathcal{P}_s 1$ is either constant equal to 1 or to 0, depending on whether $\mathcal{P}_s$ encodes some cancellation or not. Thus, with $F_{s,(x,\tau)} := \tilde{Q}_s f \cdot (\mathcal{P}_s g - \mathcal{P}_s 1 \cdot g(x, \tau))$,
\[
\bar{C}^2(f, g)(x, \tau) = \bar{C}^1(\tilde{\Pi}^b_g(f)(x, \tau) - g(x, \tau) \cdot \bar{C}^1(f)(x, \tau)) = \int_0^1 \bar{C}^1(\tilde{Q}_s^* F_{s,(x,\tau)}) (x, \tau) \frac{ds}{s}.
\]
As before, we can use that $g \in C^3_{\omega_2}$ with $\beta \in (0, 1)$. We have for $(x, \tau)$, $(y, \sigma) \in \mathcal{M}$ and $s > 0$
\[
\omega_2(x, \tau)^{-1} |g(y, \sigma) - g(x, \tau)| \lesssim \rho((x, \tau), (y, \sigma))^{\beta} \|g\|_{C^\beta_{\omega_2}},
\]
and therefore, using the “Gaussian bounds” for $\mathcal{P}_s$,
\[
\omega_2(x, \tau)^{-1} |\mathcal{P}_s g(y, \sigma) - \mathcal{P}_s 1(y, \sigma) \cdot g(x, \tau)| \lesssim (s + \rho((x, \tau), (y, \sigma))^2)^{\beta} \|g\|_{C^\beta_{\omega_2}}.
\]
As done in the proof of Proposition 20 (see Proposition 3.6), we introduce an intermediate quantity of the form
\[
S(f, u, v) = \int_0^1 \mathcal{P}_t (Q_t f \cdot Q_t v \cdot \mathcal{P}_t u) \frac{dt}{t},
\]
and write
\[
\bar{C}^1(\tilde{Q}_s^* F_{s,(x,\tau)}) (x, \tau) = \Pi^b(\tilde{\Pi}^b_0(\tilde{Q}_s^* F_{s,(x,\tau)}), v) (x, \tau) - S((\tilde{Q}_s^* F_{s,(x,\tau)}, u, v)) (x, \tau)
\]
\[
+ S(\tilde{Q}_s^* F_{s,(x,\tau)}, u, v) (x, \tau) - u(x, \tau) \cdot \Pi^b(\tilde{Q}_s^* F_{s,(x,\tau)}, v) (x, \tau)
\]
(3.9)
\[
=: I_1(s) + I_2(s).
\]
We start with the estimate for \( I_2 \). Following the proof of Proposition 20 one can then write (with generic notations for the resonant term \( \Pi^{(b)} \))

\[
(S(f, u, v) - u \cdot \Pi^{(b)}(f, v))(x, \tau) = \int_0^1 \mathcal{P}_t \left( \mathcal{Q}_t f \cdot \mathcal{Q}_t v \cdot (\mathcal{P}_t u - u(x, \tau)) \right)(x, \tau) \frac{dt}{t},
\]

and it was shown there that the integrand is pointwisely bounded by \( t^{-\frac{\alpha + \gamma + \delta}{2}} \). Since this argument only uses pointwise estimates, we can replace \( f \) by \( Q^*_{s} F_{s,(x,\tau)} \). Therefore, by writing

\[
\int_0^1 I_2(s) \frac{ds}{s} = \int_0^1 \int_0^1 \mathcal{P}_t \left( \mathcal{Q}_t \tilde{Q}_{s}^* F_{s,(x,\tau)} \cdot \mathcal{Q}_t v \cdot (\mathcal{P}_t u - u(x, \tau)) \right)(x, \tau) \frac{dt}{t} \frac{ds}{s}
\]

and using with \( h = F_{s,(x,\tau)} \)

\[
(3.10) \quad \| (\omega_1 \omega_2)^{-1} Q_t \tilde{Q}_s h \|_{L^\infty(\mathcal{M})} \lesssim \left( \frac{st}{(s + t)^2} \right)^{3/2} \| (\omega_1 \omega_2)^{-1} h \|_{L^\infty(\mathcal{M})},
\]

we obtain

\[
\left\| \omega^{-1} \int_0^1 I_2(s) \frac{ds}{s} \right\|_{L^\infty(\mathcal{M})} \lesssim \int_0^1 \int_0^1 \left\| (x, \tau) \rightarrow \mathcal{P}_t \left( \mathcal{Q}_t \tilde{Q}_{s}^* F_{s,(x,\tau)} \cdot \mathcal{Q}_t v \cdot (\mathcal{P}_t u - u(x, \tau)) \right)(x, \tau) \right\|_{L^\infty(\omega^{-1})} \frac{dt}{t} \frac{ds}{s} \\
\lesssim \| f \|_{C_{C_1}^\alpha} \| g \|_{C_{C_2}^\beta} \| u \|_{C_{C_4}^\gamma} \| v \|_{C_{C_4}^\delta} \times \int_0^1 \int_0^1 \left( \frac{st}{(s + t)^2} \right)^{\frac{3}{2}} \mathcal{G}_{t+s}((x, \tau), (y, \sigma)) \left( s + \rho((x, \tau), (y, \sigma))^2 \right)^{\frac{\sigma + \gamma + \delta}{2}} \frac{ds}{s} \frac{dt}{t} \\
\lesssim \| f \|_{C_{C_1}^\alpha} \| g \|_{C_{C_2}^\beta} \| u \|_{C_{C_4}^\gamma} \| v \|_{C_{C_4}^\delta} \int_0^1 \int_0^1 \left( \frac{st}{(s + t)^2} \right)^{\frac{3}{2}} s^{\alpha/2} (s + t)^{\beta/2} \frac{ds}{s} \frac{dt}{t} \\
\lesssim \| f \|_{C_{C_1}^\alpha} \| g \|_{C_{C_2}^\beta} \| u \|_{C_{C_4}^\gamma} \| v \|_{C_{C_4}^\delta},
\]

since \( \alpha + \beta + \gamma + \delta > 0 \). Let us now estimate the regularity of \( I_2(s) \). Let \( (x, \tau), (y, \sigma) \in \mathcal{M} \) with \( \rho((x, \tau), (y, \sigma)) \leq 1 \). We split the integral in \( t \) into two parts, corresponding to \( t < \rho((x, \tau), (y, \sigma))^2 \) or \( t > \rho((x, \tau), (y, \sigma))^2 \). In the first case, note that

\[
\int_0^{\rho((x,\tau),(y,\sigma))^2} t^{(\alpha + \beta + \gamma + \delta)/2} \frac{dt}{t} \lesssim \rho((x, \tau), (y, \sigma))^{\alpha + \beta + \gamma + \delta},
\]
so that by repeating the arguments above, we get the desired estimate. In the case $t > \rho^2$ with $\rho := \rho((x, \tau), (y, \sigma))$, write for $s \in (0, 1)\text{)}$

\[
\int_{\rho^2}^1 \left\{ \mathcal{P}_t \left( Q_t \tilde{Q}_s F_s(x, \tau) \cdot Q_t v \cdot (u(x, \tau) - \mathcal{P}_t u) \right) (x, \tau) \right. \\
- \mathcal{P}_t \left( Q_t \tilde{Q}_s F_{s(y, \sigma)} \cdot Q_t v \cdot (u(y, \sigma) - \mathcal{P}_t u) \right) (y, \sigma) \left. \right\} \frac{dt}{t} \\
= \int_{\rho^2}^1 \left\{ \mathcal{P}_t \left( Q_t \tilde{Q}_s F_s(x, \tau) \cdot Q_t v \cdot (u(x, \tau) - \mathcal{P}_t u) \right) (x, \tau) \right. \\
- \mathcal{P}_t \left( Q_t \tilde{Q}_s F_{s(x, \tau)} \cdot Q_t v \cdot (u(x, \tau) - \mathcal{P}_t u) \right) (y, \sigma) \left. \right\} \frac{dt}{t} \\
+ (g(x, \tau) - g(y, \sigma)) \int_{\rho^2}^1 \mathcal{P}_t \left( Q_t \tilde{Q}_s \tilde{Q}_s f \cdot Q_t v \cdot (u(y, \sigma) - \mathcal{P}_t u) \right) (y, \sigma) \frac{dt}{t} \\
(3.11) \\
- (u(x, \tau) - u(y, \sigma)) \int_{\rho^2}^1 \mathcal{P}_t \left( Q_t \tilde{Q}_s F_{s(y, \sigma)} \cdot Q_t v \right) (y, \sigma) \frac{dt}{t}.
\]

For the second and third term, we can assume $s \approx t$ by (3.10). One obtains

\[
\omega(x, \tau)^{-1} |g(x, \tau) - g(y, \sigma)| \int_{\rho^2}^1 \left\{ \mathcal{P}_t \left( Q_t \tilde{Q}_s \tilde{Q}_s f \cdot Q_t v \cdot (u(y, \sigma) - \mathcal{P}_t u) \right) (y, \sigma) \right\} \frac{dt}{t} \\
\lesssim \|f\|_{C^0_{c_1}} \|g\|_{C^0_{c_2}} \|u\|_{C^0_{c_3}} \|v\|_{C^0_{c_4}} \rho^{\beta} \int_{\rho^2}^1 t^{\frac{\alpha+\beta+\gamma+\delta}{2}} \frac{dt}{t} \\
\lesssim \|f\|_{C^0_{c_1}} \|g\|_{C^0_{c_2}} \|u\|_{C^0_{c_3}} \|v\|_{C^0_{c_4}} \rho^{\alpha+\beta+\gamma+\delta},
\]

since $\alpha + \gamma + \delta$ is negative, and

\[
\omega(x, \tau)^{-1} |u(x, \tau) - u(y, \sigma)| \int_{\rho^2}^1 \left\{ \mathcal{P}_t \left( Q_t \tilde{Q}_s F_{s(y, \sigma)} \cdot Q_t v \right) (y, \sigma) \right\} \frac{dt}{t} \\
\lesssim \|f\|_{C^0_{c_1}} \|g\|_{C^0_{c_2}} \|u\|_{C^0_{c_3}} \|v\|_{C^0_{c_4}} \rho^{\gamma} \int_{\rho^2}^1 t^{\frac{\alpha+\beta+\gamma+\delta}{2}} \frac{dt}{t} \\
\lesssim \|f\|_{C^0_{c_1}} \|g\|_{C^0_{c_2}} \|u\|_{C^0_{c_3}} \|v\|_{C^0_{c_4}} \rho^{\alpha+\beta+\gamma+\delta},
\]

since $\alpha + \beta + \delta$ is also negative. For the first term in (3.11), we now repeat the arguments of the proof of Proposition 20 which rely on the Lipschitz regularity of the heat kernel as well as the fact that $\alpha + \beta + \gamma + \delta \in (0, 1)$. Summarising the above, we have shown that for $(x, \tau), (y, \sigma) \in \mathcal{M}$ with $\rho((x, \tau), (y, \sigma)) \leq 1$

\[
\left| \int_0^1 (I_2(s)(x, \tau) - I_2(s)(y, \sigma)) \frac{ds}{s} \right| \\
\lesssim \omega(x, \tau) \rho((x, \tau), (y, \sigma))^{\alpha+\beta+\gamma+\delta} \|f\|_{C^0_{c_1}} \|g\|_{C^0_{c_2}} \|u\|_{C^0_{c_3}} \|v\|_{C^0_{c_4}}.
\]

Let us now come to $I_1(s)$ as defined in (3.9). We write with $h = \tilde{Q}_s F_s(x, \tau)$

\[
|\Pi^{(b)}(\tilde{\Pi}^{(b)}(h), v) - S(h, u, v)| \leq \int_0^1 |\mathcal{P}_t (A_t(h, u) \cdot Q_t v) | \frac{dt}{t}
\]

with

\[
A_t(h, u) = \mathcal{Q}_t \left( \int_0^1 \mathcal{P}_t \tilde{\mathcal{Q}}_s \left( \tilde{Q}_s h \cdot \mathcal{P}_r u \right) \frac{dr}{r} - \mathcal{P}_t u \mathcal{P}_t h \right).
\]
Following the proof of Proposition 20 and using (3.10), one obtains
\[ \left\| (\omega_1^2 \omega_2) A_t (\tilde{Q}_x^* F_{s, \tau}) u \right\|_{L^\infty(M)} \leq \int_0^1 \left( \frac{rt}{(r+t)^2} \right)^{\frac{3}{2}} \left( \frac{sr}{(s+r)^2} \right)^{\frac{3}{2}} s^{\frac{\alpha + \beta}{2}} (r + t)^{\frac{\gamma + \delta}{2}} \frac{dr}{r} \| f \|_{C^0_{s_1}} \| g \|_{C^0_{s_2}} \| u \|_{C^0_{s_3}}, \]

hence
\[ \left\| \omega^{-1} \int_0^1 I_1(s) \frac{ds}{t} \right\|_{L^\infty(M)} \leq \| f \|_{C^0_{s_1}} \| g \|_{C^0_{s_2}} \| u \|_{C^0_{s_3}} \| v \|_{C^0_{s_4}} \times \int_0^1 \int_0^1 \int_0^1 \left( \frac{rt}{(r+t)^2} \right)^{\frac{3}{2}} \left( \frac{sr}{(s+r)^2} \right)^{\frac{3}{2}} s^{\frac{\alpha + \beta}{2}} (r + t)^{\frac{\gamma + \delta}{2}} \frac{dr}{r} \frac{ds}{s} \frac{dt}{t}, \]

and the triple integral is finite since \( \alpha + \beta + \gamma + \delta \) is positive.

For the regularity estimate of \( I_1(s) \), consider
\[ \int_0^1 \left\{ P_t (A_t (\tilde{Q}_x^* F_{s, \tau}) u) \cdot Q_t(y) \right\} (x, \tau) - P_t (A_t (\tilde{Q}_x^* F_{s, \sigma}) u) \cdot Q_t(y, \sigma) \right\} dt. \]

The estimate of this expression is similar, though simpler, compared to the one for \( I_2(s) \), as here \( (x, \tau) \) is frozen only in one spot. As before, one deals with this terms using the heat kernel regularity of \( P_t \) and the regularity estimate for \( g \).

One proves the continuity of the iterated corrector \( C^3 \) along the same lines of reasoning; details are left to the reader.

**Proposition 22.** Given \( b \geq 6 \) and space-time weights \( (\omega_i)_{i=1,5} \), set \( \omega := \prod_{i=1}^5 \omega_i \). Let \( \alpha, \beta, \gamma, \delta, \eta \) be Hölder regularity exponents with \( \alpha \in (-3, 3), \beta, \gamma, \delta \in (0, 1) \) and \( \eta \in (-\infty, 0) \). Then, if \( \alpha + \beta + \gamma + \delta < 3 \) and
\[ \sigma := \alpha + \beta + \gamma + \delta + \eta \in (0, 3), \quad \alpha + \beta + \gamma + \delta + \eta < 0, \quad \alpha + \gamma + \delta + \eta < 0, \quad \alpha + \beta + \delta + \eta < 0, \]
the iterated corrector \( C^\sigma \) is a continuous linear map from \( C^\alpha_{\omega_1} \times C^\beta_{\omega_2} \times C^\gamma_{\omega_3} \times C^\delta_{\omega_4} \times C^\eta_{\omega_5} \) to \( C^\sigma_\omega \).

**Parabolic Anderson model equation in a 3-dimensional background**

We use in this section all the tools developed above to study the parabolic Anderson model equation (PAM)
\[ (\partial_t + L)u = u \cdot \zeta \]
in a possibly unbounded space \( M \), with a sub-Laplacian operator \( L \) satisfying the assumptions **Conditions** put forward in Section 2.1. We assume the metric measure space \((M, d, \mu)\) is doubling, with homogeneous dimension \( \nu = 3 \). Refer to Section 2.1 for concrete examples. What makes the study of this equation non-trivial is that we want to make sense of it and solve it uniquely in case \( \zeta \) is a distribution with negative Hölder index, say \( \alpha - 2 \), with \( 0 < \alpha < 1 \), so \( u \) is expected to have Hölder regularity \( \alpha \), as a consequence of Schauder estimates, which will not be sufficient to make sense of the product \( u \zeta \), since \( 2\alpha - 2 \) is negative in the case we are interested in. The way out of this quandary is to recast the equation in an enriched framework where the ill-defined product \( u \zeta \) can be defined and where we still have some sort of regularizing effect of the heat semigroup, as quantified by Schauder’s theorem. This is what both Hairer’s theory of regularity structures and
Gubinelli, Imkeller, Perkowski’s theory of paracontrolled distributions do with different tools. We show here how the above results allow us to extend the paracontrolled approach to deal with the PAM equation in a 3-dimensional setting where it is natural to assume $\alpha < \frac{1}{2}$, in an unbounded space. (Note that Hairer and Labbé [25] have very recently tackled the same problem in the setting of regularity structures, in $\mathbb{R}^3$ with its canonical Laplace operator. The present work offers a non-trivial generalization of their result, to start with the fact that the theory of regularity structures has not been developed so far in a manifold setting.)

To give the reader a taste of what is going on we explain first in simple terms the idea of paracontrolled calculus, such as developed in [19]. Our multicontrolled extension will be best understood via a parallel with rough paths theory explained in Section 4.2. The analysis of the PAM equation will be performed in Section 4.3 and comments on the stochastic PAM equation are given in Section 4.4.

### 4.1. Paracontrolled calculus

Both Hairer’s theory of regularity structures and Gubinelli, Imkeller, Perkowski’s paracontrolled calculus have as a starting point the fact that as one expects $u$ to be of positive Hölder regularity, and $\zeta$ has negative parabolic Hölder regularity index, the ill-defined product $u\zeta$ should behave at small scale as $\zeta$, so $u$ should somehow look like the solution $Z$ to the equation

$$(\partial_t + L)Z = \zeta.$$ 

While this rough description of $u$ is given full force in the theory of regularity structures by describing $u$ in terms of a kind of Taylor expansion in a formal basis of distributions built from $\zeta$, the heat kernel and the natural differentiation operators, the paracontrolled approach introduced in [19] uses a paraproduct as a means of making sense of the sentence “$u$ looks like $Z$ at small scale”, such as given in the following definition, given here with $Z$ as a reference distribution.

**Definition.** Let $\beta > 0$ be given. A pair of distributions $(f, g) \in \mathcal{C}^\alpha \times \mathcal{C}^\beta$ is said to be **paracontrolled by** $Z$ if

$$(f, g)^\sharp := f - \Pi_g(Z) \in \mathcal{C}^{\alpha+\beta}.$$

The norm

$$\|(f, g)\| := \|g\|_\alpha + \|(f, g)^\sharp\|_{\alpha+\beta},$$

turns the space of distributions controlled by $Z$ into a Banach space.

The twist offered by this definition, as far as the multiplication problem of $u$ by $\zeta$ is concerned, is the following. For a distribution $(u, u')$ controlled by $Z$ say, with $\beta = \alpha$, the formal manipulation (where $(2\alpha)$ stands for some $\mathcal{C}^{2\alpha}$-function)

$$u\zeta = \Pi_u^{(b)}(\zeta) + \Pi_u^{(b)}(u) + \Pi^{(b)}(u, \zeta)$$

$$= \Pi_u^{(b)}(\zeta) + \Pi_u^{(b)}(u) + \Pi^{(l)}(\Pi_u^{(b)}(Z), \zeta) + \Pi^{(b)}((2\alpha), \zeta)$$

$$= \Pi_u^{(b)}(\zeta) + \Pi_u^{(b)}(u) + \Pi^{(l)}(Z, \zeta) + \Pi^{(l)}(Z, \zeta) + \Pi^{(b)}((2\alpha), \zeta),$$

together with the fact that $3\alpha - 2$ is positive for $\alpha > \frac{2}{3}$, in which case the commutator $\Pi^{(l)}(Z, u', \zeta)$ is well-defined, shows that the only a priori undefined term in that case is the diagonal term $\Pi^{(b)}(Z, \zeta)$ and its product with $u'$.

One simply postulates in the setting of paracontrolled calculus that the diagonal term $\Pi(Z, \zeta)$ is given as a well-defined element of $\mathcal{C}^{2\alpha-2}$, in which case the product $u'.\Pi(Z, \zeta)$ makes sense if $\alpha > \frac{2}{3}$. 

4.2. A blend of rough paths theory

This situation is reminiscent of rough paths theory \cite{24,17}, where given an $\mathbb{R}^d$-valued path $(X_t)_{0 \leq t \leq 1}$ that is $\frac{1}{p}$-Hölder for $2 \leq p$, one cannot make sense of the integral

\begin{equation}
X_{ts} := \int_s^t (X_u - X_s) \otimes dX_u, \quad 0 \leq s \leq t \leq 1,
\end{equation}

in a canonical way, and one needs to postulate the existence of a $\frac{2}{p}$-Hölder function $X$ that has some awaited algebraic properties as a function of its two indices $(s,t)$. Objects playing the role of higher-iterated integrals need to be introduced if $p \geq 3$. Given some regular enough vector fields $V_i$ on $\mathbb{R}^d$, the enriched signal $X := (X,X)$ is used to make sense and solve some controlled differential equation

\begin{equation}
dz_t = V(z_t) \, dX_t = \sum_i V_i(z_t) \, dX^i_t,
\end{equation}

where the same multiplication problem $V(z_t) \, dX_t$ happens, as a solution path $z$ to that equation is expected to be $\frac{1}{p}$-Hölder, while $\frac{2}{p} - 1$ is negative. This problem is by-passed here by defining a solution path to the rough differential equation (4.2) as an $\mathbb{R}^d$-valued path $(z_t)_{0 \leq t \leq 1}$ for which one has a uniform second order Euler-Taylor expansion

\begin{equation}
z_t = z_s + (X^1_t - X^1_s)V_1(z_s) + X^j_{ts} (V_j V_k \text{Id})(z_s) + O\left( |t-s|^\frac{3}{p} > 1 \right)
\end{equation}

at all times $0 \leq s < 1$. The expression $(X^1_t - X^1_s)V_1(z_s) + X^j_{ts} (V_j V_k \text{Id})(z_s)$ somehow defines an integrated version of the product $V(z_s) \, dX_s$. (Replacing $X$ by a smooth control for which formula (4.1) makes plain sense, equation (4.3) is really the second order Euler-Taylor expansion of the well-defined solution of the ordinary controlled differential equation (4.2) in that case.)

In our singular PDE setting where the parabolic operator $(\partial_t + L)$ plays the role of the time increment operator $d$ in equation (4.2), it happens to be more convenient to define directly a product like $u \zeta$ rather than its “integrated” version $\mathcal{R}(u \zeta)$, even though we shall eventually consider the latter as well.

Equation (4.3) provides a nice guide as to how recast the problem in the rough differential equation (4.2) is rougher, say $3 \leq p < 4$; write $X = (X^1, X^2, X^3)$ in that case without paying too much attention to where these objects live as this is not so important here. It suffices to say here that they are formally defined in terms of $X^1 = X$ only and the integration operator, under the form of iterated integrals. Setting $x_{ts} = x_t - x_s$ for the increment of a vector valued path $x$, the structure of equation (4.3) is encoded in the system

\begin{equation}
\begin{aligned}
z_{ts} &= z^{(1)}_{ts} X^1_{ts} + z^{(2)}_{ts} X^2_{ts} + z^{(3)}_{ts} X^3_{ts} + O\left( |t-s|^\frac{4}{p} \right) \\
z^{(1)}_{ts} &= z^{(2)}_{ts} X^1_{ts} + z^{(3)}_{ts} X^2_{ts} + O\left( |t-s|^\frac{3}{p} \right) \\
z^{(2)}_{ts} &= z^{(3)}_{ts} X^1_{ts} + O\left( |t-s|^\frac{2}{p} \right) \\
z^{(3)}_{ts} &= O\left( |t-s|^\frac{1}{p} \right),
\end{aligned}
\end{equation}

with the appropriate objects in the role of $z^{(1)}$, $z^{(2)}$ and $z^{(3)}$. We approach below the deterministic parabolic Anderson model equation exactly in those terms.
4.3. The deterministic PAM equation

Building on the intuition provided by rough paths theory, we look for a solution to the parabolic Anderson model equation (PAM)

\[ \mathcal{L}(u) = u\zeta, \quad u|_{t=0} = u_0. \]

that somehow has a Taylor expansion of the form (4.4), with reference objects \((Z_i)_{1 \leq i \leq 3}\) formally built only from the signal \(\zeta\), in place of the \(X^3\). Given \(0 < \alpha < \frac{1}{2}\) and \(0 < \beta \leq \alpha\), we thus introduce the set \(\mathcal{S}_{\alpha,\beta}\)

\[ u = \Pi^{(b)}_{\alpha_1}(Z_1) + \Pi^{(b)}_{\alpha_2}(Z_2) + \Pi^{(b)}_{\alpha_3}(Z_3) + \|3\alpha + \beta\|_0 \]

\[ u_1 = \Pi^{(b)}_{\alpha_2}(Z_2) + \Pi^{(b)}_{\alpha_3}(Z_3) + \|2\alpha + \beta\|_0 \]

\[ u_2 = \Pi^{(b)}_{\alpha_3}(Z_3) + \|\alpha + \beta\|_0 \]

\[ u_3 = \|\beta\|_0. \]

where given \(\gamma \in \mathbb{R}\) and a non-negative integer \(c\), we denote by \(\|\gamma\|_c\) an element of \(\mathcal{C}^c_{\alpha,\beta}\).

We shall give below assumptions under which the parabolic reference objects \(Z_1, Z_2, Z_3\) satisfy

\[ \|Z_i\|_{\mathcal{C}^c_{\alpha,\beta}} \leq 1. \]

It will thus follow from the defining relation (4.6) and Proposition 14 that \(u \in \mathcal{C}^c_{\alpha,\beta} \cap \mathcal{C}^{c-a}_{\alpha,\beta}\).

We turn the space \(\mathcal{S}_{\alpha,\beta}\) into a Banach space by defining its norm as

\[ \| (u, u_1, u_2, u_3) \|_{\alpha,\beta} := \sum_{i=0}^{3} \| i\alpha + \beta \|_0 \| \mathcal{C}^c_{\alpha,\beta}. \]

Think of relation (4.6) as a Taylor expansion of \(u\) in the Hölder scale, in terms of the "monomials" \(Z_i\). The latter are defined from elementary "bricks" \(Y_1, Y_2, Y_3, Y_4\) whose definition involves only \(\zeta\), the resolution operator \(Y\) and the paraproduct \(\Pi^{(b)}\), for some high enough parameter \(b - \) this could be compared with the formal definition of \(X^1, X^2, X^3\) in the above rough paths setting, as formal iterated integrals; the resolution operator \(Y\) plays the role of the integration operator from time 0 to the current time. We define recursively \(Z_i = Y_i\), where

\[ Y_1 := \zeta, \]

\[ Y_2 := \Pi^{(b)}(Z_1) + \Pi^{(b)}(\zeta, Z_1), \]

\[ Y_3 := \Pi^{(b)}(Z_2) + \Pi^{(b)}(\zeta, Z_2) + C^1(Z_1, Z_1, \zeta) + R^1(Z_1, Z_1, \zeta) + \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1)(Z_1)) + \Pi^{(b)}(Z_1, \Pi^{(b)}(Z_1, \zeta)), \]

\[ Y_4 := \Pi^{(b)}(Z_3) + \Pi^{(b)}(\zeta, Z_3) + R^1(Z_1, Z_2, \zeta) + R^1(Z_2, Z_1, \zeta) + R^2(Z_1, Z_1, \zeta) \]

\[ + \Pi^{(b)}(C^1(Z_1, Z_1, \zeta)) + \Pi^{(b)}(Z_1, C^1(Z_1, Z_1, \zeta)) + C^2(Z_1, Z_1, Z_1, \zeta) + C^1(Z_2, Z_1, \zeta) \]

\[ + \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_2)(Z_1)) + \Pi^{(b)}(Z_2, \Pi^{(b)}(Z_1, \zeta)) + C^1(Z_1, Z_2, \zeta) \]

\[ + C^1(Z_1, Z_1, \Pi^{(b)}(\zeta, Z_1)) + \Pi^{(b)}(Z_1, \Pi^{(b)}(Z_1, \Pi^{(b)}(\zeta, Z_1))) + \Pi^{(b)}(\Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1)(Z_1)), Z_2) \]

\[ + \Pi^{(b)}(Z_2, \Pi^{(b)}(\zeta, Z_1)) + R^1(\Pi^{(b)}(\zeta, Z_1), Z_1, Z_1) + \Pi^{(b)}(\Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1)(Z_1)) \Pi^{(b)}(\Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1)(Z_1)) (Z_2)). \]
Proof – Starting from the a priori formal expression

\[
\begin{align*}
    u\zeta &= \Pi_{u}(\zeta) + \Pi_{\zeta}(u) + \Pi_{b}(u, \zeta)
\end{align*}
\]

for the product of \(u\) and \(\zeta\), we show that the resonant term \(\Pi_{b}(u, \zeta)\) is well-defined analytically if \(u\) satisfies the Ansatz \((4.6)\), by making a number of formal algebraic computations involving the commutators and correctors introduced and studied in Section 3.2. This is performed in Step 3. We actually prove that one has

\[
\begin{align*}
    u\zeta &= \Pi_{u}(Y_{1}) + \Pi_{w_{1}}(Y_{2}) + \Pi_{w_{2}}(Y_{3}) + \Pi_{w_{3}}(Y_{4}) + \|4\alpha + \beta - 2\|_{4},
\end{align*}
\]

relying exclusively on algebraic computations and the regularity results of Section 3.2. Again, a stronger statement for Schauder estimate may allow to consider an expansion which does not involve the tricky term \(Y_{4}\). Our version of Schauder theorem is not

\[
\begin{align*}
    \text{Dans notre cas où } \zeta \text{ sera un bruit blanc ... tous les termes ne sont pas à renormaliser. Certains seront définis presque surement de manière directe, d'autres (lorsque l'on a un C ou une resonante en basse fréquence) sont normalisés des que la basse fréquence est renormalisée.}
\end{align*}
\]
sufficient for working in this optimal way but suffices for our purposes though. Although somewhat lengthy, this proof is elementary. All the claims about the regularity of some paraproduct term are consequences of Proposition 11; we use repeatedly the continuity results on the commutator/corrector and their iterated versions proved in Section 3.2 without mentioning it all the time.

**Step 1.** We leave the first term \( \Pi^{(b)}_\zeta (\zeta) = \Pi^{(b)}_{u_1}(Y_1) \) in the decomposition (4.10) untouched, since it is well-defined in \( C^{\alpha - 2}_{\text{par}} \) and corresponds to the first term in (4.11).

**Step 2.** The second term \( \Pi^{(b)}_\zeta (u) \) is also well-defined in \( C^{2\alpha - 2}_{\text{par}} \). We need however to rewrite it under a form consistent with equation (4.11). We first use the Ansatz (4.6) on \( u \) to decompose this term as follows

\[
\Pi^{(b)}_\zeta (u) = \Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_1} (Z_1) \right) + \Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_2} (Z_2) \right) + \Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_3} (Z_3) \right) + \Pi^{(b)}_\zeta (\|3\alpha + \beta\|_0),
\]

and study separately each term.

(a) The term \( \Pi^{(b)}_\zeta (\|3\alpha + \beta\|_0) \) belongs to \( C^{4\alpha + \beta - 2}_{\text{par}} \).

(b) The third term \( \Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_3} (Z_3) \right) \) belongs to \( C^{4\alpha - 2}_{\text{par}} \). So using the modified commutator \( R^1 \), we have

\[
\Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_3} (Z_3) \right) = \Pi^{(b)}_{u_3} \left( \Pi^{(b)}_\zeta (Z_3) \right) + R^1 (Z_3, u_3, \zeta)
\]

\[
= \Pi^{(b)}_{u_3} \left( \Pi^{(b)}_\zeta (Z_3) \right) + \|4\alpha + \beta - 2\|_2,
\]

where we used Proposition 16.

(c) The second term \( \Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_2} (Z_2) \right) \) has a parabolic regularity of order \( 3\alpha - 2 \). Although we can write

\[
\Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_2} (Z_2) \right) = \Pi^{(b)}_{u_2} \left( \Pi^{(b)}_\zeta (Z_2) \right) + R^1 (Z_2, u_2, \zeta),
\]

the modified commutator \( R^1(Z_2, u_2, \zeta) \) only belongs to \( C^{4\alpha - 2}_{\text{par}} \). We thus use the Ansatz (4.6) to decompose \( u_2 \) and get

\[
R^1 (Z_2, u_2, \zeta) = R^1 \left( Z_2, \tilde{\Pi}^{(b)}_{u_3} (Z_1), \zeta \right) + R^1 (Z_2, \|\alpha + \beta\|_0, \zeta)
\]

\[
= R^1 \left( Z_2, \tilde{\Pi}^{(b)}_{u_3} (Z_1), \zeta \right) + \|4\alpha + \beta - 2\|_2
\]

\[
= \Pi^{(b)}_{u_3} \left( R^1 (Z_2, Z_1, \zeta) \right) + R^2 (u_3, Z_2, Z_1, \zeta) + \|4\alpha + \beta - 2\|_2;
\]

where we have \( R^2 (u_3, Z_2, Z_1, \zeta) \in C^{4\alpha + \beta - 2}_{\text{par}} \). So we have

\[
\Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_2} (Z_2) \right) = \Pi^{(b)}_{u_2} \left( \Pi^{(b)}_\zeta (Z_2) \right) + \Pi^{(b)}_{u_3} \left( R^1 (Z_2, Z_1, \zeta) \right) + \|4\alpha + \beta - 2\|_3.
\]

(d) The first term \( \Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_1} (Z_1) \right) \) is of parabolic regularity \( 2\alpha - 2 \). We proceed as above and start writing

\[
\Pi^{(b)}_\zeta \left( \tilde{\Pi}^{(b)}_{u_1} (Z_1) \right) = \Pi^{(b)}_{u_1} \left( \Pi^{(b)}_\zeta (Z_1) \right) + R^1 (Z_1, u_1, \zeta)
\]
and $R^1(Z_1, u_1, \zeta)$ has a regularity of order $3\alpha - 2$; another decomposition is needed. By Ansatz (4.6), the decomposition of $u_1$ yields

$$R^1(Z_1, u_1, \zeta) = R^1\left(Z_1, \tilde{\Pi}^{(b)}_u(Z_1), \zeta\right) + R^1\left(Z_1, \tilde{\Pi}^{(b)}_{u_3}(Z_2), \zeta\right) + R^1\left(Z_1, \|2\alpha + \beta\|_0, \zeta\right).$$

$$= R^1\left(Z_1, \tilde{\Pi}^{(b)}_u(Z_1), \zeta\right) + R^1\left(Z_1, \tilde{\Pi}^{(b)}_{u_3}(Z_2), \zeta\right) + \|4\alpha + \beta - 2\|_2$$

$$= \Pi^{(b)}_{u_1}\left(R^1(Z_1, Z_1, \zeta)\right) + \Pi^{(b)}_{u_3}\left(R^2(u_2, Z_1, Z_1, \zeta)\right)$$

$$+ R^2(u_3, Z_1, Z_1, \zeta) + \|4\alpha + \beta - 2\|_2;$$

we have $R^2(u_3, Z_1, Z_1, \zeta) \in C^{4\alpha + \beta - 2}_{\text{par}}$. It remains to decompose the term $u_2$ in $R^2(u_2, Z_1, Z_1, \zeta)$, which can be done as follows

$$R^2(u_2, Z_1, Z_1, \zeta) = R^2\left(\tilde{\Pi}^{(b)}_{u_1}(Z_1), Z_1, Z_1, \zeta\right) + R^2\left(\|\alpha + \beta\|_0, Z_1, Z_1, \zeta\right)$$

$$= R^2\left(\tilde{\Pi}^{(b)}_{u_1}(Z_1), Z_1, Z_1, \zeta\right) + \|4\alpha + \beta - 2\|_3$$

$$= \Pi^{(b)}_{u_1}\left(R^2(Z_1, Z_1, \zeta)\right) + \|4\alpha + \beta - 2\|_4.$$

At the end, we have

$$\Pi^{(b)}_{\zeta}\left(\tilde{\Pi}^{(b)}_{u_1}(Z_1)\right) = \Pi^{(b)}_{u_1}\left(\Pi^{(b)}_{\zeta}(Z_1)\right) + \Pi^{(b)}_{u_2}\left(R^1(Z_1, Z_1, \zeta)\right) + \Pi^{(b)}_{u_3}\left(R^2(Z_1, Z_1, Z_1, \zeta)\right)$$

$$+ R^1(Z_1, Z_2, \zeta) + \|4\alpha + \beta - 2\|_4,$$

giving

$$\Pi^{(b)}_{\zeta}(u) = \Pi^{(b)}_{u_1}\left(\Pi^{(b)}_{\zeta}(Z_1)\right) + \Pi^{(b)}_{u_2}\left(\Pi^{(b)}_{\zeta}(Z_2) + R^1(Z_1, Z_1, \zeta)\right)$$

$$+ \Pi^{(b)}_{u_3}\left(\Pi^{(b)}_{\zeta}(Z_3) + R^1(Z_1, Z_2, \zeta) + R^1(Z_2, Z_1, \zeta) + R^2(Z_1, Z_1, Z_1, \zeta)\right)$$

$$+ \|4\alpha + \beta - 2\|_4.$$

**Step 3.** The diagonal term $\Pi^{(b)}(\zeta, u)$ also has to be split in the same way, using correctors and iterated correctors this time.

(a) The term $\Pi^{(b)}(\zeta, \|3\alpha + \beta\|_0)$ is regular of order $4\alpha + \beta - 2 > 0$.

(b) For $\Pi^{(b)}(\zeta, \tilde{\Pi}^{(b)}_{u_3}(Z_3))$, we use a corrector to write

$$\Pi^{(b)}(\zeta, \tilde{\Pi}^{(b)}_{u_3}(Z_3)) = u_3,\left\{\Pi^{(b)}(\zeta, Z_3)\right\} + C^1(u_3, Z_3, \zeta)$$

and since $4\alpha + \beta - 2$ is positive, we have

$$\Pi^{(b)}(\zeta, \tilde{\Pi}^{(b)}_{u_3}(Z_3)) = u_3,\left\{\Pi^{(b)}(\zeta, Z_3)\right\} + \|4\alpha + \beta - 2\|_2.$$
(c) For $\Pi^{(b)} \left( \zeta, \tilde{\Pi}^{(b)}_{u_2} (Z_2) \right)$, we first write

$$\Pi^{(b)} \left( \zeta, \tilde{\Pi}^{(b)}_{u_2} (Z_2) \right) = u_2 \left\{ \Pi^{(b)} (\zeta, Z_2) \right\} + C^1 (u_2, Z_2, \zeta),$$

and use the decomposition of $u_2$ given by the Ansatz (4.6) to get

$$C^1 (u_2, Z_2, \zeta) = C^1 \left( \tilde{\Pi}^{(b)}_{u_3} (Z_1), Z_2, \zeta \right) + C^1 (\| \alpha + \beta \|_0, Z_2, \zeta)$$

$$= C^1 \left( \tilde{\Pi}^{(b)}_{u_3} (Z_1), Z_2, \zeta \right) + [4\alpha + \beta - 2]_2$$

$$= u_3 \left\{ C^1 (Z_1, Z_2, \zeta) \right\} + C^2 (u_3, Z_1, Z_2, \zeta) + [4\alpha + \beta - 2]_2$$

$$= u_3 \left\{ C^1 (Z_1, Z_2, \zeta) \right\} + [4\alpha + \beta - 2]_3,$$

$$= \Pi^{(b)}_{u_3} \left[ C^1 (Z_1, Z_2, \zeta) \right] + [4\alpha + \beta - 2]_3.$$

We then decompose

$$u_2 \left\{ \Pi^{(b)} (\zeta, Z_2) \right\} = \Pi^{(b)}_{u_2} \left( \Pi^{(b)} (\zeta, Z_2) \right) + \Pi^{(b)} \left( u_2, \Pi^{(b)} (\zeta, Z_2) \right) + \Pi^{(b)}_{\Pi^{(b)} (\zeta, Z_2)} \left( \tilde{\Pi}^{(b)}_{u_3} (Z_1) \right)$$

$$+ [4\alpha + \beta - 2]_2$$

$$= \Pi^{(b)}_{u_2} \left( \Pi^{(b)} (\zeta, Z_2) \right) + \Pi^{(b)} \left( u_2, \Pi^{(b)} (\zeta, Z_2) \right) + \Pi^{(b)}_{\Pi^{(b)} (\zeta, Z_2)} \left( \tilde{\Pi}^{(b)}_{u_3} (Z_1) \right)$$

$$+ [4\alpha + \beta - 2]_2.$$

Then by using a corrector, we have

$$\Pi^{(b)} \left( u_2, \Pi^{(b)} (\zeta, Z_2) \right) = \Pi^{(b)}_{u_2} \left( \Pi^{(b)} (\zeta, Z_2) \right) + [4\alpha + \beta - 2]_2$$

$$= u_3 \left\{ \Pi^{(b)} (Z_1, \Pi^{(b)} (\zeta, Z_2)) \right\} + [4\alpha + \beta - 2]_2$$

$$= \Pi^{(b)}_{u_3} \left( \Pi^{(b)} (Z_1, \Pi^{(b)} (\zeta, Z_2)) \right) + [4\alpha + \beta - 2]_2,$$

since we assume that $\Pi^{(b)} \left( Z_1, \Pi^{(b)} (\zeta, Z_2) \right)$ is well-defined in $C^{4\alpha - 2}_{\mathcal{V}(\mathcal{P})}$. At the end, we have

$$\Pi^{(b)} \left( \zeta, \tilde{\Pi}^{(b)}_{u_2} (Z_2) \right) = \Pi^{(b)}_{u_2} \left( \Pi^{(b)} (\zeta, Z_2) \right)$$

$$+ \Pi^{(b)}_{\Pi^{(b)} (\zeta, Z_2)} (Z_1) + \Pi^{(b)} (Z_1, \Pi^{(b)} (\zeta, Z_2)) + C^1 (Z_1, Z_2, \zeta)$$

$$+ [4\alpha + \beta - 2]_3.$$

(d) For the term $\Pi^{(b)} \left( \zeta, \tilde{\Pi}^{(b)}_{u_1} (Z_1) \right)$, we follow the same reasoning with an extra iteration. So starting from the identity

$$\Pi^{(b)} \left( \zeta, \tilde{\Pi}^{(b)}_{u_1} (Z_1) \right) = u_1 \left( \Pi^{(b)} (\zeta, Z_1) \right) + C^1 (u_1, Z_1, \zeta),$$
and using the Ansatz [4.6], it comes
\[ C^1(u_1, Z_1, \zeta) = C^1\left(\tilde{\Pi}^{(b)}_{u_2}(Z_1), Z_1, \zeta\right) + C^1\left(\tilde{\Pi}^{(b)}_{u_3}(Z_2), Z_1, \zeta\right) + \|4\alpha + \beta - 2\|_2 \]
\[ = C^2(u_2, Z_1, Z_1, \zeta) + u_2, \left\{ C^1(Z_1, Z_1, \zeta) \right\} + C^2(u_3, Z_2, Z_1, \zeta) \]
\[ + u_3, \left\{ C^1(Z_2, Z_1, \zeta) \right\} + \|4\alpha + \beta - 2\|_2 \]
\[ = C^2(u_2, Z_1, Z_1, \zeta) + u_2, \left\{ C^1(Z_1, Z_1, \zeta) \right\} + \Pi^{(b)}_{u_3}\left( C^1(Z_2, Z_1, \zeta) \right) \]
\[ + \|4\alpha + \beta - 2\|_3 \]
where we used that \( C^1(Z_2, Z_1, \zeta) \) is well-defined in \( C^{4\alpha - 2}_{\text{wpa}} \) and Proposition 21 for the boundedness of the corrector. We need again to decompose \( u_2 \) in \( C^2 \) so we have
\[ C^2(u_2, Z_1, Z_1, \zeta) = \Pi^{(b)}_{u_2}\left( C^1(Z_1, Z_1, \zeta) \right) \]
\[ + \Pi^{(b)}_{u_3}\left( \Pi^{(b)}_{C^1(Z_1, Z_1, \zeta)}(Z_1) + \Pi^{(b)}(Z_1, C^1(Z_1, Z_1, \zeta)) \right) + \|4\alpha + \beta - 2\|_4 \]
since \( C^1(u_3, Z_1, C^1(Z_1, Z_1, \zeta)) \) is of positive regularity \( 4\alpha + \beta - 2 \). At the end, we deduce that
\[ \Pi^{(b)}(\zeta, \tilde{\Pi}^{(b)}_{u_1}(Z_1)) = u_1, \left( \Pi^{(b)}(\zeta, Z_1) \right) + \Pi^{(b)}_{u_2}\left( C^1(Z_1, Z_1, \zeta) \right) \]
\[ + \Pi^{(b)}_{u_3}\left( \Pi^{(b)}(Z_1, C^1(Z_1, Z_1, \zeta)) + C^2(Z_1, Z_1, Z_1, \zeta) \right) \]
\[ + C^1(Z_2, Z_1, \zeta) + \Pi^{(b)}_{C^1(Z_1, Z_1, \zeta)}(Z_1) \right) + \|4\alpha + \beta - 2\|_4 \]
It remains us to decompose the first product, starting with
\[ u_1, \left( \Pi^{(b)}(\zeta, Z_1) \right) = \Pi^{(b)}_{u_1}\left( \Pi^{(b)}(\zeta, Z_1) \right) + \Pi^{(b)}_{\Pi^{(b)}(\zeta, Z_1)}(u_1) + \Pi^{(b)}(u_1, \Pi^{(b)}(\zeta, Z_1)). \]
We follow the same reasoning as previously and we let the reader check that combining corrector / commutators estimates with Ansatz [4.6], we get
\[ \Pi^{(b)}_{\Pi^{(b)}(\zeta, Z_1)}(u_1) = \Pi^{(b)}_{\Pi^{(b)}(\zeta, Z_1)}\left( \tilde{\Pi}^{(b)}_{u_2}(Z_1) \right) + \Pi^{(b)}_{\Pi^{(b)}(\zeta, Z_1)}\left( \tilde{\Pi}^{(b)}_{u_3}(Z_2) \right) + \|4\alpha + \beta\|_3 \]
\[ = \Pi^{(b)}_{u_2}\left( \Pi^{(b)}_{\Pi^{(b)}(\zeta, Z_1)}(Z_1) \right) + \Pi^{(b)}_{u_3}\left( R^1(\Pi^{(b)}(\zeta, Z_1), Z_1, Z_1) \right) \]
\[ + \Pi^{(b)}_{u_3}\left( \Pi^{(b)}_{\Pi^{(b)}(\zeta, Z_1)}(Z_2) \right) + \|4\alpha + \beta\|_4 \]
and
\[ \Pi^{(b)}\left( u_1, \Pi^{(b)}(\zeta, Z_1) \right) = \Pi^{(b)}\left( \tilde{\Pi}^{(b)}_{u_2}(Z_1), \Pi^{(b)}(\zeta, Z_1) \right) + \Pi^{(b)}\left( \tilde{\Pi}^{(b)}_{u_3}(Z_2), \Pi^{(b)}(\zeta, Z_1) \right) + \|4\alpha + \beta\|_3 \]
\[ = u_2, \Pi^{(b)}(Z_1, \Pi^{(b)}(\zeta, Z_1)) + C^1(u_2, Z_1, \Pi^{(b)}(\zeta, Z_1)) \]
\[ + \Pi^{(b)}_{u_3}\left( \Pi^{(b)}(Z_2, \Pi^{(b)}(\zeta, Z_1)) \right) + |4\alpha + \beta|_4. \]
So splitting again \( u_2 \) and the product gives
\[
C^1(u_2, Z_1, \Pi(b)(\zeta, Z_1)) = C^1(\tilde{\Pi}^{(b)}(Z_1), Z_1, \Pi(b)(\zeta, Z_1)) + \|4\alpha + \beta\|_4
\]
and
\[
u_2.\Pi(b)(Z_1, \Pi(b)(\zeta, Z_1)) = \Pi^{(b)}(\Pi^{(b)}(Z_1, \Pi(b)(\zeta, Z_1)) + \Pi^{(b)}(\nu_2. \Pi^{(b)}(Z_1, \Pi(b)(\zeta, Z_1)))
\]
\[
+ \Pi^{(b)}(\Pi^{(b)}(Z_1, \Pi(b)(\zeta, Z_1)))^{(u_2)}
\]
\[
= \Pi^{(b)}(\Pi^{(b)}(Z_1, \Pi(b)(\zeta, Z_1))) + \Pi^{(b)}(\Pi^{(b)}(Z_1, \Pi(b)(\zeta, Z_1)))
\]
\[
+ \Pi^{(b)}(\Pi^{(b)}(Z_1, \Pi(b)(\zeta, Z_1)))^{(Z_1)} + \|4\alpha + \beta\|_4.
\]
Consequently, we deduce that
\[
\Pi^{(b)}(\zeta, u) = \Pi^{(b)}(\Pi(b)(\zeta, Z_1))
\]
\[
+ \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_2) + C^1(Z_1, Z_1, \zeta) + \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1)))Z_1 + \Pi^{(b)}(Z_1, \Pi^{(b)}(Z_1, \zeta))
\]
\[
+ \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1))^{(Z_1)} + \|4\alpha + \beta - 2\|_4,
\]
with
\[
* := \Pi^{(b)}(Z_1, C^1(Z_1, Z_1, \zeta)) + C^2(Z_1, Z_1, Z_1, \zeta) + C^1(Z_2, Z_1, \zeta)
\]
\[
+ \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_2))Z_1 + \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_2))Z_1 + C^1(Z_1, Z_2, \zeta) + \Pi^{(b)}(\zeta, Z_1, \zeta) + \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1, \zeta))Z_1
\]
\[
+ C^1(Z_1, Z_1, \Pi^{(b)}(\zeta, Z_1)) + \Pi^{(b)}(Z_1, \Pi^{(b)}(Z_1, \Pi^{(b)}(\zeta, Z_1)))Z_1 + \Pi^{(b)}(\Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1)))Z_1
\]
\[
+ \Pi^{(b)}(Z_2, \Pi^{(b)}(\zeta, Z_1)) + \Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1, Z_1)) + \Pi^{(b)}(\Pi^{(b)}(\Pi^{(b)}(\zeta, Z_1)))Z_2.
\]

**Step 4 - Definition of \( u^\zeta \) and conclusion.** The above computations tell us that we can define the product \( u^\zeta \) as in equation (4.11) by the formula
\[
u^\zeta = \Pi^{(b)}(Y_1) + \Pi^{(b)}(Y_2) + \Pi^{(b)}(Y_3) + \Pi^{(b)}(Y_4) + \|4\alpha + \beta - 2\|_4.
\]
with the \( Y_i \) defined above. The defining relation
\[
\mathcal{R}^{(b)} = \tilde{\Pi}^{(b)}\mathcal{R}
\]
implies then that one has
\[
v = \mathcal{R}(u^\zeta) = \tilde{\Pi}^{(b)}(Z_1) + \tilde{\Pi}^{(b)}(Z_2) + \tilde{\Pi}^{(b)}(Z_3) + \tilde{\Pi}^{(b)}(Z_4) + \|4\alpha + \beta - 8\alpha\|_0
\]
where we used Schauder estimates for the remainder term and the fact that \( c^{4\alpha + \beta - 2} < C^{4\alpha + \beta - 2}(\varpi p_\alpha) \), since \( 4\alpha + \beta - 2 > 0 \); see Proposition [9]. Since \( Z_4 \in C^{4\alpha}_\varpi \), then we have \( \Pi_{u_3}^{(b)}(Z_4) \in C^{3\alpha + \beta}_\varpi \), by Proposition [12], and we conclude that the tuple \( (v, u, u_1, u_2) \) also satisfies the structure equation (4.6), since \( 8\alpha \leq \beta < \alpha \).

The estimate (4.9) is a direct consequence of Schauder estimates. Since \( Y_4 \) is assumed to be in \( (L_\varpi^\infty C^{4\alpha + \beta - 2})(\varpi p_\alpha) \) then
\[
\left\| \Pi^{(b)}_{u_3}(Z_4) \right\|_{C^{3\alpha + \beta}} \lesssim \kappa^{-\theta} \|u_3\|_{C^\alpha} \|Z_4\|_{C^{3\alpha}_\varpi}
\]
where we used that \( 2(\alpha + \theta) \leq \alpha - \beta \). Similar estimates hold for the different remaining terms, which imply estimate (4.9).
\[\triangleright\]
Given \( u_0 \in C^{4\alpha}_{\bar{\varpi}} \), write \( S_{\alpha,\beta}^{u_0} \) for the set of tuples \((u, u_1, u_2, u_3)\) satisfying the structure relation \((4.6)\), with \( u_{\tau=0} = u_0 \). As the function \((x, \tau) \mapsto (e^{-\tau L})(u_0)(x)\) belongs to \( C^{4\alpha}_{\bar{\varpi}} \), we define a map \( \Phi \) from \( S_{\alpha,\beta}^{u_0} \) to itself setting

\[
\Phi(u, u_1, u_2, u_3) = (R(u\zeta), u, u_1, u_2).
\]

Note that the map \( \Phi \) depends continuously on the additional a priori data introduced in assumption \((H)\) – all the diagonal terms in the formal definition of the \( Y_i \). We call \( \zeta \), together with this additional data a rough distribution, and denote it by \( \hat{\zeta} \); it lives in the product space described in assumption \((H)\), equipped with its product topology. The next well-posedness result is then a direct consequence of Theorem 23.

**Theorem 24.** Given \( \alpha \in \left( \frac{2}{5}, \frac{1}{2} \right) \) and \( \beta \in (0, \alpha) \), one can choose a positive parameter \( \kappa \) in the definition of the special weight \( \varpi \) large enough to have the following conclusion. If the regularity assumption \((H)\) holds with a parameter \( a < \frac{\alpha - \beta}{2} \), and \( u_0 \in C^{4\alpha}_{\bar{\varpi}} \), then the map \( \Phi \) has a unique fixed point in \( S_{\alpha,\beta}^{u_0} \), which depends continuously on the rough distribution \( \hat{\zeta} \), and the former satisfies the identity

\[
u = \tilde{\Pi}_u^{(b)}(Z_1) + \tilde{\Pi}_u^{(b)}(Z_2) + \tilde{\Pi}_u^{(b)}(Z_3) + \|3\alpha + \beta\|_0.
\]

Let us emphasize that the lower bound \( \frac{2}{5} < \alpha \) is not really important. Indeed, as soon as \( \alpha > 0 \), we let the reader check that by sufficient iterations of the correctors / commutators we can extend the previous reasoning and obtain the same results. The important property is that \( \alpha > 0 \), which allows us to gain at each iteration.

If the ambient space \( M \) is bounded, then we do not have to take care of the infinity in the space variable (mainly in the weight \( \varpi \)) and by considering the weight \( \varpi(x, \tau) = e^{\kappa \tau} \) with a large enough parameter \( \kappa \), we can prove a global (in time) result.

### 4.4. The stochastic PAM equation

Recall the time-independent white noise over the measure space \((M, \mu)\) is the centered Gaussian process \( \xi \) indexed by \( L^2(M) \), with covariance

\[
\mathbb{E}[\xi(f)^2] = \int f^2(x) \mu(dx).
\]

It can be proved \([3]\) to have a modification with values in the spacial Hölder space \( C^{\frac{5}{2} - \varepsilon}_0 \), for all positive constants \( \varepsilon \) and \( a \). We still denote it by the same letter \( \xi \). We are interested in the 3-dimensional case where \( \nu = 3 \). The stochastic parabolic Anderson model equation (PAM) is the equation

\[
(\partial_t + L)u = u\xi,
\]

with ill-defined right hand side since \( u \) is expected to be of Hölder regularity \( -\frac{\nu}{2} - \varepsilon + 2 \) and \( -\nu - 2\varepsilon + 2 = -1 - 2\varepsilon \) is non-positive in our case. Regularizing the distribution \( \xi \) and solving the associated equation, one does not get a converging family of solutions in any reasonable space as the regularizing parameter goes to 0.

We saw in the previous section that we can get out of these difficulties by adding to the initial datum \( \xi \) a number of additional data, which can be used to make sense of and solve this equation for any fixed regular enough realization of \( \xi \). It does not make sense however to work with any enhancement of \( \xi \). Starting from \( \xi \), the natural approach consists in regularizing it, defining \( Y_i, Z_i \) accordingly and letting the regularizing parameter tend to 0. The resonant terms in the \( Y_i \) have no reason to converge in the awaited functional spaces as \( \varepsilon \) goes to 0 however, and one is left with the hope of getting some converging quantities by removing the ‘diverging part’ of each term. While there is no chance that this works for
a given deterministic distribution, this can be done in a probabilistic setting, with a white noise $\xi$, if one is ready to use a weaker notion of convergence for the above ‘recentered’ otherwise diverging quantities. Convergence in probability will make the job. This purely probabilistic step for defining the extra data is called renormalization. Given $\varepsilon > 0$, set $\xi^\varepsilon := (e^{-L\varepsilon}) (\xi)$, and define $(Y_i^\varepsilon)_{i=1,4}, (Z_i^\varepsilon)_{i=1,4}$ by formula (4.8), with $\xi^\varepsilon$ in place of $\xi$. Write $Z^\varepsilon$ for the tuple $(Z_1^\varepsilon, Z_2^\varepsilon, Z_3^\varepsilon)$.

**Theorem 25** (Renormalization). Assume that the space is Ahlfors regular of dimension 3. For each resonant term $\Pi^{(b)}(\xi)$ or corrector term $C^1, C^2$ in the definition of the $Y_i^\varepsilon$, generically written $F(\xi^\varepsilon, Z^\varepsilon)$, there exists a time-independent and deterministic function $f^\varepsilon$ such that $F(\xi^\varepsilon, Z^\varepsilon) - f^\varepsilon$ is uniformly bounded and converging in $L_T^\infty C^a_{p_a}$ for every $a \in (0,1)$, as $\varepsilon$ goes to 0.

**Remark 4.1.** More precisely, we claim the following: for almost every realization of the white noise,

- $\xi^\varepsilon$ is uniformly (with respect to $\varepsilon$) bounded in $L_T^\infty C^{a_2}_{p_a}$,
- in $Y_\varepsilon^\varepsilon$, $\Pi^{(b)}(Z_1^\varepsilon)$ is uniformly (with respect to $\varepsilon$) bounded in $L_T^\infty C^{2a-2}_{p_a}$ and only the term

$$(\Xi_2) \quad \Pi^{(b)}(\xi^\varepsilon, Z_1^\varepsilon)$$

has to be renormalized to be uniformly bounded in $(L_T^\infty C^{2a-2}_{p_a})$,

- in $Y_\varepsilon^\varepsilon$, only the term

$$(\Xi_3) \quad \Pi^{(b)}(\xi^\varepsilon, Z_2^\varepsilon) + C^1(Z_1^\varepsilon, Z_1^\varepsilon, \xi^\varepsilon) + \Pi^{(b)}(Z_3^\varepsilon, \Pi^{(b)}(\xi^\varepsilon, Z_2^\varepsilon))$$

has to be renormalized to be uniformly bounded in $(L_T^\infty C^{3a-2}_{p_a})$. The other terms are directly uniformly bounded in $(L_T^\infty C^{3a-2}_{p_a})$.

- in $Y_\varepsilon^\varepsilon$, only the term

$$(\Xi_4) \quad \Pi^{(b)}(\xi^\varepsilon, Z_3^\varepsilon) + C^2(Z_1^\varepsilon, Z_1^\varepsilon, \xi^\varepsilon) + C^1(Z_2^\varepsilon, Z_2^\varepsilon, \xi^\varepsilon) + C^1(Z_2^\varepsilon, Z_2^\varepsilon, \Pi^{(b)}(\xi^\varepsilon, Z_2^\varepsilon)) + \Pi^{(b)}(Z_1^\varepsilon, \Pi^{(b)}(\xi^\varepsilon, Z_1^\varepsilon)) + \Pi^{(b)}(Z_2^\varepsilon, \Pi^{(b)}(\xi^\varepsilon, Z_2^\varepsilon))$$

has to be renormalized in $(L_T^\infty C^{4a-2}_{p_a})$.

The function $f^\varepsilon$ plays the role of that ‘diverging part’ which needs to be removed to get a converging recentered distribution. We do not prove this renormalization result here so as to keep this work at a reasonable size, and given the fact that this is mainly a technical question with no new ideas needed along the way. Full details will be given in a forthcoming work. Note that the term $\Pi^{(b)}(\xi^\varepsilon, Z^\varepsilon)$ was essentially already treated in [3], where we used time-independent paraproduct/resonant operators. The use of a space-time paraproduct here seems necessary in this 3-dimensional setting if one wants to avoid the inefficient use of a commutator between a space-paraproduct and the resolution operator $\mathcal{R}$. Remark here that as $\xi$ is time-independent, the $Z_i$ will be in Hölder space with a positive regularity exponent. So to renormalize a generic term $F(\xi^\varepsilon, Z^\varepsilon)$, one expects the regularity considerations in the time variable to be easily dealt with and the spatial regularity issue to be dealt with as in [3].

Theorem 23 together with the renormalization result, Theorem 25 have the following result as a direct consequence.
**Theorem 26.** Let $\alpha \in \left(\frac{2}{5}, \frac{1}{2}\right)$ be given. One can choose a large enough parameter $\kappa$ in the definition of the special weight $\varpi$ for the following to hold. There exists a sequence of time-independent and deterministic functions $(c^\varepsilon)_{0<\varepsilon\leq 1}$ such that if $u^\varepsilon$ stands for the solution of the renormalized equation

\begin{equation}
\partial_t u^\varepsilon + Lu^\varepsilon = u^\varepsilon (\xi^\varepsilon - c^\varepsilon), \quad u^\varepsilon(0) = u_0
\end{equation}

with initial condition $u_0 \in C^4_{\text{loc}}$, then $u^\varepsilon$ converges in probability to a function $u \in C^\alpha$. Moreover the deterministic function $c^\varepsilon$ is obtained as the sum of the three deterministic functions which allows us to renormalize $\Xi_2$, $\Xi_3$ and $\Xi_4$.

**Proof –** The proof is the direct combination between Theorems 23 and 25 together with the following observation.

For almost every realization of the white noise, we have seen in Remark 4.1 that some of the quantities in $Y_\varepsilon^\varepsilon$ have to be renormalized, in order to get uniformly (with respect to $\varepsilon$) bounded distributions. It remain also to see how this renormalization step interacts in the product $(u, \xi^\varepsilon)$. As it could be seen by tracking the suitable modifications in the proof of Theorem 24

- Renormalizing $\Xi_2$ by $\Xi_2 - \lambda_2^\varepsilon$ (for some sequence of deterministic function $(\lambda_2^\varepsilon)_\varepsilon$) in $L_T^\infty C^{2\alpha-2}$ yields that for $(u, u_1, u_2, u_3)$ satisfying ansatz (4.6) (with $Z_2^\varepsilon$), the quantity $u\xi^\varepsilon - u_1\lambda_2^\varepsilon$ is well-defined, and setting $v := R(u\xi^\varepsilon - u_1\lambda_2^\varepsilon)$, the tuple $(v, u, u_1, u_2)$ still satisfies ansatz (4.6), uniformly with respect to $\varepsilon > 0$.

- Similarly, in $Y_3^\varepsilon$, renormalizing in $(L_T^\infty C^{4\alpha-2}_{\text{loc}})$ the quantity $\Xi_3$ with a deterministic sequence of functions $(\lambda_3^\varepsilon)_\varepsilon$ implies the following: for $(u, u_1, u_2, u_3)$ satisfying ansatz (4.6) (with $Z_3^\varepsilon$), the quantity $u\xi^\varepsilon - u_2\lambda_3^\varepsilon$ is well-defined, and setting $v := R(u\xi^\varepsilon - u_2\lambda_3^\varepsilon)$, the tuple $(v, u, u_1, u_2)$ still satisfies ansatz (4.6), uniformly with respect to $\varepsilon > 0$.

- in $Y_4^\varepsilon$, renormalizing $\Xi_4$ in $(L_T^\infty C^{4\alpha-2}_{\text{loc}})$ with a sequence of deterministic functions $(\lambda_4^\varepsilon)_\varepsilon$ yields the following: for $(u, u_1, u_2, u_3)$ satisfying ansatz (4.6) (with $Z_4^\varepsilon$), the quantity $u\xi^\varepsilon - u_3\lambda_4^\varepsilon$ is well-defined, and setting $v := R(u\xi^\varepsilon - u_3\lambda_4^\varepsilon)$, the tuple $(v, u, u_1, u_2)$ still satisfies ansatz (4.6), uniformly with respect to $\varepsilon > 0$.

With this observation, the proof is obtained as the combination of Theorems 23 and 25 and we get the result with

\[ c^\varepsilon = \lambda_2^\varepsilon + \lambda_3^\varepsilon + \lambda_4^\varepsilon, \]

since for the solution of the equation, which correspond to the fixed point, we have $u = u_1 = u_2 = u_3$.

\[ \square \]

4.5. The stochastic 3D Burger equation  
We still work, under the previous setting, in a 3-dimensional space $M$. To simplify, we consider the collection of 3 “vector fields” $X_j$ so $\ell_0 = 3$ and we denote $X = (X_1, X_2, X_3)$. We assume that these three operators commute, which is the case in the Euclidean setting with the canonical vector fields for example.

We are now interested in the stochastic 3D Burger system, which takes the following form: we look for a solution $u := M \times [0, T] \rightarrow \mathbb{R}^3$ with

\[ \partial_t u + Lu + (u \cdot X)u = u\xi \]

where $\xi$ is now a white noise with values in $\mathbb{R}^3$ (indeed $\xi = (\xi_1, \xi_2, \xi_3)$ with three independent scalar white noise $\xi^j$). That means that for every coordinate function $u^j$, $j = 1, 2, 3$
we ask for
\[ \partial_t u^j + Lu^j + \sum_{i=1}^{3} u^i X_i(u^j) = u^j \dot{\xi}^j. \]

As previously, we shall work here in the regime \( \frac{2}{5} < \alpha < \frac{1}{2} \). We first obtained a deterministic result, to deal with the new nonlinearity:

**Theorem 27.** Assume (H) holds with a small enough parameter \( a < \frac{5\alpha}{2} - 1 \) such that there exists \( \beta \in (0, \alpha) \) and \( \theta \in (0, \frac{\alpha}{10}) \) with
\[ 4\alpha + \beta - 2 > 0 \quad \text{and} \quad 12(a + \theta) \leq 2\alpha + \beta - 1. \]
Assume also that for \( i, j \in \{1, 2, 3\} \) the quantity \( \pi^{(b)}(Z_i^1, X_i^1) \) is well-defined in \( L^\infty C^{2\alpha-1} \) (where \( \pi^{(b)} \) is the spatial paraproduct, freezing the time variable). Let \( (u, u_1, u_2, u_3) \) be a tuple satisfying the structure equation (4.6) (strictly speaking the structure (4.6) is satisfied in the three coordinates since \( u, u_i \) have values in \( \mathbb{R}^3 \), with \( u \in C^{\alpha-2\alpha-2\theta} \). Then the nonlinearity \( (u \cdot X)u \) is well-defined and
\[ \mathcal{R}[(u \cdot X)u] \in C^{3\alpha+\beta}. \]

**Proof** – Indeed, we only use the “first order” expansion of the functions \( u^j \). From the ansatz (4.6), we know that for every \( j = 1, 2, 3 \)
\[ u^j = \Pi_{u_1^j}^{(b)}(Z_i^1) + \|2\alpha - 1\|_2^2, \]
and so
\[ X_i(u^j) = X_i\Pi_{u_1^j}^{(b)}(Z_i^1) + \|2\alpha - 1\|_2^2, \]
where \( \|\gamma\|_c \) denotes an element of \( L_T^{\infty} C^{\gamma}_{\varphi_p \alpha} \) for \( \gamma < 0 \) and a non negative integer \( c \). Since the vector fields commute then, an easy computation relying on the Leibniz rule shows that
\[ X_i\Pi_{u_1^j}^{(b)}(Z_i^1) = \Pi_{u_1^j}^{(b)}(X_i Z_i^1) + \Pi_{u_1^j}^{(b)}(Z_i^1) = \Pi_{u_1^j}^{(b)}(X_i Z_i^1) + \|2\alpha - 1\|_3^2, \]
where we used Proposition (11) for the last step. Indeed
\[ \Pi_{X_i u_1^j}^{(b)}(Z_i^1) = \mathcal{R}\left[ \Pi_{X_i u_1^j}^{(b)}(\zeta^j) \right] \]
and since \( \zeta^j \) is independent with respect to the time variable and \( X_i u_1^j \in L_{T}^{\infty} C^{\gamma}_{\varphi_p \alpha} \) then we have by repeating the proof of Proposition (11) that
\[ \Pi_{X_i u_1^j}^{(b)}(\zeta^j) \in L_T^{\infty} C^{2\alpha-3}_{\varphi_p \gamma}, \]
At the end, we conclude that
\[ (4.13) \quad X_i(u^j) = \Pi_{u_1^j}^{(b)}(X_i Z_i^1) + \|2\alpha - 1\|_2^2. \]

Then we expand the product \( u^j X_i(u^j) \) using paraproducts. Since we aim to work with \( L_T^{\infty} C^{\gamma} \) spaces for some \( \gamma < 0 \), then it is more convenient to use spatial paraproducts (as defined in (3)) and not space-time paraproducts. So let us denote \( \pi^{(b)} \) the spatial paraproduct, which consists to replace \( \varphi \ast \varphi \) by the dirac distribution \( \delta_0 \) (and so \( \psi \) by

\(^4\text{Since } \zeta^j \text{ does not depend on time, then in the paraproduct } \Pi_{X_i u_1^j}^{(b)}(\zeta^j), \text{ the contribution is null for the terms associated with a cancellation in the time-variable and so we only have to deal with cancellation in space.} \)
0 and for every \( t \in (0,1] \) \((\varphi \ast \varphi)_t^2 = \text{Id}) in the definition of paraproducts. Then we have the decomposition (where the time is now fixed)
\[
u^i . X_i(u^j) = \pi_{\nu^i}(X_i(u^j)) + \pi_{X_i(u^j)}(u^i) + \pi_{\nu^i}(u^i, X_i(u^j)).
\]
Using the Hölder continuity of the spatial paraproducts (see [3] which corresponds to remove the time in Proposition 11) then, it comes that uniformly in time, the two first paraproducts belong to \( C_{\text{par}}^{2\alpha} \) and so
\[
\pi_{\nu^i}(X_i(u^j)) + \pi_{X_i(u^j)}(u^i) = \|\alpha - 1\|_2^2.
\]
In the third resonant term, we expand \( u^i \) and \( X_i(u^j) \) (with \([4,13]\)) and we have that
\[
\pi_{\nu^i}(u^i, X_i(u^j)) = \pi_{\nu^i}(\tilde{\Pi}_{u^i}(Z_i^1), \Pi_{u^i}(X_i Z_i^1)) + \|3\alpha - 1\|_5^2.
\]
We know that
\[
\tilde{\Pi}_{u^i}(Z_i^1) = \mathcal{R}\left[ \Pi_{u^i}(\xi^i) \right]
= \mathcal{R}\left[ \pi_{u^i}(\xi^i) + \|2\alpha - 2\|_2^2 \right]
= \mathcal{R}\left[ \pi_{u^i}(\xi^i) \right] + \|2\alpha\|_2^2
\]
and similarly
\[
\tilde{\Pi}_{u^i}(X_i Z_i^1) = \mathcal{R}\left[ \pi_{u^i}(X_i \xi^i) \right] + \|2\alpha - 1\|_5^2,
\]
where we used the comparison between spatial paraproducts and space-time paraproducts, see [19 Lemma 5.1] for more details in the Euclidean framework (similar computations can be done in this setting). So we have, since \( \alpha > 1/3 \) (and so \( 3\alpha - 1 > 0 \) which allows us to use boundedness of the resonant part on remainder terms)
\[
\pi_{\nu^i}(u^i, X_i(u^j)) = \pi_{\nu^i}(\tilde{\Pi}_{u^i}(Z_i^1), \tilde{\Pi}_{u^i}(X_i Z_i^1)) + \|3\alpha - 1\|_6^2.
\]
Hence, using commutator estimates, as done in [19 Section 5] in the Euclidean setting or in [3] (commutator estimates and Schauder estimates for paracontrolled distributions), we can obtain
\[
\pi_{\nu^i}(u^i, X_i(u^j)) = (u^i_1 u^i_2) \pi_{\nu^i}(Z_i^1, X_i Z_i^1) + \|3\alpha - 1\|_6^2.
\]
So at the end, we have
\[
u^i . X_i(u^j) = (u^i_1 u^i_2) \pi_{\nu^i}(Z_i^1, X_i Z_i^1) + \|\alpha - 1\|_6^2.
\]
Then since we assume that \( \pi_{\nu^i}(Z_i^1, X_i Z_i^1) \) is well-defined in \( L^\infty C_{2\alpha}^{2\alpha - 1} \) and \( u_1 \) is in \( L^\infty C_{\text{par}}^{2\alpha} \), we conclude that \( (u^i_1 u^i_2) \pi_{\nu^i}(Z_i^1, X_i Z_i^1) \) is well-defined in \( L^\infty C_{2\alpha}^{2\alpha - 1} \) since \( 3\alpha - 1 > 0 \). So we deduce that for every \( i, j \in \{1, 2, 3\} \) then \( u^i . X_i(u^j) = \|\alpha - 1\|_6^2 \) and so \( (u \cdot X)u = \|\alpha - 1\|_6^2 \). The proof is then completed by using Schauder estimates.

By the previous result, we deduce that \( \mathcal{R}\left[ (u . X)u \right] \) is sufficiently to regular to be taken into account in the 'rest'. So by combining Theorem 23 and Theorem 27 we get the following result:

**Theorem 28.** Assume (H) holds for a parameter \( \alpha < 1 \) small enough such that there exists \( \beta \in (0,\alpha) \) and \( \theta \in (0, \frac{\alpha}{10}) \) with
\[4\alpha + \beta - 2 > 0 \quad \text{and} \quad 12(\alpha + \theta) \leq 2\alpha + \beta - 1\]
\[ 2(\alpha + \theta) \leq \alpha - \beta. \]

Let \( (u, u_1, u_2, u_3) \) be a tuple satisfying the (3D valued) structure equation \((4.6)\), with \( u \in C^{\alpha - 2a - \theta}_{\infty} \). Then the product \( u\zeta \) and the nonlinearity \((u \cdot X)u\) are well-defined, and setting \( v := \mathcal{R}(u\zeta - (u \cdot X)u) \), the tuple \((v, u, u_1, u_2)\) satisfies the structure equation \((4.6)\), and

\[
\left\| (v, u, u_1, u_2) \right\|_{\alpha, \beta} \lesssim \kappa^{-\theta} \left\| (u, u_1, u_2, u_3) \right\|_{\alpha, \beta}. \tag{4.14}
\]

Then, up to a renormalization step, we may deduce the following result on the stochastic 3D Burger system:

**Theorem 29.** Let \( \alpha \in \left( \frac{3}{2}, \frac{7}{4} \right) \) be given. One can choose a large enough parameter \( \kappa \) in the definition of the special weight \( \varpi \) for the following to hold. There exists sequences of time-independent and deterministic functions \((v^\varepsilon)_{0 < \varepsilon \leq 1}\) and \((d_{i,j}^\varepsilon)_{0 < \varepsilon \leq 1}\) for \( i, j \in \{1, 2, 3\} \) such that if \( u^\varepsilon = (u_{i,j}^\varepsilon, u_{i,j}^\varepsilon, u_{i,j}^\varepsilon) \) stands for the solution of the renormalized equation: for \( j = 1, 2, 3 \)

\[
\partial_t u_{i,j}^\varepsilon + Lu_{i,j}^\varepsilon + \sum_{i=1}^{3} u_{i,j}^\varepsilon \cdot X_i (u_{i,j}^\varepsilon) = u_{i,j}^\varepsilon (\xi_{i,j}^\varepsilon - \xi^\varepsilon) - \sum_{i=1}^{3} u_{i,j}^\varepsilon u_{i,j}^\varepsilon d_{i,j}^\varepsilon \quad u_{i,j}^\varepsilon(0) = u_{0,i,j}^j \tag{4.15}
\]

with initial condition \( u_0 \in C^{\alpha \lambda}_{\infty} \), then \( u^\varepsilon \) converges in probability to a function \( u \in C^\alpha \). Moreover the deterministic function \( c^\varepsilon \) is the same as the one for PAM equation and \( d_{i,j}^\varepsilon \) correspond to the deterministic functions which allows us to renormalize \( \pi^{(b)}(Z_1, X_i Z_i) \).

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**References**


**Institut de Recherche Mathématiques de Rennes**, 263 Avenue du General Leclerc, 35042 Rennes, France

*E-mail address*: ismael.bailleul@univ-rennes1.fr

**CNRS - Université de Nantes, Laboratoire de Mathématiques Jean Leray**, 2, Rue de la Houssinière 44322 Nantes Cedex 03, France

*E-mail address*: frederic.bernicot@univ-nantes.fr

**CNRS - Université Paris-Sud, Laboratoire de Mathématiques, UMR 8628, 91405 Orsay, France**

*E-mail address*: dorothee.frey@univ-nantes.fr