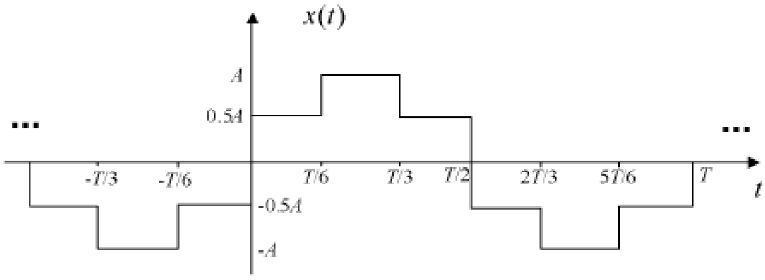
Fourier analysis and applications

Fourier series

1. Find the complex exponential Fourier series representation of $x(t) = \frac{4}{\pi}(\sin t + \sin 3t)$.

2. Digital Sine Wave Generator (long!)

A programmable digital signal generator generates a sinusoidal waveform by filtering the staircase approximation to a sine wave shown in the figure:



(a) Find the complex Fourier series coefficients c_k of the periodic signal $x(t)$. Show that the even harmonics vanish. Express $x(t)$ as a Fourier series.

(b) Write $x(t)$ using the real form of the Fourier series.

$$x(t) = a_0 + 2\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right] \equiv a_0 + 2\sum_{k=1}^{\infty}\left[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right]$$

(c) Design an ideal lowpass filter that will produce the perfect sinusoidal waveform $y(t) = \sin\left(\frac{2\pi}{T}t\right)$ at its output with $x(t)$ as its input. Sketch its frequency response and specify its gain K and cutoff frequency ω_c.

Fourier transforms

3. Sketch the following signal and find its Fourier transform: $x(t) = (1 - e^{-|t|})[u(t+1) - u(t-1)]$. Show that $X(j\omega)$ is real and even.

4. Show that the Fourier transform of a Gaussian pulse in the time domain $x(t) = e^{-\pi t^2}$ is a Gaussian pulse in the frequency domain.

Filters

5. Consider the periodic function of the example in section 4 of the lecture notes where $x(t) = t^2$, $-1 < t < 1$ was extended on the real line. Suppose this signal is passed through an ideal low pass filter described by

$|H(\omega)| = \begin{cases} 1 & |\omega| \le 3\pi \\ 0 & |\omega| > 3\pi \end{cases}$. Find the output signal $y(t)$ for this filter. Consider what happens if the cutoff frequency is reduced to $\omega_c = 2\pi$.

Hint: in the lectures, we found that the Fourier series expansion of the input signal was given by

$x(t) = \frac{1}{3} - \frac{4}{\pi^2}\cos \pi t + \frac{1}{\pi^2}\cos 2\pi t - \frac{4}{9\pi^2}\cos 3\pi t + \ldots$. If you can, plot the original function, and the various unfiltered and filtered Fourier series representations to compare them.