Analysis of the Anderson operator

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Abstract. We consider the continuous Anderson operator $H = \Delta + \xi$ on a two dimensional closed Riemannian manifold $\mathcal{S}$. We provide a short self-contained functional analysis construction of the operator as an unbounded operator on $L^2(\mathcal{S})$ and give almost sure spectral gap estimates under mild geometric assumptions on the Riemannian manifold. We prove a sharp Gaussian small time asymptotic for the heat kernel of H that leads amongst others to strong norm estimates for quasimodes. We introduce a new random field, called Anderson Gaussian free field, and prove that the law of its random partition function characterizes the law of the spectrum of H. We also give a simple and short construction of the polymer measure on path space and relate the Wick square of the Anderson Gaussian free field to the occupation measure of a Poisson process of loops of polymer paths. We further prove large deviation results for the polymer measure and its bridges.

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1 – Introduction

Let $\mathcal{S}$ be a two dimensional closed Riemannian manifold with metric g and associated volume measure μ. White noise on $\mathcal{S}$ is a $\mathcal{D}'(\mathcal{S})$-valued random variable ξ with Gaussian law with null mean and covariance

$$\mathbb{E}\big[\xi(\varphi_1)\,\xi(\varphi_2)\big] = \int_{\mathcal{S}} \varphi_1\varphi_2\, d\mu,$$

for φ_1, φ_2 smooth functions on $\mathcal{S}$. Almost surely it takes values in the Besov space $\mathcal{B}^{\alpha-2}_{\infty\infty}(\mathcal{S})$, for any $\alpha < 1$, a distribution space, and its law depends only on the metric g on $\mathcal{S}$. Let $h \in C^\infty(\mathcal{S})$ be a smooth function. Denote by $M_{h\xi}$ the multiplication operator by $h\xi$, and by Δ the Laplace-Beltrami operator associated with the Riemannian metric on $\mathcal{S}$. The Anderson Hamiltonian is the random operator

$$H := \Delta + M_{h\xi}, \tag{1.1}$$

perturbation of the Laplace-Beltrami operator by a distribution-valued potential. The smooth function h plays the role of a modulator for the noise, a position dependent coupling constant. The operator H arises naturally as the scaling limit of a number of microscopic discrete operators of interest in statistical physics. The study of the Anderson Hamiltonian presents an additional difficulty compared to its discrete counterparts. Unlike what happens for the Laplace-Beltrami operator Δ or its perturbations by smooth potentials, the low regularity of ξ prevents a straightforward definition of H as a continuous operator from the Sobolev space $H^2(\mathcal{S})$ into $L^2(\mathcal{S})$ since

$$M_{h\xi}(f) = fh\xi$$

is not an element of $L^2(\mathcal{S})$ for a generic $f \in H^2(\mathcal{S})$. One had to wait for the recent development of the theory of paracontrolled calculus and regularity structures before appropriate functional settings were introduced for the study of the Anderson Hamiltonian – corresponding to $h = 1$. Let $\mathbb{T}^2$ stand for the two dimensional flat torus. Allez and Chouk [1] first used paracontrolled calculus

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to define a random domain for H and proved that one can define H as an unbounded self-adjoint operator on $L^2(\mathbb{T}^2)$, with discrete spectrum $\lambda_n(\widehat{\xi})$ tending to $+\infty$ and eigenvalues $\lambda_n(\widehat{\xi})$ that are continuous functions of a measurable functional $\widehat{\xi}$ of ξ taking values in a Banach space. The basic mechanics at work in [1] was improved in Gubinelli, Ugurcan & Zachhuber's recent work [24] in which a similar result on the three dimensional torus was proved, amongst others. Labbé was also able in [29] to use the tools of regularity structures to get similar results. We refer to these works for detailed accounts of related matters and extensive references to the litterature. All these works are set in the torus. The very recent work of Mouzard [31] used the tools of the high order paracontrolled calculus developed by Bailleul & Bernicot in [4, 5, 6] to study Anderson Hamiltonian on a two dimensional manifold, simplifying a number of technical points compared to [1, 24] and proving that the random spectrum of H satisfies the same Weyl asymptotic law as the spectrum of the Laplace-Beltrami operator.

○ *Anderson operator.* We give in this work a self-contained construction of the Anderson operator that is different from the previous constructions. It relies on the direct construction of the resolvent operator via a fixed point equation where the analytic Fredholm theory can be used efficiently. We note in particular that the only point from paracontrolled calculus that we use is the fundamental continuity estimate on the corrector first proved by Gubinelli, Imkeller & Perkowski in the flat torus [22], later extended to a manifold (and possibly parabolic) setting by Bailleul & Bernicot in [4]. Recall that h is the coupling function that appears in front of the noise in the definition (1.1) of Anderson operator. Given a positive regularization parameter r let $\xi_r = e^{-r\Delta}(\xi)$ stand for the heat regularized white noise. The family of operators $\Delta + M_{h\xi_r} - \frac{|\log r|}{4\pi}\, h^2$ converges in probability as r goes to 0 to a limit random unbounded self-adjoint operator H which is a quadratic functional of the coupling function h and has a discrete spectrum $\sigma(H)$ tending to $+\infty$. This random operator is called *Anderson operator.* We give in Section 3 a short and self-contained construction of that operator that does not need a fine description of the domain of the Anderson operator to construct it, unlike the previous works [1, 24, 31]. It uses the language of paracontrolled calculus but requires nothing more than the absolute minimum on the subject. Our construction is essentially functional analytic.

We give in Theorem 17 a detailed description of the solution to the parabolic Anderson equation with singular initial conditions, giving back in particular the heat kernel $p_t(x,y)$ of H. Our main point here is that a fine description of $p_t(x,y)$ actually contains a lot of information on the operator H itself. As a direct illustration we recover in Proposition 29 Mouzard's Weyl law for the spectrum of H from a Tauberian point of view. Information on different norms of the eigenfunctions or quasi-modes of H can also be recovered from a good control of the heat semigroup. Denote by $(u_n)_{n\geq 0}$ the sequence of L^2 normalized eigenfunctions of H with corresponding eigenvalues $\lambda_n(\widehat{\xi})$. Recall $\alpha - 2 < -1$ stands for the almost sure Hölder regularity of white noise ξ.

Theorem 1 – *For every $\beta' > 1$ there exists a positive random variable C such that the following two facts hold true almost surely.*

- *One has for all $n \geq 0$ such that $|\lambda_n(\widehat{\xi})| \geq 1$ the n-uniform estimate*

$$\|u_n\|_{C^{2\alpha-1}} \leq C\big|\lambda_n(\widehat{\xi})\big|^{\frac{\beta'}{2}}. \tag{1.2}$$

- *For every $\Lambda \in \mathbb{R}$ and every $u \in \operatorname{span}\big(u_n\,;\,\lambda_n(\widehat{\xi}) \leq \Lambda\big)$ with unit L^2 norm one has*

$$\|u\|_{H^\alpha} \leq C\Lambda^{1/2}.$$

We are able to obtain in Proposition 25 lower and upper Gaussian bounds for $p_t(x,y)$, which imply an interesting parabolic Harnack estimate for $(\partial_t + H)$-harmonic functions. Somewhat independently of the good control on the heat kernel from Theorem 17 we are also able to quantify the spectral gap of H in terms of some isoperimetric constant of the Riemannian manifold $(\mathcal{S}, g)$ generalizing Cheeger's Poincaré inequalities to our setting and also under the assumption that the Riemannian volume form μ satisfies a log-Sobolev inequality – the definitions of the different quantities below will be recalled in Section 4.3. The eigenfunction u_0 – the ground state, is associated with the smallest eigenvalue $\lambda_0(\widehat{\xi})$ of H.

Theorem 2 – *One has the following two almost sure estimates on the spectral gap of H.*

- *Denote by $C(\mathcal{S},g)>0$ the Cheeger constant of the Riemannian manifold $(\mathcal{S},g)$. Then one has the spectral gap estimate*
$$\big|\lambda_0(\widehat{\xi})-\lambda_1(\widehat{\xi})\big|\geq\left(\frac{\min u_0}{\max u_0}\right)^4\frac{C(\mathcal{S},g)^2}{4}>0.$$
- *Assume that the Riemannian volume measure μ satisfies a* log*-Sobolev inequality with constant C_{LS}. Then one has the spectral gap estimate*
$$\big|\lambda_0(\widehat{\xi})-\lambda_1(\widehat{\xi})\big|\geq\left(\frac{\min u_0}{\max u_0}\right)^2\frac{\left(\max u_0^4+\max u_0^{-4}\right)^{-1}}{2\,C_{\mathrm{LS}}}>0.$$

∘ *Anderson Gaussian free field.* We introduce and study the Anderson Gaussian free field in Section 5. This doubly random field ϕ on $\mathcal{S}$ is defined from the L^2 spectral decomposition of the random operator H in the same way as Gaussian free field is defined from the L^2 spectral decomposition of Δ. It thus has two layers of randomness. Like the usual Gaussian free field it is almost surely of regularity 0^-. One can define the Wick square $:\phi^2:$ of ϕ as a doubly random variable; its distribution $\mathcal{L}(:\phi^2:)$ depends on H so it is random. The following result is proved in a more precise form in Theorem 37 and Corollary 38.

Theorem 3 – *The law of the random spectrum of H is characterized by the law of $\mathcal{L}(:\phi^2:)$.*

∘ *The polymer measure.* The polymer measure provides a mathematical model for the random motion of a particle subject to a thermal motion in an extremely disordered potential modeled by white noise. From Feynman-Kac representation formula it is the non-negative measure Q formally defined at a generic point $w\in C([0,1],\mathcal{S})$ by its density
$$\exp\left(\int_0^1\xi(w_t)dt\right)$$
with respect to the Wiener measure $P_{\mathscr{W}}$ on path space over $\mathcal{S}$, up to a multiplicative normalization constant. The pointwise evaluation of the distribution ξ is however meaningless, which motivates a definition of the polymer measure Q as a limit as $r>0$ goes to 0 of the measures $Q^{(r)}$ obtained from a regularized noise ξ_r setting
$$\frac{dQ^{(r)}}{dP_{\mathscr{W}}}(w)\sim\exp\left(\int_0^1\left(\xi_r+\frac{|\log r|}{4\pi}\right)(w_t)dt\right). \tag{1.3}$$
Note that the measures $Q^{(r)}$ and the limit measure Q are random, as the white noise environment is random. (Both $Q^{(r)}$ and Q depend implicitly on the starting point of the path w, that may be fixed or random, possibly independently of the environment.) This measure was first constructed in the flat setting of the two dimensional torus by Cannizzaro & Chouk in [10] using the then newly developed tools of paracontrolled calculus. We give here the first construction of this measure on a closed Riemannian manifold. Our construction is different from that of Cannizzaro & Chouk and we construct the random measure Q as the law of a Markov process with transition probability $e^{-t(H-\lambda_0(\widehat{\xi}))}$. The sharp small time asymptotic that we obtain on the kernel of that operator, or the Gaussian bound proved for that kernel, allow for a straightforward use of Kolmogorov's criterion to construct the polymer measure on a space of Hölder paths. It is singular with respect to Wiener measure on $C([0,1],\mathcal{S})$ although it has support in all the spaces $C^\gamma([0,1],\mathcal{S})$, for $\gamma<1/2$ like Brownian motion. Following a long tradition going back to the work of Symanzik on constructive quantum field theory in the 60's, we relate in Section 6.2 the distribution of the square of the Anderson Gaussian free field and the distribution of the renormalized occupation measure $\mathcal{O}_{1/2}$ of a certain Poisson point process of polymer loops in $\mathcal{S}$. The notations will be defined in Section 6.2.

Theorem 4 – *The renormalized occupation measure $\mathcal{O}_{1/2}$ has the same distribution as the Wick square $:\phi^2:$ of the Anderson Gaussian free field.*

Finally we prove large deviation results for the free end point path and bridge polymer measures, for small traveling time. Given a point $x\in\mathcal{S}$ write Q_x for the polymer measure started from x.

Given $0 < r \leq 1$ and $0 < \gamma < 1/2$, denote by $Q_x^{(r)}$ the law under Q_x on $C^\gamma([0,1], S)$ of the process $(w_{sr})_{0 \leq s \leq 1}$; this is the law of the Markov process with generator $r\big(H - \lambda_0(\widehat{\xi}\,)\big)$ started from x. Given another point $y \in \mathcal{S}$ denote by $Q_{x,y}^{(r)}$ the law of the polymer path conditioned on starting from x and ending up in y at time r, *after linear reparametrization of the time interval* $[0,r]$ *by the fixed interval* $[0,1]$. Set

$$\mathscr{I}(w) := \int_0^1 |\dot{w}_s|_g^2 \, ds \tag{1.4}$$

for $w \in H^1([0,1], \mathcal{S})$, and $\mathscr{I}(w) = \infty$, otherwise. One proves the following large deviation result for the polymer measure and its bridges, where $d(x,y)$ stands for the Riemannian distance between x and y. Recall that $Q_x^{(r)}$ and $Q_{x,y}^{(r)}$ are families of random measures.

Theorem 5 – *Fix two points $x \neq y$ in $\mathcal{S}$ and $0 < \gamma < 1/2$. The following happens almost surely.*

- *The family $\big(Q_x^{(r)}\big)_{0<r\leq 1}$ satisfies in $C^\gamma([0,1], \mathcal{S})$ a large deviation principle with good rate function $\mathscr{I}(\cdot)$.*
- *The family $\big(Q_{x,y}^{(r)}\big)_{0<r\leq 1}$ satisfies in $C^\gamma([0,1], \mathcal{S})$ a large deviation principle with good rate function $\mathscr{I}(\cdot) - d(x,y)^2$.*

So the polymer measure on free and fixed endpoints paths satisfies the same large deviation principle as Wiener measure and the rate function does not see the effect of the white noise potential.

We have organized this work by gathering in Section 2 a number of elementary facts that we use in the remainder of the work. Section 3 provides a short self-contained functional analytic construction of the Anderson operator H. Section 4 provides a fine description of the heat kernel of H and applications to the spectral gap and eigenfunction estimates of H amongst others. Section 5 introduces the Anderson Gaussian free field and studies some of its properties. We relate in particular the distribution of the Wick square Anderson Gaussian free field to the distribution of the spectrum of H. Section 6 deals with the polymer measure, its construction and properties, its link with the Anderson Gaussian free field and the large deviation results for this measure and its bridges. The introduction of each section gives more details on its content. Appendix A contains a proof of a parametric version of meromorphic Fredholm theory. Appendix B gives a number of elements on the geometric Littlewood-Paley decomposition that we use, and Appendix C presents an elementary probabilistic derivation of a Faber-Krahn type lower bound of the spectral radius of the Laplace-Beltrami operator.

Notations. We collect here a number of notations that are used throughout the text.

- *We denote by μ the Riemannian volume measure.*
- *We use the notation $C^\gamma(\mathcal{S})$ for the Hölder spaces, and $H^\gamma(\mathcal{S})$ for the Sobolev sapces, for any $\gamma \in \mathbb{R}$, both defined as Besov spaces over $\mathcal{S}$.*
- *The notation $\mathcal{B}(E,F)$ stands for the space of continuous linear maps from a Banach space E into a Banach space F, with operator norm $\|\cdot\|_{\mathcal{B}(E,F)}$.*
- *For a constant $z \in \mathbb{C}$, we will stick to the usual convention that z stands for the multiplication operator M_z in an identity involving operators.*
- *The notation $O_E(1)$ stands for a bounded E-valued function.*

2 – Tools for the analysis

We will use in the sequel a number of elementary facts on paraproducts and meromorphic Fredholm theory. We recall here what we need from them and refer the reader to [2, 22, 5, 31] for basics and non-basics on paraproduct and resonant operators.

• **Paraproduct, resonant operator and corrector –** Recall from Littlewood-Paley theory that one can decompose an arbitrary distribution f on the d-dimensional torus as a sum of smooth functions

$$f = \sum_{n \geq -1} P_n f$$

approximately localized in frequency space in annuli of size 2^n. This allows to decompose formally the product of two distributions into

$$fg = \sum_{i<j-1} (P_i f)(P_j g) + \sum_{j<i-1} (P_i f)(P_j g) + \sum_{|i-j|\leq 1} (P_i f)(P_j g),$$

with the first two quantities always converging. Based on that model, and set in our 2-dimensional setting, one can decompose the product of any two smooth functions f, g on $\mathcal{S}$ under the form

$$fg = \mathsf{P}_f g + \mathsf{P}_g f + \mathsf{\Pi}(f,g), \tag{2.1}$$

with paraproduct and resonant operators P and $\mathsf{\Pi}$ with the following continuity properties.

– *For any $\alpha_1, \alpha_2 \in \mathbb{R}$ the paraproduct operator*

$$\mathsf{P} : (f,g) \mapsto \mathsf{P}_f g,$$

maps continuously $C^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S})$ into $C^{\alpha_1 \wedge 0 + \alpha_2}(\mathcal{S})$, and the space $C^{\alpha_1}(\mathcal{S}) \times H^{\alpha_2}(\mathcal{S})$ and $H^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S})$ into $H^{\alpha_1 \wedge 0 + \alpha_2}(\mathcal{S})$.

– *The resonant operator*

$$\mathsf{\Pi} : (f,g) \mapsto \mathsf{\Pi}(f,g),$$

is symmetric and well-defined as a continuous operator from $C^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S})$ into $C^{\alpha_1+\alpha_2}(\mathcal{S})$, and from $C^{\alpha_1}(\mathcal{S}) \times H^{\alpha_2}(\mathcal{S})$ into $H^{\alpha_1+\alpha_2}(\mathcal{S})$, iff $\alpha_1 + \alpha_2 > 0$.

Identity (2.1) thus makes sense for all $f \in C^{\alpha_1}(\mathcal{S}), g \in C^{\alpha_2}(\mathcal{S})$, or $f \in C^{\alpha_1}(\mathcal{S}), g \in H^{\alpha_2}(\mathcal{S})$, provided $\alpha_1 + \alpha_2 > 0$. The reader will find more details on these paraproduct and resonant operators in Appendix B. The next fundamental result is the backbone of Gubinelli, Imkeller & Perkowski' seminal work [22] on singular stochastic PDEs. Its extension to a manifold setting was worked out in Bailleul & Bernicot's work [4] in a general parabolic setting – see Mouzard's work [31] for the mixed elliptic Sobolev/Hölder result.

– *The trilinear operator*

$$\mathsf{C}(a,b,c) := \mathsf{\Pi}(\mathsf{P}_a b, c) - a\mathsf{\Pi}(b,c)$$

is continuous from $C^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S}) \times C^{\alpha_3}(\mathcal{S})$ into $C^{\alpha_1+\alpha_2+\alpha_3}(\mathcal{S})$, and from $H^{\alpha_1}(\mathcal{S}) \times C^{\alpha_2}(\mathcal{S}) \times C^{\alpha_3}(\mathcal{S})$ into $H^{\alpha_1+\alpha_2+\alpha_3}(\mathcal{S})$, if $\alpha_2 + \alpha_3 < 0$ and $\alpha_1 + \alpha_2 + \alpha_3 \in (0,1)$.

It is well-known that space white noise takes almost surely its values in the Besov space $B^{\alpha'-2}_{\infty,\infty}(\mathcal{S})$, for any $\alpha' < 1$. The reader can then think of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which it is defined as $\Omega = B^{\alpha'-2}_{\infty,\infty}(\mathcal{S})$, for an ad hoc regularity exponent. Fix

$$0 < 2 - 2\alpha' < \alpha < \alpha' < 1,$$

and let ξ stand for white noise on $\mathcal{S}$. Fix also a smooth real valued 'coupling' function h on $\mathcal{S}$. Denote now by $\sigma(\Delta)$ the spectrum of the Laplace-Beltrami operator Δ. Given $z_0 \notin \sigma(\Delta)$, we will use occasionally the paraproduct-like operator $\overline{\mathsf{P}}$ defined by the intertwining relation

$$(\Delta - z_0)\,\overline{\mathsf{P}}_f g := \mathsf{P}_f\big((\Delta - z_0)g\big).$$

It was proved in Bailleul and Bernicot's work [5] that this operator has the same regularity properties as the operator P. (Strictly speaking, the work [5] deals with the more general parabolic situation; see [31] for the elliptic setting.) This operator $\overline{\mathsf{P}}$ depends on z_0, which will be fixed throughout, so we do not record it in the notation. It was proved in [5] that the (modified) corrector

$$\overline{\mathsf{C}}(a,b,c) := \mathsf{\Pi}(\overline{\mathsf{P}}_a b, c) - a\mathsf{\Pi}(b,c) \tag{2.2}$$

enjoys the same continuity property as C. Set

$$\mathsf{M}^-(f) := \mathsf{P}_f(h\xi), \qquad \mathsf{M}^+(f) := \mathsf{P}_{h\xi} f + \mathsf{\Pi}(f, h\xi). \tag{2.3}$$

While the operator M^- is well-defined and sends continuously $H^\gamma(\mathcal{S})$ into $H^{\gamma\wedge 0+\alpha'-2}(\mathcal{S})$ for any $\gamma\in\mathbb{R}$ the operator M^+ is only defined on the spaces $C^\gamma(\mathcal{S})$ and $H^\gamma(\mathcal{S})$ for $\gamma>2-\alpha'$, due to the resonant operator in the definition of M^+. Set

$$\delta f := f + (\Delta - z_0)^{-1}\mathsf{M}^-(f) = f + \overline{\mathsf{P}}_f(X_h),$$

where

$$X_h := (\Delta - z_0)^{-1}(h\xi).$$

The operator δ is well-defined on all of $\mathcal{D}'(\mathcal{S})$. Pick

$$2-2\alpha' < s < \alpha < \frac{\alpha+1}{2} < \alpha' < 1. \tag{2.4}$$

We single out here an elementary fact whose proof is left to the reader.

Lemma 6 – *For every regularity exponent $\gamma\in\mathbb{R}$ and every positive η there exists a positive constant m_η such that for every real parameter $z_0\le m_\eta$ one has*

$$\big\|(\Delta - z_0)^{-1}\big\|_{\mathcal{B}(H^\gamma(\mathcal{S}),H^{\gamma+2-\eta}(\mathcal{S}))} < \eta,$$

and the continuous map

$$(\Delta - z_0)^{-1}\mathsf{M}^- : H^\gamma(\mathcal{S}) \to H^{\gamma\wedge 0+\alpha}(\mathcal{S})$$

has a norm smaller than 1.

We use the fact that $\xi\in C^{\alpha'-2}(\mathcal{S})$ and $\alpha<\alpha'$, in the proof of the second item of the lemma. It follows that, for every $0<\beta\le\alpha$, the map δ from $H^\beta(\mathcal{S})$ into itself is invertible for z_0 negative and large enough. Taking z_0 even larger if needed, the map δ is also invertible as a map from $C^\beta(\mathcal{S})$ into itself, for all $0<\beta\le\alpha$.

• **Meromorphic Fredholm theory with a parameter** – The analytic Fredholm theory provides conditions under which one can invert a family of Fredholm operators acting on some Hilbert space. Let U be a connected open subset of the complex plane $\mathbb{C}$. Let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be a Hilbert space. Recall that a family $\big(A(z)\big)_{z\in U}$ of linear maps from $\mathcal{H}$ into itself is said to be holomorphic iff the map A is $\mathbb{C}$-differentiable in U. This is equivalent to requiring that the $\mathbb{C}$-valued function $z\mapsto\big\langle y,\big(A(z)x\big)\big\rangle$ is holomorphic for any $x,y\in\mathcal{H}^2$. The family $\big(A(z)\big)_{z\in U}$ is said to be *finitely meromorphic* if for any $z\in U$, there exists a finite collection of operators $(A_j)_{1\le j\le n_0}$ *of finite rank* and a holomorphic family $A_0(\cdot)$, defined near z, such that one has

$$A(z') = A_0(z') + (z'-z)^{-1}A_1 + \cdots + (z'-z)^{-n_0}A_{n_0},$$

near z. We shall need a version with parameters of the meromorphic Fredholm Theorem where $A(z,\mathbf{a})$ depends continuously on a parameter $\mathbf{a}$, element of a metric space.

Theorem 7 – *Let $U\subset\mathbb{C}$ be a connected open subset of the complex plane. Let $(\mathcal{A},d)$ be a metric space and $\big(K(z,\mathbf{a})\big)_{z\in U,\mathbf{a}\in\mathcal{A}}$ be a finitely meromorphic family of compact operators depending continuously on $\mathbf{a}\in\mathcal{A}$. If for every $\mathbf{a}_0\in\mathcal{A}$ the operator $\big(\mathrm{Id}-K(z,\mathbf{a})\big)^{-1}$ exists at some point $z\in U$ for all $\mathbf{a}$ in a neighborhood of $\mathbf{a}_0$ then the family*

$$(z'\in U)\mapsto\big(\mathrm{Id}-K(z',\mathbf{a})\big)^{-1}$$

is a well-defined meromorphic family of operators with poles of finite rank which depends continuously on $\mathbf{a}\in\mathcal{A}$.

A proof of this statement is given in Appendix A. Recall here that a sequence $(h_n)_{n\ge0}$ of Banach space-valued meromorphic functions, defined on a common open subset of $\mathbb{C}$, converge to a limit meromorphic function h if h_n converges uniformly to h on every compact set that does not contain any pole of h.

3 – A construction of the Anderson operator

Let ξ stand for space white noise on the Riemannian manifold $\mathcal{S}$ and let h stand for a smooth real valued function on $\mathcal{S}$. We denote by Δ the Laplace-Beltrami operator associated with the

Riemannian metric on $\mathcal{S}$, and recall that one can construct ξ as a random series $\sum_{n\geq 0}\gamma_n f_n$, where the f_n are the eigenfunctions of the Laplace-Beltrami operator and the γ_n are a family of centered Gaussian random variables with unit variance all independent. We *define* in this section the unbounded operator $H = \Delta + M_{h\xi}$ on $L^2(\mathcal{S})$ by its resolvent map $R(z)$, a meromorphic function of z. We identify R as the unique solution of a fixed point equation. The naive formulation of the fixed point equation involves however a multiplication problem that is the signature of the singular character of the operator H. A renormalization process is needed to make sense of it, that is, we smoothen the noise ξ with the heat kernel $e^{-r\Delta}$ and add r-dependent diverging terms in the operator to make the resolvent associated with this modified operator converge as r tends to 0. The resolvent R is then defined from a renormalized version of a naive fixed point equation using the meromorphic Fredholm theory.

A reader already familiar with one of the previous constructions of the Anderson operator [1, 29, 24, 31] may skip this section and keep in mind that we construct the resolvent of this operator as a meromorphic function defined on all of $\mathbb{C}$. This plays a role in the sequel.

To disentangle the multiplication problem involved in the definition of the operator H and its resolvent, it turns out to be useful to split the multiplication operator $M_{h\xi}$ into

$$M_{h\xi} = \mathsf{M}^- + \mathsf{M}^+,$$

using the operators M^- and M^+ from (2.3). This allows to separate well-defined terms of low regularity from ill-defined terms of a priori better regularity. This approach allows to get around the tricky use of strongly paracontrolled distributions from [1, 24], and to avoid the use of the subtle quasi-duality between paraproduct and resonant operators from [24, 31].

Pick z_0 negative and big enough. We will tune it later to make some ξ-dependent quantities small using Lemma 6.

3.1 Definition and approximation of the resolvent

We first formulate in Section 3.1.1 a fixed point equation for the resolvent that involves an ill-defined term, as expected from the singular nature of the Anderson operator. This analytically ill-defined term only involves the noise and it can be given sense by a renormalization procedure of Wick type described in Proposition 8. This is where the fact that the noise is random is put to work as the renormalized term is constructed by probabilistic means as a random variable. Rewriting in Section 3.1.2 the fixed point equation with the ill-defined term replaced by its well-defined counterpart provides an equation that can be solved uniquely in an appropriate space of meromorphic operator valued functions. The renormalization procedure is interpreted in Section 3.1.3 as giving an r-indexed family of resolvent operators associated with a diverging r-indexed family of operators.

3.1.1 – The naive fixed point equation for the resolvent. One has at a formal level

$$\begin{aligned} R(z) = \big(\Delta + M_\xi - z\big)^{-1} &= \Big(\mathrm{Id} + (\Delta - z_0)^{-1}\mathsf{M}^- + (\Delta - z_0)^{-1}(\mathsf{M}^+ - z + z_0)\Big)^{-1}(\Delta - z_0)^{-1} \\ &= \Big(\mathrm{Id} + \delta^{-1}(\Delta - z_0)^{-1}\big(\mathsf{M}^+ - z + z_0\big)\Big)^{-1}\delta^{-1}(\Delta - z_0)^{-1} \\ &= \delta^{-1}(\Delta - z_0)^{-1} - R(z)\left(\mathsf{M}^+ - z + z_0\right)\delta^{-1}(\Delta - z_0)^{-1}. \end{aligned}$$

This is the raw version of the fixed point equation that should define $R(z)$. Recall from (2.4) the constraints on the exponents s, α and α'. We spot a problem in the term $\mathsf{M}^+\delta^{-1}$ as the resonant term in

$$\mathsf{M}^+(\delta^{-1}u) = \mathsf{P}_\xi(\delta^{-1}u) + \mathsf{\Pi}\big(\delta^{-1}u, h\xi\big),$$

is not well-defined as δ^{-1} takes its values at best in $C^{\alpha'}(\mathcal{S})$ and $\alpha' + (\alpha' - 2) < 0$. The identity

$$\delta^{-1} = \mathrm{Id} - (\Delta - z_0)^{-1}\mathsf{M}^-\delta^{-1}$$

allows to rewrite the fixed point equation for $R(z)$ as

$$R(z) = \delta^{-1}(\Delta - z_0)^{-1} - R(z)\left(\mathsf{M}^+ - \mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-\delta^{-1} - (z - z_0)\delta^{-1}\right)(\Delta - z_0)^{-1}, \quad (3.1)$$

and to isolate precisely the problem in the expression $\mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-$, that is in the resonant term

$$\Pi\Big(\overline{\mathsf{P}}_u X_h, h\xi\Big), \qquad u \in H^s(\mathcal{S}) \tag{3.2}$$

that comes from the M^+ operator. Using the corrector $\overline{\mathsf{C}}$ from (2.2) one has

$$\mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-(u) = \mathsf{P}_{h\xi}(\overline{\mathsf{P}}_u X_h) + u\,\Pi(X_h, h\xi) + \overline{\mathsf{C}}(u, X_h, h\xi).$$

We isolate in the u-independent, noise-dependent, term $\Pi(X_h, h\xi)$ the only ill-defined term – the sum of the regularity exponents of X_h and $h\xi$ add up to a negative constant. An elementary renormalization process allows however to give a proper meaning to such a term. Set

$$\xi_r := e^{-r\Delta}$$

for the heat regularized space white noise and

$$X_{h,r} := (\Delta - z_0)^{-1}(h\xi_r).$$

The next statement identifies the singular part of the diverging resonant term

$$\Pi\big(X_{h,r}, h\xi_r\big).$$

Proposition 8 – *We have the exact expression*

$$\mathbb{E}\big[\Pi(X_{h,r}, h\xi_r)\big] = \frac{|\log r\,|h^2}{4\pi} + O_{\mathcal{C}^{2\alpha'-2}}(1) \tag{3.3}$$

and the random variables

$$\Pi\big(X_{h,r}, h\xi_r\big) - \frac{|\log r\,|h^2}{4\pi}$$

converge in probability in the space $\mathcal{C}^{2\alpha'-2}$, as $\varepsilon > 0$ goes to 0, to a limit random variable denoted by

$$\mathscr{R}\big\{\Pi(X_h, h\xi)\big\}.$$

Moreover $\mathscr{R}\big\{\Pi(X_h, h\xi)\big\}$, which implicitly depends on z_0, goes to 0 in probability in the space $\mathcal{C}^{2\alpha'-2}(\mathcal{S})$ as $z_0 < 0$ diverges to $-\infty$.

The letter '$\mathscr{R}$' is chosen for '*renormalized*'. Identity (3.3) improves upon the corresponding statement in [31] by showing that the singular part of the resonance is a constant when $h \equiv 1$, rather than a function. A similar fact was proved in the closely related work [13] of Dahlqvist, Diehl & Driver on the parabolic Anderson model equation in a closed two dimensional Riemannian manifold. (They developed for their purpose a first order version of regularity structures in that setting, rather than using paracontrolled calculus.) Identity (3.3) improves upon [13] by showing that the singular part is a local functional of the coupling function h, in the sense that for any test functions $(h_1, h_2) \in C^\infty(S)$ with disjoint supports we have

$$\delta_{h_1}\delta_{h_2}\left(\frac{|\log r\,|\,h_1h_2}{4\pi}\right) = 0. \tag{3.4}$$

(We denote here by δ_h the functional derivative with respect to h.) As a matter of fact we already have here

$$\frac{|\log r\,|\,h_1h_2}{4\pi} = 0$$

if h_1 and h_2 have disjoint supports. The proof of Proposition 8 follows the usual pattern for similar Wick renormalization proofs; it is given in Appendix B.

3.1.2 – The renormalized fixed point equation for the resolvent. We define the renormalized version of the operator $\mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-$ setting for all $u \in H^s(\mathcal{S})$

$$\mathscr{R}\big\{\mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-\big\}(u) := \mathsf{P}_\xi\big((\Delta - z_0)^{-1}\mathsf{M}^- u\big) + u\,\mathscr{R}\big\{\Pi(X_h, h\xi)\big\} + \overline{\mathsf{C}}(u, X_h, h\xi).$$

The assumptions (2.4) on the regularity exponents guarantee that the operator $\mathscr{R}\big\{\mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-\big\}$ is linear continuous from $L^2(\mathcal{S})$ into $H^{2\alpha-2}(\mathcal{S})$. The renormalized counterpart of the fixed point equation (3.1) for $R(z)$ reads

$$R(z) = \delta^{-1}(\Delta - z_0)^{-1} - R(z)\Big(\mathsf{M}^+ - \mathscr{R}\big\{\mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-\big\}\delta^{-1} - (z - z_0)\delta^{-1}\Big)(\Delta - z_0)^{-1},$$

that is

$$R(z)\left\{\mathrm{Id}+\Big(\mathsf{M}^{+}-\mathscr{R}\{\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\}\delta^{-1}-(z-z_0)\delta^{-1}\Big)(\Delta-z_0)^{-1}\right\}=\delta^{-1}(\Delta-z_0)^{-1} \quad (3.5)$$

Choosing $z_0<0$ random and big enough ensures with Lemma 6 the bound

$$\left\|\Big(\mathsf{M}^{+}-\mathcal{R}\left(\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\right)\delta^{-1}\Big)(\Delta-z_0)^{-1}\right\|_{\mathcal{B}(E,H^{2\alpha-2}(\mathcal{S}))}<1, \quad (3.6)$$

with $E=L^2(\mathcal{S})$ or $H^{2\alpha-2}(\mathcal{S})$. One notes further that the operator in the preceding inequality is *compact* in $\mathcal{B}(H^{2\alpha-2}(\mathcal{S}),H^{2\alpha-2}(\mathcal{S}))$ as it actually maps $H^{2\alpha-2}(\mathcal{S})$ into $H^{2\alpha'-2}(\mathcal{S})$, and $\alpha<\alpha'$. Equation (3.5) then defines a map

$$R(z_0)=\delta^{-1}(\Delta-z_0)^{-1}\left\{\mathrm{Id}+\Big(\mathsf{M}^{+}-\mathscr{R}\{\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\}\delta^{-1}\Big)(\Delta-z_0)^{-1}\right\}^{-1}, \quad (3.7)$$

and the meromorphic Fredholm theory applied to the holomorphic family of compact operators acting on $H^{2\alpha-2}(\mathcal{S})$

$$\mathrm{Id}+\left(\mathsf{M}^{+}-\mathscr{R}\{\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\delta^{-1}\}-(z-z_0)\delta^{-1}\right)(\Delta-z_0)^{-1}\in\mathcal{B}\big(H^{2\alpha-2}(\mathcal{S}),H^{2\alpha-2}(\mathcal{S})\big)$$

allows to define

$$R(z)=\delta^{-1}(\Delta-z_0)^{-1}\left\{\mathrm{Id}+\left(\mathsf{M}^{+}-\mathscr{R}\Big\{\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\Big\}\delta^{-1}-(z-z_0)\,\delta^{-1}\right)(\Delta-z_0)^{-1}\right\}^{-1}$$

as a meromorphic function of $z\in\mathbb{C}$ with values in $\mathcal{B}\big(H^{2\alpha-2}(\mathcal{S}),\delta^{-1}\big(H^{2\alpha}(\mathcal{S})\big)\big)$. Since $H^{2\alpha}(\mathcal{S})$ is continuously embedded into $C^{2\alpha-1}(\mathcal{S})$, the restriction of R to $L^2(\mathcal{S})$ defines a meromorphic function with values in $\mathcal{B}\big(L^2(\mathcal{S}),C^{2\alpha-1}(\mathcal{S})\big)$. We invite the reader to check that the assumptions of Theorem 7 on meromorphic Fredholm theory with a parameter are met, with $\widehat{\xi}\in C^{\alpha'-2}(\mathcal{S})\times C^{2\alpha'-2}(\mathcal{S})$ in the role of the parameter. The meromorphic operators $R(\cdot)$ are thus continuous functions of $\widehat{\xi}$.

3.1.3 – The regularized renormalized fixed point equation. The convergence result of Proposition 8 and the fixed point equation giving the meromorphic function $R(\cdot)$ can be put together to provide approximations of $R(z)$ by the resolvent of bounded operators. It is convenient for that purpose to use Skorohod representation theorem for weak convergence (hence convergence in probability) and assume that the convergence in Proposition 8 is almost sure. This can be done by a change of probability space Ω on which white noise is defined – see e.g. Theorem 4.30 in Kallenberg's book [26] for Skorohod theorem. Denote by Ω_1 the measurable subset of Ω of probability 1 where the almost sure convergence holds. Since we are only interested in almost sure statements what happens on the null set $\Omega\backslash\Omega_1$ is irrelevant.

Given a positive regularization parameter r, set

$$\delta_r(f):=f+(\Delta-z_0)^{-1}\mathsf{P}_f(h\xi_r)$$

and

$$\mathsf{M}_r^{-}(f):=\mathsf{P}_f(h\xi_r),\qquad \mathsf{M}_r^{+}(f):=\mathsf{P}_{h\xi_r}f+\mathsf{\Pi}(f,h\xi_r).$$

One proves the following statement in Appendix B.

Lemma 9 – *For $r>0$ the operator M_r^{-} is a smoothing operator and the operator M_r^{+} is a pseudodifferential operator of order* 0.

The operator δ_r^{-1} is also a pseudo-differential operator of order 0. Denote here by

$$c_{h,r}:=\frac{|\log r|\,h^2}{4\pi}$$

the diverging part of $\mathsf{\Pi}(X_{h,r},h\xi_r)$ – this is a function on $\mathcal{S}$ whose associated multiplication operator is denoted by $M_{c_{h,r}}$. Set

$$\mathscr{R}_r\big\{\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\big\}(u):=\mathsf{M}_r^{+}(\Delta-z_0)^{-1}\mathsf{M}_r^{-}u-c_{h,r}u.$$

The convergence result from Proposition 8 implies that the map $\mathscr{R}_r\big\{\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\big\}$ is converging to the map $\mathscr{R}\big\{\mathsf{M}^{+}(\Delta-z_0)^{-1}\mathsf{M}^{-}\big\}$ in $\mathcal{B}\big(L^2(\mathcal{S}),H^{2\alpha-2}(\mathcal{S})\big)$, for all chance elements $\omega\in\Omega_1$.

It follows that for all $\omega \in \Omega_1$, the (ω-dependent) operators

$$R_r(z) := \delta_r^{-1}(\Delta - z_0)^{-1}\left\{\mathrm{Id} + \left(\mathsf{M}^+ - \mathscr{R}_r\left\{\mathsf{M}^+(\Delta - z_0)^{-1}\mathsf{M}^-\right\}\delta_r^{-1} - (z - z_0)\,\delta_r^{-1}\right)(\Delta - z_0)^{-1}\right\}^{-1}$$

converge as r goes to 0 to the (ω-dependent) operator $R(z)$ in $\mathcal{B}\big(L^2(\mathcal{S}), C^{2\alpha-1}(\mathcal{S})\big)$, as a meromorphic function of z by the analytic Fredholm theory. Rewinding the algebraic process that led to this expression of $R_r(z)$ requires the use of the following elementary statement, whose proof is given in Appendix B.

Lemma 10 – *Pick $a \in \mathbb{R}$. Let P be an invertible elliptic pseudo-differential operator of order a and $Q(z)$ a pseudo-differential operator of positive order b, depending holomorphically on $z \in \mathbb{C}$. If there exists z_0 such that $\big(\mathrm{Id} + P^{-1}Q(z_0)\big)$ and $\big(\mathrm{Id} + Q(z_0)P^{-1}\big)$ are invertible from $H^a(\mathcal{S})$ into itself then we have*

$$\big(P + Q(z)\big)^{-1} = \big(\mathrm{Id} + P^{-1}Q(z)\big)^{-1}P^{-1} = P^{-1}\big(\mathrm{Id} + Q(z)P^{-1}\big)^{-1} \tag{3.8}$$

for all $z \in \mathbb{C}$, where both sides of each equality are Fredholm operators from $H^a(\mathcal{S})$ into itself depending meromorphically on $z \in \mathbb{C}$.

Lemma 9 and Lemma 10 justify that we write

$$\begin{aligned}
&\Big(\Delta - z_0 + \mathsf{M}_r^-\Big)^{-1}\Big\{\mathrm{Id} + \Big(\mathsf{M}_r^+ - \Big(\mathsf{M}_r^+(\Delta - z_0)^{-1}\mathsf{M}_r^- - M_{c_{h,r}}\Big)\,\delta_r^{-1} - (z - z_0)\,\delta_r^{-1}\Big)(\Delta - z_0)^{-1}\Big\}^{-1} \\
&= \Big(\Delta - z_0 + \mathsf{M}^-\Big)^{-1} \circ \\
&\qquad \Big\{\mathrm{Id} + \mathsf{M}_r^+(\Delta - z_0)^{-1} - \Big(\mathsf{M}_r^+(\Delta - z_0)^{-1}\mathsf{M}_r^- - M_{c_{h,r}} - (z - z_0)\Big)(\Delta - z_0 + \mathsf{M}_r^-)^{-1}\Big\}^{-1} \\
&= \Big\{\Delta - z_0 + \mathsf{M}_r^- + \mathsf{M}_r^+(\Delta - z_0)^{-1}\Big(\Delta - z_0 + \mathsf{M}_\varepsilon^-\Big) - \Big(\mathsf{M}_r^+(\Delta - z_0)^{-1}\mathsf{M}_r^- - M_{c_{h,r}} - (z - z_0)\Big)\Big\}^{-1} \\
&= \Big(\Delta - z + \mathsf{M}_r^- + \mathsf{M}_r^+ + M_{c_{h,r}}\Big)^{-1} = \Big(\Delta - z + M_{h\xi_r} + M_{c_{h,r}}\Big)^{-1},
\end{aligned}$$

by the usual composition in the pseudo-differential calculus. So $R_r(z)$ is the resolvent of the operator $\Delta + M_{h\xi_r + c_{h,r}}$, perturbation of the Laplace-Beltrami operator Δ by the r-diverging smooth potential $h\xi_r + c_{h,r}$.

Proposition 11 – *The meromorphic maps $R_r(\cdot)$, with values in $\mathcal{B}\big(L^2(\mathcal{S}), C^{2\alpha-1}(\mathcal{S})\big)$, converge to the meromorphic map $R(\cdot)$ as $r > 0$ goes to 0, and $R(\cdot)$ has real poles in a half-plane $\big\{\mathrm{Re}(z) > m\big\}$, for m negative large enough.*

Proof – The R_r have real poles as the potentials ξ_r and $c_{h,r}$ are real valued. The poles of R are limits of the poles of R_r. We see from (3.6) and (3.7) that R has no poles in the half-place $\big\{\mathrm{Re}(z) \leqslant m\big\}$, for m negative large enough. ▷

We used Skorohod representation theorem to represent a convergence in probability as an almost sure convergence on a different probability space. The reader should keep in mind that the resolvent of the regularized and renormalized operator $\Delta + h\xi_r - \frac{\ln r}{4\pi}h^2$ is only converging *in probability* to a limit resolvent.

3.2 Construction of the operator H

We can construct an operator associated with the map R.

Theorem 12 – *The map R is the resolvent of a closed unbounded self-adjoint operator H on $L^2(\mathcal{S})$ with real discrete spectrum bounded below.*

Proof – Pick a real number z_1 which is not a pole of the limit family $R(\cdot)$. For $r_0 > 0$ small enough, z_1 is not a pole of the resolvent $R_r(\cdot)$ for all $r \in [0, r_0]$, so $R(z_1)$ is the limit in operator norms of the family $R_r(z_1)$ of self-adjoint operators acting on $L^2(\mathcal{S})$, as r goes to 0. This implies that $R(z_1)$ itself is compact self-adjoint as an operator on $L^2(\mathcal{S})$. Denote by

$$\sigma\big(R(z_1)\big) = \big\{(\lambda_n - z_1)^{-1}\big\}_{n\geq 0} \subset \mathbb{R}$$

its spectrum, with $\lambda_n \leq \lambda_{n+1}$ for all n, and by $(u_n)_{n\geq 0}$ its eigenvalues – they form an orthonormal system of L^2. Also the meromorphic family of operators $R(z)$ satisfies the resolvent identity

$$R(z) = R(z_1)\big(\mathrm{Id} + (z - z_1)R(z_1)\big)^{-1}, \tag{3.9}$$

for any z that is not a pole of $R(\cdot)$, where the term $\big(\mathrm{Id}+(z-z_1)R(z_1)\big)^{-1}$ exists by meromorphic Fredholm theory in $\mathcal{B}\big(L^2(\mathcal{S}), L^2(\mathcal{S})\big)$ relying on the compactness of $R(z_1) \in \mathcal{B}\big(L^2(\mathcal{S}), H^{2\alpha}(\mathcal{S})\big)$. (This identity is obtained by passing to the limit in the corresponding identity satisfied by R_r using the convergence of R_r to R.) The resolvent identity (3.9) implies that the range of $R(z_1)$ does not depend on z_1. Define the z-independent vector space

$$\mathfrak{D}(H) := R(z)\big(L^2(\mathcal{S})\big).$$

By the resolvent equation (3.9), the meromorphic family of operators $R(\cdot)$ has poles contained in $(\lambda_n)_{n\geq 0}$ and satisfies for all $n \geq 0$ the eigenvalue equation

$$R(z)u_n = (z - \lambda_n)^{-1}u_n.$$

This implies that we can define an unbounded operator $H - z$ on $L^2(\mathcal{S})$, with domain $\mathfrak{D}(H)$, in such a way that $(H - z)R(z)$ is the identity map on $L^2(\mathcal{S})$.

The spectrum of H is bounded below since its resolvent $R(\cdot)$ has no poles in the half-plane $\{\mathrm{Re}(z) \leqslant m\}$, for m negative large enough. Last the operator $H : \mathfrak{D}(H) \subset L^2(\mathcal{S}) \mapsto L^2(\mathcal{S})$ is self–adjoint, hence closed since $\mathfrak{D}(H) = R(z)\big(L^2(\mathcal{S})\big)$, and $(H-z)R(z) = textrmId : L^2(\mathcal{S}) \mapsto L^2(\mathcal{S})$ and $R(z_1)$ is bounded self–adjoint. $\rhd$

Remarks – 1. *Since*

$$\big(\Delta + M_{h\xi_r} + M_{c_{h,r}}\big)R_r$$

is the identity map on $L^2(\mathcal{S})$, and R is the limit of the R_r, one can think of H as the limit of the operators $\Delta + M_{h\xi_r} + M_{c_{h,r}}$.

2. *One has*

$$\mathfrak{D}(H) = \mathrm{Im}\big(R(z_1)\big) \subset C^{2\alpha-1}(\mathcal{S}),$$

with elements $f \in \mathfrak{D}(H)$ such that $f + \overline{\mathsf{P}}_f X_h \in H^{2\alpha}(\mathcal{S})$. This property of elements in the domain of H was the starting point of the constructions of the Anderson operator in [1, 24, 31]*. A regularity structures picture is given in* [29]*. (Note that we learn from the explicit description of $\mathfrak{D}(H)$ in* [31] *that the domain of H is not an algebra.) The operator H and its domain are the objects of primary interest in these works and one has first to 'guess' the domain and check its density in an appropriate space before proving a number of functional inequalities satisfied by H. A fixed point argument is used in* [1, 29] *to construct the inverse of $H + c$, for c positive and big enough, while the Babuska-Lax-Milgram theorem is used as a substitute in* [24, 31]*. The interest of working with the meromorphic resolvent on the entire complex plane will appear for instance in the functional analytic proof of Proposition 15 on the continuous dependence of the spectral data of H on the enhanced noise $\widehat{\xi}$.*

It follows from the spectral theorem for unbounded self-adjoint operators with compact resolvent that one has the following spectral representation of the heat kernel of H

$$e^{-tH} = \sum_{n\geq 0} e^{-t\lambda_n}\, u_n \otimes u_n.$$

We emphasize the dependence of the eigenvalues λ_n of H on $\widehat{\xi}$ by writing $\lambda_n(\widehat{\xi})$. We will see in Proposition 15 below that the eigenvalues and their associated eigen-projectors are continuous functions of the enhanced noise $\widehat{\xi}$.

4 – Heat operator for the Anderson operator

The main result of this section, Theorem 17, provides a sharp asymptotic Gaussian estimate for the Schwartz kernel $p_t(x,y)$ of e^{-tH}. The existence, regularity and strict positivity of p_t are proved in Section 4.1, with a number of consequences. The sharp asymptotic of p_t obtained in Section 4.2 gives a direct access in Section 4.3 to a proof of Weyl's law for the distribution of the eigenvalues of H and different estimates for its eigenfunctions. We also prove in that section some Gaussian upper and lower bounds on $p_t(x,y)$ and give almost sure lower bounds on the spectral gap of H under different kinds of geometric assumptions on $(\mathcal{S},g)$.

4.1 Heat kernel and properties of H

It is elementary to get qualitative informations on the Schwartz kernel of the heat operator of H. Thinking of α as 1^- the regularity exponent $(2\alpha-1)$ that appears in the next statement is also of the form 1^-.

Proposition 13 – *The heat semigroup e^{-tH} of the Anderson Hamiltonian H has a positive kernel $p_t(x,y)$ with respect to the Riemannian volume measure on $\mathcal{S}$ that is a continuous function of all its arguments that is an $(2\alpha-1)$-Hölder function of its arguments (x,y) on any finite time interval $[t_0,t_1]$, for $0<t_0\leq t_1<\infty$.*

Proof – *Existence of the heat kernel.* We follow the classical approach, as exposed for instance in Section 5.2 of Davies' textbook [14]. Recall that the graph norm of H on its domain $\mathfrak{D}(H)$ is defined by

$$\|u\|_H^2 := \|u\|_{L^2}^2 + \|Hu\|_{L^2}^2,$$

and that it turns $\mathfrak{D}(H)$ into a Hilbert space. We note first that for $f\in L^2(\mathcal{S})$, the element $e^{-tH}f$ belongs to the domain $\mathfrak{D}(H)$ of H, for all $t>0$, by the spectral theorem, so $\big(e^{-tH}f\big)(x)$ is a $(2\alpha-1)$-Hölder function of $x\in\mathcal{S}$ for each $t>0$. Since $t\mapsto e^{-tH}f$, is a continuous function of t on the half plane $\{\mathrm{Re}(t)>0\}$, with values in the Hilbert space $\big(\mathfrak{D}(H),\|\cdot\|_H\big)$, we have that $(t,x)\mapsto\big(e^{-tH}f\big)(x)$, is a continuous function on $[t_0,t_1]\times\mathcal{S}$, for each compact interval $[t_0,t_1]\subset(0,\infty)$. As the linear form $f\mapsto\big(e^{-tH}f\big)(x)$, is bounded on $L^2(\mathcal{S})$ for each $t>0$ and $x\in\mathcal{S}$, there exists $a(t,x)\in L^2(\mathcal{S})$ such that

$$\big(e^{-tH}f\big)(x) = \big\langle f, a(t,x)\big\rangle_{L^2}.$$

The map

$$\Big((t,x)\in(0,1]\times\mathcal{S}\Big)\mapsto a(t,x)\in L^2(\mathcal{S}),$$

being weakly continuous is norm continuous – a consequence of the uniform boundedness principle. We then have for all test functions $h_1,h_2\in C^\infty(\mathcal{S})$

$$\begin{aligned}\big\langle e^{-tH}h_1,h_2\big\rangle_{L^2} &= \Big\langle e^{-Ht/2}h_1, e^{-Ht/2}h_2\Big\rangle_{L^2}\\ &= \int p_t(x,y)h_1(x)h_2(y)\,dxdy\end{aligned}$$

with

$$p_t(x,y) := \big\langle a(t/2,x), a(t/2,y)\big\rangle_{L^2}$$

a continuous function of its arguments. One gets the $(2\alpha-1)$-Hölder regularity of $p_t(x,y)$ as a function of x, for t,y fixed, noting that since the map $(x\in\mathcal{S})\mapsto a(t,x)\in L^2(\mathcal{S})$ is weakly $(2\alpha-1)$-Hölder continuous it is also norm $(2\alpha-1)$-Hölder continuous – here again a consequence of the uniform boundedness principle. The joint regularity of $p_t(x,y)$ as a function of (x,y) follows, for $0<t_0\leq t\leq t_1<\infty$.

Positivity. The fact that $p_t(x,y)$ is non-negative and non-null comes from the fact that e^{-tH} is the strong limit in operator norm of the semigroup e^{-tH_r} of the renormalised Hamiltonian H_r. The non-negativity of the approximating operators e^{-tH_r} is straightforward from their Feynman-Kac representation. One proves that $p_t(\cdot,\cdot)$ is positive for all positive times t using the strong maximum principle as Cannizzaro, Friz & Gassiat in their proof of Theorem 5.1

in [11]. Note that their proof works only for a continuous initial condition while we need the result for any initial condition in $L^2(\mathcal{S})$. We conclude from the fact that e^{-tH} sends continuously $L^2(\mathcal{S})$ into $\mathfrak{D}(H) \subset C(\mathcal{S})$ for any $t > 0$, using their result after an arbitrary positive time. We let the reader check that their proof works verbatim in our manifold setting as it only uses a crude estimate on the heat kernel of the Laplace operator that holds in our Riemannian manifold setting as well. $\rhd$

We note here that Dahlqvist, Diehl and Driver only considered in [13] the parabolic Anderson model equation *with smooth initial condition*, so their results do not provide any insight on the heat kernel of the Anderson operator. A reader who has seen the parabolic paracontrolled structure used to solve the parabolic Anderson model equation may be puzzled by the fact that $e^{-tH}f$ is in the domain of H for any $f \in L^2(\mathcal{S})$ at positive times t, while it is essentially given by a seemingly different structure $(\partial_t + \Delta)^{-1}(\mathsf{P}_u\xi)$, for some u, up to a remainder term. Commuting the paraproduct and the resolution operator $(\partial_t + \Delta)^{-1}$ produces a remainder term, so the elliptic paracontrolled structure pops out from the parabolic structure as a consequence of the identity

$$(\partial_t + \Delta)^{-1}(\xi)(t) = \int_0^t e^{-(t-s)\Delta}\xi\, ds = \int_0^t e^{-r\Delta}\xi\, dr = \Delta^{-1}\xi - \int_t^\infty e^{-r\Delta}\xi\, dr. \tag{4.1}$$

We take profit here from the fact that the noise ξ is time-independent and the integral over (t, ∞) is a smooth remainder term when $t > 0$.

The next statement follows from the positivity of the heat kernel of H and the Krein-Rutman theorem [38, Thm A.1 p. 123].

Corollary 14 – *Almost surely the lowest eigenvalue $\lambda_0(\widehat{\xi})$ of H is simple with a positive eigenvector.*

(Note that this question was also considered in Chouk & van Zuijlen's work [12], however their proof seems incomplete since they used Cannizzaro, Friz & Gassiat' strong maximum principle [11] which requires a continuous initial condition rather than an arbitrary initial condition in $L^2(\mathcal{S})$. Proceeding as in the 'Positivity' paragraph of the proof of Proposition 13 fixes that point.) We now state another corollary of Proposition 13 that will be important for us later. It makes a crucial use of our construction in Section 3 of the resolvent of H as a meromorphic function defined on all of $\mathbb{C}$.

Proposition 15 – *The eigenvalues and their associated spectral projectors in $L^2(\mathcal{S})$ are continuous functions of $\widehat{\xi}$. The spectral projectors are further continuous functions of $\widehat{\xi}$ as elements of $\mathcal{B}\big(L^2(S), \mathcal{C}^{2\alpha-1}(S)\big)$.*

We have in particular that the ground state $u_{0,r}$ of H_r is converging in $\mathcal{C}^{2\alpha-1}(\mathcal{S})$ to the ground state u_0 of H as r goes to 0. We note before giving the proof of Proposition 15 that the continuity of $\lambda_n(\widehat{\xi})$ as a function of $\widehat{\xi}$ was already proved in Allez & Chouk' seminal work [1] in their setting. The continuity of the spectral projectors was somehow proved by Labbé in Theorem 1 of [29].

Proof – Pick an eigenvalue λ of H and a small disc D around λ with intersection with $\sigma(H)$ equal to $\{\lambda\}$. Since the regularized and renormalized resolvent R_r converges to R in the sense of Fredhom analytic operators and $R(z)$ is invertible for $z \in \partial D$, we know that for r small enough, the operator $R_r(z)$ is well-defined and invertible for $z \in \partial D$. Moreover it follows from the uniform convergence of $R_r(z)$ to $R(z)$ on ∂D that the family of spectral projectors

$$\Pi_r^D := \frac{i}{2\pi} \int_{\partial D} R_r(z)\, dz$$

is well-defined for $r > 0$ small enough and converges in $\mathcal{B}\big(L^2(S), H^{2\alpha-1}(\mathcal{S})\big)$, so the limit operator reads

$$\Pi_\lambda := \frac{i}{2\pi} \int_\gamma R(z)dz : L^2(\mathcal{S}) \mapsto H^{2\alpha-1}(\mathcal{S}).$$

We know from Rouché's Theorem [15, Thm C.12] applied to the operator valued meromorphic function $\big(\mathrm{Id} + (z - z_1)R_r(z_1)\big)^{-1}$, $z_1 \notin \mathbb{R}$, (this meromorphic Fredholm operator has same poles with multiplicity as $R(z)$) that $\sigma(H_r) \cap D$ has fixed multiplicity for r small enough since the poles of R_r and R contained in the disc D have the same multiplicity. Furthermore, as Π_r^D is a self-adjoint spectral projector, one has $\Pi_r^D \circ \Pi_r^D = \Pi_r^D$. It follows that $\Pi_\lambda^2 = \Pi_\lambda$ and Π_λ is a self-adjoint projector such that one has for any $n \geq 0$

$$\Pi_\lambda u_n = \frac{i}{2\pi}\int_{\partial D} R(z)u_n dz = \frac{i}{2\pi}\int_{\partial D} \big(\lambda_n(\widehat{\xi}) - z\big)^{-1} u_n dz = \delta_\lambda^{\lambda_n(\widehat{\xi})} u_n.$$

This implies that Π_λ acts as the identity when restricted on the eigenspace of λ and vanishes on all eigenfunctions u_n of eigenvalue $\lambda_n(\widehat{\xi}) \neq \lambda$. By continuity of $\Pi_\lambda \in \mathcal{B}(L^2(\mathcal{S}), L^2(\mathcal{S}))$ this implies that Π_λ vanishes on the orthogonal of the eigenspace of λ hence Π_λ is the orthogonal projector on the eigenspace of λ.

As a consequence of this discussion $\lambda_0(\widehat{\xi_r})$ and $\lambda_1(\widehat{\xi_r})$ are both converging to $\lambda_0(\widehat{\xi})$ and $\lambda_1(\widehat{\xi})$. By construction the lowest eigenvalues $\lambda_0(\widehat{\xi_r})$ are simple for all r small enough, including $r = 0$. Using the regularizing property of the operators e^{-H_r} and e^{-H} stated in Proposition 13, and the convergence of the kernel of e^{-H_r} to the kernel of e^{-H} in the space $\mathcal{B}(L^2(S), \mathcal{C}^{2\alpha-1}(S))$ that we will later prove below in Section 4.2, we see that if one picks a small disc $D_0(\widehat{\xi})$ with center $\lambda_0(\widehat{\xi})$ so that $D_0(\widehat{\xi}) \cap \sigma(H) = \{\lambda_0(\widehat{\xi})\}$, one has the convergence of $\Pi_r^0 = e^{\lambda_0(\widehat{\xi}_r)} e^{H_r} \Pi_r^{D_0(\widehat{\xi})}$ to $\Pi_{\lambda_0(\widehat{\xi})} = e^{\lambda_0(\widehat{\xi})} e^H \Pi_{\lambda_0(\widehat{\xi})}$ in $\mathcal{B}\big(L^2(\mathcal{S}), \mathcal{C}^{2\alpha-1}(\mathcal{S})\big)$. ▷

The image νe^{-tH} by e^{-tH} of a Borel finite measure ν on $\mathcal{S}$ has density

$$\int_{\mathcal{S}} p_t(x, \cdot)\nu(dx)$$

with respect to the Riemannian volume measure on $\mathcal{S}$. One says that ν is invariant by the semigroup $(e^{-tH})_{t>0}$ if $\nu e^{-tH} = \nu$, for all $t > 0$.

Corollary 16 – *Each random variable $\lambda_n(\widehat{\xi})$ has a law that is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, with a positive density. So the kernel of H is almost surely trivial and the semigroup $(e^{-tH})_{t>0}$ has no invariant Borel probability measure.*

Proof – The first point comes from Proposition 15 and the fact that the laws of ξ and $\xi + c$ are equivalent for all constants c.

Since the unbounded operator H is symmetric in $L^2(\mathcal{S})$, the heat kernel of H is a symmetric function of its space arguments. So a Borel invariant probability measure has a non-negative density with respect to the Riemannian volume measure, which is in the domain of H and in its kernel. Conversely, a non-null element of the kernel of H defines an invariant Borel signed measure.

The previous absolute continuity result implies that any eigenvalue of H has null probability of being null. Recall from Section 3.2 that we denote by u_n the eigenvectors of H; they form an orthonormal system of $L^2(\mathcal{S})$. An element

$$f = \sum_{n\geq 0} c_n u_n$$

of $L^2(\mathcal{S})$ such that $e^{-tH}f = f$ satisfies then

$$e^{-t\lambda_n(\widehat{\xi})} c_n = c_n$$

for all $n \geq 0$. Since all the $\lambda_n(\widehat{\xi})$ are almost surely different from 0, this can happen only if $c_n = 0$ for all n, that is if $f = 0$. ▷

It is not clear however that tuples of k eigenvalues have a law that is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^k$.

4.2 A sharp asymptotic for the heat kernel of Anderson operator

The qualitative estimate on the heat kernel p of H provided by Proposition 13 is not sufficient for our needs, which are quantitative. Fix a finite positive time horizon T. Let $(E, |\cdot|)$ be a Banach space. For $\gamma < 0$ and a regularity exponent $\beta \in (0,1)$ set

$$t^\gamma C^\beta\big((0,T],E\big) := \left\{ v \in C\big((0,T],E\big)\,;\ \sup_{0<s\leq t\leq T} s^{|\gamma|}\,\frac{\big|v(t)-v(s)\big|}{|t-s|^\beta} < \infty \right\}.$$

The above supremum defines the norm $\|v\|_{t^\gamma C^\beta}$ of an element of that space. This norm turns this space into a Banach space. Denote by $p_t^\Delta(x,y)$ the Schwartz kernel of the usual heat operator $e^{-t\Delta}$. Given $0<\alpha<\alpha'<1$ pick a positive constant ε such that

$$\alpha+\varepsilon>1.$$

Pick also a regularity exponent β so that

$$(\alpha'-2)+\beta>0$$

and write

$$\beta' := \beta + 2\varepsilon + 2\eta,$$

for a (small) positive constant

$$0<\eta<\frac{\alpha'-\alpha}{2}.$$

To fix the ideas one can think of α,α' as 1^- and β,β' as 1^+.

Theorem 17 – *Define formally the map*

$$(\star) : u_0 \mapsto \Big\{ (t,x) \mapsto \big\langle u_0(\cdot), (p_t - p_t^\Delta)(x,\cdot)\big\rangle \Big\}.$$

Almost surely

(1) the map $(\star)$ sends continuously $B^{-\varepsilon}_{1,\infty}(\mathcal{S})$ into $t^{-\frac{\beta'}{2}}C\big((0,T],C^\alpha(\mathcal{S})\big)$,

(2) the map $(\star)$ sends continuously $H^{-2\alpha'}(\mathcal{S})$ into $t^{-\alpha'}C\big((0,T],H^\alpha(\mathcal{S})\big)$.

These two functions depend continuously on $\widehat{\xi}$.

Note that while all elements in $L^2(\mathcal{S})$ of the form $e^{-tH}f$, for $f\in L^2(\mathcal{S})$, are in the domain of H, the Dirac distributions δ_y are not elements of $L^2(\mathcal{S})$ so one does not expect the $p_t(\cdot,y)$ to be elements of the domain of H. The above time weighted spaces are the natural spaces where to look at the classical heat kernel p^Δ, as a consequence of the classical sharp estimates

$$\|e^{-t\Delta}v\|_{C^{\beta_2}} \lesssim t^{\frac{\beta_1-\beta_2}{2}}\|v\|_{C^{\beta_1}} \qquad (\beta_1,\beta_2\in\mathbb{R})$$

and similar estimates in the Sobolev scale, satisfied by p^Δ. So it is not surprising to see these spaces pop out here. Item *(1)* of Theorem 17 says in particular that $p-p^\Delta$ explodes near $t=0^+$ at worst as $t^{-\frac{\beta'}{2}}$. A dimensional analysis of the second term in Duhamel's heuristic picture

$$p_t = p_t^\Delta + \int_0^t e^{-(t-s)\Delta}M_{h\xi}e^{-s\Delta}\,ds + (\cdots)$$

shows that it actually scales as $t^{-1/2}$ so the result stated in Theorem 17 is indeed sharp as $\beta'>1$ can be chosen arbitrarily close to 1. We will use item *(2)* of Theorem 17 in Section 4.4, in our proof of size estimates for the eigenvectors of H.

Proof – We already constructed p as a continuous function on $\mathcal{S}\times(0,T]\times\mathcal{S}$ in the proof of Proposition 13. We show here that it has the regularity properties given by the two items of Theorem 17 by showing that it is the unique solution in some spaces of some equation obtained as a variant of the Duhamel equation. Set

$$\mathbf{L} := \partial_t + \Delta, \quad \text{and} \quad (\mathcal{F}u_0)(s) := e^{-s\Delta}(u_0),$$

with the letter '$\mathcal{F}$' chosen for 'free propagation'. We rewrite the parabolic Anderson model equation

$$(\partial_t+\Delta)p = M_{h\xi}(p)$$

satisfied formally by p – with y argument 'fixed', under the form of an equation on $v := p-p^\Delta$

$$v = \mathbf{L}^{-1}\big(M_{h\xi}(v)\big) + \mathbf{L}^{-1}\big(M_{h\xi}(\mathcal{F}u_0)\big). \tag{4.2}$$

We first concentrate on the free propagation term in the right hand side of this equation and set

$$u_{(0)} := \mathsf{L}^{-1}\big(M_{h\xi}(\mathcal{F}u_0)\big).$$

Lemma 18 – *For all $u_0 \in B^{-\varepsilon}_{1,\infty}(\mathcal{S})$ one has $u_{(0)} \in t^{-2\varepsilon}C\big((0,T], C^\alpha(\mathcal{S})\big)$.*

Proof – Note first that one has

$$\big\|\mathsf{M}^-(e^{-s\Delta}u_0)\big\|_{C^{\alpha'-2}} \lesssim s^{-2\varepsilon}\|u_0\|_{B^{-\varepsilon}_{1,\infty}}. \tag{4.3}$$

and for all $\rho > 0$

$$\big\|e^{-(t-s)\Delta}\mathsf{M}^-(e^{-s\Delta}u_0)\big\|_{C^\rho} \lesssim |t-s|^{\frac{\alpha'-2-\rho}{2}} s^{-2\varepsilon}\|u_0\|_{B^{-\varepsilon}_{1,\infty}}.$$

For $0 < t_1 < t_2 \leq T$ one has

$$\begin{aligned}
&\bigg\|\int_0^{t_1} e^{-(t_1-s)\Delta}\mathsf{M}^-(e^{-s\Delta}u_0)\,ds - \int_0^{t_2} e^{-(t_2-s)\Delta}\mathsf{M}^-(e^{-s\Delta}u_0)\,ds\bigg\|_{C^\alpha} \\
&\leq \bigg\|\int_{t_1}^{t_2} e^{-(t_2-s)\Delta}\mathsf{M}^-(e^{-s\Delta}u_0)\,ds\bigg\|_{C^\alpha} + \bigg\|\int_0^{t_1}\Big(e^{-(t_1-s)\Delta} - e^{-(t_2-s)\Delta}\Big)\,\mathsf{M}^-(e^{-s\Delta}u_0)\,ds\bigg\|_{C^\alpha} \\
&\leq \|u_0\|_{B^{-\varepsilon}_{1,\infty}}\int_{t_1}^{t_2}|t_2-s|^{\frac{\alpha'-2-\alpha}{2}}s^{-2\varepsilon}\,ds \\
&\quad + \big\|\mathrm{Id} - e^{-(t_2-t_1)\Delta}\big\|_{\mathcal{B}(C^{\alpha+2\eta},C^\alpha)}\bigg\|\int_0^{t_1} e^{-(t_1-s)\Delta}\mathsf{M}^-(e^{-s\Delta}u_0)\,ds\bigg\|_{C^{\alpha+2\eta}} \\
&\leq (1) + (2)\times(3).
\end{aligned}$$

Now one has

$$(1) \lesssim O\Big(|t_2-t_1|^{\frac{\alpha'-\alpha}{2}}t_1^{-2\varepsilon}\Big),$$

and for the term (2)

$$\big\|\mathrm{Id} - e^{-(t_2-t_1)\Delta}\big\|_{\mathcal{B}(C^{\alpha+2\eta},C^\alpha)} \lesssim |t_2-t_1|^\eta.$$

For the term (3) one has

$$(3) \lesssim \|u_0\|_{B^{-\varepsilon}_{1,\infty}}\int_0^{t_1}|t_1-s|^{\frac{\alpha'-2-(\alpha+2\eta)}{2}}s^{-2\varepsilon}\,ds = O\Big(t_1^{\frac{\alpha'-\alpha}{2}-\eta-2\varepsilon}\Big).$$

Similar computations with M^+ in place of M^- conclude the proof. We leave these computations to the reader and note here that $\mathsf{L}^{-1}\big(\mathsf{M}^+(\mathcal{F}u_0)\big) \in t^{-2\varepsilon}C\big((0,T], C^{2\alpha}(\mathcal{S})\big)$. ▷

The very same proof shows that $u_{(0)} \in t^{-\alpha'}C\big((0,T], H^\alpha(\mathcal{S})\big)$ if $u_0 \in H^{-2\alpha}(\mathcal{S})$, with

$$\mathsf{L}^{-1}\big(\mathsf{M}^+(\mathcal{F}u_0)\big) \in t^{-\alpha'}C\big((0,T], H^{2\alpha}(\mathcal{S})\big).$$

The proof actually shows that $u_{(0)}$ takes values in a space of the form $t^{-\alpha'}C^\eta\big((0,T], E\big)$ under assumption *(1)* or *(2)*.

Unlike its counterpart in $u_{(0)}$ the multiplication operation $M_{h\xi}(v)$ in (4.2) is ill-posed for the natural class of functions v. To deal with this term requires a renormalization step and to work with a space of v's with a special paracontrolled structure. The next statement deals with the renormalization step. Recall from Section 3.1 that we use the notation $c_{h,r}$ for the function $\frac{|\log r|}{4\pi}h^2$.

Lemma 19 – *Let $o(1) \in (0, \alpha' - \alpha)$ be a small positive constant. Consider the operators $\mathsf{M}_r^+\mathsf{L}^{-1}\mathsf{M}_r^- - M_{c_{h,r}}$ as elements of the space of continuous linear maps from $t^{-\frac{\beta'}{2}}C\big((0,T], C^\alpha(\mathcal{S})\big)$ with values in the sum space*

$$t^{-\frac{\beta'}{2}}C\big((0,T], C^{3\alpha-2}(\mathcal{S})\big) + t^{-\frac{\beta'+o(1)}{2}}C\big((0,T], C^{2\alpha'-2}(\mathcal{S})\big). \tag{4.4}$$

They converge in probability as r goes to 0 to a limit random operator denoted by $\mathscr{R}(\mathsf{M}^+\mathsf{L}^{-1}\mathsf{M}^-)$.

Proof – Given $u\in t^{-\frac{\beta'}{2}}C\big((0,T],C^\alpha(\mathcal{S})\big)$ and $3\alpha-2>0$ one uses the (space) corrector C to isolate from the resonant term in M_r^+ the term

$$\begin{aligned}\Pi\big(h\xi_r,\mathbf{L}^{-1}\mathsf{P}_u(h\xi_r)\big)-c_{h,r}u &= \mathsf{C}\big(u,h\xi_r,\mathbf{L}^{-1}(h\xi_r)\big)+u\Big\{\Pi\big(h\xi_r,\mathbf{L}^{-1}(h\xi_r)\big)-c_{h,r}\Big\}\\ &\in t^{-\frac{\beta'}{2}}C\big((0,T],C^{3\alpha-2}(\mathcal{S})\big)+t^{-\frac{\beta'+o(1)}{2}}C\big((0,T],C^{2\alpha'-2}(\mathcal{S})\big).\end{aligned}$$

The corrector term is converging almost surely as r goes to 0, and the term $\Pi\big(h\xi_r,\mathbf{L}^{-1}(h\xi_r)\big)-c_{h,r}$ is converging in probability in $t^{-\frac{\beta'+o(1)}{2}}C\big((0,T],C^{2\alpha-2}(\mathcal{S})\big)$, as r goes to 0. Indeed, since the map

$$t\in(0,T]\mapsto\int_t^\infty e^{-s\Delta}(\cdot)\,ds\in\mathcal{B}\big(C^{\alpha'-2}(\mathcal{S}),C^{2\alpha'-2}(\mathcal{S})\big)$$

is continuous, the relation (4.1) between $\mathbf{L}^{-1}$ and Δ^{-1} shows that the limit process is a continuous function of $t\in(0,T]$. However one needs to subtract $\frac{\log t}{4\pi}\,h^2$ to the resonant term for

$$\Pi\Big(h\xi,\big(\mathbf{L}^{-1}(h\xi)\big)(t)\Big)-\frac{\log t}{4\pi}\,h^2$$

to converge in $C^{2\alpha-2}(\mathcal{S})$ as t goes to 0. This $\log t$ term explains why we need to introduce the $o(1)$ exponent in (4.4), so that the second term takes values in $t^{-\frac{\beta'+o(1)}{2}}C\big((0,T],C^{2\alpha'-2}(\mathcal{S})\big)$. ▷

We will trade below the explosion factor $t^{-\frac{o(1)}{2}}$ against some space regularity, which is why we are insisting to have $2\alpha'-2$ rather than $2\alpha-2$ as a space regularity exponent for the corresponding term. The fact that space time white noise is almost surely γ-Hölder regular for all $\gamma<-1$ gives us that freedom. We note now the following elementary fact that will be useful.

Lemma 20 – *One has*

$$\mathbf{L}^{-1}\Big(t^{-\frac{\beta'+o(1)}{2}}C\big((0,T],C^{2\alpha'-2}(\mathcal{S})\big)\Big)\subset t^{-\frac{\beta'}{2}+\frac{o(1)}{2}}C\big((0,T],C^{2\alpha}(\mathcal{S})\big).$$

Proof – This is mainly the proof of Lemma 18 with the additional trick that consists in decomposing the analogue here of the term (1) therein under the form

$$\begin{aligned}(1)&=\|v\|\int_{t_1}^{t_2}|t_2-s|^{\eta+o(1)/2-1}s^{-o(1)/2}s^{-\frac{\beta'}{2}+\frac{o(1)}{2}}\,ds\\ &\lesssim t_1^{-\frac{\beta'}{2}+\frac{o(1)}{2}}\int_{t_1}^{t_2}|t_2-s|^{\eta+o(1)-1}s^{-o(1)/2}\,ds.\end{aligned}$$

The result then boils down to the fact that the integral

$$\int_{t_1}^{t_2}|t_2-s|^{\eta+o(1)/2-1}s^{-o(1)/2}\,ds$$

is of size $|t_2-t_1|^\eta$; a fact that can be seen by a change of variable. We apply a similar trick in the term corresponding to (2) × (3). ▷

Definition – *We say that a function $v\in t^{-\frac{\beta'}{2}}C\big((0,T],C^\alpha(\mathcal{S})\big)$ has a* **paracontrolled structure** *if*

$$v=\mathbf{L}^{-1}\mathsf{M}^-(v')+v^\sharp,$$

for $v'\in t^{-\frac{\beta'}{2}}C\big((0,T],C^\alpha(\mathcal{S})\big)$ and $v^\sharp\in t^{-\frac{\beta'}{2}+\frac{o(1)}{2}}C\big((0,T],C^{2\alpha}(\mathcal{S})\big)$. The sum of the natural norms of v' and $v^\sharp$ endows the space $\mathcal{V}$ of paracontrolled functions with a norm that turns it into a Banach space.

For v paracontrolled one can rewrite the formal equation (4.2) under the renormalized form

$$\begin{aligned}v&=\mathbf{L}^{-1}\big(\mathsf{M}^-(v+\mathcal{F}u_0)\big)+\mathbf{L}^{-1}\big(\mathsf{M}^+v+\mathsf{M}^+\mathcal{F}u_0\big)\\ &=\mathbf{L}^{-1}\big(\mathsf{M}^-(v+\mathcal{F}u_0)\big)+\Big\{\mathbf{L}^{-1}\big(\mathscr{R}(\mathsf{M}^+\mathbf{L}^{-1}\mathsf{M}^-)(v')\big)+\mathbf{L}^{-1}\big(\mathsf{M}^+v^\sharp\big)+\mathbf{L}^{-1}\big(\mathsf{M}^+\mathcal{F}u_0\big)\Big\}\end{aligned}$$

and formulate it as a fixed point equation in the space of paracontrolled v as

$$\begin{aligned} v' &= v + \mathcal{F}u_0 \\ v^\sharp &= \mathsf{L}^{-1}(\mathsf{M}^+ v) + \mathsf{L}^{-1}\big(\mathsf{M}^+ \mathcal{F}u_0\big) \end{aligned}$$

that is

$$\begin{aligned} v' &= \mathsf{L}^{-1}\mathsf{M}^-(v') + v^\sharp + \mathcal{F}u_0 \\ v^\sharp &= \mathsf{L}^{-1}\big(\mathscr{R}(\mathsf{M}^+\mathsf{L}^{-1}\mathsf{M}^-)(v')\big) + \mathsf{L}^{-1}\big(\mathsf{M}^+ v^\sharp\big) + \mathsf{L}^{-1}\big(\mathsf{M}^+\mathcal{F}u_0\big), \end{aligned}$$

or

$$(v', v^\sharp) =: \Phi(v', v^\sharp).$$

Lemma 21 – *The map Φ is a contraction of $\mathcal{V}$ for T small enough.*

Proof – We use Banach fixed point theorem. The proof is elementary and consists in exploiting the fact that Φ naturally takes values in a space of functions with smaller explosion exponent; this allows to gain the contracting factor is small time. We use for that purpose the fact that ξ has almost surely Hölder regularity arbitrarily close to -1 to gain in the well-defined terms $\mathsf{M}^- v'$ and $\mathsf{M}^+ v^\sharp$ a small positive regularity exponent that is turned by L^{-1} into a small positive 'explosion' exponent as in the proof of Lemma 18. We leave the details to the reader as all the ingredients have been spelled out above explicitly and that kind of reasoning is now classical in the litterature on singular stochastic PDEs. ▷

The continuity of the fixed point as a function of the renormalized operator $\mathscr{R}(\mathsf{M}^+\mathsf{L}^{-1}\mathsf{M}^-)$ is automatic in a fixed point scheme. This gives the continuous dependence of v on $\widehat{\xi}$ and concludes the proof of item *(1)* of Theorem 17. Together with Lemma 19 it shows that if $p^{(r)}$ stands for the heat kernel of regularized and renormalized operator $H_r - c_{h,r}$ then $p^{(r)} - p^\Delta$ is converging to $p - p^\Delta$ in the appropriate space. As usual with linear equations, one sees that the lifetime T of the solution does not depend on the initial condition u_0. This gives global in time well-posedness. The proof of item *(2)* is similar and left to the reader. ▷

The next statement gives a property of the operator $p - p^\Delta$ that we will use later.

Corollary 22 – *Let A be a continuous linear map from $B^{-\varepsilon}_{1,\infty}(\mathcal{S})$ into $t^{-\frac{\beta'}{2}}C\big((0,T], C^\alpha(\mathcal{S})\big)$. Then*

- *the operator A has a well-defined Schwartz kernel $A(x,(t,y))$ such that*

$$\sup_{x\in\mathcal{S}}\ \sup_{0<t\leq T}\ \sup_{y_1\neq y_2} t^{-\frac{\beta'}{2}}\,\frac{\big|A(x,(t,y_1)) - A(x,(t,y_2))\big|}{|y_1-y_2|^\alpha} < \infty, \tag{4.5}$$

- *for all $t>0$ the operator $A(t)$ is trace class in $L^2(\mathcal{S})$ and one has*

$$\mathrm{tr}_{L^2}\big(A(t)\big) \leq O\big(t^{-\frac{\beta'}{2}}\big).$$

Proof – *Step 1 - Localization by partition of unity.* Choose a finite cover $\cup_j U_j$ of $\mathcal{S}$ where each open set U_j is diffeomorphic to a ball of $\mathbb{R}^2$, and a partition of unity $\sum_j \chi_j = 1$, with $\chi_j \in C^\infty_c(U_j)$ subordinated to that cover. Consider the decomposition

$$\overline{A}f = \sum_{j,k} \chi_k\big(\overline{A}(\chi_j f)\big) =: \sum_{j,k} \overline{A}_{jk} f.$$

It suffices to prove that each operator $\overline{A}_{jk}(t)$ is trace class on L^2 and satisfies the bound (4.5). We may assume we have maps $\Phi_j : V_j \subset \mathbb{T}^2 \to U_j = \Phi_j(V_j) \subset \mathcal{S}$ that map diffeomorphically a subset V_j of $\mathbb{T}^2$ onto U_j. Set $\Psi_j \in C^\infty_c(V_j)$ be test functions on $\mathbb{T}^2$ such that $\Psi_j = 1$ on $\Phi_j^{-1}(\mathrm{supp}(\chi_j))$. Define the operator

$$B_{jk}(t) = \Psi_k \Phi_k^* \overline{A}_{jk}(t) \Phi_j^{-1*} \Psi_j : L^2(\mathbb{T}^2) \mapsto L^2(\mathbb{T}^2).$$

This map is well–defined since the map Φ_j^{-1*} is well–defined on the support of Ψ_j and for all functions $\varphi \in L^2(S)$, the function $\overline{A}_{jk}(\varphi)$ has its image supported in $\mathrm{supp}(\chi_k)$ hence the pull-back $\Phi_k^*\overline{A}_{jk}(t)$ is always well–defined and is supported in V_k. It suffices to prove that the operator $B_{jk}(t)$ defined above is trace class on $L^2(\mathbb{T}^2)$ endowed with the Haar measure of $\mathbb{T}^2$

since the push-forward Riemannian measure from $\mathcal{S}$ has smooth density with respect to the Haar measure on $\mathbb{T}^2$ and trace class elements form an ideal in $\mathcal{B}\big(L^2(\mathcal{S}), L^2(\mathcal{S})\big)$. Note that by boundedness of the pull-back by smooth diffeomorphisms acting on Besov spaces, each operator B_{ij} maps continuously $B^{-\varepsilon}_{1,\infty}(\mathbb{T}^2)$ into $t^{-\frac{\beta'}{2}}C\big((0,T], C^\alpha(\mathbb{T}^2)\big)$.

Step 2 - Existence of Schwartz kernels as densities. We work with the operators B_{jk} which live on the torus and can be identified with compactly supported operators on $\mathbb{R}^2\times[0,T]\times\mathbb{R}^2$. Recall we denote by $\mu(dx)$ the Riemannian volume measure. We are looking for a function $B_{ij}(x,(t,y))$ such that one has

$$B_{jk}(u)(t,y) = \int_{\mathbb{T}^2} B_{jk}(x,(t,y))u(x)\,\mu(dx) \in t^{-\frac{\beta'}{2}}C\big((0,T], C^\alpha(\mathbb{T}^2)\big)$$

for all $u\in C^\infty_c(V_j)$. Let $\chi:\mathbb{R}^2\to\mathbb{R}$ be a smooth non-negative function with compact support whose integral equals 1. Since

$$u_s := \int_{\mathbb{R}^2} s^{-2}\chi\Big(\frac{\cdot - x}{s}\Big)u(x)\,\mu(dx)$$

converges to u in $B^{-\varepsilon}_{1,\infty}(\mathbb{T}^2)$ as $s>0$ goes to 0, the following limit exists

$$B_{jk}(u)(t,y) = \lim_{s\to 0} B_{jk}(u_s)(t,y) = \lim_{s\to 0}\int_{\mathbb{R}^2} B_{jk}\Big(s^{-2}\chi\big(\frac{\cdot - x}{s}\big)\Big)(t,y)u(x)\,\mu(dx).$$

Now observe that the family $s^{-2}\chi(\frac{\cdot - x}{s})$ converges in $B^{-\varepsilon}_{1,\infty}(\mathbb{T}^2)$ to δ_x when s goes to 0, uniformly in $x\in V_j$. This implies that

$$\lim_{s\to 0} B_{jk}\Big(s^{-2}\chi\big(\frac{\cdot - x}{s}\big)\Big)(t,y)$$

exists for all $(t,y)\in(0,T]\times V_j$, and the previous quantity is bounded by a constant C_K, uniformly in (t,y) in any compact subset K of $(0,T]\times V_j$ and $x\in V_k$. Dominated convergence can then be used and gives

$$B_{jk}(u)(t,y) = \int_{\mathbb{R}^2}\lim_{s\to 0} B_{jk}\Big(s^{-2}\chi\big(\frac{\cdot - x}{s}\big)\Big)(t,y)u(x)\,\mu(dx)$$

for all $(t,y)\in K$. So

$$B_{jk}(x,(t,y)) := \lim_{s\to 0} B_{jk}\Big(s^{-2}\chi\big(\frac{\cdot - x}{s}\big)\Big)(t,y)$$

is the Schwartz kernel of B_{jk}. It is uniformly bounded in (x,y) for each $t>0$, and it satisfies

$$\sup_{x\in V_k}\ \sup_{0<t\leq T}\ \sup_{(y_1,y_2)\in V_j^2, y_1\neq y_2} t^{-\frac{\beta'}{2}}\frac{\big|B_{jk}(x,(t,y_1)) - B_{jk}(x,(t,y_2))\big|}{|y_1-y_2|^\alpha} < \infty.$$

Step 3 - Fourier bounds and L^2–traces. The trick consists in writing, for $\ell\in\mathbb{Z}^2$,

$$\big\langle e^{i\ell\cdot}, B_{jk}(t)e^{i\ell\cdot}\big\rangle_{L^2} = t^{-\frac{\beta'}{2}}\langle\ell\rangle^{\alpha+\beta}\Big\langle \langle\ell\rangle^\alpha e^{i\ell\cdot}\ ,\ t^{\frac{\beta'}{2}}B_{jk}(t)\big(\langle\ell\rangle^\varepsilon e^{i\ell\cdot}\big)\Big\rangle_{L^2}, \tag{4.6}$$

and noting that since the ℓ-indexed family $\langle\ell\rangle^\varepsilon e^{i\ell\cdot}$ is bounded in $B^{-\varepsilon}_{1,\infty}(\mathbb{T}^2)$ and the family $\langle\ell\rangle^\alpha e^{i\ell\cdot}$ is bounded in $C^{-\alpha}(\mathbb{T}^2)$, uniformly in ℓ, the fact that B_{jk} be a continuous linear map from $B^{-\varepsilon}_{1,\infty}(\mathbb{T}^2)$ into $t^{-\frac{\beta'}{2}}C\big((0,T], C^\alpha(\mathbb{T}^2)\big)$ implies that the big bracket term above is bounded by a constant independent of ℓ. The bound $\alpha+\beta>1$ gives a converging sum when summing (4.6) over $\ell\in\mathbb{Z}^2$, so the operator $B_{jk}(t)$ has indeed a finite trace, of order $t^{-\frac{\beta'}{2}}$. ▷

By taking β' slightly bigger we can assume without loss of generality that

$$\lim_{t\to 0^+} t^{\frac{\beta'}{2}}A(t,\cdot) = 0.$$

4.3 Moment bounds for the heat kernel and spectral gap

The sharp description of the heat kernel of H provided by Theorem 17 has a number of useful and non-trivial consequences. We first prove moment bounds that will be useful in Section 6 in our construction of the polymer measure. We prove two sided Gaussian estimates for the heat kernel of H. Building on the proof of this fact that we give in Proposition 25 we are able to provide in Theorem 28 an almost sure spectral estimate for H in terms of u_0 only under a mild geometric assumption on the Riemannian manifold. In Theorem 27, we give an estimate on the spectral gap in terms of isoperimetric constants and the ground state of H which holds true for any Riemannian surface $\mathcal{S}$.

We start by proving a moment bound on $A(t)$ that is a direct consequence of Corollary 22 and leads in Proposition 24 to a useful moment bound on e^{-tH}.

Lemma 23 – *Let A be a continuous linear operator from $B^{-\varepsilon}_{1,\infty}(\mathcal{S})$ into $t^{-\frac{\beta'}{2}}C\big((0,T],C^\alpha(\mathcal{S})\big)$ such that $\lim_{t\to 0^+} t^{\frac{\beta'}{2}}A(t,\cdot)=0$. Then for all $k\geq 1$, for $0\leq t\leq T$, the Schwartz kernel $A(t,x,y)$ of A satisfies the bound*

$$\sup_{x\in\mathcal{S}}\left|\int_{\mathcal{S}} A(x,(t,y))\,d(x,y)^k\,dy\right|\lesssim k!\,t^{\frac{k}{2}+\frac{\alpha-\beta'}{2}} \tag{4.7}$$

with an implicit multiplicative constant independent of k.

Proof – We use the previous notations from Corollary 22. Using the fact that the Euclidean distance function induced by smooth charts and the Riemannian distance function are equivalent, it suffices to prove an estimate of the form

$$\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2}\big|B_{ij}(x,(t,y))\big||x-y|^k\,dy\lesssim k!\,t^{\frac{\alpha-\beta'+k}{2}}.$$

Observe that the family of distributions $\big(\delta(\cdot-x)\big)_{x\in V_j}$ is uniformly bounded in $B^{-2\varepsilon}_{1,\infty}(V_j)$, therefore by continuity of $B_{jk}:B^{-2\varepsilon}_{1,\infty}(\mathbb{R}^2)\mapsto t^{-\frac{\beta'}{2}}\mathcal{C}^\eta\big([0,T],C^\alpha(V_j)\big)$, the family of functions

$$(t,y)\in[0,T]\times V_j\mapsto t^{\frac{\beta'}{2}}B_{jk}\big(\delta(\cdot-x)\big)(t,y),$$

is bounded in $\mathcal{C}^\eta\big([0,T],C^\alpha(V_j)\big)$, uniformly in $x\in V_k$, and compactly supported in the variable $y\in V_j$ since B_{jk} is defined using cut–off functions. From the Hölder regularity and the fact that $t^{\frac{\beta'}{2}}A(t,\cdot)$ vanishes at time 0 we deduce that for all $\lambda\in(0,1)$, we have a scaling bound

$$\sup_{t\in(0,T]}\sup_{x\in V_k} t^{\frac{\beta'}{2}}\big|B_{jk}\big(x,(t,\lambda(z-x)+x)\big)\big|\lesssim\lambda^\alpha.$$

From this we deduce by compactness of the supports of B_{jk} that if one chooses test functions $(\psi_j,\psi_k)\in C^\infty_c(\mathbb{R}^2)^2$ such that $\psi_j,\psi_k\leqslant 1$ are equal to 1 on V_j and V_k respectively then one has

$$\begin{aligned}\int_{\mathbb{R}^2}\big|B_{jk}(x,(t,y))\big||x-y|^k\,dy&=\psi_j(x)\int_{\mathbb{R}^2}\psi_k(y)\big|B_{jk}(x,(t,y))\big||x-y|^k\,\mu(dy)\\&=\psi_j(x)t\int_{\mathbb{R}^2}\psi_k\big(\sqrt{t}(z-x)+x\big)\big|B_{jk}\big(x,(t,\sqrt{t}(z-x)+x)\big)\big||\sqrt{t}(z-x)|^k\,\mu(dz)\\&\lesssim t^{1+\frac{k-\beta'+\alpha}{2}}\int_{\mathbb{R}^2}\psi_k\big(\sqrt{t}(z-x)+x\big)|z-x|^k\,\mu(dz)\lesssim k!\,t^{\frac{k-\beta'+\alpha}{2}},\end{aligned}$$

from a change of variable. The implicit multiplicative constant in the last inequality depends on ψ_k; it can be chosen independent of ψ_k in (4.7) as only finitely many B_{jk} are involved in the decomposition of A in the proof of Lemma 22. ▷

It is a well-known elementary fact that the heat kernel of the Laplace operator satisfies the estimate (4.7) with the exponent $(\alpha-\beta'+k)/2$ in the upper bound replaced by $k/2$. The following Kolmogorov type bound follows as a consequence; it will play an important role in our construction of the polymer measure in Section 6.1. Recall from Proposition 13 that the heat kernel p_t of H is positive.

Proposition 24 – *For all positive exponents a one has the moment estimate*

$$\sup_{x\in\mathcal{S}}\left(\int_{\mathcal{S}} p_t(x,y)\,d(x,y)^k\,\mu(dy)\right)^{1/k} \lesssim (k!)^{1/k}\,t^{\frac{1}{2}+o_k(1)}, \tag{4.8}$$

for an exponent $o_k(1)$ that goes to 0 as k goes to $+\infty$.

Bounds of the form (4.8), with $1/2$ in place of $1/2+o_k(1)$, are typical of Gaussian type kernels. We will use the moment estimate (4.8) below in our study of the polymer measure, in Section 6. It is not clear from our analysis that p_t has a Gaussian bound as it is not clear that operators A with the properties of Lemma 23 satisfy such bounds. It is possible to prove such bounds. We emphasize that except from the results of this section, the rough estimate provided by Proposition 24 suffices for all the other results of the present work.

We now give a proof of Gaussian upper and lower bounds for the heat kernel of H..

Proposition 25 – *There exists constants m and c that depend only on the ground state u_0 of H such that one has*

$$\frac{e^{-t\lambda_0(\widehat{\xi})}}{mct}\exp\left(-\frac{cd(y,x)^2}{t}\right) \leq p(t,x,y) \leq \frac{mce^{-t\lambda_0(\widehat{\xi})}}{t}\exp\left(-\frac{d(y,x)^2}{ct}\right) \tag{4.9}$$

for all $0<t\leq 1$.

The bound (4.9) gives back the moment estimate(4.8) from Proposition 24. For a positive regularization parameter r set

$$H_r f := \Delta f + h\xi_r f + c_{h,r} f$$

with $c_{h,r} = -\frac{|\log r|}{4\pi}\,h^2$. We justify Proposition 25 by proving an r-uniform similar estimate for the heat kernel $p_r(t,x,y)$ of H_r. The continuity of $p-p^\Delta$ as a function of $\widehat{\xi}$ in item *(i)* of Theorem 17 allows us to pass to the limit in the corresponding inequalities for each fixed positive t. For a fixed positive r we use the idea of conjugating the operator to a simpler operator for which one can use well-known heat kernel bounds with good control on its parameters as functions of r. The reader will find in Section 1.1 of [34] more references on works about diffusions with distributional drifts.

Proof – Pick $1<\beta<2$. Fix $r>0$ and denote by $u_{0,r}$ 'the' ground state of H_r, with associated eigenvalue $\lambda_{0,r}$; it is a positive function. It will turn out to be crucial to allow for a normalization different from the usual unit L^2-norm normalization; we will fix this normalization below. The conjugated operator

$$M_{u_{0,r}^{-1}}\big(H_r-\lambda_{0,r}\big)M_{u_{0,r}} = \Delta - 2\nabla(\log u_{0,r})\,\nabla \tag{4.10}$$

is known to have a heat kernel with Gaussian lower and upper bounds depending only on the oscillation $\mathrm{osc}(u_{0,r}^2) := \max u_{0,r}^2 - \min u_{0,r}^2$ of $u_{0,r}^2$, as this is a conservative perturbation of the Laplace-Beltrami operator. See e.g. Section 4.3 and Section 6.4 of Stroock's book [40]. So there is a continuous positive fonction $c(\cdot)$ of $\mathrm{osc}(u_{0,r}^2)$ with $c(0)=1$ such that setting

$$c_r := c\big(\mathrm{osc}(u_{0,r}^2)\big), \qquad m_r := \frac{\max u_{0,r}}{\min u_{0,r}},$$

one has

$$\frac{e^{-t\lambda_{0,r}}}{m_r c_r t}\exp\left(-\frac{c_r d(y,x)^2}{t}\right) \leq p_r(t,x,y) \leq \frac{m_r c_r e^{-t\lambda_{0,r}}}{t}\exp\left(-\frac{d(y,x)^2}{c_r t}\right) \tag{4.11}$$

for all $0<t\leq 1$ and $x,y\in\mathcal{S}$. We now see that one can take the constants $\lambda_{0,r}, m_r$ and c_r uniform in $r\in(0,1]$. One gets from Proposition 15 the continuous dependence of $\lambda_{0,r}$ and $u_{0,r}\in C^{2\alpha-1}(\mathcal{S})$ as functions of r. The bounds (4.9) follow from that continuity and the fact that the limit u_0 is positive. ▷

It is well-known from Fabes & Stroock work [16] that the above two sided Gaussian bounds are all we need to prove a parabolic Harnack principle which takes here the following form. Denote by $B(x,\rho)$ the closed geodesic ball of $\mathcal{S}$ of center x and radius ρ.

Corollary 26 – *Pick $0 < k_1 < k_2 < 1$ and $k_3 \in (0,1)$. There exists a constant c depending only on k_1, k_2, k_3 such that for all non-negative $(\partial_t - H)$ harmonic function u on a domain of $(0,1] \times \mathcal{S}$ of the form $[s-\rho, s] \times B(x,\rho)$, one has*

$$u(t,y) \leq c\, u(s,x),$$

for all $(t,y) \in [s - k_2\rho^2, s - k_1\rho^2] \times B(x, k_3\rho)$.

The conjugation trick used in the proof of Proposition 25 together with a continuity argument turns out to be useful to give lower bounds on the spectral gap of H that seem to be hard to obtain otherwise. We do that under two kinds of assumptions, geometric and functional analytic.

• *Isoperimetric estimate on the spectral gap.* Let ν be a smooth volume measure on $\mathcal{S}$. Given a subset A of $\mathcal{S}$ and $\kappa > 0$ denote by $A^{(\kappa)} := \{m \in \mathcal{S}\,;\, d(m,A) \leq \kappa\}$ its κ-enlargement and set

$$\sigma_\nu(\partial A) := \liminf_{\kappa \searrow 0} \frac{\nu(A^{(\kappa)}) - \nu(A)}{\kappa}.$$

The *Cheeger constant* of the Riemannian manifold $(\mathcal{S}, g)$ associated with the smooth volume measure ν is defined as

$$C(\nu) := \inf_{A \subset \mathcal{S}} \frac{\sigma_\nu(\partial A)}{\min\big\{\nu(A), \nu(\mathcal{S}\backslash A)\big\}}.$$

We do not emphasize the dependence on $(\mathcal{S}, g)$ in the notation as the manifold $\mathcal{S}$ and its Riemannian structure g are fixed in almost all of this work. Recall we denote by μ the Riemannian volume measure on $\mathcal{S}$.

Theorem 27 – *One has almost surely the following estimate on the spectral gap*

$$\big|\lambda_0(\widehat{\xi}\,) - \lambda_1(\widehat{\xi}\,)\big| \geq \frac{C(u_0^2\mu)^2}{4}.$$

This formula gives back in particular the almost sure lower bound $\Big(\frac{\min u_0}{\max u_0}\Big)^4 \frac{C(\mu)^2}{4}$ for the spectral gap of H, in terms of the Cheeger constant $C(\mu)$ of $(\mathcal{S}, g)$; this lower bound is positive. The constant $C(\mu)$ was denoted by $C(\mathcal{S}, g)$ in Theorem 2. It is equal to $2/L$ for a flat torus of size L.

Proof – Proceeding as in the proof of Proposition 25, we see that it suffices to prove that the spectral gap $\lambda_1(\widehat{\xi_r}) - \lambda_0(\widehat{\xi_r})$ of the conjugated regularized operator $\Delta - 2(\nabla \log u_{0,r})\nabla$ is bounded below by $C(u_{0,r}^2\mu)^2/4$, for the convergence of $u_{0,r}$ to u_0 in $C^{2\alpha-1}(\mathcal{S})$ proved in Proposition 15 implies that $C(u_{0,r}^2\mu)$ is converging to $C(u_0^2\mu)$ as r goes to 0.
The Cheeger lower bound on $\lambda_1(\widehat{\xi_r}) - \lambda_0(\widehat{\xi_r})$ is classical in Riemannian geometry and we give a self–contained proof adapted to our context as follows. We use the notation $\nu_{0,r}$ for the volume measure $u_{0,r}^2\mu$. The point is to see that for all smooth functions $f \in C^\infty(\mathcal{S})$, with median value $m_{0,r}(f)$ with respect to $\nu_{0,r}$, one has

$$\int_{\mathcal{S}} \|\nabla f\|\, d\nu_{0,r} \geqslant C(\nu_{0,r}) \int_{\mathcal{S}} \big|f - m_{0,r}(f)\big|\, d\nu_{0,r}. \tag{4.12}$$

If one takes (4.12) for granted for a moment one can apply this inequality to the function $f|f|$ where f is rescaled in such a way that it has unit $L^2(\nu_{0,r})$-norm and $f^{-1}(0)$ and $(f|f|)^{-1}(0)$ have equal $\nu_{0,r}$-measure $\nu_{0,r}(\mathcal{S})/2$, so $f|f|$ has null median. This yields

$$\int_{\mathcal{S}} \big\|\nabla\left(f|f|\right)\big\|\, d\nu_{0,r} = 2\int_{\mathcal{S}} \|f\nabla f\|\, d\nu_{0,r} \geqslant C(\nu_{0,r}) \int_{\mathcal{S}} |f|^2\, d\nu_{0,r} = C(\nu_{0,r}),$$

and we get from Cauchy-Schwartz inequality that

$$C(\nu_{0,r}) \leq 2\|\nabla f\|_{L^2(\nu_{0,r})}.$$

In the general case if $f \in C^\infty(S, \mathbb{R})$ is such that $\int_{\mathcal{S}} f d\nu_{0,r} = 0$ and $\int_{\mathcal{S}} f^2 d\nu_{0,r} = 1$, one can use the inequality

$$\int_{\mathcal{S}} (f+c)^2 d\nu_{0,r} = \int_{\mathcal{S}} (f^2 + c^2)\, d\nu_{0,r} \geqslant \int_{\mathcal{S}} f^2\, d\nu_{0,r}$$

to possibly add a constant to f and trade the assumption that $\int_{\mathcal{S}} f d\nu_{0,r} = 0$ for the assumption that $f^{-1}(0)$ cuts $\mathcal{S}$ in two pieces of equal $\nu_{0,r}$ measure. Applying the above arguments to $\frac{f+c}{\|f+c\|_{L^2(\nu_{0,r})}}$ yields

$$C(\nu_{0,r}) \leq 2\frac{\|\nabla f\|_{L^2(\nu_{0,r})}}{\|f+c\|_{L^2(\nu_{0,r})}} \leq 2\frac{\|\nabla f\|_{L^2(\nu_{0,r})}}{\|f\|_{L^2(\nu_{0,r})}}.$$

The representation of the spectral gap of $\Delta + 2(\nabla \log u_{0,r})\nabla$ as a Rayleigh quotient

$$\lambda_1(\widehat{\xi}_r) - \lambda_0(\widehat{\xi}_r) = \inf_{\int_{\mathcal{S}} f\, d\nu_{0,r}=0} \frac{\int_{\mathcal{S}} \|\nabla f\|^2\, d\nu_{0,r}}{\int_{\mathcal{S}} |f|^2\, d\nu_{0,r}}$$

then makes it clear that

$$\lambda_1(\widehat{\xi}_r) - \lambda_0(\widehat{\xi}_r) \geq \frac{C(\nu_{0,r})^2}{4}.$$

It remains to prove formula (4.12). Recall from the coarea formula that one has

$$\int_{\mathcal{S}} \|\nabla f\|\, d\nu_{0,r} = \int_{\mathbb{R}} \sigma_{\nu_{0,r}}\big(\{f=t\}\big)\, dt.$$

From the isoperimetric inequality

$$\sigma_{\nu_{0,r}}(\partial A) \geqslant C(\nu_{0,r}) \min\big(\nu_{0,r}(A), \nu_{0,r}(\mathcal{S}\setminus A)\big)$$

we deduce that if 0 is a median of f we have the bounds

$$\begin{aligned}
\int_{\mathcal{S}} \|\nabla f\|\, d\nu_{0,r} &= \int_{f\leq 0} |\nabla f|\, d\nu_{0,r} + \int_{f>0} |\nabla f|\, d\nu_{0,r} \\
&= \int_{-\infty}^{0} \sigma_{\nu_{0,r}}\left(\{f=t\}\right) dt + \int_{0}^{\infty} \sigma_{\nu_{0,r}}\left(\{f=t\}\right)\, dt \\
&\geqslant C(\nu_{0,r}) \left(\int_{-\infty}^{0} \nu_{0,r}(\{f\leq t\}) dt + \int_{0}^{\infty} \nu_{0,r}(\{f>t\})\, dt \right) \\
&\geqslant C(\nu_{0,r}) \int_{\mathcal{S}} |f|\, d\nu_{0,r}
\end{aligned}$$

where we used integration by parts for the last step and disintegration of the volume $\nu_{0,r}$ along level sets of f. ▷

• *Log-Sobolev estimate on the spectral gap.* Let ν be a non-negative measure on $\mathcal{S}$. Recall that the ν-entropy of a positive integrable function f such that $\int_{\mathcal{S}} f|\log f|\, d\nu < \infty$ is the quantity

$$\mathrm{Ent}_\nu(f) := \int_{\mathcal{S}} f \log f\, d\nu - \left(\int_{\mathcal{S}} f\, d\nu\right) \log\left(\int_{\mathcal{S}} f\, d\nu\right).$$

Recall also that we say that a measure ν on $\mathcal{S}$ satisfies a log-Sobolev inequality with constant C_{LS} with respect to the Dirichlet form associated with the Riemannian gradient operator ∇ if

$$\mathrm{Ent}_\nu(f^2) \leq 2C_{\mathrm{LS}} \int_{\mathcal{S}} |\nabla f|^2\, d\nu$$

for all functions f in the domain of the Dirichlet form. Such an inequality is known to imply a Poincaré inequality with constant $1/C_{\mathrm{LS}}$ and a corresponding spectral gap. Bakry, Gentil & Ledoux's monograph [7] presents a number of geometric conditions ensuring that μ satisfies a log-Sobolev inequality.

Theorem 28 – *Assume that the Riemannian volume form μ satisfies a* log-*Sobolev inequality with constant C_{LS}. Then the spectral gap of H satisfies almost surely the lower bound*

$$\big|\lambda_0(\widehat{\xi}) - \lambda_1(\widehat{\xi})\big| \geq \left(\frac{\min u_0}{\max u_0}\right)^2 \frac{\left(\max u_0^4 + \max u_0^{-4}\right)^{-1}}{2C_{\mathrm{LS}}}.$$

Proof – Fix a regularization parameter $r>0$. Denote by m_r the spectral gap of H_r in $L^2(\mu)$ and by m_r' the spectral gap of H_r in $L^2(u_{0,r}^2\mu)$. Then m_r' is equal to the spectral gap of the conjugated operator $\Delta + 2\nabla(\log u_{0,r})\nabla$ and

$$m_r \geq m'_r \left(\frac{\min u_{0,r}}{\max u_{0,r}}\right)^2.$$

As in the proof of Theorem 27 we recognize in the conjugated operator the Dirichlet form of the Riemannian gradient operator with respect to the weighted Riemannian volume form $u_{0,r}^2\mu$. As Holley & Stroock well-known stability argument for log-Sobolev inequality ensures that the weighted measure $u_{0,r}^2\mu$ satisfies, under the assumption of the statement, a log-Sobolev inequality with constant $2C_{\mathrm{LS}}\big(\max u_{0,r}^4 + \max u_{0,r}^{-4}\big)$ we see that

$$m'_r \geq \frac{\big(\max u_{0,r}^4 + \max u_{0,r}^{-4}\big)^{-1}}{2C_{\mathrm{LS}}}.$$

(See e.g. Proposition 5.1.6 in [7] for a proof of the stability argument.) We thus have the lower bound

$$\big|\lambda_{0,r} - \lambda_{1,r}\big| = m_r \geq \left(\frac{\min u_{0,r}}{\max u_{0,r}}\right)^2 \frac{\big(\max u_{0,r}^4 + \max u_{0,r}^{-4}\big)^{-1}}{2C_{\mathrm{LS}}}.$$

We conclude by using the continuity of the eigenvalues as functions of $\widehat{\xi}_r$ and the convergence in $L^\infty(\mathcal{S})$ of $u_{0,r}$ to u_0 – Proposition 15. ▷

Note that the lower bounds on the spectral gap of H of Theorem 27 and Theorem 28 both involve only the ground state u_0.

4.4 Bounds for the eigenvalues and eigenfunctions of H

The sharp description of p_t given by Theorem 17 gives a direct access to quantitative informations on the spectrum of H and its eigenfunctions. Recall we denote by μ the Riemannian volume measure.

∘ Pick any $t > 0$. As e^{-tH} is symmetric non-negative and its continuous kernel p_t has finite 'trace'

$$\int_{\mathcal{S}} p_t(x,x)\,\mu(dx) < \infty,$$

it follows from a well-known fact that e^{-tH} is trace class in $L^2(\mathcal{S})$, with trace equal to the previous integral – see e.g. the Lemma at the bottom of p.64 in [35], Section XI.4. The spectral resolution of the self-adjoint operator H tells us that one further has

$$\mathrm{tr}_{L^2}\big(e^{-tH}\big) = \sum_{\lambda\in\sigma(H)} e^{-t\lambda}.$$

As we also have

$$\mathrm{tr}_{L^2}\big(e^{-tH}\big) = \mathrm{tr}_{L^2}\big(e^{-t\Delta}\big) + \mathrm{tr}_{L^2}\big(A(t)\big),$$

where A satisfies the assumptions of Lemma 22, we have the asymptotic

$$\mathrm{tr}_{L^2}\big(e^{-tH}\big) = \mathrm{tr}_{L^2}\big(e^{-t\Delta}\big) + \mathrm{tr}_{L^2}\big(A(t)\big) = \frac{\mu(\mathcal{S})}{4\pi t} + O\big(t^{-\frac{\beta'}{2}}\big). \tag{4.13}$$

The following statement was first proved by Mouzard in [31] by using a fine description of the domain of H and minimax representations for the eigenvalues, based on the link between the operators H and Δ. This statement follows here from the small time equivalent (4.13) for the heat kernel by Karamata's Tauberian Theorem.

Proposition 29 – *We have almost surely the equivalent*

$$\sharp\Big\{\lambda\in\sigma(H)\,;\,\lambda\leq a\Big\} \underset{a,+\infty}{\sim} \frac{\mu(\mathcal{S})}{4\pi}\,a. \tag{4.14}$$

One thus has almost surely the equivalent

$$\lambda_n(\widehat{\xi}\,) \sim \lambda_n(0) \sim \frac{4\pi}{\mu(\mathcal{S})}\,n$$

as n goes to ∞, with $\lambda_n(0)$ the n^{th} eigenvalue of the Laplace-Beltrami operator Δ. Note that one cannot get this estimate from the Gaussian upper bound (4.9). We note further that since there is a random variable c_1 such that one has

$$\mathrm{tr}_{L^2}(e^{-tH}) \leq \frac{c_1(\widehat{\xi}\,)}{t} \tag{4.15}$$

for all $0 < t \leq 1$, and the λ_k are non-decreasing, we have for all $k \geq 1$

$$ke^{-\lambda_k(\widehat{\xi}\,)t} \leq \frac{c_1(\widehat{\xi}\,)}{t},$$

so taking $t = 1/|\lambda_k(\widehat{\xi}\,)|$ when this quantity is less than 1 gives the non-asymptotic lower bound

$$\big|\lambda_k(\widehat{\xi}\,)\big| \geq \frac{e}{c_1(\widehat{\xi}\,)}\,k,$$

for all eigvenvalues such that $|\lambda_k(\widehat{\xi}\,)| \geq 1$. The function

$$F_1(x) := \mathbb{P}\big(c_1(\widehat{\xi}\,) \geq x\big)$$

has thus the property that

$$\mathbb{P}\big(1 \leq |\lambda_k(\widehat{\xi}\,)| \leq a\big) \leq F_1\Big(\frac{ek}{a}\Big)$$

for all $k \geq 1$ and $a \geq 1$. The analysis of the proof of Theorem 17 shows that one can choose $c_1(\widehat{\xi}\,)$ of the form

$$c_1(\widehat{\xi}\,) = e^{c\|\widehat{\xi}\|},$$

for a positive constant c, and

$$\widehat{\xi} \in \Big(h\xi, \mathscr{R}\big\{\sf\Pi(X_h, h\xi)\big\}\Big) \in \mathcal{C}^{\alpha'-2}(\mathcal{S}) \times \mathcal{C}^{2\alpha'-2}(\mathcal{S}).$$

As we know that ξ has a Gaussian tail and $\mathscr{R}\big\{\sf\Pi(X_h, h\xi)\big\}$ has an exponential tail – see e.g. Proposition 2.2 in [31], there exists a positive constant b such that

$$F_1(x) \lesssim \frac{1}{x^b}.$$

We record these facts as a statement.

Proposition 30 – *One has*

$$\mathbb{P}\big(1 \leq |\lambda_k(\widehat{\xi}\,)| \leq a\big) \lesssim \Big(\frac{a}{k}\Big)^b$$

for all $k \geq 1$ and $a \geq 1$.

This kind of statement is somewhat 'orthogonal' to the exponential tail bounds from Allez & Chouk [1] and Labbé [29]; they take here the form

$$e^{-b_1(k)a} \lesssim \mathbb{P}\big(\lambda_k(\widehat{\xi}\,) < -a\big) \lesssim e^{-b_2(k)a} \tag{4.16}$$

when $a > a_k$ is large, for some positive constants $b_1(k), b_2(k)$ on which we have relatively poor control as functions of k. We also infer from the bound (4.15) that if $n_k(\widehat{\xi}\,)$ stands for the multiplicity of the eigenvalue $\lambda_k(\widehat{\xi}\,)$ then one has

$$n_k(\widehat{\xi}\,) \leq e\,c_1(\widehat{\xi}\,)\,\big|\lambda_k(\widehat{\xi}\,)\big|$$

for all eigenvalues for which $|\lambda_k(\widehat{\xi}\,)| \geq 1$. The following elementary bound

$$n_k(\widehat{\xi}\,) \leq c_1(\widehat{\xi}\,)\,e^{\lambda_k(\widehat{\xi}\,)}$$

can be interesting for negative eigenvalues. Since $n_0(\widehat{\xi}\,) = 1$ we infer from that bound that

$$\lambda_0(\widehat{\xi}\,) \geq -\ln c_1(\widehat{\xi}\,) \gtrsim -\|\widehat{\xi}\|.$$

(This lower bound is consistent with what one can infer from (3.6) and (3.7).) We recover from the integrability properties of $\|\widehat{\xi}\|$ the upper bound of (4.16) for $\lambda_0(\widehat{\xi})$. We conjecture that H has almost surely a simple spectrum.

The zeta function $\zeta_H(s)$ of H is defined by the formula

$$\zeta_H(s) := \sum_{\lambda\in\sigma(H)} |\lambda|^{-s}.$$

It follows from Weyl law that it defines an analytic function of s on the open half space $\{\mathrm{Re}(s) > 1\}$. One has in that domain

$$\begin{aligned}\zeta_H(s) &= \sum_{\lambda\in\sigma(H)\cap(-\infty,0)} \left(|\lambda|^{-s} - \frac{1}{\Gamma(s)}\int_0^1 e^{-r\lambda} r^{s-1}\,dr\right) + \frac{1}{\Gamma(s)}\int_0^1 \mathrm{tr}_{L^2}(e^{-rH})\, r^{s-1}\,dr \\ &\quad + \sum_{\lambda\in\sigma(H)\cap(0,\infty)} \frac{1}{\Gamma(s)}\int_1^\infty e^{-r\lambda} r^{s-1}\,dr. \\ &=: (1) + (2) + (3).\end{aligned}$$

As the set $\sigma(H)\cap(-\infty,0)$ is (almost surely) finite, the sum (1) defines an entire function of $s\in\mathbb{C}$. So does the sum (3), as $\min\{\lambda\,;\,\lambda\in\sigma(H)\cap(0,\infty)\} > 0$. As for the term (2) we know from the trace asymptotic (4.13) that

$$\frac{1}{\Gamma(s)}\int_0^1 \mathrm{tr}_{L^2}(e^{-rH})\, r^{s-1}\,dr = \frac{1}{\Gamma(s)}\left\{\frac{\mu(\mathcal{S})}{4\pi(s-1)} + \int_0^1 O\big(r^{-\frac12-\delta}\big)\, r^s\,\frac{dr}{r}\right\}.$$

Proposition 31 – *The function $\zeta_H(\cdot)$ is a well-defined meromorphic function on the half-plane $\big\{\mathrm{Re}(s) > 1/2\big\}$.*

The Duhamel formula tells us that

$$p_t = p_t^\Delta + \int_0^t p_{t-s}^\Delta M_\xi\, p_s^\Delta\,ds + \int_{0\le s_2\le s_1\le t} p_{t-s_1}^\Delta M_\xi p_{s_1-s_2}^\Delta M_\xi\, p_{s_2}\,ds_2 ds_1.$$

A dimensional analysis of the first integral on the right hand side tells us that it behaves as $t^{-1/2}$ as t goes to 0, up to log corrections. (The distribution of this centered Gaussian random variable Λ_t satisfies for all test functions φ the identity $\mathbb{E}\big[\Lambda_t(\varphi)^2\big] = t^{-1}\ell(\varphi)$ for an explicit function $\ell(\cdot)$ of φ.) The second integral on the other hand behaves almost surely as t^{0^-}. It follows that our meromorphic extension of ζ_H to the half-plane $\{\mathrm{Re}(s) > 1/2\}$ is sharp. It is possible to prove that $\mathbb{E}\big[\zeta_H(s)\big]$ has a meromorphic extension to all of $\mathbb{C}$. We do not give here a proof of that fairly non-trivial result.

$\circ$ Recall $(u_n)_{n\ge0}$ stands for the orthonormal basis of $L^2(\mathcal{S})$ made up of eigenvectors of H, with corresponding eigenvalues in non-decreasing order. Recall also that the constants $\alpha,\alpha',\beta,\beta'$ were chosen at the beginning of Section 4.2 before Theorem 17 so that $0<\alpha<\alpha'<1$, there is $\varepsilon>0$ such that $\alpha+\varepsilon>1, \beta$ satisfy $(\alpha'-2)+\beta>0$ and $\beta' := \beta+2\varepsilon+2\eta$, for a (small) positive constant $0<\eta<\frac{\alpha'-\alpha}{2}$.

Theorem 32 – *One has for all $n\ge0$ such that $|\lambda_n(\widehat{\xi})|\ge1$ the n-uniform estimate*

$$\|u_n\|_{C^{2\alpha-1}} \lesssim \big|\lambda_n(\widehat{\xi})\big|^{\frac{\beta'}{2}}, \tag{4.17}$$

and for $p\ge2$

$$\|u_n\|_{L^p} \lesssim \big|\lambda_n(\widehat{\xi})\big|^{(\frac12-\frac1p)\beta'}. \tag{4.18}$$

The optimal L^∞ bound for the Laplacian eigenfunctions involves an exponent 1/2, so (4.17) is not far from being optimal as β' can be chosen arbitrarily close to 1. A similar form of L^p bound (4.18) for the eigenfunctions of H, for $2<p<\infty$, was previously proved by Mouzard & Zacchuber in [32] as an application of their Strichartz estimates for the Anderson operator. Hölder estimates of the type (4.17) cannot be proved by their method.

Our proof of Theorem 32 is amazingly simple.

Proof – As one knows from the qualitative Proposition 13 that each map e^{-tH} sends continuously $L^2(\mathcal{S})$ into $C^{2\alpha-1}(\mathcal{S})$ one has

$$e^{-1}\|u_n\|_{C^{2a-1}} = \big\|e^{-\frac{1}{\lambda_n(\widehat{\xi})}H}u_n\big\|_{C^{2a-1}} \leq \big\|e^{-\frac{1}{\lambda_n(\widehat{\xi})}H}\big\|_{\mathcal{B}(L^2,C^{2a-1})}.$$

Now it is classical and elementary that one has for all $\kappa > 0$ and all $t > 0$ the continuity estimate

$$\big\|e^{-t\Delta}f\big\|_{C^{2a-1}} \lesssim_\kappa t^{\frac{1}{2}-a-\kappa}|f|_{L^2}.$$

Denote by A_t the operator associated with the kernel $p_t - p_t^\Delta$. One knows from the proof of item *(i)* of Theorem 17 that $A_t \in \mathcal{B}\big(L^2(\mathcal{S}), C^a(\mathcal{S})\big)$ has norm bounded above by a constant multiple of $t^{-\beta'/2}$, as $L^2(\mathcal{S})$ is continuously embedded in $B^{-\varepsilon}_{1,\infty}(\mathcal{S})$. Taking κ small and using that $2\alpha-1 \leq \alpha$ we obtain the estimate

$$\big\|e^{-\frac{1}{\lambda_n(\widehat{\xi})}H}\big\|_{\mathcal{B}(L^2,C^{2a-1})} \leq |\lambda_n|^{\frac{\beta'}{2}} + |\lambda_n|^{\kappa+\alpha-\frac{1}{2}} \lesssim |\lambda_n|^{\frac{\beta'}{2}}.$$

The $L^p(\mathcal{S})$ bound follows from the $L^\infty(\mathcal{S})$ bound (4.17) by interpolation. ▷

Corollary 33 – *The heat semigroup $(e^{-tH})_{t>0}$ is hypercontractive.*

Proof – It suffices to notice that for any $2 < p$ one has from Theorem 32

$$\big\|e^{-t(H-\lambda_0(\widehat{\xi}))}f\big\|_{L^p} \lesssim \|f\|_{L^2}\sum_{n\geq 0} e^{-t(\lambda_n(\widehat{\xi})-\lambda_0(\widehat{\xi}))}|\lambda_n|^{\frac{\beta'}{2}-\frac{1}{p}} \leq C_1(t)\|f\|_{L^2}$$

for any positive time t, for a finite positive constant C_1, from Weyl estimate. This is known to entail that the semigroup satisfies a log-Sobolev inequality with constants $\frac{2pt}{p-2}$ and $\frac{2p\log C_1(t)}{p-2}$ – see e.g. Theorem 5.2.5 in [7], from which the full hypercontractivity property follows. ▷

The proof of Theorem 32 is tailor made to get estimates on eigenfunctions. We use item *(ii)* of Theorem 17 to obtain estimates on eigen clusters or quasimodes in $H^\alpha(\mathcal{S})$ rather than in a Hölder space. Recall the exponents $0 < \alpha < \alpha' < 1$ are fixed as in the beginning of Section 4.2; think of them as 1^-. Given $a \in \mathbb{R}$ denote by

$$\pi_{\leq a} : L^2(\mathcal{S}) \to L^2(\mathcal{S})$$

the spectral projector

$$\pi_{\leq a}(f) := \sum_{\lambda_n \leq a} (f, u_n)\, u_n,$$

with (f, u_n) standing for the L^2 scalar product of f and u_n.

Theorem 34 – *One has for all $a \in \mathbb{R}_+$ and all $f \in L^2(\mathcal{S})$ the upper bound*

$$\|\pi_{\leq a}(f)\|_{H^\alpha} \lesssim a^{\frac{1}{2}}\,\|f\|_{L^2}, \tag{4.19}$$

and all $2 \leq p < \infty$

$$\|\pi_{\leq a}(f)\|_{L^p} \lesssim a^{(\frac{1}{2}-\frac{1}{p})\frac{1}{\alpha}}\|f\|_{L^2}. \tag{4.20}$$

The L^p bound is a direct consequence of the $H^\alpha(\mathcal{S})$ bound. The $H^\alpha(\mathcal{S})$ estimate (4.19) is new and seems out of reach of the methods of [32]. The estimate (4.20) improves much on the corresponding estimate from [32] on the eigenfunction cluster $\sum_{\lambda_n\in[a,a+1]}(f, u_n)\, u_n$. Mouzard & Zachhuber's estimate involves an exponent $1 - 1/p$ rather than $1/2 - 1/p$. In any case, it is expected that one could prove sharper L^p bounds if one could control the small time behaviour of a parametrix of the wave operator associated to H. In the case of the Laplace operator Δ this gives sharp exponents $1/2 - 2/p$, for $p \geq 6$, for the L^p size of the eigenfunctions of Δ. This loss in the exponent is reminiscent of the corresponding loss of regularity in the Strichartz inequality on a compact manifold proved first by Burq, Gérard & Tzvetkov in [9] and Staffilani & Tataru [39].

Proof – The technical heart of the argument is given by item *(2)* of Theorem 17. Pick $c > |\lambda_0(\widehat{\xi})|$ so that $H+c$ is positive. We prove in a quantitative way that the Sobolev norm of finite linear combinations $u = \sum_{n=0}^N c_n u_n$ of eigenfunctions of H is equivalent to some seminorm defined in terms of the positive operator $H + c$. Since the operator $H + c : \mathfrak{D}(H) \subset L^2(\mathcal{S}) \mapsto$

$L^2(\mathcal{S})$ is positive and self-adjoint the functional calculus gives the representation

$$(H+c)^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}\left(e^{-t(\Delta+c)} + e^{-tc}A(t)\right)dt + \frac{1}{\Gamma(\alpha)}\int_1^\infty t^{\alpha-1}e^{-t(H+c)}\,dt, \tag{4.21}$$

where the identity holds, at first, in the sense of bounded operators from $L^2(\mathcal{S})$ into itself. We next show that the previous equality allows to extend the action of $(H+c)^{-\alpha}$ to $\Delta^\alpha u$ for u a finite linear combination of eigenfunctions. In fact, we both make sense of $(H+c)^{-\alpha}\,\Delta^\alpha u$ and prove some bound of the form

$$\big\|\,(H+c)^{-\alpha}\,\Delta^\alpha u\big\|_{L^2} \lesssim \|u\|_{L^2}.$$

- For the integral over $[1,\infty)$ we use that

$$e^{-tH} = e^{-\frac{1}{2}H}e^{-(t-\frac{1}{2})H}.$$

It follows from item *(2)* of Theorem 17 and $\alpha<\alpha'$ that $e^{-\frac{1}{2}(H+c)}(\Delta+1)^\alpha u$ is well–defined and satisfies some bound of the form

$$\big\|e^{-\frac{1}{2}(H+c)}(\Delta+1)^\alpha u\big\|_{L^2} \leqslant C\|u\|_{L^2}$$

for a positive constant C independent of u. As

$$\big\|e^{-(t-\frac{1}{2})(H+c)}\big\|_{\mathcal{B}(L^2,L^2)} \lesssim e^{-(t-\frac{1}{2})(m-\lambda_0(\widehat{\xi}))}$$

we see that $(\Delta+1)^\alpha\frac{1}{\Gamma(\alpha)}\int_1^\infty e^{-t(H+c)}t^{\alpha-1}u$ is well–defined and we have

$$\|\frac{1}{\Gamma(\alpha)}\int_1^\infty e^{-t(H+c)}t^{\alpha-1}(\Delta+1)^\alpha u dt\|_{L^2} \lesssim \|u\|_{L^2}\int_1^\infty t^{\alpha-1}e^{-(t-\frac{1}{2})(c+\lambda_0(\widehat{\xi}))}\,dt,$$

with a finite integral

- For the integral over $(0,1]$ note first that

$$\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}e^{-t(\Delta+c1)}\,dt = (\Delta+c)^{-\alpha} - \frac{1}{\Gamma(\alpha)}\int_1^\infty t^{\alpha-1}e^{-t(\Delta+c)}\,dt,$$

where the second integral converges absolutely again by the spectral gap argument, and is smoothing since we deal with the usual heat kernel. So one has

$$\|\int_0^1 t^{\alpha-1}e^{-t(\Delta+c)}(\Delta+1)^\alpha u\,dt\|_{L^2} \lesssim \|u\|_{L^2} + \|\int_1^\infty e^{-t(\Delta+c)}(\Delta+1)^\alpha u\,dt\|_{L^2} \lesssim \|u\|_{L^2}.$$

We use the continuity property of the map $A : H^{-2\alpha'}(\mathcal{S}) \mapsto t^{-\alpha'}C((0,T],H^\alpha(\mathcal{S}))$ to write

$$\|\int_0^1 t^{\alpha-1}e^{-tc}A(t)(\Delta+1)^\alpha u\,dt\|_{L^2} \lesssim \|u\|_{L^2}\int_0^1 t^{\alpha-\alpha'}dt < \infty,$$

as $0<\alpha<\alpha'<1$. This shows that one has indeed $\big\|\,(H+c)^{-\alpha}\,\Delta^\alpha u\big\|_{L^2} \lesssim \|u\|_{L^2}$.

We conclude the proof of Theorem 34 with the identity

$$\begin{aligned}\left\langle\Delta^\alpha u,u\right\rangle_{L^2} &= \left\langle(H+c)^\alpha\,(H+c)^{-\alpha}\,\Delta^\alpha u,u\right\rangle_{L^2} = \left\langle(H+c)^{-\alpha}\,\Delta^\alpha u,(H+c)^\alpha u\right\rangle_{L^2} \\ &\leqslant \big\|\,(H+c)^{-\alpha}\,\Delta^\alpha u\big\|_{L^2}\big\|(H+c)^\alpha u\big\|_{L^2}.\end{aligned}$$

The term $\|(H+c)^\alpha u\|_{L^2}$ is finite since the quasimode u is in the domain of H. It is further of size $O(a^\alpha)$ for $u\in\pi_{\leq a}(L^2(\mathcal{S}))$, which gives the conclusion. ▷

5 – Anderson Gaussian free field

We fix throughout this section a random variable

$$c = c(\omega) > \big|\lambda_0(\widehat{\xi})\big|,$$

with $\omega\in\Omega$ the probability space on which the space white noise ξ is defined. The operator $H+c$ is thus positive and one defines a distribution valued Gaussian field with covariance $(H+c)^{-1}$. We call it the *Anderson Gaussian free field.* It is denoted by ϕ and defined by the formula

$$\phi := \sum_{n\geq 0} \gamma_n \big(\lambda_n(\widehat{\xi}) + c\big)^{-1/2} u_n, \tag{5.1}$$

where the γ_n are independent, identically distributed, real-valued random variables with law $\mathcal{N}(0,1)$, defined on a probability space Ω' with expectation operator $\mathbb{E}$. This random variable has thus two independent layers of randomness, one coming from H, that is ξ, and the other coming from the γ_n. A notation emphasizing that fact would be

$$\phi(\omega,\omega') = \sum_{n\geq 0} \gamma_n(\omega')\, \frac{u_n(\omega)}{\big(\lambda_n(\omega)+c(\omega)\big)^{1/2}},$$

for two chances elements (ω,ω') in the product space $\Omega\times\Omega'$. The environment, or chance element ω, is fixed from now on until Corollary 38. We do not keep track of the dependence on c in the notation for ϕ. We start by giving an almost sure regularity estimate for the Anderson Gaussian free field. As the classical Gaussian free field in dimension 2 it turns out to have regularity 0^-. Then we construct the Wick square of ϕ in Theorem 37 and prove in Theorem 38 that the law of the spectrum of H is encoded in the law of the random partition function of the Wick square of ϕ.

We first show that the random field ϕ is (ω,ω') almost surely essentially 0^- regular. Recall one can think of $\alpha' < 1$ as arbitrarily close to 1.

Theorem 35 – *The Anderson Gaussian free field is almost surely in $H^{-\nu}(\mathcal{S})$, for every $\nu > 1-\alpha'$.*

Proof – Use the fact that the L^2-trace does not depend on the choice of an orthonormal basis of $L^2(\mathcal{S})$ to write

$$\begin{aligned}\mathbb{E}\Big[\big\|\,(\Delta+1)^{-\frac{\nu}{2}}\,\phi\big\|_{L^2}^2\Big] &= \sum_{n\geq 0}\frac{1}{\lambda_n+c}\Big\langle u_n, (\Delta+1)^{-\nu}u_n\Big\rangle_{L^2} = \sum_{n\geq 0}\frac{1}{\lambda_n+c}\|u_n\|_{H^{-\nu/2}}^2 \\ &= \mathrm{tr}\big((\Delta+1)^{-\nu}(H+c)^{-1}\big).\end{aligned}$$

We check that the operator $(\Delta+1)^{-\nu}(H+c)^{-1}$ is trace class. The decomposition

$$(H+c)^{-1} = (\Delta+c)^{-1} + \int_0^1 e^{-tc}A(t)\,dt + \int_1^\infty \big(e^{-t(H+c)} - e^{-t(\Delta+c)}\big)\,dt$$

and the properties of $A(t)$ proved in item *(1)* of Theorem 17 ensure that the kernel K of the operator $(\Delta+1)^{-\nu}(H+c)^{-1}$ is continuous and such that

$$\int_{\mathcal{S}} K(x,x)\,dx < \infty.$$

It remains to prove that the Schwartz kernel K is nonnegative. Note that the Schwartz kernel of $(H+c)^{-1}$ is positive since it is defined by the convergent integral $\int_0^\infty e^{-t(H+c)}dt$ where $e^{-t(H+c)}$ has nonnegative kernel and that $(\Delta+1)^{-\nu}$ also has a nonnegative kernel by the Hadamard–Schwinger–Fock formula $\frac{1}{\Gamma(\nu)}\int_0^\infty e^{-t(\Delta+1)}t^{\nu-1}dt$ where $\Gamma(\nu) > 0$ and again the heat kernel $e^{-t(\Delta+1)}$ is positive. Therefore the composite Schwartz kernel K is also nonnegative. It follows then that the operator $(\Delta+1)^{-\nu}(H+c)^{-1}$ is trace class, with trace equal to $\int_{\mathcal{S}} K(x,x)\,dx$ by the Lemma at the bottom of p.65 in [35], Section XI.4, for a proof of this well-known fact. ▷

The above statement gives both the well-defined character of ϕ and its regularity. The usual proof of this result for the Gaussian free field uses the fact that the operator $(\Delta+1)^{-1}$ increases regularity by 2, so one can use the fact that operators that increase regularity by 2^+ in the Sobolev scale are trace class. We cannot resort to that mechanism here as $(H+c)^{-1}$ only sends $L^2(\mathcal{S})$ into $H^\alpha(\mathcal{S})$, so the usual reasoning only gives regularity -1^- for ϕ. As $\alpha' < 1$ can be chosen arbitrarily close to 1 we see that ϕ is almost surely in all the spaces $H^{-\nu}(\mathcal{S})$, for $\nu > 0$.

We note from the fact that the operator H is not conformally invariant (in law) that one cannot expect the random field ϕ to be conformally invariant.

The Cameron-Martin space of the Gaussian law of ϕ is the Hilbert space

$$\mathsf{CM} := \Big\{ h_a := \sum_{n\geq 0} \frac{a_n}{(\lambda+c)^{1/2}}\, u_n \,;\, a\in \ell^2(\mathbb{N}) \Big\},$$

with norm

$$\|h_a\|_{\mathsf{CM}} := \|a\|_{\ell^2}.$$

For $s\in(0,1)$ define the operator $(H+c)^s$ by its spectral action and the operator $(H+c)^{-s}$ on $L^2(\mathcal{S})$ by functional calculus

$$(H+c)^{-s} = \frac{1}{\Gamma(\frac{s-1}{2})} \int_0^\infty e^{-t(H+c)} t^{s-1}\, ds.$$

As one has

$$\big\|(H+c)^s h_a\big\|_{L^2}^2 = \sum_{n\geq 0} \frac{a_n^2}{\lambda_n^{1-2s}} < \infty$$

for all $0<s<1/2$, one has the continuous inclusion

$$\mathsf{CM} \subset (H+c)^{-s}\big(L^2(\mathcal{S})\big).$$

It follows from item *(1)* of Theorem 17 and real interpolation that the maps

$$e^{-t(H+c)} : L^2(\mathcal{S}) \to H^\nu(\mathcal{S})$$

have norms bounded above by $t^{-(\nu\beta')/2}$, for $0<t\leq 1$. By decomposing the integral giving $(H+c)^{-s}$ into an integral over $(0,1]$ and an integral over $(1,\infty)$, and using in the analysis of this second integral the same regularizing effect of $e^{-\frac12 (H+c)}$ as in the proof Theorem 34, one sees that $(H+c)^{-s}$ sends $L^2(\mathcal{S})$ into $H^{\nu^+}(\mathcal{S})$, for all $\nu^+ < \frac{1}{\beta'}$. So we have the continuous inclusion

$$\mathsf{CM} \subset H^{\nu^+}(\mathcal{S}), \tag{5.2}$$

with $\nu<\nu^+$. (The inequality $\nu > 1-\alpha'$ places no real constraint on ν^+ since β' is close to 1 while $(1-\alpha')$ is close to 0.)

We prove below that the Wick square $:\!\phi^2\!:$ of ϕ can be defined as a random element of $H^{-2\nu}(\mathcal{S})$. Its distribution depends on the enhanced noise $\widehat{\xi}$ since H does, so it is random. Theorem 37 below shows that the law of the spectrum of H is characterized by the law of the distribution of the random law of $:\!\phi^2\!:$. We need an intermediate result before stating and proving it. We choose below the letter 'G' for 'Green function'.

Lemma 36 – *The operator $(H+c)^{-1}$ has a Schwartz kernel $G(x,y)$ that is continuous outside the diagonal and such that*

$$G(x,y) \lesssim \big|\log d(x,y)\big|,$$

for an implicit constant independent of $x,y\in\mathcal{S}$.

Proof – The proof is a direct application of the integral representation (4.21) of $(H+c)^{-1}$, of the fact that $t^{-\frac{\alpha'}{2}}$ is integrable on $(0,1]$, and of the fact that the Green function of $\Delta+1$ has an upper bound of the form $\log d(x,y)$. ▷

For $n\geq 2$, set

$$a_n := \int \prod_{i=1}^n G(x_i, x_{i+1})\, dx_1 \dots dx_n,$$

with the convention that $x_{n+1}=x_1$ in the integral. Lemma 36 ensures that all the a_n are well-defined, for $n\geq 2$. One has actually, for $n\geq 2$,

$$a_n = \mathrm{tr}_{L^2}\big((H+c)^{-n}\big).$$

Here again it is not the (poor) regularizing property of $(H+c)^{-1}$ that ensures that $(H+c)^{-n}$ is trace class but rather Weyl estimates, Corollary 29. The quantity a_n is purely spectral as we have from Lidskii's theorem

$$a_n = \sum_{k\geq 0} \big(\lambda_k(\widehat{\xi}\,) + c\big)^{-n}. \tag{5.3}$$

Given a positive regularization parameter r denote by

$$\phi_r = e^{-r\Delta}(\phi)$$

the heat regularized Anderson Gaussian free field. We define the regularized Wick square $:\phi_r^2:$ of ϕ_r setting

$$:\phi_r^2: := \phi_r^2 - \mathbb{E}\big[\phi_r^2\big].$$

(Recall the enhanced noise $\widehat{\xi}$ is fixed and $\mathbb{E}$ stands for the expectation operator on the probability space where the γ_n are defined.) It will be crucial in the proof of the next statement that while $(H+c)^{-1}$ is not trace class, the Weyl law stated in Corollary 29 ensures that $(H+c)^{-1}$ is Hilbert-Schmidt.

Theorem 37 – *The regularized Wick square $:\phi_r^2:$ of Anderson Gaussian free field converges in law as r goes to 0, as a random variable on Ω' with values in $H^{-2\nu}(\mathcal{S})$, to a limit random variable denoted by $:\phi^2:$ and such that one has for all $\lambda\in\mathbb{C}$ sufficiently small*

$$Z(\lambda) :=\ \mathbb{E}\left[e^{-\lambda:\phi^2:(\mathbf{1})}\right] = \det{}_2\Big(\mathrm{Id} + \lambda(H+c)^{-1}\Big)^{-1/2} = \exp\left(\sum_{n\geq 2}\frac{(-\lambda)^n a_n}{2n}\right).$$

This function of λ has an analytic extension to all of $\mathbb{C}$.

Proof – We first take care of the probabilistic convergence of $:\phi_r^2:$ before looking at the partition function.

• Fix a large integer p. We first prove the convergence in $L^2(\Omega',\mathbb{E})$ of $:\phi_r^2:$ as a random variable with values in $B^{-2\nu}_{2p,2p}$; we conclude with Besov embedding and the fact that $\nu > 1-\alpha'$ can actually be chosen arbitrarily close to $1-\alpha'$.

For $0<r_1,r_2\leq 1$ hypercontractivity ensures that we have

$$\mathbb{E}\Big[\| :\phi_{r_1}^2: - :\phi_{r_2}^2: \|^{2p}_{B^{-2\nu}_{2p,2p}}\Big] \lesssim \sum_{j\geq -1} 2^{2pj(-2\nu)}\left(\int_{\mathcal{S}}\mathbb{E}\Big[P_j\big(:\phi_{r_1}^2: - :\phi_{r_2}^2: \big)(x)^2\Big]\,dx\right)^p,$$

so it suffices to see that one has an x-uniform bound

$$\mathbb{E}\Big[P_j\big(:\phi_{r_1}^2: - :\phi_{r_2}^2: \big)(x)^2\Big] = o_{r_1,r_2}(1), \tag{5.4}$$

as r_1 and r_2 go to 0. Using the definition of Littlewood–Paley blocks from the Appendix B, we get

$$\begin{aligned}
&\mathbb{E}\Big[P_j\big(:\phi_{r_1}^2: - :\phi_{r_2}^2: \big)(x)^2\Big] \\
&\quad= \int_{\mathcal{S}\times\mathcal{S}} \Big\{ 2\big(e^{-r_1\Delta}(H+c)^{-1}e^{-r_1\Delta}(z_1,z_2)\big)^2 + 2\big(e^{-r_2\Delta}(H+c)^{-1}e^{-r_2\Delta}(z_1,z_2)\big)^2 \\
&\qquad\qquad - 2\big(e^{-r_1\Delta}(H+c)^{-1}e^{-r_2\Delta}(z_1,z_2)\big)^2 - 2\big(e^{-r_2\Delta}(H+c)^{-1}e^{-r_1\Delta}(z_1,z_2)\big)^2\Big\} \\
&\qquad\quad \times P_j(x,z_1)P_j(x,z_2)\,dz_1dz_2.
\end{aligned}$$

We first start from the decomposition

$$e^{-r_1\Delta}(H+c)^{-1}e^{-r_2\Delta}(x,y) = e^{-r_1\Delta}\left(\int_0^1 e^{-t(H+c)}dt\right)e^{-r_2\Delta} + e^{-r_1\Delta}\left(\int_1^\infty e^{-t(H+c)}dt\right)e^{-r_2\Delta}.$$

Writing

$$\int_1^\infty e^{-t(H+c)}dt = e^{-\frac14(H+c)}\left(\int_1^\infty e^{-(t-\frac12)(H+c)}dt\right)e^{-\frac14(H+c)}$$

with

$$e^{-(t-\frac12)(H+c)} : L^2(\mathcal{S})\mapsto L^2(\mathcal{S})$$

with operator norm bounded by $e^{-(t-\frac12)k}$ for $k>0$, we see that

$$\int_1^\infty e^{-(t-\frac12)(H+c)}dt = O_{\mathcal{B}(L^2,L^2)}(1).$$

Since the operator $e^{-\frac14(H+c)}$ has continuous positive kernel the map

$$x \in S \mapsto e^{-\frac{1}{4}(H+c)}(x,.) \in L^2(\mathcal{S})$$

is continuous therefore we deduce that the composite operator

$$e^{-\frac{1}{4}(H+c)}\left(\int_1^\infty e^{-(t-\frac{1}{2})(H+c)}dt\right)e^{-\frac{1}{4}(H+c)}$$

has continuous Schwartz kernel. This means that one has the convergence

$$e^{-r_1\Delta}\left(\int_1^\infty e^{-t(H+c)}dt\right)e^{-r_2\Delta} \underset{r_1,r_2\to 0}{\longrightarrow} \int_1^\infty e^{-t(H+c)}dt \in C^0(\mathcal{S}\times\mathcal{S}).$$

Consider now the term $\int_0^1 e^{-t(H+c)}\,dt$ which decomposes as

$$\int_0^1 e^{-t(H+c)}\,dt = \int_0^1 \left(e^{-t(\Delta+c)} + A(t)e^{-tc}\right)dt.$$

Since $A(t,x,y) = O\big(t^{-\frac{\beta'}{2}}\big)$ the function $\int_0^1 A(t)e^{-tc}dt \in C^0(\mathcal{S}\times\mathcal{S})$ converges with a continuous kernel and

$$e^{-r_1\Delta}\left(\int_0^1 A(t)e^{-tc}dt\right)e^{-r_2\Delta} \underset{r_1,r_2\to 0}{\longrightarrow} \int_0^1 A(t)e^{-tc}dt \in C^0(\mathcal{S}\times\mathcal{S}).$$

It remains to observe that since the only 'singular' term in

$$\begin{aligned}B_{r_1,r_2}(z_1,z_2) := {}& 2\big(e^{-r_1\Delta}(H+c)^{-1}e^{-r_1\Delta}(z_1,z_2)\big)^2 + 2\big(e^{-r_2\Delta}(H+c)^{-1}e^{-r_2\Delta}(z_1,z_2)\big)^2\\ &- 2\big(e^{-r_1\Delta}(H+c)^{-1}e^{-r_2\Delta}(z_1,z_2)\big)^2 - 2\big(e^{-r_2\Delta}(H+c)^{-1}e^{-r_1\Delta}(z_1,z_2)\big)^2\end{aligned}$$

is of the form $\int_0^1 e^{-(t+r_1+r_2)\Delta}(z_1,z_2)dt$, we have the convergence

$$\lim_{r_1,r_2\to 0} B_{r_1,r_2}(z_1,z_2) = 0$$

in $C^0(\mathcal{S}\times\mathcal{S})$. We recall in identity (B.3) of Appendix B that the kernels P_j satisfy identities of the form

$$P_j(x,y) = 2^{j(\frac{d}{2}-1)}K_j(x, 2^{\frac{j}{2}}(x-y))$$

in well-chosen charts $U\times U$, where K_j is a bounded family of smooth functions. It follows that one has

$$\left|\int_{U\times U} B_{r_1,r_2}(z_1,z_2)P_j(x,z_1)P_j(x,z_2)d^2z_1d^2z_2\right| \leq C2^{-2j}\|B_{r_1,r_2}\|_{C^0(\mathcal{S}\times\mathcal{S})} \underset{r_1,r_2\to 0}{\longrightarrow} 0$$

where a positive constant C independent of j, r_1, r_2. This concludes the proof of the bound (5.4).

• Define the joint variable

$$\mathbf{X}(\phi) := \big(\phi, :\phi^2:\big) \in H^{-\nu}(\mathcal{S})\times H^{-2\nu}(\mathcal{S}),$$

and equip the product space $H^{-\nu}(\mathcal{S})\times H^{-2\nu}(\mathcal{S})$ with the norm

$$(\!|(a,b)|\!) := \|a\|_{H^{-\nu}} + \|b\|^{1/2}_{H^{-2\nu}}.$$

We consider $\mathbf{X}$ as a measurable function of ϕ. The Cameron-Martin embedding (5.2) implies that almost surely one has for all $h \in \mathsf{CM}$

$$\mathbf{X}(\phi+h) = \mathbf{X}(\phi) + 2h\phi + h^2,$$

with a well-defined product $h\phi$ since $\nu^+ - \nu > 0$. The function $(\!|\mathbf{X}(\cdot)|\!)$ satisfies then ϕ-almost surely the estimate

$$(\!|\mathbf{X}(\phi)|\!) \lesssim (\!|\mathbf{X}(\phi-h)|\!) + \|h\|_{\mathsf{CM}} \tag{5.5}$$

for all $h \in \mathsf{CM}$, for an absolute implicit multiplicative constant in the inequality. One then gets from Friz & Oberhauser generalized Fernique's theorem [18] that the random variable $(\!|\mathbf{X}(\phi)|\!)$ has a Gaussian tail. The random variable $\exp\big(-\lambda :\phi^2:(\mathbf{1})\big)$ is thus integrable for $\lambda \in \mathbb{C}$ small enough.

If one defines similarly

$$\mathsf{X}_r(\phi) := \big(\phi_r, :\phi_r^2:\big) \in H^{-\nu}(\mathcal{S}) \times H^{-2\nu}(\mathcal{S}),$$

then he function $(\!|\mathsf{X}_r(\cdot)|\!)$ also satisfies the estimate

$$(\!|\mathsf{X}_r(\phi)|\!) \lesssim (\!|\mathsf{X}_r(\phi - h)|\!) + \|h\|_{\mathsf{CM}}$$

with the same implicit constant as in (5.5). The conclusion of Fernique's generalized theorem is actually quantitative and can be written in terms of the $\overline{\mathrm{erf}}$ function

$$\overline{\mathrm{erf}}(z) = 1 - \mathrm{erf}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-a^2/2} da.$$

If one sets

$$\mu_{a,r} := \mathbb{P}\big((\!|\mathsf{X}_r(\phi)|\!) \le a\big), \qquad a'_r := \mathrm{erf}^{-1}(\mu_{a,r}),$$

for a fixed $a > 0$ such that $0 < \mu_{a,r} < 1$, then

$$\mathbb{P}\big((\!|\mathsf{X}_r(\phi)|\!) > m\big), \le \overline{\mathrm{erf}}(a'_r + \sigma m),$$

for a positive constant σ that depends only on a and the implicit constant in (5.5). As $(\!|\mathsf{X}_r(\cdot)|\!)$ is converging in $L^2(\Omega', \mathbb{E})$ to $(\!|\mathsf{X}(\cdot)|\!)$ one can choose a constant a such that $\mathbb{P}\big((\!|\mathsf{X}(\cdot)|\!) \le a\big)$ is also in $(0,1)$. It is thus possible to find an a' such that one has

$$\sup_{0<r\le 1} \mathbb{P}\big((\!|\mathsf{X}_r(\phi)|\!) > m\big) \le \overline{\mathrm{erf}}(a' + \sigma m).$$

It follows from that estimate that the family of random variables $\exp\big(-\lambda :\phi_r^2:(\mathbf{1})\big)$, for $0 < r \le 1$ and λ in a small ball of $\mathbb{C}$, is uniformly integrable; so it converges in $L^1(\Omega', \mathbb{E})$ to $\exp\big(-\lambda :\phi^2:(\mathbf{1})\big)$.

• Denote by $\|\cdot\|_{\mathrm{HS}}$ the Hilbert-Schmidt norm. One knows from Proposition 9.3.1 in Glimm & Jaffe's book [19] and the elementary properties of the Gohberg-Krein $\det_2$ determinant on the space of Hilbert-Schmidt operators that one has the equality of analytic functions

$$\mathbb{E}\left[\exp\big(-\lambda\ :\phi_r^2:(\mathbf{1})\big)\right] = \det{}_2\Big(\mathrm{Id} + \lambda e^{-2r\Delta}(H+c)^{-1}\Big)^{-1/2} \tag{5.6}$$

on the disc $|\lambda| < \|e^{-2r\Delta}H^{-1}\|_{\mathrm{HS}}$ of the complex plane. For $r > 0$ fixed the analytic continuation property of the Gohberg-Krein determinant tells us that both sides of the equation extend as a meromorphic function over all of $\mathbb{C}$.

We see the convergence of $e^{-2r\Delta}(H+c)^{-1}$ to $(H+c)^{-1}$ in the space of Hilbert-Schmidt operators by noting first that the operators $(H+c)^{-1}e^{-s\Delta}(H+c)^{-1}$ are indeed trace class for all $s \in [0,1]$ as they are symmetric non-negative and their kernels $K_s(x,y)$ satisfy the estimate

$$\int_{\mathcal{S}} K_s(x,x)\, dx < \infty$$

uniformly in $s \in [0,1]$, from the log estimate on G in Lemma 36. As in the proof of Theorem 35, it follows that

$$\begin{aligned}
\mathrm{tr}_{L^2}&\Big((H+c)^{-1}\big(e^{-2r\Delta} - 1\big)\big(e^{-2r\Delta} - 1\big)(H+c)^{-1}\Big) \\
&= \mathrm{tr}_{L^2}\Big((H+c)^{-1}e^{-4r\Delta}H^{-1}\Big) - 2\,\mathrm{tr}_{L^2}\Big(H^{-1}e^{-2r\Delta}(H+c)^{-1}\Big) + \mathrm{tr}_{L^2}\big((H+c)^{-2}\big) \\
&= \int_{\mathcal{S}} G(x,y)\, p_{4r}^{\Delta}(y,z) G(z,x)\, dzdydx - 2\int_{\mathcal{S}} G(x,y)\, p_{2r}^{\Delta}(y,z)\, G(z,x)\, dzdydx + \int_{\mathcal{S}} G(x,y)^2\, dx
\end{aligned}$$

is converging to 0.

The continuity of the $\det_2$ function on the ideal of Hilbert-Schmidt operators on $L^2(\mathcal{S})$ implies then the equality

$$\mathbb{E}\left[\exp\big(-\lambda\ :\phi^2:(\mathbf{1})\big)\right] = \lim_{r,0} \mathbb{E}\left[\exp\big(-\lambda\ :\phi_r^2:(\mathbf{1})\big)\right] = \det{}_2\Big(\mathrm{Id} + \lambda(H+c)^{-1}\Big)^{-1/2}.$$

Since the analytic continuation to all of $\mathbb{C}$ of the locally defined function $\lambda \mapsto \det_2\big(\mathrm{Id} + \lambda(H+c)^{-1}\big)$ has its zero set equal to $\big\{ -z^{-1}\,;\, z \in \sigma((H+c)^{-1})\big\}$ we see that the partition function $Z(\cdot)$ determines the spectrum of $H+c$, hence the spectrum of H. The formula involving the a_n comes from identity (5.6) and the general identity

$$\det_2(1+\lambda A) = \exp\left(-\sum_{n\geq 2} \frac{(-\lambda)^n}{n}\,\mathrm{tr}(A^n)\right),$$

valid for any Hilbert-Schmidt operator A on $L^2(\mathcal{S})$. ▷

The proof of Theorem 37 actually tells us that for every non-negative function f in $B^{1/p}_{p,\infty}(\mathcal{S})$ with $1/p < 2\nu$, one has the formula

$$Z(f) := \mathbb{E}\left[e^{-:\phi^2:(f)}\right] = \det_2\Big(\mathrm{Id} + M_{f^{1/2}}(H+c)^{-1}M_{f^{1/2}}\Big)^{-1/2}. \tag{5.7}$$

Indicators of subsets of $\mathcal{S}$ with finite perimeter are elements of the spaces $B^{1/p}_{p,\infty}(\mathcal{S})$ with $1/p < 2\nu$ – see e.g. Theorem 2 in Sickel's survey [36]. We emphasize here that the real valued quantities $Z(\lambda)$ and c_n are random and their law depend on the Riemannian metric space $(\mathcal{S}, g)$ by writing $Z(\lambda)(\mathcal{S}, g)$ and $c_n(\mathcal{S}, g)$. The next statement gives a characterization of the law of the spectrum of H, a function of $(\mathcal{S}, g)$, in terms of the law of the $c_n(\mathcal{S}, g)$. Write here $H(\mathcal{S}, g)$ to emphasize this dependence.

Corollary 38 – *Let $(\mathcal{S}_1, g_1)$ and $(\mathcal{S}_2, g_2)$ be two Riemannian closed surfaces. Then the spectra of the operators $H(\mathcal{S}_1, g_1)$ and $H(\mathcal{S}_2, g_2)$ have the same law iff the sequences $\big(c_n(\mathcal{S}_1, g_1)\big)_{n\geq 2}$ and $\big(c_n(\mathcal{S}_2, g_2)\big)_{n\geq 2}$ have the same law.*

Either condition is equivalent to the fact that the functions $Z(\cdot)(\mathcal{S}_1, g_1)$ and $Z(\cdot)(\mathcal{S}_2, g_2)$ have the same law.

Proof – Use Skorohod representation theorem to turn equality in law into almost sure equality on a different probability space.
If the two sequences $\big(c_n(\mathcal{S}_1, g_1)\big)_{n\geq 2}$ and $\big(c_n(\mathcal{S}_2, g_2)\big)_{n\geq 2}$ are equal the two functions $Z(\cdot)(\mathcal{S}_1, g_1)$ and $Z(\cdot)(\mathcal{S}_2, g_2)$ are equal, and the functions $\det_2\big(1+\lambda H(\mathcal{S}_1, g_1)\big)$ and $\det_2\big(1+\lambda H(\mathcal{S}_2, g_2)\big)$ of λ coincide on a small disk, hence on all of $\mathbb{C}$. Given the relation between the zero set of these functions and the spectrum of the operators $H(\mathcal{S}_1, g_1)$ and $H(\mathcal{S}_2, g_2)$ these spectra need to coincide. The function Z is determined by the spectrum of H since the a_n have that property from (5.3). ▷

Corollary 38 somehow says that the law of the partition function of $:\phi^2:$ determines the law of the spectrum of H.

6 – The polymer measure

The polymer measure describes the evolution of a Brownian particle in a white noise environment. Section 6.1 is dedicated to the construction of the polymer measure and the proof of some of its properties. We relate in Section 6.2 the occupation measure of a Poisson point process of polymer loops with the Wick square of the Anderson Gaussian free field. We prove Theorem 5 in Section 6.3; it provides large deviation results for the polymer measure and its bridges.

The polymer measure on path space over the 2-dimensional torus was first constructed by Cannizzaro & Chouk in [10]. Their approach consists in building the polymer measure on a time interval $[0, T]$ as the law of the solution to a stochastic differential equation of the form

$$dX_t = \nabla h(T-t, X_t)\,dt + dB_t$$

with B a Brownian motion and h a solution of a KPZ-type singular stochastic partial differential equation

$$(\partial_t - \Delta)h = |\nabla h|^2 + \xi$$

with space white noise ξ. Note that the drift in the dynamics of X needs to be a time-dependent distribution. They develop a paracontrolled approach to the study of such (partial or stochastic) equations in the setting of a 2 or 3 dimensional torus. They further proved that the law of the polymer measure is singular with respect to the law of Brownian motion; we get back that property in our setting in Proposition 40. One can find in [28] variations of the approach by Cannizzaro & Chouk – and much more.

We work throughout this section with a coupling function h identically equals to 1.

6.1 Construction and properties of the polymer measure

We construct the polymer measure in Section 6.1.1 from the semigroup $e^{-t(H-\lambda_0(\widehat{\xi}))}$ and provide a conditional spectral gap estimate for H. We show in Section 6.1.2 that the polymer diffusion has a deterministic quadratic variation process and reprove in Section 6.1.3 that the polymer measure is singular with respect to Wiener measure.

6.1.1 – Construction of the polymer measure. We construct the polymer measure as the law of the time homogeneous Markov process with generator $H - \lambda_0(\widehat{\xi})$. Shifting H by $\lambda_0(\widehat{\xi})$ will produce probability measures on path space and will be more convenient than working with time dependent measures. This makes no essential difference as long as we work on a finite time interval. It follows from the 'scaling' bound (4.8) on the heat kernel of H and Kolmogorov regularity criterion that this random process (has a modification that) takes values in the space of γ-Hölder paths, for any $\gamma < 1/2$. We denote by Q_x the polymer measure on $C^\gamma([0,T],\mathcal{S})$, for $0<\gamma<1/2$, corresponding to an initial starting point x for the (doubly) random process. It is a random measure that depends on the enhancement $\widehat{\xi}$ of the white noise ξ used in the definition of Anderson operator and its heat kernel. We call the random process associated with the polymer measure the *polymer diffusion.* When working on the infinite time interval $[0,\infty)$ the family of probability $(Q_x)_{x\in\mathcal{S}}$ turns the canonical coordinate process $X_t : \omega \mapsto \omega_t$ into a Markov process that enjoys the strong Markov property. All the elements in the domain $\mathfrak{D}(H)$ of H are in the domain of the generator of the Markov process. It follows in particular from Dynkin's formula that if u stands for an eigenfunction of H with eigenvalue λ then the process

$$e^{(\lambda-\lambda_0(\widehat{\xi}))t}u(X_t) \tag{6.1}$$

is a martingale – with respect to the universal completion of the canonical filtration, under any probability measure Q_x. This elementary fact has the following non-trivial consequence on the spectral gap of H when $\mathcal{S}$ has small volume. Recall from Proposition 25 the almost sure estimate

$$\frac{1}{mct}\exp\left(-\frac{cd(y,x)^2}{t}\right) \le p(t,x,y) \le \frac{mc}{t}\exp\left(-\frac{d(y,x)^2}{ct}\right), \qquad (0<t\le 1), \tag{6.2}$$

for the kernel of $e^{-t(H-\lambda_0(\widehat{\xi}))}$, with $c := c(\mathrm{osc}(u_0)) \ge 1$ and $m := \frac{\max u_0}{\min u_0} = 1 + \frac{\mathrm{osc}(u_0)}{\min u_0}$ – so $mc \ge 1$.

Proposition 39 – *Assume $\mu(\mathcal{S}) < 1$. Then the spectral gap of H is bigger than $\frac{1}{mc\mu(\mathcal{S})}\log\left(\frac{1}{mc\mu(\mathcal{S})}\right)$ on the event $\big\{mc < 1/\mu(S)\big\}$.*

The interest of the above conditional and rough lower bound on the spectral gap of H is that it only depends on the volume of $\mathcal{S}$ and requires no geometric assumption on $(\mathcal{S}, g)$ unlike the almost sure spectral gap results of Theorem 27 and Theorem 28. We will denote in the proof below by $\mathbb{E}_x$ the expectation operator associated with the probability measure Q_x.

Proof – Let u_1 stand for an eigenfunction of H with eigenvalue $\lambda_1(\widehat{\xi})$. Without loss of generality, and trading possibly u_1 for $-u_1$, one can assume ε small enough so that we have

$$0 < \mu\big(\{u_1 > \varepsilon\}\big),$$

so the hitting time τ of the set $\{u_1 > \varepsilon\}$ is almost surely finite under any Q_x. We know from the Krein-Rutman theorem applied to the compact and positivity improving operator $e^{-(H-\lambda_0(\widehat{\xi}))}$ that the ground state is the only eigenfunction that has constant sign; any other eigenfunction changes sign. This is particular the case of u_1, so $\mu\big(\{u_1 > \varepsilon\}\big) < \mu(\mathcal{S})$. Let

$x \in \mathcal{S}$ be a point where u_1 is null. Would the expectation $\mathbb{E}_x\big[e^{|\lambda_1(\widehat{\xi})-\lambda_0(\widehat{\xi})|\tau}\big]$ be finite we could use the optional stopping theorem on the martingale (6.1) and write

$$0 = u_1(x) = \mathbb{E}_x\Big[e^{|\lambda_1(\widehat{\xi})-\lambda_0(\widehat{\xi})|\tau}u_1(X_\tau)\Big] = \varepsilon\,\mathbb{E}_x\big[e^{|\lambda_1(\widehat{\xi})-\lambda_0(\widehat{\xi})|\tau}\big] > 0;$$

a contradiction. As one always has

$$\mathbb{E}_x\Big[e^{|\lambda_1(\widehat{\xi})-\lambda_0(\widehat{\xi})|\tau}\Big] \leq \sum_{n\geq 1} e^{|\lambda_1|n}\,Q_x(\tau \geq n-1),$$

a geometric bound $Q_x(\tau \geq n-1) \lesssim b^n$, entails that $e^{|\lambda_1(\widehat{\xi})-\lambda_0(\widehat{\xi})|}\,b \geq 1$, that is $\big|\lambda_1(\widehat{\xi}) - \lambda_0(\widehat{\xi})\big| \geq \ln\frac{1}{b}$. Now given $t \in (0,1]$ and integers n,k such that $n-1 > kt$, one has from the heat kernel bound (6.2) the estimate

$$\begin{aligned} Q_x\big(\tau \geq n-1\big) &\leq \int \overline{p}_t(x,y_1)\cdots\overline{p}_t(y_{n-2},y_k)\mathbf{1}_{\{u_1\leq\varepsilon\}}(y_1)\dots\mathbf{1}_{\{u_1\leq\varepsilon\}}(y_k)\,dy_1\dots dy_k \\ &\leq \Big(\frac{mc\mu(\mathcal{S})}{t}\Big)^{n/t} \end{aligned}$$

so the result follows by taking $t = mc\mu(\mathcal{S}) < 1$. ▷

Appendix C shows how the reasoning used in the preceding proof gives back some nice spectral gap bounds of Faber-Krahn type for the Laplace-Beltrami operator on Riemannian manifolds of arbitrary dimension and finite volume subject to a mild heat kernel bound.

6.1.2 – Quadratic variation process. We prove here that the quadratic variation of the canonical process on path space is a well-defined random variable under Q_x. This means that

$$\sum_{i=0}^{n} d(w_{t_{i+1}}, w_{t_i})^2$$

converges in $L^2(Q_x)$ to (the constant random variable) t, for each t when the mesh of a partition $0 < t_1 < \cdots < t_n < t$ of an interval $[0,t]$, with $t_0 := 0$ and $t_{n+1} := 1$, goes to 0. (Do not mingle the fact for a process to have a finite quadratic variation process and the property of its sample paths to be almost surely of finite 2-variation. Brownian motion has for instance a finite quadratic variation process on any finite interval but has almost surely an infinite 2-variation on any finite interval.) To prove the preceding convergence in probability it suffices to notice that the fine asymptotic from Theorem 17 for the heat kernel of H gives

$$\mathbb{E}_x\big[d(w_{t_{i+1}}, w_{t_i})^2\big] = t_{i+1} - t_i + O(t_{i+1}-t_i)^b \tag{6.3}$$

for a constant $b > 1$, and that

$$\mathbb{E}_x\big[d(w_{t_{i+1}}, w_{t_i})^4\big] = O(t_{i+1}-t_i)^b,$$

from the 'scaling' bound (4.8) – or the Gaussian upper bound (4.9). Chebychev inequality then gives the result. We note here for later purpose that for each t there is a sequence of partitions of the interval $[0,t]$ such that the corresponding sum of squared increments converges almost surely to t. The quadratic variation process thus depends only on the equivalence class of a finite non-negative measure on path space under the equivalence relation given by reciprocal absolute continuity.

Note that the Gaussian lower and upper estimates on the heat kernel p_t proved in Proposition 25 are not sufficient to get back the exact scaling relation (6.3). One really needs the result of Theorem 17 for that purpose.

6.1.3 – Singularity with respect to Wiener measure. The Wiener measure $P_{\mathscr{W},x}$ on $\mathcal{S}$ is the law of the Brownian motion started from x. Given a positive time horizon T it is convenient to denote by Q_x^T and $P_{\mathscr{W},x}^T$ the restrictions to $C^\alpha([0,T],\mathcal{S})$ of the measures Q_x and $P_{\mathscr{W},x}$. We denote by $\mathbb{E}_x^T$ and $\mathbb{E}_{\mathscr{W},x}^T$ their associated expectation operators. We can follow Cannizzaro & Chouk [10] to prove the following result. We define the measure $Q_{r,x}^T$ by its density

$$D_r(w) := \frac{dQ_{r,x}^T}{dP_{\mathscr{W},x}}(w) := \exp\left(-\int_0^T \Big(\xi_r + \frac{\log r}{4\pi}\Big)(w_t)dt\right)$$

with respect to $P_{\mathscr{W},x}$ – it is associated with the renormalized regularized Anderson operator $\Delta + \xi_r + \frac{\log r}{4\pi}$.

Proposition 40 – *Pick $x \in \mathcal{S}$. The polymer measure Q_x^T is $\mathbb{P}$-almost surely singular with respect to the Wiener measure $P_{\mathscr{W},x}^T$.*

Proof – The proof proceeds as in the proof of Theorem 1.4 of [10] given in Section 7.2 of the work; we recall the main points of the details for the reader's convenience. Pick a sequence $(r_n)_{n\geq 0}$ decreasing to 0 and look at the event $\limsup_n \big\{Y_{r_n} < 1\big\}$. We show that it has $P_{\mathscr{W},x}^T$-probability 1 and Q_x^T-probability 0.

- First, we have

$$\mathbb{E}_{\mathscr{W},x}^T\big[D_{r_n}^{1/2}\big] = \mathbb{E}_{\mathscr{W},x}^T\left[e^{-\frac{1}{2}\int_0^T\left(\xi_{r_n}+\log r/(4\pi)\right)(w_t)\,dt}\right] = \left(e^{-T(\Delta+\xi_{r_n}/2+(\log r_n)/(8\pi))}\mathbf{1}\right)(x).$$

One has

$$\left(e^{-T(\Delta+\xi_{r_n}/2+(\log r_n)/(8\pi))}\mathbf{1}\right)(x) = e^{T(\log r_n)/(16\pi)}\left(e^{-T(\Delta+\xi_{r_n}/2+(\log r_n)/(16\pi))}\mathbf{1}\right)(x)$$

where the last term converges as n goes to infinity as it involves the semigroup of the Anderson operator with noise $\xi/2$ – recall the quadratic dependence of the renormalization constant on the coupling constant. So

$$\lim_{n\to\infty}\mathbb{E}_{\mathscr{W},x}^T\left[D_{r_n}^{1/2}\right] = 0$$

and $\mathbb{P}_{\mathscr{W},x}^T(Y_n > 1)$ tends to 0 from Chebychev inequality. One has as a consequence

$$\mathbb{P}_{\mathscr{W},x}^T\Big(\limsup_n\big\{D_{r_n} < 1\big\}\Big) \geq \limsup_n\,\mathbb{P}_{\mathscr{W},x}^T(D_{r_n} < 1) = 1.$$

- Now for a fixed $k \geq 1$ we have

$$Q_x^T(D_{r_k} < 1) \leq \liminf_n\, Q_{r_n,x}^T(D_{r_k} < 1),$$

and

$$Q_{r_n,x}^T(D_{r_k} < 1) = \mathbb{E}_{\mathscr{W},x}^T\left[e^{-\int_0^T(\xi_{r_n}+(\log r_n)/(4\pi))(w_t)\,dt}\,D_{r_k}^{1/2-1/2}\mathbf{1}_{D_{r_k}<1}\right]$$

$$\leq \mathbb{E}_{\mathscr{W},x}^T\left[e^{-\int_0^T[\xi_{r_n}+(\log r_n)/(4\pi)-1/2(\xi_{r_k}+(\log r_k)/(4\pi))](w_t)\,dt}\right]$$

$$\leq e^{-T(\log r_k)/(16\pi)}\,\mathbb{E}_{\mathscr{W},x}^T\left[e^{-\int_0^T[\xi_{r_n}+(\log r_n)/(4\pi)-(1/2\xi_{r_k}+(\log r_k)/(16\pi))](w_t)\,dt}\right].$$

As

$$\Pi\Big(X_{r_n}+\frac{1}{2}X_{r_k},\xi_{r_n}+\frac{1}{2}\xi_{r_k}\Big) - \frac{\log r_n}{4\pi} + \frac{5}{4}\,\frac{\log r_k}{4\pi}$$

is converging in probability in $C^{2\alpha-2}(\mathcal{S})$, under $\mathbb{P}_{\mathscr{W},x}^T$, as n goes to ∞ then k goes to ∞, one sees that the quantity

$$\mathbb{E}_{\mathscr{W},x}^T\left[e^{-\int_0^T[\xi_{r_n}+(\log r_n)/(4\pi)-(1/2\xi_{r_k}+(\log r_k)/(16\pi))-(\log r_k)/(4\pi)](w_t)\,dt}\right]$$

is converging as n goes first to ∞ then k goes to ∞. It follows that

$$Q_{r_n,x}^T(D_{r_k} < 1) \lesssim e^{\frac{3T}{4}\,\frac{\log r_k}{4\pi}}$$

uniformly in n and k, so

$$Q_x^T(D_{r_k} < 1) \lesssim e^{\frac{3T}{4}\,\frac{\log r_k}{4\pi}}.$$

Choosing a sequence r_k that decreases sufficiently fast to 0 provides then an upper bound for $Q_{r_n,x}^T(D_{r_k} < 1)$ that allows to conclude with Borel-Cantelli lemma that

$$Q_x^T\Big(\limsup_k\{D_{r_k} < 1\}\Big) = 0.$$

(The speed of convergence of r_k to 0 will depend on T.) ▷

6.2 Wick square of Anderson Gaussian free field and polymer measure

The study of the links between Markov fields and Poissonian ensembles of Markov loops goes back to Symanzik' seminal work [41]. It was elaborated in a large number of works and we take advantage here of the general result proved by Le Jan in [30], giving a correspondance between the occupation measure of a loop ensemble and Wick square of some Gaussian free field – see Section 9 therein. It allows at no cost to relate (a measure built from) the polymer measure to the Wick square of the Anderson free field that was the object of Theorem 37. We dress the table before bringing the dish.

Rather than working with the polymer measure built from the operator $H - \lambda_0(\widehat{\xi})$ we pick a positive constant a and work with the measure built from $H - \lambda_0(\widehat{\xi}) + a$. With the notations of Section 5 one takes here $c = -\lambda_0(\widehat{\xi}) + a$. This choice ensures that the Green function of the corresponding operator is finite and has the properties stated and used in Section 5. This amounts to add killing at constant rate for the process built in Section 6.1.1. This does not change its properties and we have in particular that the corresponding polymer paths have an associated quadratic variation process equal to the traveling time and defined on the random lifetime interval $[0,\zeta)$. Denote by $\overline{p}_t(x,y)$ the transition density of the process built from $H - \lambda_0(\widehat{\xi}) + a$ and denote by $\overline{P}^t_{x,x}$ the unnormalized excursion measure of duration t started from $x \in \mathcal{S}$. It is characterized by the identity

$$\overline{P}^t_{x,x}\Big(X_{t_1} \in dx_1, \dots X_{t_k} \in dx_k\Big) = \overline{p}_{t_1}(x,x_1)\overline{p}_{t_2-t_1}(x_1,x_2)\dots\overline{p}_{t-t_k}(x_k,x)\,dx_1\dots dx_k$$

for all $0 \leq t_1 \leq \cdots \leq t_k \leq t$. This non-negative measure has finite mass equal to $\overline{p}_t(x,x)$. A standard argument using the symmetry of $\overline{p}_t(x,y)$ as a function (x,y) shows that the measure $\overline{P}$ is supported on (rooted) loops of Hölder regularity strictly less than 1/2. The loop measure is defined as

$$\mathscr{M}(\cdot) := \int_{\mathcal{S}}\int_0^\infty \frac{1}{t}\,\overline{P}^t_{x,x}(\cdot)\,dt\,\mu(dx).$$

It follows from the result of Section 6.1.2 that the factor $1/t$ in this integral accounts for the intrinsic lifetime of the loop – so this non-negative measure is indeed a measure on unrooted loops. Note that it has an infinite mass that comes from the mass of small loops. Denote by $\mathbb{E}_{\mathscr{M}}$ the expectation operator associated with $\mathscr{M}$ and by $\zeta(\ell)$ the lifetime of a loop ℓ. For such a loop we define a measure on $\mathcal{S}$ setting

$$\widehat{\ell}(\cdot) := \int_0^{\zeta(\ell)} \delta_{\ell(s)}(\cdot)\,ds.$$

One has for any non-negative function f on $\mathcal{S}$ and all $n \geq 1$

$$\mathbb{E}_{\mathscr{M}}\big[\widehat{\ell}(f)^n\big] = (n-1)!\int_{\mathcal{S}^n} G(x_1,x_2)f(x_2)G(x_2,x_3)f(x_3)\cdots G(x_n,x_1)f(x_1)\,dx_1\dots dx_n, \tag{6.4}$$

and

$$\mathbb{E}_{\mathscr{M}}\Big[e^{-z\widehat{\ell}(f)} + z\widehat{\ell}(f) - 1\Big] = -\log\det{}_2\Big(\mathrm{Id} + zM_{f^{1/2}}GM_{f^{1/2}}\Big),$$

from an elementary series expansion and the preceding equality. (We used here the same notation for the Green kernel G and its associated operator. Le Jan's proof [30] of identity (6.4) applies verbatim here.)

Given $\gamma \geq 0$ denote by Λ_γ a Poisson process on the space of (unrooted) loops over $\mathcal{S}$ with intensity $\gamma\mathscr{M}$. It is characterized by its characteristic function

$$\mathbb{E}\big[e^{i\Lambda_\gamma(F)}\big] = \exp\left(\gamma\int\big(e^{iF(\ell)} - 1\big)\,\mathscr{M}(d\ell)\right),$$

for all functions F on loop space that are null on loops of sufficiently small lifetime – so the resulting quantity $\Lambda_\gamma(F)$ is almost surely well-defined. Denote by A_γ the support of Λ_γ, so $\Lambda_\gamma = \sum_{\ell\in A_\gamma}\delta_\ell$. The regularized renormalized occupation measure of Λ_γ is defined for each $r > 0$ as the non-negative measure on $\mathcal{S}$

$$\mathcal{O}^r_\gamma(f) := \sum_{\ell\in A_\gamma} \mathbf{1}_{\eta(\ell)>r}\,\widehat{\ell}(f) - \gamma\,\mathbb{E}_{\mathscr{M}}\big[\mathbf{1}_{\eta(\ell')>r}\,\widehat{\ell'}(f)\big];$$

the expectation is over ℓ' and f is a generic non-negative continuous function on $\mathcal{S}$. For γ and f fixed the continuous time random process $\gamma\mapsto\mathcal{O}^r_\gamma(f)$ is actually a Lévy process with positive jumps with characteristic function

$$\mathbb{E}\Big[e^{-\mathcal{O}^r_\gamma(f)}\Big] = \exp\Big(-\gamma\,\mathbb{E}_{\mathscr{M}}\Big[\mathbf{1}_{\zeta(\ell')>r}\big(e^{-\widehat{\ell}(f)}+\widehat{\ell}(f)-1\big)\Big]\Big)$$

converging to its natural limit as r goes to 0. The limit Lévy process is denoted by $(\mathcal{O}_\gamma(f))_{\gamma\geq 0}$. (All this is explained in detail in Le Jan's work [30].) The following result follows from the preceding analysis and formula (5.7) for the partition function of the Wick square of the Anderson Gaussian free field.

Theorem 41 – *One has for every continuous function f on $\mathcal{S}$ the identity*

$$\mathbb{E}\big[e^{-\mathcal{O}_{1/2}(f)}\big] = \mathbb{E}\big[e^{-:\phi^2:(f)}\big].$$

One deduces from this identity that the renormalized occupation measure of the loop measure of polymer paths has the same distribution as the Wick square of the Anderson Gaussian free field. It has in particular a version that has almost surely regularity -2η in the Sobolev scale. This identification does not tell us that $\mathcal{O}_{1/2}$ is a measure, despite its name.

6.3 Large deviation principles for the polymer measure and its bridges

We prove in this section that the polymer measure on free end paths and bridges satisfies the same large deviation results as Wiener measure and its induced bridge measures. These results were stated as Theorem 5 in the introduction. The effect on these measures of the white noise environment is thus evanescent as the traveling time goes to 0. On a technical level one can trace this fact back to Theorem 17. This statement implies in particular that the effect of the random environment is contained in the correction term to the Riemannian heat kernel. The conclusion will follow from the fact that large deviation results are essentially driven by the dominant term in the small time heat kernel expansion – the proof below will make that point clear.

Our proof of the large deviation results of Theorem 5 follows partly the proofs of the analogue statements for Wiener measure on $\mathcal{S}$ and its bridges. We give some details on the large deviation result for Q_x, as we give a non-classical proof, and give the essential ingredients of the proof of the corresponding result for the bridges of polymer paths. Pick $0<\gamma<1/2$. Given $0<r\leq 1$ let $Q^{(r)}_x$ be the image measure of the restriction to $C^\gamma([0,r],\mathcal{S})$ by the time change map $s\in[0,1]\mapsto sr$ – this is a non-negative finite measure on $C^\gamma([0,1],\mathcal{S})$ for all $0<r\leq 1$.

6.3.1 – Large deviation principle for $Q^{(r)}_x$. Pick $x\in\mathcal{S}$. Most proofs of the large deviation principle for the Wiener measure $P_{\mathscr{W},x}$ use its dynamical description as the law of a diffusion process solution of a stochastic differential equation, for which one can resort to Freidlin & Wentzell theory of large deviations. (Rough paths theory provides an economical way of understanding the large deviation principles obtained in this way from a unique large deviation principle satisfied by the Brownian rough path.) We cannot proceed similarly here as stochastic differential equations cannot be used to describe the typical dynamics of a polymer path.

We use a different way of proving a large deviation principle, by proving a large deviation principle for the finite dimensional time marginals of the process and proving that the family of measures is ($\mathbb{P}$-almost surely) exponentially tight. One can then resort to the general theory, such as exposed for instance in Section 4.7 of Feng & Kurtz textbook [17], to conclude. The identification of the (good) rate function as the function $\mathscr{I}(\cdot)$ from (1.4) comes from the fact that the finite dimensional large deviation principle involves the squared geodesic distance, a consequence of the asymptotic behaviour of the heat kernel of H stated in item *(i)* of Theorem 17.

Proposition 42 – *Fix $0<s_1<\cdots<s_n\leq 1$ and subsets $A_1,\dots,A_n$ of $\mathcal{S}$. One has*

$$\limsup_{r\to 0^+} r\log Q^{(r)}_x\big(w_{s_1}\in\mathring{A}_1,\dots,w_{s_n}\in\mathring{A}_n\big)$$
$$\geq \inf\left\{\sum_{i=0}^n (s_{i+1}-s_i)\,d(x_{i+1},x_i)^2\,;\,x_0=x,x_1\in\mathring{A}_1,\dots,x_n\in\mathring{A}_n\right\},$$

and

$$\liminf_{r\to 0^+} r\log Q_x^{(r)}\big(w_{t_1}\in\overline{A_1},\dots,w_{t_n}\in\overline{A_n}\big)$$
$$\leq \inf\left\{\sum_{i=0}^{n}(s_{i+1}-s_i)\,d(x_{i+1},x_i)^2\,;\,x_0=x,x_1\in\overline{A_1},\dots,x_n\in\overline{A_n}\right\}.$$

We recognize in the infimum the rate function satisfied by the finite dimensional marginals of Brownian motion on $\mathcal{S}$.

Proof – This is a direct consequence of the exact formula

$$\int p_{s_1r}(x,x_1)\mathbf{1}_{x_1\in B_1}p_{(s_2-s_1)r}(x_1,x_2)\cdots p_{(s_n-s_{n-1})r}(x_{n-1},x_n)\mathbf{1}_{x_n\in B_n}\,dx_n\cdots dx_1$$

for

$$Q_x^{(r)}\big(w_{s_1}\in B_1,\dots,w_{s_n}\in B_n\big),$$

valid for any subsets $B_1,\dots,B_n$ of $\mathcal{S}$, the sharp Gaussian asymptotic giving p_t as a $O(t^{-\beta'/2})$ perturbation of the heat kernel of the Laplace operator, and an elementary change of variable. ▷

Here again we note that lower and upper Gaussian estimates on p_t would not be sufficient to prove Proposition 42. Pick $0<\gamma<\frac{1}{2}$. We obtain the exponential tightness of the family $\big(Q_x^{(r)}\big)_{0<r\leq 1}$ by proving that the γ-Hölder norm $\|w\|_\gamma$ of a typical polymer path has a Gaussian moment. We denote by $E_x^{(r)}$ the expectation operator associated with the finite non-negative measure $Q_x^{(r)}$.

Proposition 43 – *There is a positive constant* c_0 *such that one has*

$$E_x^{(r)}\big[\exp(c_0\|w\|_\gamma^2)\big]<\infty, \tag{6.5}$$

uniformly in $0<r\leq 1$ *and* $x\in\mathcal{S}$.

Proof – We use Besov inequality

$$\|w\|_\gamma^{2k}\lesssim_\gamma \int_0^1\int_0^1\left(\frac{d(w_t,w_s)}{|t-s|^{1/2}}\right)^{2k}dsdt,$$

valid for any continuous path w, any integer $k\geq 1$ and $0<a<1/2$, to get from the scaling bound (4.8) and a time change of variable the bound

$$E_x^{(r)}\big[\|w\|_\gamma^{2k}\big]\lesssim_\gamma \int_0^1\int_0^1|t-s|^{-k}\,E_x^{(r)}\big[d(w_t,w_s)^{2k}\big]\,dsdt\lesssim_\gamma r^{(1+o(1))k}2^k k! \tag{6.6}$$

– the conclusion follows. ▷

The exponential tightness of the family $\big(Q_x^{(r)}\big)_{0<r\leq 1}$ of finite non-negative measures on the space $C^\gamma([0,1],\mathcal{S})$ and the identification of the large deviation principle satisfied by its finite dimensional marginals entail that the family $\big(Q_x^{(r)}\big)_{0<r\leq 1}$ satisfies itself a large deviation principle in $C([0,1],\mathcal{S})$ with rate function determined by the rate function of the finite dimensional large deviation principle – see for instance Theorem 4.30 in [17]. As the latter rate function is identical to the rate function of the large deviation principle satisfied by the finite dimensional marginals of Brownian motion, this leads to the identification of the rate function $\mathscr{I}(\cdot)$ as the functional (1.4). This is a good rate function. This proves the first item of Theorem 5.

6.3.2 – Large deviation principle for the bridge probability measures $Q_{x,x,\cdot}^{(r)}$. The proof of the large deviation result for the bridges of polymers follows from the large deviation result for $Q_x^{(r)}$ proved in Section 6.2.1 and the following two analytic estimates that are consequences of our estimates on the heat kernel of H. One has

$$\lim_{r\searrow 0} r\log p_r(x,y)=-\frac{d(x,y)^2}{2}, \tag{6.7}$$

uniformly in $x, y \in \mathcal{S}$, and

$$p_r(x,y) \leq cr^{-1}, \tag{6.8}$$

for a positive constant c and all $x, y \in \mathcal{S}$ and $0 < r \leq 1$. The pattern of proof was devised in [25] by E. P. Hsu in his study of the large deviation principle for the bridges of Brownian motion. As it works almost verbatim here we will only sketch the lines of the reasoning, refering to [25] for the details. We fix for the remainder of this section two distinct points x, y of $\mathcal{S}$. Recall the notations of Section 1.

$\triangleright$ *Step 1. Exponential tightness of the $Q^{(r)}_{x,y}$ in* $C^\gamma([0,1],\mathcal{S})$. We describe below how to prove this fact. As the inclusion of $C^\gamma([0,1],\mathcal{S})$ into $C^0([0,1],\mathcal{S})$ is continuous it suffices, by the inverse contraction principle, to prove that the probability measures $Q^{(r)}_{x,y}$ satisfy a large deviation principle in $C([0,1],\mathcal{S})$ with good rate function $\mathscr{I}(\cdot) - d^2(x,y)$, to prove the second point of Theorem 5. This is the object of *Step 2*. Set

$$\Omega_{x,y} := \Big\{\omega \in C([0,1],\mathcal{S})\,;\, \omega(0) = x,\, \omega(1) = y\Big\};$$

Given an integer $n \geq 1$ and $k_n \in \mathbb{N}\backslash\{0\}$ to be fixed later, the formula

$$C^n_{x,y} := \left\{\omega \in \Omega_{x,y}\,;\, \sup_{\substack{s,t\in[0,1]\\ 0<t-s\leq 1/n}} \frac{|\omega_t - \omega_s|}{|t-s|^\gamma} \leq 1\right\}$$

defines a compact subset of both $C([0,1],\mathcal{S})$ and $C^\gamma([0,1],\mathcal{S})$. We prove that one has the exponential tightness estimate

$$\overline{\lim_{r\downarrow 0}}\, r\log Q^{(r)}_{x,y}\big(\Omega_{x,y}\backslash C^n_{x,y}\big) \leq -n^{1-2\gamma}.$$

It is convenient for that purpose to introduce the two sets

$$C^{n,1}_{x,y} := \left\{\omega \in \Omega_{x,y}\,;\, \sup_{\substack{s,t\in[0,2/3]\\ 0<t-s\leq 1/n}} \frac{|\omega_t - \omega_s|}{|t-s|^\gamma} \leq 1\right\}, \qquad C^{n,2}_{x,y} := \left\{\omega \in \Omega_{x,y}\,;\, \sup_{\substack{s,t\in[1/3,1]\\ 0<t-s\leq 1/n}} \frac{|\omega_t - \omega_s|}{|t-s|^\gamma} \leq 1\right\}$$

and prove separately

$$\overline{\lim_{r\downarrow 0}}\, r\log Q^{(r)}_{x,y}\big(\Omega_{x,y}\backslash C^{n,i}_{x,y}\big) \leq -n^{1-2\gamma}, \tag{6.9}$$

for $i \in \{1,2\}$. One can concentrate on the $i = 1$ case as one gets the estimate for $i = 2$ from the estimate for $i = 1$ by using the symmetry of H to say that

$$Q^{(r)}_{x,y}\big(\Omega_{x,y}\backslash C^{n,2}_{x,y}\big) = Q^{(r)}_{y,x}\big(\Omega_{y,x}\backslash C^{n,1}_{y,x}\big).$$

First, the inequality

$$(\star)_r := Q^{(r)}_{x,y}\big(\Omega_{x,y}\backslash C^{n,1}_{x,y}\big) \leq \frac{n}{3} \sup_{0\leq s_0\leq 2/3} Q^{(r)}_{x,y}\left(\sup_{s_0\leq t_1<t_2\leq s_0+2/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2-t_1|^\gamma} > 1\right),$$

guarantees by (6.8) that one has

$$\begin{aligned}(\star)_r &\lesssim n \sup_{0\leq s'_0\leq (2r)/3} E_x\left[\frac{p_{r-s'_0-(2r)/n}\big(\omega_{r(s'_0+(2r)/n)}, y\big)}{p_r(x,y)}\,;\, \sup_{s'_0\leq t_1<t_2\leq s'_0+(2r)/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2-t_1|^\gamma} > 1\right] \\ &\lesssim \frac{nr^{-1}}{p_r(x,y)} \sup_{z\in\mathcal{S}} P_z\left(\sup_{0\leq t_1<t_2\leq (2r)/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2-t_1|^\gamma} > 1\right).\end{aligned} \tag{6.10}$$

If one rewrites the bound (6.6) under the form

$$E^{(r_1)}_z\Big[\exp\big(c_0 r_1^{-1}\|\omega\|^2_\gamma\big)\Big] \lesssim 1$$

for an implicit multiplicative constant uniform in $0 < r_1 \leq 1$ sufficiently small and $z \in \mathcal{S}$, one can use the exponential form of Chebychev inequality to estimate the term

$$P_z\left(\sup_{0\leq t_1<t_2\leq (2r)/n} \frac{|\omega_{t_2} - \omega_{t_1}|}{|t_2-t_1|^\gamma} > 1\right) = P^{(2r/n)}_z\Big(\|\omega\|_\gamma > (2r/n)^\gamma\Big)$$

in (6.10) and get from (6.7) the estimate

$$r\log(\star)_r \lesssim -r\log p_r(x,y) + r\log(nr^{-1}) + r\log\Big(\sup_z P_z(\cdots)\Big)$$
$$\lesssim \frac{d(x,y)^2}{2} + o_r(1) - (n/r)^{1-2\gamma}.$$

As $0<\gamma<1/2$ this proves (6.9) for $i=1$. (Remark that the only thing that matters here in the term $r\log p_r$ is the fact that it is uniformly bounded in r on $\mathcal{S}^2$. The precise asymptotic has no importance here, while it is fundamental in the details of the proof of the upper and lower bounds in *Step 2*.)

$\triangleright$ *Step 2. Upper and lower bounds for the large deviation principle.* The proofs of the upper and lower bounds for the large deviation principle satisfied by the $Q^{(r)}_{x,y}$ follow verbatim Hsu's proof [25] of the corresponding principle for the Brownian bridge measure, as the only ingredients he uses are the heat kernel estimates (6.7) and (6.8) and the Brownian equivalent of the exponential tightness result established in the first step. We do not repeat the proof here and refer the reader to Hsu's proof, pp. 109-112. (Hsu works in an unbounded complete Riemannian manifold. The details of [25] were reworked in the simpler setting of a compact manifold, for hypoelliptic diffusions, in Section 2 of [3].)

A – Meromorphic Fredholm theory with a parameter

We prove Theorem 7 in this section. As a guide to the subject of this appendix the reader will find in Appendix D of Zworski's book [42] an elementary account of the usual, parameter free, meromorphic Fredholm theory.

Proof – Our proof follows closely the proof given by Borthwick in Theorem 6.1 of [8]. It suffices to prove the result near any $z_0\in U$ which contains only finitely many poles of K. With this assumption, we may decompose

$$K(z,\mathbf{a}) = A(z,\mathbf{a}) + F(z,\mathbf{a}),$$

where $F(z,\mathbf{a})$ is a meromorphic family of finite-rank operators for $z\in U$ and $A(z,\mathbf{a})$ is a holomorphic family of compact operators. Both operators depend continuously on the parameter $\mathbf{a}$. Using the approximation of the compact operator $A(z_0,\mathbf{a})$ by finite-rank operators, and assuming U is sufficiently small and that we choose a sufficiently small neighborhood of $\mathbf{a}_0$, we can find a fixed finite-rank operator B such that

$$\big\|A(z,\mathbf{a}) - B\big\| < 1$$

for all $z\in U$. Note that implies that $\mathrm{Id} - A(z,\mathbf{a}) + B$ is holomorphically invertible for $z\in U$, by the usual Neumann series as

$$\big(\mathrm{Id} - A(z,\mathbf{a}) + B\big)^{-1} = \sum_{k=1}^{\infty}\big(A(z,\mathbf{a}) - B\big)^k.$$

Since the Neumann series converges absolutely in $\mathcal{B}(\mathcal{H},\mathcal{H})$ uniformly in $(z,\mathbf{a})$ in some neighborhood of $(z_0,\mathbf{a}_0)$ and each term $\big(A(z,\mathbf{a}) - B\big)^k$ is continuous in u, it follows that the map

$$\mathbf{a}\mapsto \big(\mathrm{Id} - A(z,\mathbf{a}) + B\big)^{-1} \in \mathcal{B}(\mathcal{H},\mathcal{H})$$

is continuous. Thus if we set

$$G(z,\mathbf{a}) := \big(F(z,\mathbf{a}) + B\big)\big(\mathrm{Id} - K(z,\mathbf{a}) + B\big)^{-1}$$

then we can write

$$\mathrm{Id} - K(z,\mathbf{a}) = \big(\mathrm{Id} - G(z,\mathbf{a})\big)\big(\mathrm{Id} - K(z,\mathbf{a}) + B\big)^{-1}.$$

It is immediate that $G(z,\mathbf{a})$ has finite rank and depends continuously on $\mathbf{a}$ by its construction involving the finite rank operators $F(z,\mathbf{a}), B$. We already know that $\big(\mathrm{Id} - K(z,\mathbf{a}) + B\big)^{-1}$ is holomorphic in z near z_0 and depends continuously on $\mathbf{a}$, so the problem is reduced to proving the meromorphic invertibility of $\big(\mathrm{Id} - G(z,\mathbf{a})\big)$ and the continuity with respect to the parameter

a. Recall that $G(z,\mathbf{a})$ is meromorphic in z, continuous in $\mathbf{a}$, with finite rank, so we can always represent it as

$$G(z,\mathbf{a}) = \sum_{1\leqslant i,j\leqslant p} a_{ij}(z,\mathbf{a})\, |\varphi_i >< \psi_j|$$

where the coefficients $a_{ij}(z,\mathbf{a})$ are meromorphic in z, continuous in $\mathbf{a}$ and $(\varphi_i)_{i=1}^p$ is a finite family of linearly independent vectors in $\mathcal{H}$. To solve $\big(\mathrm{Id} - G(z,\mathbf{a})\big)v = w$ where w is given, we make the ansatz $v = w + \sum_{i=1}^p b_i\varphi_i$ therefore the equation becomes

$$\big(\mathrm{Id} - G(z,\mathbf{a})\big)v = \big(\mathrm{Id} - G(z,\mathbf{a})\big)\left(w + \sum_{i=1}^p b_i\varphi_i\right)$$

$$= w + \sum_{i=1}^p b_i\varphi_i - \sum_{1\leqslant i,j\leqslant p,k} b_k a_{ij}(z,\mathbf{a})\varphi_i \langle\psi_j,\varphi_k\rangle - \sum_{1\leqslant i,j\leqslant p} a_{ij}(z,\mathbf{a})\varphi_i \langle\psi_j,w\rangle$$

that simplifies to the simpler relation

$$\sum_{i=1}^p b_i\varphi_i - \sum_{1\leqslant i,j\leqslant p,k} b_k a_{ij}(z,u)\varphi_i \langle\psi_j,\varphi_k\rangle = \sum_{1\leqslant i,j\leqslant p} a_{ij}(z,\mathbf{a})\varphi_i \langle\psi_j,w\rangle\,.$$

By linear algebra, the above equation can be solved on the complement of the zero locus of the polynomial

$$\det\left(\delta_{ik} - \sum_j a_{ij}(z,\mathbf{a})\,\langle\psi_j,\varphi_k\rangle\right)$$

which depends meromorphically on z and continuously on $\mathbf{a}$. So away from the zero locus of the determinant we can meromorphically invert $\mathrm{Id}-G(z,\mathbf{a})$ hence $\mathrm{Id}-K(z,\mathbf{a})$ and everything depends continuously on the parameter $\mathbf{a}$. The fact that the poles have finite rank comes from the fact that they only appear through the finite rank operator $G(z,\mathbf{a})$. ▷

B – Geometric Littlewood-Paley decomposition

We recall from Klainerman & Rodnianski's work [27] the basics of Littlewood-Paley decomposition in a manifold setting. We use if to provide a self-contained proof of Proposition 8 on the renormalization of $\Pi(h\xi_r, X_{h,r})$, and Lemma 9 and Lemma 10, both used in the construction of the resolvent of H in Section 3.1.

Theorem 44 – *Given $\ell\in\mathbb{N}$ there exists a Schwartz function m such that*

$$\int_0^\infty t^{k_1}\partial_t^{k_2} m(t)\,dt = 0 \qquad (\forall (k_1,k_2),\ k_1+k_2\leqslant \ell) \tag{B.1}$$

and such that the self–adjoint smoothing operators

$$P_k = \int_0^\infty 2^{2k} m(2^{2k}t) e^{-t\Delta}\,dt \qquad (k\in\mathbb{N}\cup\{-1\}) \tag{B.2}$$

enjoy the following properties.

(a) **Resolution of the identity.** *One has* $\sum_{k\geq -1} P_k = \mathrm{Id}$.

(b) **Bessel inequality.** *One has*

$$\sum_{k\geq 0} \|P_k f\|_{L^2} \lesssim \|f\|_{L^2}.$$

(c) **Finite band property.** *One has*

$$\|\Delta P_k f\|_{L^p} \lesssim 2^{2k}\|f\|_{L^p},$$

and

$$\|P_k f\|_{L^p} \lesssim 2^{-2k}\|\Delta f\|_{L^p};$$

also we have the dual estimate $\|P_k\nabla f\|_{L^2} \lesssim 2^k\|f\|_{L^2}$,

(d) **Flexibility property.** *There exists a function* $\tilde{m}$ *satisfying* (B.1) *such that* $\Delta P_k = 2^{2k}\tilde{P}_k$ *and the family* $(\tilde{P}_k)_k$ *is a Littlewood–Paley decomposition which might not satisfy the resolution of identity equation.*

We quickly recall the main features of the heat calculus we shall use in the sequel. The heat calculus is a way to encode the salient features of the Euclidean heat kernel $(4\pi t)^{-\frac{d}{2}}e^{-\frac{\|x-y\|^2}{4t}}$ and of the first approximation of the heat kernel on manifolds $K_1(t,x,y) = (4\pi t)^{\frac{d}{2}}e^{-\frac{\|x-y\|^2_{g(y)}}{4t}}$, which are

- the prefactor $t^{-\frac{d}{2}}$,
- the exponential factor, which is a smooth function of $X = \frac{x-y}{\sqrt{t}}$ and y, exponentially decaying as $\|X\| \to +\infty$.

This motivates the following definition, in which the notation $C^\infty([0,+\infty)_{\frac{1}{2}})$ stands for the set of functions $f(t)$ which are smooth as functions of $\sqrt{t}$, for $t \geqslant 0$.

Definition 45 – *Pick a non-positive index* γ*. The space* Ψ^γ_H *is defined to be the set of functions in* $C^\infty((0,+\infty)\times S^2)$ *satisfying the following axioms*

- *A is smooth, if* $x \neq y$ *then* $A(t,x,y) = O(t^\infty)$*,*
- *For any* $p \in M$*, there exists a chart* U *containing* p *and* $\tilde{A} \in C^\infty\big([0,+\infty)_{\frac{1}{2}} \times U \times \mathbb{R}^d\big)$ *such that for* $(x,y) \in U^2$ *one has*

$$A(t,x,y) = t^{-\frac{d+2}{2}-\gamma}\widetilde{A}\Big(\sqrt{t}, \frac{x-y}{\sqrt{t}}, y\Big)$$

where $\widetilde{A}$ *has rapid decay in the second variable*

$$\big\|D^\gamma_{\sqrt{t},X,y}\tilde{A}\big\| = O\big(\|X\|^{-\infty}\big)$$

when $\|X\| \to +\infty$*.*

The use of the heat calculus gives a familiar form to the operators P_k. Set

$$M(t) := \int_0^t m(s)ds$$

and use the presentation of the heat calculus in the chart from definition 45 to write

$$\int_0^\infty 2^j m(2^j t)e^{-t\Delta}(x,y)dt = 2^{-j}\int_0^\infty M(t)2^{k\frac{d}{2}}t^{-\frac{d}{2}}\tilde{A}\Big(2^{-k}t, x, 2^{\frac{k}{2}}\frac{x-y}{\sqrt{t}}\Big)\,dt.$$

Then for any pair of test functions χ_1, χ_2

$$\begin{aligned}\widehat{(P_k\chi_1)\chi_2}(\xi,\eta) &= 2^{-k}\int_{U\times\mathbb{R}^2}\chi_1(x)\chi_2(h)e^{i(\xi.x+h.\eta)}\int_0^\infty M(t)2^{k\frac{d}{2}}t^{-\frac{d}{2}}\tilde{A}\Big(2^{-k}t, x, 2^{\frac{k}{2}}\frac{h}{\sqrt{t}}\Big)\,dtdx2h\\ &= 2^{-k}\int_{U\times\mathbb{R}^2}\chi_1(x)\chi_2(2^{-\frac{k}{2}}h)e^{i(\xi.x+2^{-\frac{k}{2}}h.\eta)}\int_0^\infty M(t)t^{-\frac{d}{2}}\tilde{A}\Big(2^{-k}t, x, \frac{h}{\sqrt{t}}\Big)\,dtdxdh.\end{aligned}$$

Using the rapid decay in the h variable for all values of $t \in [0,+\infty)$, $x \in U$

$$\sup_{x\in U}\Big|\tilde{A}\Big(2^{-k}t, x, \frac{h}{\sqrt{t}}\Big)\Big| \leqslant C_N\big(1+|h|\big)^{-N}$$

and the fact that $\chi_1(x)\chi_2(2^{-\frac{k}{2}}h)\int_0^\infty M(t)t^{-\frac{d}{2}}\tilde{A}\big(2^{-k}t, x, \frac{h}{\sqrt{t}}\big)\,dt$ is bounded in $C^\infty(U\times\mathbb{R}^2)$ uniformly in the parameter k, we have an estimate of the form

$$\big|\widehat{(P_k\chi_1)\chi_2}(\xi,\eta)\big| \leqslant C_N 2^{-k}\left(1+|\xi|+2^{-\frac{k}{2}}|\eta|\right)^{-N}$$

In position space, in the local chart $U\times U$ from definition 45, the estimate reads

$$P_k(x,y) = 2^{-k}2^{k\frac{d}{2}}K_k\big(x, 2^{\frac{k}{2}}(x-y)\big), \tag{B.3}$$

where the $(K_k)_k$ form a bounded family of smooth functions in $C^\infty\big(U\times\big\{|h|\leqslant 1\big\}\big)$.

Let P and $\tilde{P}$ be a family of geometric Littlewood-Paley projectors built from functions m and $\tilde{m}$ that vanish at $t = 0$. It will be convenient in the proof of Proposition 8 to control the kernel

$\sum_{i,j\geqslant 0}\big((\Delta^\alpha P_i)P_j\big)(x,y)$ in terms of α. We know from p.140 of [27] that we have the exact identity

$$\big(\widetilde{P}_i P_j\big)(x,y) = -2^{-2|i-j|}\int_0^\infty\int_0^\infty\int_0^1 e^{-(t_1+st_2)\Delta}\widetilde{m}_i(t_1)t_2 m_j(t_2)\,dsdt_1dt_2.$$

Using the structure of the heat kernel which follows from the heat calculus we may write in local coordinate chart $x\in U, h\in\mathbb{R}^2$

$$\big(\widetilde{P}_i P_j\big)(x,x+h) =: 2^{-2|i-j|}K_{ij}(x,h)$$

where

$$\sup_{x\in U}\Big|\partial_h^\beta\partial_x^\alpha K_{ij}(x,h)\Big| \leqslant C_{\alpha,\beta}\,2^{-i}2^{i\frac{d}{2}}2^{i\frac{|\beta|}{2}} \tag{B.4}$$

uniformly in (i,j). These are actually the seminorms for the topology of distributions whose wave front set is concentrated on the conormal bundle of the diagonal.

Now in [27] we also find that $\Delta^\alpha P_i\tilde{P}_j = 2^{2i\alpha}Q_iP_j$, where $(Q_i)_i$ is an admissible family of LP projectors. We deduce from this observation an estimate of the form

$$(\Delta^\alpha P_i)P_j(x,x+h) = 2^{2i\alpha}2^{-2|i-j|}K_{ij}(x,h),$$

where the kernel K_{ij} satisfies the same estimate B.4. This is all we need to prove the following technical lemma.

Lemma 46 – *Let the Littlewood-Paley projectors $(P_i)_i$ be constructed from a function m that vanishes at $t=0$. Fix $k\geq 1$ and $(\alpha_1,\dots,\alpha_k)\in\mathbb{Z}^k$. The series of Schwartz kernels*

$$\sum_{i_1,\dots,i_k,j_1,\dots,j_k}\ \sum_{|i_1-i_2|\leq 1,\dots,|i_1-i_k|\leqslant 1}\big((\Delta^{\alpha_1}P_{i_1})P_{j_1}\big)(x,y)\dots\big((\Delta^{\alpha_k}P_{i_k})P_{j_k}\big)(x,y) \tag{B.5}$$

converges absolutely in the space of pseudodifferential kernels of order $2(\alpha_1+\dots+\alpha_k)+(k-1)\frac{d}{2}$.

Proof – Using the above discussion we may rewrite

$$\begin{aligned}\big((\Delta^{\alpha_1}P_{i_1})P_{j_1}\big)(x,y)&\dots\big((\Delta^{\alpha_k}P_{i_k})P_{j_k}\big)(x,y)\\ &= 2^{2(i_1\alpha_1+\dots+i_k\alpha_k)}2^{-2(|i_1-j_1|+\dots+|i_k-j_k|)}K_{i_1j_1}(x,y)\dots K_{i_kj_k}(x,y)\end{aligned}$$

where the smooth functions $K_{i_nj_n}(x,y)$ satisfy the estimate (B.4). So one has for all tuples $(i_1,\dots,i_k,j_1,\dots,j_k)$ such that $|i_1-i_2|\leq 1,\dots,|i_1-i_k|\leqslant 1$ an estimate of the form

$$\begin{aligned}&\Big|\partial_h^b\partial_x^a K_{i_1,j_1}(x,x+h)\dots K_{i_k,j_k}(x,x+h)\Big|\\ &\qquad\leqslant C_{ab}\,2^{-(i_1+\dots+i_k)}2^{(i_1+\dots+i_k)\frac{d}{2}}2^{2\inf(i_1,\dots,i_k,j_1,\dots,j_k)\frac{|b|}{2}}\end{aligned}$$

where the constant C_{ab} does not depend on the indices $(i_1,\dots,i_k,j_1,\dots,j_k)$. This estimate ensures that the sum (B.5) convergences in the space of conormal distributions of order $2(\alpha_1+\dots+\alpha_k)+(k-1)\frac{d}{2}$. ▷

We give here the proof of Proposition 8 performing the Wick renormalization of the resonant term $\Pi(h\xi,X_h)$.

Proof – *Step 1 – Singular part.* Since the two paraproduct terms in the decomposition of the product $h\xi_rX_{h,r}$ are converging as r goes to 0 the quantities $\mathbb{E}\big[\Pi(h\xi_r,X_{h,r})\big]$ and $\mathbb{E}\big[h\xi_rX_{h,r}\big]$ differ by a converging quantity. Use now the Markov property of the heat operator and the definition of white noise to see that

$$\mathbb{E}\Big[\Big((\Delta-z_0)^{-1}h\xi_r\Big)(x)h\xi_r(x)\Big] = h^2(x)\big(e^{-2r\Delta}(\Delta-z_0)^{-1}\big)(x,x).$$

An immediate computation yields

$$e^{-2r\Delta}(\Delta-z_0)^{-1} = \int_{2r}^1 e^{(z_0-2r)s}e^{-s\Delta}(\mathrm{Id}-\pi_0)ds + \int_1^\infty e^{-s\Delta}(\mathrm{Id}-\pi_0)e^{z_0s}ds$$

where π_0 is the orthogonal projector on the subspace of constant functions. Recall that z_0 is large and negative so the integral over $[1,\infty)$ converges absolutely and defines a smoothing operator; it does not contribute to the singular part of $\big(e^{-2r\Delta}(\Delta-z_0)^{-1}\big)(x,x)$ when r goes to 0. Now using the asymptotic expansion of the heat kernel yields the identity

$$\big(e^{-s\Delta}(\mathrm{Id}-\pi_0)\big)(x,x)=\frac{1}{4\pi s}+O(1),$$

with an error term $O(1)$ bounded in s and smooth in the x variable. It follows that

$$\big(e^{-2r\Delta}\,(\Delta-z_0)^{-1}\,\big)(x,x)=\int_{2r}^{1}e^{(z_0-2r)s}\frac{1}{4\pi s}ds+\mathcal{O}(1)=\frac{|\log(r)|}{4\pi}+O(1).$$

We see here that the singular part of $\mathbb{E}\big[h\xi_r X_{h,r}\big]$ only depends on the point x only through $h(x)$.

Step 2 – Stochastic estimates. Write

$$\mathbb{E}\left[\left(\sum_{|i-j|\leqslant 1}\Delta^{-s}:\!\big(P_i(h\xi_r)\Delta^{-1}P_j(h\xi_r)\big)\!:\right)^2\right]=I_1+I_2,$$

where I_1 equals

$$\sum_{|i_1-j_1|\leqslant 1,|i_2-j_2|\leqslant 1}\int\Delta^{-s}(x_1,y_1)\Delta^{-s}(x_1,y_2)\left(\Delta^{-2}P_{j_1}P_{j_2}\right)(y_1,y_2)\left(P_{i_1}P_{i_2}\right)(y_1,y_2)h^4(y_2)\,dy_1dy_2$$

and I_2 equals

$$\sum_{|i_1-j_1|\leqslant 1,|i_2-j_2|\leqslant 1}\int\Delta^{-s}(x_1,y_1)\Delta^{-s}(x_1,y_2)\left(\Delta^{-1}P_{j_1}P_{j_2}\right)(y_1,y_2)\left(\Delta^{-1}P_{i_1}P_{i_2}\right)(y_1,y_2)h^4(y_2)\,dy_1dy_2.$$

Lemma 46 shows that the series

$$\sum_{|i_1-j_1|\leqslant 1,|i_2-j_2|\leqslant 1}\left(\Delta^{-2}P_{j_1}P_{j_2}\right)(y_1,y_2)\left(P_{i_1}P_{i_2}\right)(y_1,y_2)$$

converges to some pseudodifferential kernel in $\Psi^{-2}(\mathcal{S})$, so I_1 is the diagonal restriction of an element in $\Psi^{-2-2s}(\mathcal{S})$, by composition of pseudodifferential operators, and is therefore bounded in $x_1\in\mathcal{S}$. Since we are in dimension 2 Lemma 46 shows that

$$\sum_{|i_1-j_1|\leqslant 1,|i_2-j_2|\leqslant 1}\left((\Delta^{-1}P_{j_1})P_{j_2}\right)^2(y_1,y_2)$$

represents a pseudodifferential kernel in $\Psi^{-2}(\mathcal{S})$ so I_2 is also the diagonal restriction of an element in $\Psi^{-2-2s}(\mathcal{S})$ and is therefore bounded in $x\in\mathcal{S}$.

We conclude using the hypercontractivity property of Gaussian measures and Besov embedding. For every integer $p\in\mathbb{N}$, one has an inequality of the form

$$\begin{aligned}\mathbb{E}\Big[\|:\!\mathsf{\Pi}(h\xi_r,X_{h,r})\!:\|^{2p}_{W^{s,2p}}\Big]&=\mathbb{E}\left[\int_S\Big((\mathrm{Id}+\Delta)^{\frac{s}{2}}:\mathsf{\Pi}(h\xi_r,X_{h,r}):\Big)^{2p}\right]\\&\lesssim_p\mathbb{E}\left[\int_S\Big((\mathrm{Id}+\Delta)^{\frac{s}{2}}:\!\mathsf{\Pi}(h\xi_r,X_{h,r})\!:\Big)^{2}\right]^p\\&\lesssim_p\mathbb{E}\Big[\|:\!\mathsf{\Pi}(h\xi_r,X_{h,r})\!:\|^2_{H^s(S)}\Big]^p.\end{aligned}$$

Sending now r to 0 the upper bound remains bounded. The same computations show moreover that

$$\mathbb{E}\Big[\|:\!\mathsf{\Pi}(h\xi_r,X_{h,r})\!:-:\!\mathsf{\Pi}(h\xi_{r'},X_{h,r'})\!:\|^{2p}_{W^{s,2p}}\Big]$$

is bounded above by a constant multiple of

$$\mathbb{E}\Big[\|:\!\mathsf{\Pi}(h\xi_r,X_{h,r})\!:-:\!\mathsf{\Pi}(h\xi_{r'},X_{h,r'})\!:\|^{2}_{H^s(S)}\Big]^p$$

with an upper bound that goes to 0 as r and r' go to 0. One can thus define the element $:\!\mathsf{\Pi}\left(h\xi,\Delta^{-1}(h\xi)\right)\!:$ as the limit of a Cauchy family in the space $W^{s,2p}(\mathcal{S})$; Besov embedding does the remaining job. ▷

The following observation will be useful in the proof of Lemma 9.

Lemma 47 – *For any bounded family of smooth functions $(A_j)_{j\in\mathbb{N}}$ in $C^\infty(S\times S)$, the series*

$$\sum_{j=0}^{\infty}(A_jP_j)(x,y)$$

converges in the space of pseudodifferential kernels in $\Psi^\varepsilon(\mathcal{S})$, for all $\varepsilon>0$, and the partial sums

$$\left(\sum_{j=0}^{N}(A_jP_j)(x,y)\right)_{N\in\mathbb{N}}$$

are bounded in $\Psi^0(\mathcal{S})$.

Proof – We would like to show that $\sum_{j=1}^\infty P_j(x,y)$ converges in the space of co-normal distributions. The convergence of $\sum_{j=1}^\infty P_j(x,y)$ as a distribution is an obvious consequence of the Bessel inequality and the fact that a bounded operator from $L^2(\mathcal{S})$ into itself has a well–defined distributional kernel. We see from the representation (B.3) of the Littlewood-Paley projectors that the series $\sum_{k=1}^\infty P_k$ converges absolutely as a co-normal distribution of the diagonal $I(N^*d_2)$ of the form $\int e^{i\xi.(x-y)}a(x;\xi)d\xi$ where the symbol a has order 0. In other words, the series $\sum_{k=1}^\infty P_k$ converges as pseudodifferential kernels in $\Psi^{+0}(\mathcal{S})$. ▷

We provide now a proof of Lemma 9, which says that for each regularization parameter $r>0$ the operator M_r^+ is a pseudodifferential operator of order 0.

Proof – As ξ_r is smooth the resonant part of M_r^+ is smoothing. The paraproduct part if given by

$$f\mapsto\sum_{j_1\leqslant j_2-2}(P_{j_1}\xi_r)\,P_{j_2}(hf).$$

Observe that the sequence $\left(\sum_{k\leqslant j}P_j\xi_r\right)_j$ converges in all Sobolev spaces since ξ_r is smooth. Moreover the family of operators $P_j\circ M_h$ is bounded in $\Psi^0(\mathcal{S})$ and the series $\sum_{j=1}^\infty P_j\circ M_h$ converges absolutely in pseudodifferential kernels in $S^a(\mathcal{S})$, for all $a>0$, therefore the product $\big((P_j\xi_r)(x)P_j(x,y)h(y)\big)_j$ also forms the general term of a convergent series in $S^a(\mathcal{S})$ for all $a>0$, by Lemma 47. ▷

We finish with the proof of Lemma 10.

Proof – Since one has $Q(z):H^s(\mathcal{S})\mapsto H^{s+b}(\mathcal{S})\subset H^s(\mathcal{S})$ the map $Q(z):H^s(\mathcal{S})\mapsto H^s(\mathcal{S})$ is compact and the operators $\big(\mathrm{Id}+P^{-1}Q(z)\big)^{-1}$ and $\big(\mathrm{Id}+Q(z)P^{-1}\big)^{-1}$ are well-defined by the meromorphic Fredholm theory. For every compact subset of the complex plane one can decompose $Q(z)$, for z in the compact set, as a sum

$$Q(z)=\Pi(z)+E(z)$$

of a finite rank part $\Pi(z):H^s(\mathcal{S})\mapsto H^s(\mathcal{S})$ that depends holomorphically on z, and a part $E(z):H^s(\mathcal{S})\mapsto H^s(\mathcal{S})$ with small operator norm. ▷

C – Spectral gap

We obtained in Theorem 17 an upper bound on the heat kernel of Anderson operator of the form c_1/t, for $0<t\leq 1$. Had we been working on an unbounded Riemannian manifold $\mathcal{S}$ we could have ended up with a similar bound holding for all positive times. We would then have been in a position to prove an elementary and interesting lower bound on the spectral radius of H. Without going into the details of the construction of Anderson operator in an unbounded setting – others are working on it, we illustrate the probabilistic mechanics of the spectral gap/radius in the classical setting of the Laplace-Beltrami operator on a d-dimensional smooth Riemannian manifold that is either complete unbounded or compact with a smooth boundary on which we put

a Dirichlet condition. Denote by μ the Riemannian volume measure. The *spectral radius* of the Laplace Beltrami operator is defined

$$\lambda_1 := \inf_{\Omega} \lambda_1(\Omega)$$

with $\lambda_1(\Omega)$ the spectral gap of the Laplace Beltrami operator on a relatively compact open subset Ω of M with Dirichlet boundary condition, and an infimum over such sub-domains Ω. It coincides with the spectral gap of the operator Δ when the manifold is compact.

Theorem 48 – *Assume the heat kernel of the Laplace-Beltrami operator satisfies the uniform 'on-diagonal' estimate*

$$p_t(x,y) \leq (Ct)^{-d/2}, \qquad \forall\, x,y \in M, \tag{C.1}$$

for all times. Then one has

$$|\lambda_1| \geq \frac{2^{2/d-1}C}{e}\,\frac{d}{\mu(M)^{2/d}}. \tag{C.2}$$

Proof – We prove that one has

$$|\lambda_1(\Omega)| \geq \frac{2^{2/d-1}C}{e}\,\frac{d}{\mu(\Omega)^{2/d}}$$

for any bounded open set $\Omega \subset M$. The heat kernel of the Laplace Beltrami operator Δ_Ω on Ω with Dirichlet condition is bounded above by p. The decreasing character of λ_1 as a function of the domain does the rest.

We know from Krein-Milman theorem applied to the compact and positivity improving operator e^{Δ_Ω} that the ground state is the only eigenvector that has constant sign; all other eigenvectors change sign. Denote by u_1 the eigenvector associated with $\lambda_1 < 0$ and write B for Brownian motion on Ω, killed on the boundary of Ω. The function

$$e^{-\lambda_1 t} u_1(B_t) = e^{|\lambda_1|t} u_1(B_t)$$

is a martingale under any $\mathbb{P}_m$. Denote by τ the hitting time of the boundary of Ω. It is almost surely finite under any $\mathbb{P}_m$. Pick $0 < \alpha < 1$. Without loss of generality, and trading possibly u_1 for $-u_1$, one can assume ε small enough so that we have

$$\mu\big(\{u_1 \geq \varepsilon\}\big) \geq \alpha\,\frac{\mu(\Omega)}{2}.$$

(We implicitly use here the fact that $\mu\big(\{u_1 = 0\}\big) = 0$ – a consequence of the unique continuation principle.) Let $x \in M$ be a point where u_1 is attains its maximum; one has necessarily $u_1(x) > 0$. Would the expectation $\mathbb{E}_x\big[e^{|\lambda_1|\tau}\big]$ be finite we could use dominated convergence to write

$$0 < u_1(x) = \mathbb{E}_x\Big[e^{|\lambda_1|\tau} u_1(B_\tau)\Big] = 0;$$

a contradiction. Now

$$\mathbb{E}_x\big[e^{|\lambda_1|\tau}\big] \leq \sum_{n\geq 1} e^{|\lambda_1| n}\,\mathbb{P}_x(\tau \geq n-1),$$

so a geometric bound $\mathbb{P}_x(\tau \geq n-1) \lesssim c^n$, entails that $e^{|\lambda_1|}c \geq 1$, that is $|\lambda_1| \geq \ln\frac{1}{c}$.

Given $t > 0$ and integers n,k such that $n-1 > kt$ one has from the decay assumption (C.1) on the heat kernel

$$\begin{aligned}\mathbb{P}_x\big(\tau \geq n-1\big) &\leq \int p_t(x,y_1)\cdots p_t(y_{n-2},y_k)\mathbf{1}_{\{u_1<\varepsilon\}}(y_1)\dots\mathbf{1}_{\{u_1<\varepsilon\}}(y_k)\,dy_1\dots dy_k \\ &\leq \left((Ct)^{-d/2}(2-\alpha)\,\frac{\mu(\Omega)}{2}\right)^k \leq \left((Ct)^{-d/2}(2-\alpha)\,\frac{\mu(\Omega)}{2}\right)^{n/t}.\end{aligned}$$

Optimizing over the choice of time t leads to choose

$$t = \frac{e}{C}\left((2-\alpha)\,\frac{\mu(\Omega)}{2}\right)^{2/d}$$

and gives the lower bound (C.2) as $\alpha < 1$ can be chosen arbitrarily close to 1, independently of λ_1. ▷

Grigor'yan's lecture notes [23] give a nice account of an analytical proof of Theorem 48 and many references on related matters – see in particular Section 6.2 of [23]. The uniform upper bound (C.1) on the heat kernel turns out to be equivalent to different functional inequalities (Nash, $H^1 - L^{2d/(d-1)}$-Sobolev, log-Sobolev and Faber-Krahn inequalities). The lower bound (C.2) is thus uniform in the class of closed Riemannian manifolds that satisfy condition (C.1). Curvature bounds of the form $\mathrm{Ric} \geq Kg$ give constants C in (C.1) that depend only on K.

Note that the above proof applies if one has any kind of decay $1/f(t)$ in (C.1) rather than a polynomial decay. For instance, with $f(t) \asymp (\ln t)^\alpha$, with $\alpha \geq 1$, one gets $\exp(-e^{1/\alpha}\mu(M)/2)$ as a lower bound for the spectral gap. With $f(t) \asymp e^{ct^\alpha}$ with $\alpha > 0$, one gets $\big(\frac{c}{1+\ln(\mu(M)/2)}\big)^{1/\alpha}$ as a lower bound for the spectral radius. We also note that our method of proof gives gives the same conclusion as in Theorem 48 when working on a manifold equipped with a positive smooth density and its associated Laplace operator, or when working on a graph or a Dirichlet space.

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