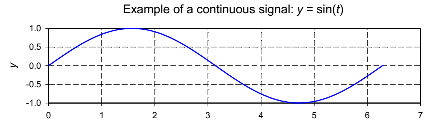
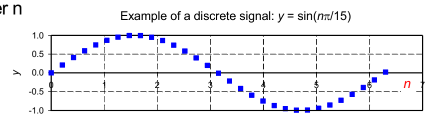
3. Data analysis 3.1. Elements of systems and signals

3.1.1 Continuous and discrete signals

Continuous signal just a regular function which can assume any value in some continuous interval (*a*, *b*) e.g. a sine wave



Discrete time signal defined only at discrete times we can label with an integer n

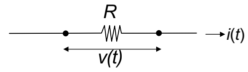


Created from continuous signal by sampling y(t) at regular intervals, T_s

$y[n] = y(nT_s)$

3.1. Elements of systems and signals

3.1.2 Energy and power in signals



Ohm's law: $v(t) = Ri(t)$

Instantaneous power: $p(t) = (v(t))^2/R = Ri((t)^2$

On a per Ohm basis: $p(t) = i(t)^2$

Total energy in Joules: $E = \int_{-\infty}^{\infty} p(t)dt = \int_{-\infty}^{\infty} i(t)^2 dt$

Average power: $P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} i(t)^2 dt$

Energy and power content in arbitrary signal $x(t)$ (may be *complex*):

use $|x(t)|^2 = x(t)x^*(t)$ where $x^*(t)$ is complex conjugate

Normalised energy content:

$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

or if signal is discrete

$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Normalised average power:

$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$

$P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$

3.1. Elements of systems and signals

3.1.3 Classification of signals

Energy signal: $0 < E < \infty$, $P = 0$

Power signal: $E = \infty$, $0 < P < \infty$

DC signal: signal with constant value for all times

Periodic signals

Common. For periodic signal ∃ some positive number T_0 (the period):

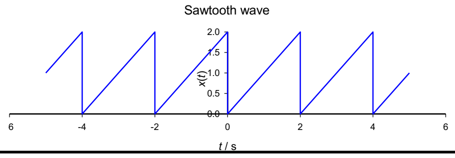
$x(t) = x(t + T_0)$

Fundamental frequency $f_0 = 1 / T_0$ in Hz

For periodic signals, consider energy content over one period, E_0

If $0 < E_0 < \infty$, power signal: $P = E_0 / T_0$

Many types of periodic function possible e.g. sawtooth wave:



by inspection

$T_0 = 2$ s

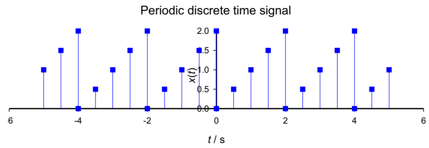
$f_0 = 1 / 2$ Hz

3.1. Elements of systems and signals

3.1.3 Classification of signals (contd)

Periodic discrete time signals also possible:

$x[n] = x[n + N]$: period is N e.g.



Sinusoidal signals

$x(t) = A\cos(\omega t + \theta)$ with A the *amplitude* and θ the *phase angle*.

Fundamental period $T_0 = 2\pi / \omega$, ω the *angular frequency* ($\omega = 2\pi f_0$)

Recall Euler's formula: $e^{\pm j\omega t} = \cos\omega t \pm j\sin\omega t$

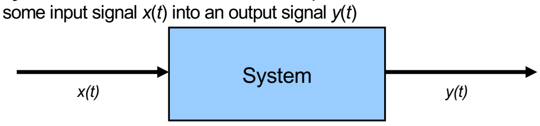
$\therefore \cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \quad \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$

(note use of $j = \sqrt{-1}$ in this subject)

3.1. Elements of systems and signals

3.1.4 Introduction to systems

system: a mathematical model that represents the transformation of some input signal $x(t)$ into an output signal $y(t)$



Represent system by a *transformation* or an *operator* $\hat{T}$ and then the action of a system on a signal: $y(t) = \hat{T}\{x(t)\}$

Examples: A resistor transforms current signal into voltage: $\hat{T} \equiv R$

$x(t) \equiv i(t)$, $y(t) \equiv v(t)$, and the system relationship is $y(t) = R\,x(t)$

Slightly more complex for a capacitor $i(t) = C\frac{dv}{dt}$: $\hat{T} = C\frac{d}{dt}$

A **continuous time system**: both $x(t)$ and $y(t)$ are continuous time signals.

Also possible to have discrete time systems.

3.1. Elements of systems and signals

3.1.5 Causal and non-causal systems

Causal system: output $y(t)$ depends only on the input at present or earlier times

- output does not anticipate future values of the input
- any real time-dependant system is causal (laws of Physics!)

Example of **non-causal system**: $y(t) = Cx(t + a)$ where $a \in \mathbb{R}$

3.1.6 Linear systems

Suppose operator acts on two input signals to produce output signals:

$\hat{T}\{x_1(t)\} = y_1(t)$ and $\hat{T}\{x_2(t)\} = y_2(t)$

Transformation is linear if for two constants *a, b*

$\hat{T}\{ax_1(t) + bx_2(t)\} = ay_1(t) + by_2(t)$

Linear system: system represented by a linear transformation.

To determine if a system is linear:

- consider 2 i/o relationships $y_1(t)$, $y_2(t)$ and form sum $ay_1(t) + by_2(t)$
- construct $T\{ax_1(t) + bx_2(t)\}$ – if equal to $ay_1(t) + by_2(t)$ for scalars *a, b* then system is linear

3.1. Elements of systems and signals

3.1.6 Linear systems (contd)

Example: determine if the following system is linear: $y(t) = \frac{d^2x}{dt^2}$

3.1.7 Time invariance

If time-shift of input signal: $x(t) \rightarrow x(t \pm \tau)$ causes same time-shift in output signal, system is *time-invariant.*

- if linear system, then called *linear time-invariant system* or LTI

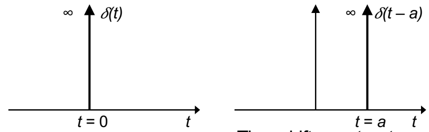
Can write $y_\tau(t) = \hat{T}\{x(t-\tau)\}$

If $y_\tau(t) = y(t-\tau)$ then system is time-invariant

3.1. Elements of systems and signals

3.1.8 Unit impulse function

Dirac delta function, defined as $\delta(t) = \lim_{\Delta\to 0} \delta_\Delta(t) : \delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases}$



Time shift: e.g. $t \rightarrow t + a$

Area under curve = 1: $\int_{-\infty}^{\infty} \delta(t)dt = \int_{-\infty}^{\infty} \delta(t \pm a)dt = 1$

Sampling property: use to pick out value at given time:

$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0)$ $\quad \int_{-\infty}^{\infty} \phi(t)\delta(t-a)dt = \phi(a)$

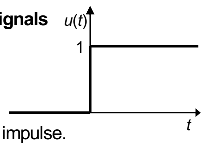
Further useful properties of unit impulse function:

$\delta(at) = \frac{1}{|a|}\delta(t)$ $\quad \delta(-t) = \delta(t)$

Any continuous time signal can be written: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$

3.1. Elements of systems and signals

3.1.9 Unit step function



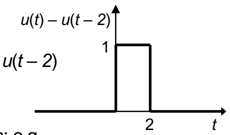
Defined as $u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$

Time shifting in same way as unit impulse.

Change limits of integration:

$\int_{-\infty}^{\infty} u(t)x(t)dt = \int_0^{\infty} x(t)dt$ $\quad$ or $\quad \int_{-\infty}^{\infty} u(t-3)x(t)dt = \int_3^{\infty} x(t)dt$

Square pulse: e.g. $u(t) - u(t-2)$



Simplification of integrals: e.g.

$\int_{-\infty}^{\infty} [u(t-1) - u(t-2)]\cos t \, dt = \int_1^2 \cos t \, dt = \sin 2 - \sin 1$

Unit impulse and unit step functions related:

$\delta(t) = \frac{du(t)}{dt}$ $\quad$ and $\quad u(t) = \int_{-\infty}^{t} \delta(t)\, dt$

3.1. Elements of systems and signals

3.1.10 Impulse response of an LTI system

Impulse response $h(t) = \hat{T}\{\delta(t)\}$

Use to determine system response to arbitrary input (*linear* system)

$y(t) = \hat{T}\{x(t)\} = \hat{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} = \int_{-\infty}^{\infty} x(\tau)\hat{T}\{\delta(t-\tau)\}d\tau$

Time invariant system ⇒ $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

convolution of input signal with impulse response $h(t)$

3.1.11 System step response

System step response $s(t) = \hat{T}\{u(t)\}$

Determine by *convolution*: $s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$

Convolution commutative (see on) so can write

$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$

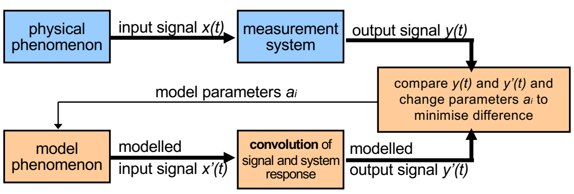
If know step response of system can find impulse response : $h(t) = \frac{ds}{dt}$

3.1. Elements of systems and signals

3.1.12 Convolution (introduction)

Very important for data analysis

- enables simulation of effect of instrument function on signal
- this can be used to fit data to physical model by 'forward convolution'



- *deconvolution* can be used (see later) to recover original signal

3.1. Elements of systems and signals

3.1.12 Convolution (introduction) (contd)

Convolution of two functions $f(t)$ and $g(t)$: $f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$

The convolution operation is

- commutative: $f(t) * g(t) = g(t) * f(t)$
- associative: $[f(t) * g(t)] * w(t) = f(t) * [g(t) * w(t)]$
- distributive: $f(t) * [g(t) + w(t)] = f(t) * g(t) + f(t) * w(t)$
- commutative with respect to multiplication by a scalar:
  $[\alpha f(t)] * g(t) = f(t) * [\alpha g(t)] = \alpha[f(t) * g(t)]$

Finally, convolution of any signal with a unit impulse leaves the signal unchanged: $f(t) * \delta(t) = f(t)$

3.1. Elements of systems and signals

3.1.12 Convolution (introduction) (contd)

Calculation of the convolution integral

First, obtain signal $h(t-\tau)$ as function of τ, then multiply by $x(\tau)$ to obtain another function $g(\tau)$ then integrate $g(\tau)$ to get $y(t)$

Step 1: sketch the time-reversed impulse response $h(-\tau)$

Step 2: shift this new function to the right by $|t|$ (time delay) for $t > 0$ to obtain $h(-(\tau - t)) = h(t-\tau)$, or to the left by $|t|$ (time advance) for $t < 0$ to obtain $h(-(\tau + |t|)) = h(t-\tau)$

Note: convolution is commutative so sometimes easier to work with $h(\tau)$ and $x(t-\tau)$ instead of $x(\tau)$ and $h(t-\tau)$

3.1. Elements of systems and signals

3.1.12 Convolution (introduction) (contd)

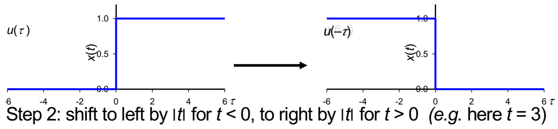
Example: given $x(t) = u(t)$ and $h(t) = \cos(\pi t)\, u(t)$, find the response $y(t)$

Note first that $h(t) = 0$ for $t < 0$ so system is causal, so can apply

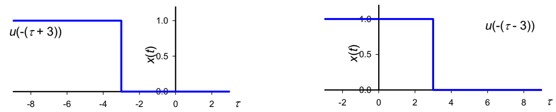
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{0}^{\infty} \cos(\pi\tau)u(t-\tau)\,d\tau$$

Rely on graphical approach to find limits of integration:

Step 1: reflect $x(\tau) = u(\tau)$ about vertical axis to give $u(-\tau)$



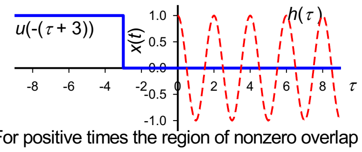
Step 2: shift to left by $|t|$ for $t < 0$, to right by $|t|$ for $t > 0$ (*e.g.* here $t = 3$)



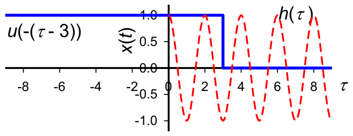
3.1. Elements of systems and signals

3.1.12 Convolution (introduction) (contd)

For negative times, no overlap as $h(t) = \cos(\pi t)\, u(t)$ is zero for $t < 0$



For positive times the region of nonzero overlap is $0 \le \tau \le t$



Therefore system response is:

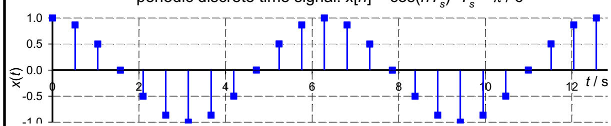
$$y(t) = \int_0^t \cos(\pi\tau)\,d\tau = \left[\frac{1}{\pi}\sin(\pi\tau)\right]_0^t = \frac{1}{\pi}\sin(\pi t)$$

3.1. Elements of systems and signals

3.1.13 Discrete time signals

In almost all physics experiments, analogue signal will be sampled and digitized, transforming continuous signal into discrete time signal.

periodic discrete time signal: $x[n] = \cos(nT_s)$ $T_s = \pi / 6$



- sampling interval: T_s
- discrete time signal $x[n] = x(nT_s)$ (n is integer) – a sequence $\{x_n\}$

For two discrete time signals $\{x_n\}$, $\{y_n\}$

- $\{z_n\} = \{x_n\} + \{y_n\} \Rightarrow z[n] = x[n] + y[n]$
- $\{z_n\} = \{x_n\}\{y_n\} \Rightarrow z[n] = x[n]\, y[n]$
- $\{z_n\} = \alpha\{x_n\} \Rightarrow z[n] = \alpha x[n]$

To plot, draw point and draw vertical line from time axis as above

3.1. Elements of systems and signals

3.1.14 Energy and power in discrete signals

Normalised energy content: $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$

Normalised average power: $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

- $0 < E < \infty$, $P = 0 \Rightarrow x[n]$ is an energy signal
- $0 < P < \infty$, $E = \infty \Rightarrow x[n]$ is a power signal

Example: Find the energy content and average power of the following signal and determine whether it is an energy signal or a power signal:

$$\{x_n\} = \left(\frac{1}{2}\right)^n \quad n \ge 0$$

3.1. Elements of systems and signals

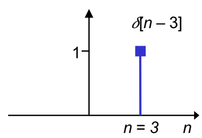
3.1.15 Unit impulse sequence

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \ne 0 \end{cases}$$

can shift the unit impulse sequence by integer k

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & n \ne k \end{cases}$$

In the figure we show $\delta[n-3]$



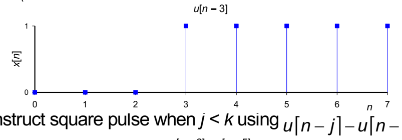
Satisfies the sampling property so $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

3.1. Elements of systems and signals

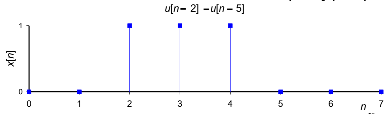
3.1.16 Unit step sequence

$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ a sequence of unit pulses starting at zero

$u[n-k] = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}$ shifted as illustrated for $k = 3$:



Can construct square pulse when $j < k$ using $u[n-j] - u[n-k]$ e.g.



Can define $\delta[n] = u[n] - u[n-1]$ and, conversely, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

3.1. Elements of systems and signals

3.1.17 Discrete linear time-invariant systems and convolution

- impulse response $h[n] = T\{\delta[n]\}$
- system is time invariant so can write $h[n-k] = T\{\delta[n-k]\}$
- remember $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- so response to arbitrary input $x[n]$ is given by: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
- this is discrete version of convolution $y[n] = x[n] * h[n]$
- commutative, associative and distributive properties apply
- since convolution is commutative we can also write

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Procedure for discrete time convolution:

- compute signals $x[k]$ and $h[n-k]$ as functions of k
- multiply them at each k
- sum all these values to yield output signal
- alternatively, use $h[k]$ and $x[n-k]$

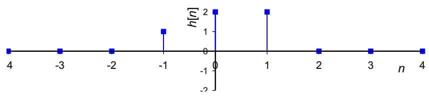
3.1. Elements of systems and signals

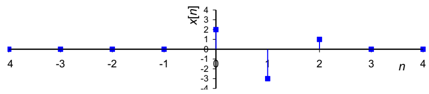
3.1.17 Discrete linear time-invariant systems (cont)

Example:

Compute $y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

for the following impulse response and input signal:





We can write: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{2} x[k]h[n-k]$ as $x[n]$ is zero everywhere else, draw resultant product for each value of k and sum.

3.2. Fourier Analysis and Applications

3.2.1 Introduction

A signal can be viewed from two different standpoints:

- the frequency domain
- the time domain

Any signal can be fully described in either of these domains

- go between the two by using a tool called the Fourier transform.

Why the frequency domain ?

- may be simpler to analyse signal in frequency domain

Fourier techniques have many applications

- optics: diffraction, interference
- audio: synthesis
- communications: filtering
- spectroscopy and dynamics: use of ultrafast lasers
- physics experiments: filtering noise, deconvolution

3.2. Fourier Analysis and Applications

3.2.2 Fourier series

Periodic signal $x(t)$ can be represented by a Fourier series expansion:

$$x(t) = a_0 + 2\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right]$$

where T_0 is the fundamental period of the signal

The cos and sin functions are used as basis functions

- obey orthogonality relations

$$\int_{-T_0/2}^{T_0/2} \cos\left(\frac{2\pi mt}{T_0}\right)\sin\left(\frac{2\pi nt}{T_0}\right)dt = 0$$

$$\int_{-T_0/2}^{T_0/2} \cos\left(\frac{2\pi mt}{T_0}\right)\cos\left(\frac{2\pi nt}{T_0}\right)dt = \begin{cases} T_0/2 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

$$\int_{-T_0/2}^{T_0/2} \sin\left(\frac{2\pi mt}{T_0}\right)\sin\left(\frac{2\pi nt}{T_0}\right)dt = \begin{cases} T_0/2 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

Integrate basis functions over single period enables determination of mean value of the signal, a_0

3.2. Fourier Analysis and Applications

3.2.2 Fourier series (cont)

Mean value of a periodic signal, a_0

$$\int_{-T_0/2}^{T_0/2} \cos\left(\frac{2\pi nt}{T_0}\right)dt = \left[\frac{T_0}{2\pi n}\sin\left(\frac{2\pi nt}{T_0}\right)\right]_{-T_0/2}^{T_0/2} = \frac{T_0}{2\pi n}\left[2\sin(\pi n)\right] = 0$$

Similar result for sin function.

So, to obtain a_0, integrate Fourier series expansion over one period:

$$\int_{-T_0/2}^{T_0/2} x(t)dt = a_0\int_{-T_0/2}^{T_0/2} dt + 2\int_{-T_0/2}^{T_0/2}\left\{\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right]\right\}dt$$

Bring integral inside sum: terms vanish

$$2\int_{-T_0/2}^{T_0/2}\left\{\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right]\right\}dt$$

$$= 2\left\{\sum_{n=1}^{\infty}\left[a_n\int_{-T_0/2}^{T_0/2}\left\{\cos\left(\frac{2\pi nt}{T_0}\right)\right\}dt + b_n\int_{-T_0/2}^{T_0/2}\left\{\sin\left(\frac{2\pi nt}{T_0}\right)\right\}dt\right]\right\} = 0$$

Thus we are left with $\int_{-T_0/2}^{T_0/2} x(t)dt = a_0\int_{-T_0/2}^{T_0/2} dt = a_0[t]_{-T_0/2}^{T_0/2} = a_0T_0$

and so obtain a_0: $a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)dt$

3.2. Fourier Analysis and Applications

3.2.2 Fourier series (cont)

Other coefficients

Multiply both sides of Fourier series expansion by $\cos\left(\frac{2\pi nt}{T_0}\right)$ and then integrate over one period:

$$a_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt$$

Similarly, multiplying by $\sin(2\pi nt / T_0)$ we obtain the remaining coeffts:

$$b_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt$$

Dirichlet conditions

Conditions to tell whether or not a periodic signal $x(t)$ can be represented by a Fourier series:

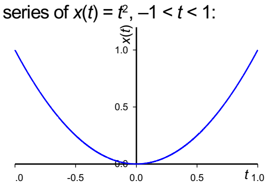
- $x(t)$ is single valued over the fundamental period
- $x(t)$ has finite number of discontinuities, minima and maxima over the fundamental period
- $\int_{-T_0/2}^{T_0/2} |x(t)|dt < \infty$ that is, $x(t)$ is *absolutely integrable*

3.2. Fourier Analysis and Applications

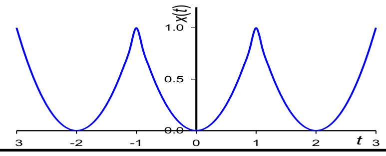
3.2.2 Fourier series (cont)

Example

Find the Fourier series of $x(t) = t^2$, $-1 < t < 1$:



Assume that function has been duplicated up and down the real line.



3.2. Fourier Analysis and Applications

3.2.3 Complex Fourier series

Using Euler's formulae: $\cos x = \frac{e^{jx} + e^{-jx}}{2}$ and $\sin x = \frac{e^{jx} - e^{-jx}}{2j}$

If write $x = 2\pi t / T_0$ then can write Fourier series expansion as

$$x(t) = a_0 + 2\sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right]$$

Now define: $c_0 = a_0$; $c_n = a_n - jb_n$; $c_{-n} = a_n + jb_n$ ⇒

$$x(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jnx} + c_{-n}e^{-jnx}\right]$$ and recall $x = 2\pi t / T_0$ then obtain

$$x(t) = \sum_{n=-\infty}^{\infty}\left[c_n \exp\left(j\frac{2\pi nt}{T_0}\right)\right]$$ complex exponential Fourier series

3.2. Fourier Analysis and Applications

3.2.3 Complex Fourier series (cont)

The coefficients are given by $c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi nt/T_0}dt$

Can write these complex coefficients as $c_n = |c_n|e^{j\phi_n}$

where $\phi_n = \arg(c_n)$ is the phase.

- amplitude spectrum – plot of $|c_n|$ against frequency
- phase spectrum – plot of ϕ_n against frequency

3.2.4 Power in periodic signals

Recall that average power of a periodic signal over one period is

$$P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2\, dt$$

It can be shown (Parseval's theorem) that if represent $x(t)$ by complex exponential Fourier series, then can write power as:

$$P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Example

Find complex exponential Fourier representation of $x(t) = 2\sin(t)\cos(t)$.

3.2. Fourier Analysis and Applications

3.2.5 Fourier transform

Conventionally we denote :

- signal in time domain $x(t)$
- signal in frequency domain $X(f)$
- can also write $X(\omega)$ where $\omega = 2\pi f$

The Fourier transform and inverse Fourier transform enable us to pass back and forth between the time and frequency domains:

$$x(t) \underset{\text{inverse Fourier transform}}{\overset{\text{Fourier transform}}{\rightleftarrows}} X(f)$$

The Fourier transform of a signal $x(t)$ is given by

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

and the inverse Fourier transform of $X(f)$ by

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

To be able to find FT of given signal $x(t)$ Dirichlet conditions are sufficient, but not strictly necessary, for example in the case of the unit impulse function:

3.2. Fourier Analysis and Applications

3.2.5 Fourier transform (cont)

Example

Find the Fourier transforms of (a) $\delta(t)$ and (b) $\delta(t - a)$.

Fourier transform pairs

Shorthand notation to denote signal in time domain and its Fourier transform – a Fourier transform pair:

$$x(t) \rightleftharpoons X(f)$$

So for the impulse functions in the last example we can write

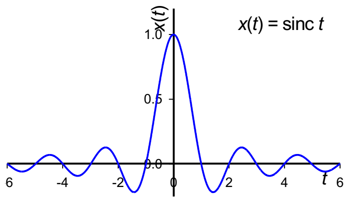
3.2. Fourier Analysis and Applications

3.2.5 Fourier transform (cont)

The sinc function

The sinc function is common in signal processing and is defined by

$$\text{sinc } t = \frac{\sin \pi t}{\pi t}$$



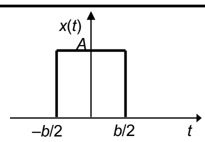
Exercise: show that sinc $t \to 1$ at origin (hint: Taylor expansion)

3.2. Fourier Analysis and Applications

3.2.5 Fourier transform (cont)

Further example

Find the Fourier transform of a square pulse :



3.2. Fourier Analysis and Applications

3.2.6 Properties of the Fourier transform

Time shifting

- if signal shifted in time domain by a, FT multiplied by $e^{-j2\pi af}$:
- $\text{FT}[x(t-a)] = e^{-j2\pi af} X(f)$

Frequency shifting

- multiplication of a signal in the time domain by $e^{j2\pi f_0 t} \equiv$ shift by f_0 in the frequency domain
- $e^{j2\pi f_0 t} x(t) \rightleftharpoons X(f-f_0)$

Time scaling

- compression in time of a signal $x(t)$ causes a broadening of frequency of $X(f)$
- broadening in time of a signal $x(t)$ causes a compression of frequency of $X(f)$
- for $x(t) \rightleftharpoons X(f)$, then $x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{f}{a}\right)$
- leads to constant time-bandwidth product, see later

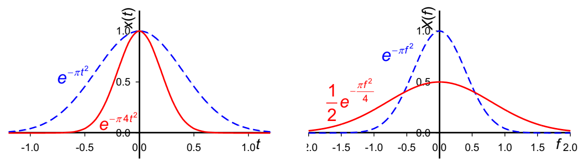
3.2. Fourier Analysis and Applications

3.2.6 Properties of the Fourier transform (cont)

Example

Investigate time scaling by finding the FT of $x(t) = e^{-\pi 4t^2}$

Time scaling factor here is two as $x(t) = e^{-\pi 4t^2} = e^{-\pi (2t)^2}$



Using time scaling property we know that since

$$x(t) = e^{-\pi t^2} \xrightarrow{FT} X(f) = e^{-\pi f^2} \Rightarrow x(2t) = e^{-\pi (2t)^2} \xrightarrow{FT} \frac{1}{2} X\left(\frac{f}{2}\right) = \frac{1}{2} e^{-\frac{\pi f^2}{4}}$$

Compression in time has resulted in expansion in frequency

3.2. Fourier Analysis and Applications

3.2.6 Properties of the Fourier transform (cont)

Superposition principle

- FT is linear $a x_1(t) + b x_2(t) \rightleftharpoons a X_1(f) + b X_2(f)$

Duality

- useful when computing FTs
- $x(t) \rightleftharpoons X(f) \quad \Rightarrow \quad X(t) \rightleftharpoons x(-f)$

Differentiation and integration

Inverse FT: $x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi f t)\, df$

$$\frac{dx}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} X(f) \exp(j2\pi f t)\, df$$

$$\frac{dx}{dt} = \int_{-\infty}^{\infty} \frac{d}{dt} X(f) \exp(j2\pi f t)\, df = j2\pi f \int_{-\infty}^{\infty} X(f) \exp(j2\pi f t)\, df$$

Thus if $x(t) \rightleftharpoons X(f) \quad \Rightarrow \quad \frac{dx}{dt} \rightleftharpoons j2\pi f X(f)$

If $X(0) = 0 \quad \Rightarrow \quad \int_{-\infty}^{t} x(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f} X(f)$

- differentiation in t domain ≡ multiplication by $j2\pi f$ in freq. domain
- integration in t domain ≡ division by $j2\pi f$ in freq. domain

3.2. Fourier Analysis and Applications

3.2.6 Properties of the Fourier transform (cont)

Convolution

- multiplying two functions together in the time domain results in convolution in the frequency domain
- $x_1(t) x_2(t) \rightleftharpoons \int_{-\infty}^{\infty} X_1(\sigma) X_2(f-\sigma)\, d\sigma$
- convolution in the time domain translates into multiplication in the frequency domain!
- $\boxed{\int_{-\infty}^{\infty} x_1(t) x_2(t-\tau)\, d\tau \rightleftharpoons X_1(f) X_2(f) \quad \textbf{The Convolution Theorem}}$

Even and odd parts of a function

Suppose

- $x(t)$ real signal decomposed into even / odd parts: $x(t) = x_e(t) + x_o(t)$
- and $X(f) = A(f) + jB(f)$

$$\Rightarrow x_e(t) \rightleftharpoons A(f) \qquad x_o(t) \rightleftharpoons jB(f) \qquad X(-f) = X^*(f)$$

- real part of the FT and even part of signal constitute a FT pair
- imaginary part of FT and odd part of signal also constitute a FT pair

3.2. Fourier Analysis and Applications

3.2.7 Spectrum plots

In general, FT of a signal $x(t)$ is a complex function, so can write $X(f)$ in polar representation, that is:

$X(f)=|X(f)|e^{j\phi}$

where $|X(f)|$ is the amplitude of $X(f)$ and $\phi=\arg(X(f))$ is the phase

- plot of $|X(f)|$ is known as the amplitude spectrum of the signal
- plot of $\phi = \arg(X(f))$ is known as the phase spectrum of the signal

Exercise

Suppose the FT of some signal is

$$X(f)=\frac{1}{2+jf}$$

Plot the amplitude and phase spectra.

3.2. Fourier Analysis and Applications

3.2.8 Parseval's theorem

- the energy content of a signal is equivalent to the energy spectral density of the signal, found by integrating $|X(f)|^2$
- $\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_{-\infty}^{\infty}|X(f)|^2\,df=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega$
- alternative statement: the total average power of a periodic signal is equal to the sum of the average powers in all of its harmonic components (cf Fourier series representation)
- $P_k=\frac{1}{T}\int_0^T\left|a_k e^{jk\omega_0 t}\right|^2 dt=\frac{1}{T}\int_0^T|a_k|^2\,dt=|a_k|^2$
- $P_k = P_{-k}$ so the total power of the kth harmonic components of the signal (i.e. the total power at frequency $k\omega_0$) is $2P_k$
- total average signal power is given in the frequency domain by the Parseval theorem therefore as
- $P=\frac{1}{T}\int_0^T|x(t)|^2\,dt=\sum_{k=-\infty}^{\infty}|a_k|^2$

3.2. Fourier Analysis and Applications

3.2.9 Frequency response

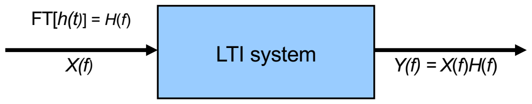
$\boxed{\int_{-\infty}^{\infty}x_1(t)x_2(t-\tau)\,d\tau \rightleftharpoons X_1(f)X_2(f)\quad \textbf{The Convolution Theorem}}$

Recall: for continuous LTI system with impulse response $h(t)$, the response $y(t)$ to any input signal $x(t)$ given by convolution:

- $y(t) = x(t)*h(t)$.

Using the convolution theorem we can write

- $Y(f) = X(f)H(f)$
- $H(f)$ is the *frequency response* or transfer function of the system



In general $H(f)$ is complex: $H(f)=|H(f)|e^{j\theta_H}$

- $|H(f)|$ – amplitude response
- $\theta_H(f) = \arg[H(f)]$ – phase response

3.2. Fourier Analysis and Applications

3.2.9 Frequency response (cont)

FT of output signal:

$y(t)=\int_{-\infty}^{\infty}Y(f)e^{j2\pi ft}\,df$

$Y(f)=|Y(f)|e^{j\theta_Y}$ $\quad X(f)=|X(f)|e^{j\theta_X}$ $\quad H(f)=|H(f)|e^{j\theta_H}$

$Y(f) \quad = X(f)H(f)$

Amplitude of output signal related to *product* of amplitudes of input signal and frequency response:

- $|Y(f)|=|X(f)||H(f)|$

Phase of output signal related to *sum* of phases of input signal and frequency response:

- $\theta_Y(f)=\theta_X(f)+\theta_H(f)$

3.3. Energy spectral density and correlation

3.3.1 Cross correlation

A way to measure the similarity between two energy signals

- measures the properties of an unknown signal by comparing it to a known signal.
- compare $x_1(t)$ to a time-delayed version of $x_2(t)$
- cross correlation function given by $R_{12}(\tau)=\int_{-\infty}^{\infty}x_1(t)^*x_2(t+\tau)\,dt$ (* denoted complex conjugate)
- reminder: functions orthogonal if $\int_{-\infty}^{\infty}x_1(t)x_2(t)\,dt=0$

Example

Show that $x_1(t) = \sin t$, $x_2(t) = \cos t$ are orthogonal and calculate their cross correlation function for $\tau = \pi$

Consider over one period:

$$\int_{-\pi}^{\pi}\sin(t)\cos(t)\,dt=\left[\frac{\sin^2(t)}{2}\right]_{-\pi}^{\pi}=\frac{\sin^2(\pi)}{2}-\frac{\sin^2(-\pi)}{2}=0$$

This will always be zero as $\sin^2 x = \sin^2(-x)$. The functions are orthogonal.

3.3. Energy spectral density and correlation

3.3.1 Cross correlation (cont) $R_{12}(\tau)\ =\int_{-\pi}^{\pi}\sin(t)\cos(t+\tau)\,dt$

However, let us now calculate the cross-correlation over one period

$$R_{12}(\tau=\pi/2)=\int_{-\pi}^{\pi}\sin(t)\cos(t+\pi/2)\,dt=\int_{-\pi}^{\pi}-\sin^2(t)\,dt=\frac{1}{2}\int_{-\pi}^{\pi}-(1+\cos(2t))\,dt$$

$$=\left[-\frac{t}{2}+\frac{\sin(2t)}{4}\right]_{-\pi}^{\pi}=-\frac{\pi}{2}-\frac{\pi}{2}+\frac{\sin(2\pi)}{4}-\frac{\sin(-2\pi)}{4}=-\pi$$

Further properties of cross correlation

- sometimes written $R_{12}(\tau)=\int_{-\infty}^{\infty}x_1(t)^*x_2(t+\tau)\,dt=x_1(t)\star x_2(t)$
- related to convolution: $x_1(t)\star x_2(t)=x_1(-t)^* * x_2(t)$
- if either is an even function, then $x_1(t)\star x_2(t)=x_1(t)*x_2(t)$
- $(x_1(t)\star x_2(t))\star(x_1(t)\star x_2(t))=(x_1(t)\star x_1(t))\star(x_2(t)\star x_2(t))$
- definition of discrete cross correlation: a series given by

$$r_{xy}[l]=\sum_{n=-\infty}^{\infty}x[n]^*y[n+l] \qquad l=0,\pm1,\pm2,\ldots$$

- cross correlation theorem (often used with FFT to compute c-cs):

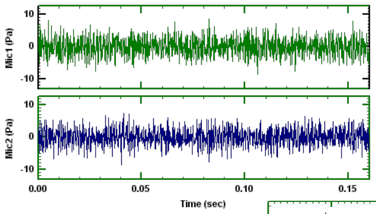
$$x_1(t)\star x_2(t)\rightleftharpoons X_1(f)^*X_2(f)$$

3.3. Energy spectral density and correlation

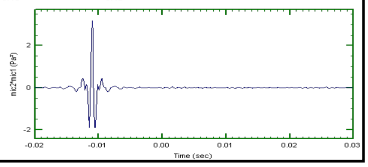
3.3.1 Cross correlation (cont)

Application: distinguishing signal from apparent noise

- two microphones at different distances from source gave signals:



- cross-correlating the two signals gives a peak at –11 ms, which corresponds to the time delay between the two microphones:



3.3. Energy spectral density and correlation

3.3.2 Autocorrelation

For a complex function $x_1(t)$ autocorrelation defined by

$$R_{11}(\tau) = x_1 \star x_1 = x_1(-\tau)^* * x_1(\tau) = \int_{-\infty}^{\infty} x_1(t)^* x_1(t+\tau)\,dt$$

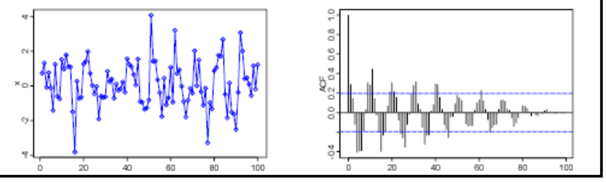
Normalised energy content obtained by setting $\tau = 0$:

$$E = R_{11}(0) = \int_{-\infty}^{\infty} x_1(t)^* x_1(t)\,dt = \int_{-\infty}^{\infty} |x_1(t)|^2\,dt$$

To find autocorrelation of a power signal compute time average:

$$\overline{R}_{11}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_1(t)^* x_1(t+\tau)\,dt \qquad \overline{R}_{11}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_1(t)^* x_1(t+\tau)\,dt \text{ (periodic sig)}$$

Example: A plot showing 100 random numbers with a "hidden" sine function, and an autocorrelation of the series on the right.



3.3. Energy spectral density and correlation

3.3.2 Autocorrelation (cont)

Applications

One application of autocorrelation is the measurement of optical spectra and the measurement of very-short-duration light pulses produced by lasers, both using optical autocorrelators.

In optics, normalized autocorrelations and cross-correlations give the degree of coherence of an electromagnetic field.

In signal processing, autocorrelation can give information about repeating events like musical beats or pulsar frequencies, though it cannot tell the position in time of the beat.

Further properties

$R_{11}(\tau) = x_1 \star x_1$ is a maximum at the origin

Autocorrelation is a Hermitian function as $R_{11}(-\tau) = R_{11}^*(\tau)$

Discrete version (real numbers)

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n+l] = r_{xx}[-l] \qquad l = 0, \pm1, \pm2, \ldots$$

3.3. Energy spectral density and correlation

3.3.3 Energy spectral density

- measures distribution of signal energy E over frequency
- found by taking the FT of the autocorrelation function
- denoted by $S_{11}(\omega)$ $\quad S_{11}(\omega) = \int_{-\infty}^{\infty} R_{11}(\tau)e^{-j\omega\tau}d\tau$
- if signal $x(t)$ is real then $S_{11}(\omega) = |X(\omega)|^2$
- can compute energy in signal

$$E = R_{11}(0) = \int_{-\infty}^{\infty} |x_1(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_1(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{11}(\omega)d\omega$$

- explains why call $S_{11}(\omega)$ energy spectral density

3.3.4 Power spectral density

- defined in a similar way to energy spectral density
- computed as the FT of the time-average autocorrelation:

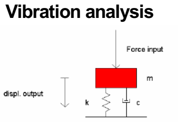
$$\overline{S}_{11}(\omega) = FT\left[\overline{R}_{11}(\tau)\right] = \int_{-\infty}^{\infty} \overline{R}_{11}(\tau)e^{-j\omega\tau}d\tau$$

- can find power in signal using this: $P = \overline{R}_{11}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{S}_{11}(\omega)d\omega$
- unit of PSD is (unit of measured quantity)²/Hz.

3.3. Energy spectral density and correlation

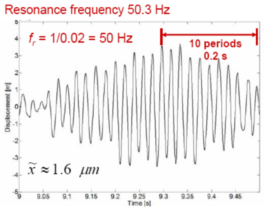
3.3.5 Applications of energy/power spectral density

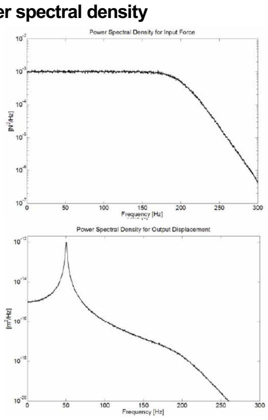
Vibration analysis



Mass	m	10 [kg]
Damping coefficient	c	100 [Ns/m]
Spring coefficient	k	1000000[N/m]

Resonance frequency 50.3 Hz

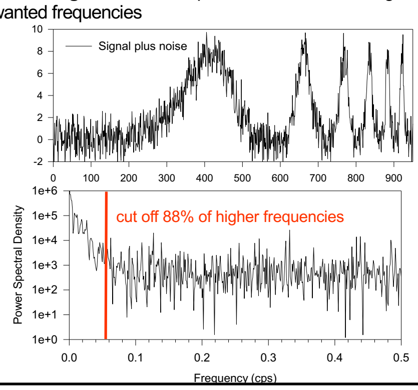




3.3. Energy spectral density and correlation

3.3.5 Applications of energy/power spectral density

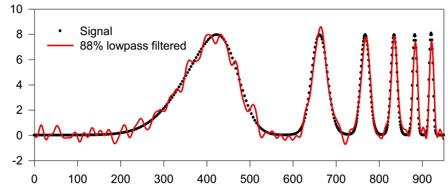
Simple Filtering : obtain PSD spectrum and use to design filter to block unwanted frequencies

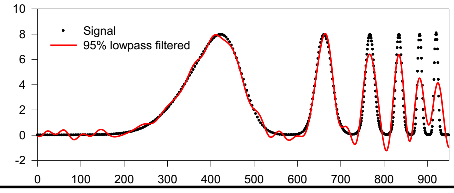


3.3. Energy spectral density and correlation

3.3.5 Applications of energy/power spectral density

Simple filtering (cont)



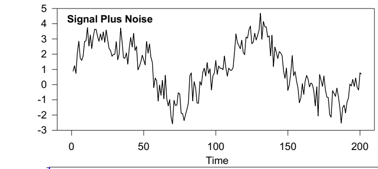


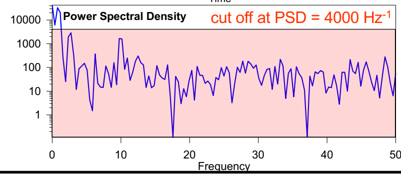
3.3. Energy spectral density and correlation

3.3.5 Applications of energy/power spectral density

Gain filter smoothing

Gain filtering filters data by removing frequency components with power spectral density magnitude less than a specified value

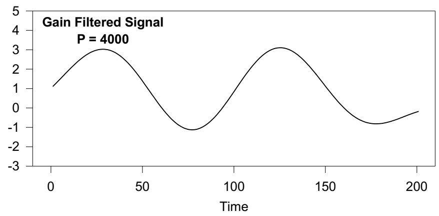




3.3. Energy spectral density and correlation

3.3.5 Applications of energy/power spectral density

Gain filter smoothing (cont)



3.4. Discrete Fourier transforms and sampling

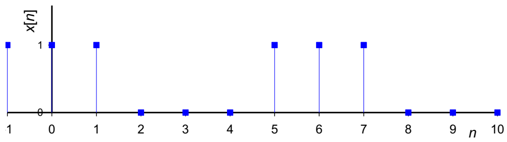
3.4.1 Discrete time Fourier series

- discrete time signal $x[n]$ with fundamental period N_0 : $x[n] = x[n + N_0]$.
- fundamental frequency $\Omega_0 = 2\pi / N_0$
- Fourier series representation of $x[n]$ is given by $x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n}$
- c_k – Fourier or spectral coefficients, given by $c_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}$
- if sum runs over any N_0 consecutive values of k: $x[n] = \sum_{k=\langle N_0\rangle} c_k e^{jk\Omega_0 n}$
- known as the *synthesis* equation.
- using same notation can express coefficients: $c_k = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n] e^{-jk\Omega_0 n}$
- sometimes called the *analysis* equation.
- spectral coefficients and sequence $x[n]$ constitute Fourier series pair $x[n] \rightleftharpoons c_k$
- average value of $x[n]$ over a period is given by: $c_0 = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n]$

3.4. Discrete Fourier transforms and sampling

3.4.1 Discrete time Fourier series (cont)

Example: find the spectral coefficients for the discrete time square wave shown below:



3.4.2 Properties of Discrete time Fourier series

For periodic discrete time signal $x[n] = x[n + N_0]$ spectral coefficients are also periodic: $c_k = c_{k+N_0}$

View members of discrete time sequence as Fourier coefficients of the c_k

$$c_k = c[k] = \frac{1}{N_0}\sum_{n=\langle N_0\rangle} x[n] e^{-jk\Omega_0 n} = \sum_{n=\langle N_0\rangle} \frac{x[n]}{N_0} e^{-jk\Omega_0 n} \quad \text{Now let } m = -n$$

$$c[k] = \sum_{m=\langle N_0\rangle} \frac{x[-m]}{N_0} e^{jk\Omega_0 m} \quad \text{Now } k \to n \text{ and } m \to k: c[n] = \sum_{k=\langle N_0\rangle} \frac{x[-k]}{N_0} e^{jk\Omega_0 n}$$

3.4. Discrete Fourier transforms and sampling

3.4.2 Properties of Discrete time Fourier series (cont)

This is just the discrete Fourier series representation for the $c[n]$. A demonstration of the *duality* property, which states

- if $x[n]$ and $c[k]$ form a Fourier series pair $x[n] \rightleftharpoons c[k]$
- then also have a Fourier series pair $c[n] \rightleftharpoons x[-k] / N_0$

Parseval's theorem for discrete Fourier series

Enables us to find the average power of a discrete time signal by summing the squared amplitudes of its harmonic components:

$$\frac{1}{N_0}\sum_{n=\langle N_0\rangle} \left|x[n]\right|^2 = \sum_{k=\langle N_0\rangle} \left|c[k]\right|^2$$

Example: demonstrate Parseval's theorem for the signal in 3.4.1

3.4. Discrete Fourier transforms and sampling

3.4.3 Fourier transform of a discrete time signal

FT of arbitrary non-periodic discrete time signal $x[n]$: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

- FT is periodic in 2π, $X[\Omega] = X[\Omega + 2\pi]$
- product $X[\Omega]e^{j\Omega n}$ also periodic in 2π
- Inverse FT – integrate over interval 2π: $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$
- FT of DT signal is linear: $ax_1[n] + bx_2[n] = aX_1[\Omega] + bX_2[\Omega]$
- time shift by n_0: $x[n-n_0] \rightleftharpoons e^{-j\Omega n_0} X(\Omega)$
- frequency shift by Ω_0: $e^{j\Omega_0 n} x[n] \rightleftharpoons X(\Omega - \Omega_0)$
- using time shifting obtain: $x[n] - x[n-1] \rightleftharpoons (1 - e^{-j\Omega}) X(\Omega)$
- accumulation property (where $|\Omega| \le 2\pi$):

$$\sum_{k=-\infty}^{\infty} x[k] \rightleftharpoons \pi X(0)\delta(\Omega) + \frac{1}{(1-e^{-j\Omega})} X(\Omega)$$

3.4. Discrete Fourier transforms and sampling

3.4.4 Discrete Fourier transform and sampling

Here we consider sampling of a continuous time signal $x(t)$ that is of finite duration.

- sample the signal at intervals of T_s called the sampling period
- total of N samples of the original signal, then we will have the sampled values $x(t)$, $x(T_s)$, $x(2T_s)$, ..., $x((N-1)T_s)$
- defines values of discrete time signal $x[n]$.

The DFT of $x[n]$ is denoted by $X[k]$ and is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

The inverse discrete FT is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

3.4. Discrete Fourier transforms and sampling

3.4.4 Discrete Fourier transform and sampling (cont)

Example

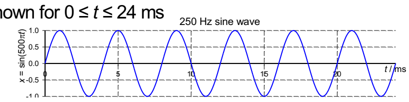
Given that $X[k] = \{0, -3-3j, -2, -3+3j\}$, use the inverse DFT to find $x[n]$

3.4. Discrete Fourier transforms and sampling

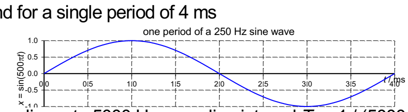
3.4.5 Sampling

Consider a sine wave with a frequency of $f = 250$ Hz.

- period $T = 1/f = 1/250 \equiv 4$ ms
- continuous time signal $x(t) = \sin(2\pi f t) = \sin\{2\pi(250)t\}$
- shown for $0 \le t \le 24$ ms



- and for a single period of 4 ms

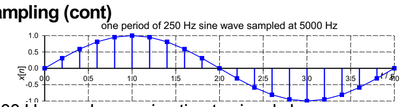


- sampling rate 5000 Hz, sampling interval $T_s = 1/(5000\text{ Hz}) \equiv 0.2$ ms
- out to 1 ms have the discrete time signal

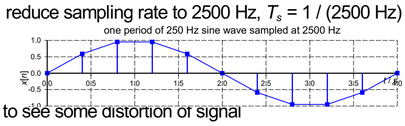
  $x[n] = \{0.0000, 0.3090, 0.5878, 0.8090, 0.9511, 1.0000\}$

3.4. Discrete Fourier transforms and sampling

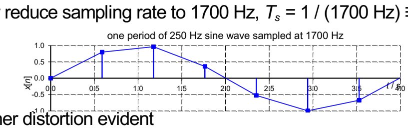
3.4.5 Sampling (cont)



- at 5000 Hz, good approximation to signal shape
- now, reduce sampling rate to 2500 Hz, $T_s = 1/(2500\text{ Hz}) \equiv 0.4$ ms



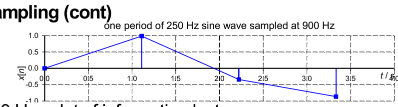
- start to see some distortion of signal
- now reduce sampling rate to 1700 Hz, $T_s = 1/(1700\text{ Hz}) \equiv 0.59$ ms



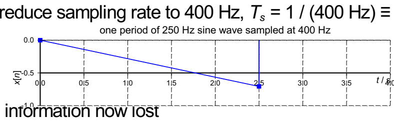
- further distortion evident
- sample signal now at 900 Hz, $T_s = 1/(900\text{ Hz}) \equiv 1.11$ ms

3.4. Discrete Fourier transforms and sampling

3.4.5 Sampling (cont)



- at 900 Hz, a lot of information lost
- now, reduce sampling rate to 400 Hz, $T_s = 1/(400\text{ Hz}) \equiv 2.5$ ms



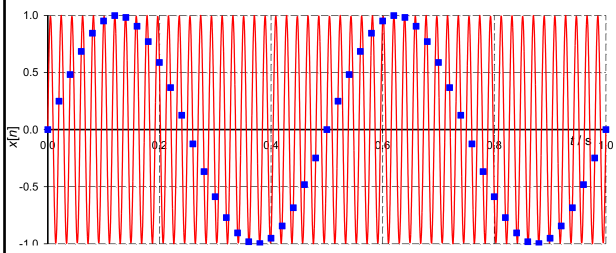
- much information now lost

Summarising:

- if signal changes rapidly in time, sampling interval T_s must be small enough to capture variations
- high frequency variation implies high frequency components in signal, requires high sampling rate
- when sampling rate not high enough / sampling interval too long to capture signal variation, we say that **aliasing** has occurred

3.4. Discrete Fourier transforms and sampling

Extreme example of aliasing

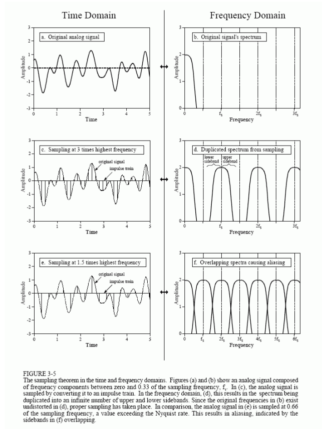


52Hz signal sampled at 50 Hz

3.4. Discrete Fourier transforms and sampling

Aliasing

Diagram from dspguide.com



3.4. Discrete Fourier transforms and sampling

3.4.5 Sampling (cont)

Nyquist/Shannon sampling theorem

To sample a signal correctly, sampling rate (ω_s rad/sec) should be at least twice the highest frequency component (ω_h) present in the signal: $\omega_s \geq 2\omega_h$

For signals band width limited to $[-\omega / 2, \omega / 2]$

- the critical sampling interval $T_s = 2\pi / \omega$,
- $\omega_c = \omega$ is the *Nyquist critical frequency*
- Nyquist critical frequency is highest frequency that can pick up
- for a sine wave, this corresponds to a minimum of two samples per period
- an arbitrary band-width limited signal $x(t)$ is completely determined by its samples $x[n]$ taken at the Nyquist critical frequency:

$$x(t) = T_s \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\left[\omega_c (t - nT_s)\right]}{\pi (t - nT_s)}$$

On the other hand, if sample a continuous function that is not bandwidth limited to less than the Nyquist critical frequency

3.4. Discrete Fourier transforms and sampling

3.4.5 Sampling (cont)

Nyquist/Shannon sampling theorem (cont)

- all of power spectral density lying outside range $(-\omega_c / 2) < \omega < (\omega_c / 2)$ is incorrectly moved into that range: aliasing

Reconstruction of sampled signals

For example, reconstruction of sound from digital recording.

A band-limited signal sampled at frequency $\omega_s = 2\pi / T_s$ gives discrete time signal $x[n] = x(nT_s)$ from which we would like to recover the original continuous time signal.

- Ideally, we would do this by constructing a train of impulses from the $x[n]$ and then filter this signal with an ideal lowpass filter

In real life, two possibilities:

Zero-order hold, interpolates signal samples with a constant line segment over a sampling period for each sample

- frequency response is a poor approximation to ideal lowpass filter's

First-order hold

- triangular impulse response,
- gives a linear interpolation between each sample

3.4. Discrete Fourier transforms and sampling

3.4.5 Sampling (cont)

Nyquist/Shannon sampling theorem (cont)

- all of power spectral density lying outside range $(-\omega_c / 2) < \omega < (\omega_c / 2)$ is incorrectly moved into that range: aliasing

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3.4. Discrete Fourier transforms and sampling

3.4.5 Sampling (cont)

Reconstruction of sampled signals (cont)

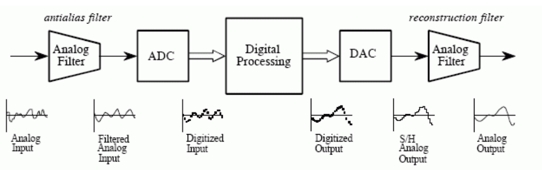

FIGURE 3-7
Analog electronic filters used to comply with the sampling theorem. The electronic filter placed before an ADC is called an *antialias filter*. It is used to remove frequency components above one-half of the sampling rate that would alias during the sampling. The electronic filter placed after a DAC is called a *reconstruction filter*. It also eliminates frequencies above the Nyquist rate, and may include a correction for the zeroth-order hold.

(Diagram from www.dspguide.com)