The Galois groupoid of P_I and its irreducibility

Let $\mathcal{F}$ be a holomorphic codimension 2 foliation of $\mathbb{C}^n$.

Définition. *$\mathcal{F}$ is reducible if there existe a sequence of differential fields*

$$\mathbb{C}(x_1, \ldots, x_n) = K_0 \subset K_1 \subset \ldots \subset K_n$$

such that

(1) • $K_{i+1} = K_i(h_1, \ldots, h_p)$ *with* $\frac{\partial h_\ell}{\partial x_j} = \sum A^k_{\ell,j} h_k$, $A^k_{\ell,j} \in K_i$,

• *or* $K_{i+1} = K_i(< h_1 >)$ *with* $dh \wedge \omega = 0$, $\omega \in K_i \otimes \Omega^1_{\mathbb{C}^n}$,

• *or* K_{i+1} *is algebraic on* K_i

(2) *there is* H_1, H_2 *in* K_n*, two independant first integrals*

Let $\mathcal{F}_1$ be the foliation of $\mathbb{C}^3$ given by

$$X_1 = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + (6y^2 + x) \frac{\partial}{\partial y'}.$$

Theorem. $\mathcal{F}_1$ *is irreducible.*

Proof :

If $\mathcal{F}_1$ is reducible then

(1) There is two independant first integrals satisfying a *big* system of pde.

(2) The holonomy of $\mathcal{F}_1$ leaves this system invariant

$\Rightarrow$ it satisfies a *big* system of pde too

(3) The bigger system of pde satisfied by the holonomy describes the Galois groupoid of $\mathcal{F}_1$

$\Rightarrow$ computation of $Gal(\mathcal{F}_1)$ and contradiction

Définition. *Let $\mathcal{I}$ be a prime differential ideal of*

$$\mathbb{C}(x_1,\ldots,x_n) < H_1,\ldots,H_p >$$

and $\mathcal{I}_\ell$ the set of order less than ℓ equations.

Let V_ℓ be the variety defined by $\mathcal{I}_\ell$ and $c_\ell = \dim_{\mathbb{C}(x_1,\ldots,x_n)} V_\ell$.

$$c_\ell \underset{\infty}{\sim} q\ell^a \text{ is called the type of } \mathcal{I}.$$

Lemma. *The type is well defined for field extension.*

Lemma. *The type of a reducible field extension is linear.*

Définition. *The Galois groupoid of $\mathcal{F}$ is the smallest algebraic Lie groupoid which contains the holonomy pseudo-group.*

(E. Vessiot and B. Malgrange)

$$\Rightarrow \; Gal(\mathcal{F}) \subset \bigcap_{\mathcal{I}\in Spec^{diff}\mathcal{O}_\mathcal{F}} Stab(\mathcal{I})$$

From E. Cartan (and S. Lie) local classification of regular pseudo-groups acting on $\mathbb{C}^2$, we obtain a global classification of algebraic Lie groupoids acting on $\mathbb{C}^n$, leaving a codimension two foliation invariant and containing the holonomy of this foliation.

In case of $\mathcal{F}_1$,

let $\gamma = i_{X_1} dx \wedge dy \wedge dy'$ the closed 2-form vanishing on $\mathcal{F}_1$,

$$Gal(\mathcal{F}_1) \subset Inv(\gamma)$$

Proposition. *We are in one of these cases :*

- *there is a rational first integal of $\mathcal{F}_1$,*
- *there is an algebraic integrable 1-form vanishing on $\mathcal{F}_1$,*
- *$\mathcal{F}_1$ is defined by two algebraic 1-foms $\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ and there is a traceless matrix of algebraic 1-form Ω such that*
$$d\Theta = \Omega \wedge \Theta \ \mathit{et} \ d\Omega = \Omega \wedge \Omega$$
- *$Gal(\mathcal{F}_1) = Inv(\gamma)$.*

$\Rightarrow$ **Theorem.** $Gal(\mathcal{F}_1) = Inv(\gamma)$.