Reducibility and Galois groupoid

... two examples

First Painlevé equation P_1

$$\frac{d^2y}{dx^2} = 6y^2 + x$$

Picard-Painlevé sixth equation PP_6

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 \\ &\quad + \left(\frac{1}{x-y} + \frac{1}{1-x} - \frac{1}{x}\right)\frac{dy}{dx} + \frac{y(y-1)}{2x(x-1)(y-x)}\end{aligned}$$

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Classical functions

Definition (Painlevé, Umemura)

A differential extension of $\mathbb{C}(x)$ is a field of classical functions if it is obtained by a tower of strongly normal extensions and algebraic extensions. Strongly normal extensions are :

- extensions by the entries of a fundamental solution of a linear ODE,
- extensions by an abelian function with classical functions as arguments.

Theorem (Nishioka, Umemura 1987)

No solution of P_1 is classical.

Theorem (Picard 1889)

PP_6 is solved by $y(x) = \wp(a\omega_1(x) + b\omega_2(x); \omega_1(x), \omega_2(x))$ with a and b two constants and $\omega_{1,2}$ two periods of
$z^2 = y(y-1)(y-x)$.

Remark (Painlevé, Mazzocco, Umemura)

Except for rational a and b, these functions are not classical.

Drach-Vessiot reducibility of a vector field on $\mathbb{C}^3$.

X is a vector field with coefficients in $\mathbb{C}(x,y,z)$.

Let C_1 and C_2 be two differential indeterminates. $\mathbb{C}(x,y,z)\{C_1,C_2\}$ is the partial diff.ring generated by C_1 and C_2 and E is the differential ideal generated by

$$XC_1 = 0$$
$$XC_2 = 0$$

<u>Definition</u>

X is Drach-Vessiot reducible if the ring

$$\mathbb{C}(x,y,z)\{C_1,C_2\}/E$$

has a non trivial differential ideal.

For a divergence free vector field, one adds the components of $dC_1 \wedge dC_2 = i_X dx \wedge dy \wedge dz$ to E.

<u>Definition</u>

Let I be a differential ideal in $\mathbb{C}(x,y,z)\{C_1,C_2\}$, $\mathbb{C}(x,y,z)\{C_1,C_2\}_n$ be the ring of order less than n equations and I_n its ideal of equations in I. Then

$$transc.deg._{/\mathbb{C}}\mathbb{C}(x,y,z)\{C_1,C_2\}_n/I_n \sim an^b$$

is the size of E.

Consequence of D.-V. reducibility on the flows of X.

The flows satisfy some differential equations:

$$\varphi_X^* X = X$$

and

$$\varphi_X^* dx \wedge dy \wedge dz = dx \wedge dy \wedge dz$$

Let $\mathbb{C}(x, y, z, \varphi_1, \varphi_2, \varphi_3)\left[\ \varphi_i^\alpha\ ; i = 1, 2, 3\ ; \alpha \in \mathbb{N}^3, \frac{1}{\det(\varphi_i^{\epsilon_j})}\right]$ with derivations

$$D_x = \frac{\partial}{\partial x} + \varphi_j^{\epsilon_1} \frac{\partial}{\partial \varphi_j} + \ldots + \varphi_j^{\alpha+\epsilon_1} \frac{\partial}{\partial \varphi_j^\alpha} \ldots ; D_y ; D_z$$

be the ring of diff. equations on germs of diffeomorphisms of $\mathbb{C}^3$

Lemma

Let I be a diff.ideal of $\mathbb{C}(x, y, z)\{C_1, C_2\}/E$ and J_I be the diff.ideal of $\mathbb{C}(x, y, z, \varphi_1, \varphi_2, \varphi_3)\{\varphi\}$ of equations satisfied by

$\{\varphi/(c_1, c_2)$ is solution of I iff $(c_1 \circ \varphi, c_2 \circ \varphi)$ is solution of $I\}$.

Then φ_X are solutions of J_I and its solutions (resp. formal) form a pseudogroup (resp. groupoid out of a hypersurface).

$\sim$ Definition (Malgrange)

The Galois groupoid of X is the groupoid (out of a hypersurface) defined by the ideal J_X in $\mathbb{C}(x, y, z, \varphi_1, \varphi_2, \varphi_3)\{\varphi\}$ of all the diff. equations satisfied by the flows of X.

Proposition ($\sim$ E. Cartan)

We are in one of these cases :

- there is an integrable 1-form vanishing on X with algebraic coefficients over $\mathbb{C}(x, y, z)$,
- there are two algebraic 1-foms $\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ vanishing on X and a traceless matrix of 1-form Ω with algebraic coefficients over $\mathbb{C}(x, y, z)$ such that
$$d\Theta = \Omega \wedge \Theta \text{ and } d\Omega = \Omega \wedge \Omega$$
- the ideal of the Galois groupoid of X is diff. generated by the components of $\varphi^* X_1 - \ X_1$
  and $\varphi^* dx \wedge dy \wedge dz - dx \wedge dy \wedge dz$

Corollary

If X is a reducible vector field with $divX = 0$ then

- $\exists \omega \in \Omega^1$ such that $dC_1 \wedge \omega = 0$
  and $dC_1 \wedge dC_2 = i_X dx \wedge dy \wedge dz$

- $\exists \Theta, \Omega$ a vector and a matrix with entries in Ω^1 such that $d\partial C = -\partial C \Omega$ and $d \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \partial C \Theta$.

Theorem

- $X_1 = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + (6y^2 + x) \frac{\partial}{\partial y'}$ is D.-V. irreducible.

- $X_6 = \frac{\partial}{\partial x} + \frac{\partial K}{\partial z} \frac{\partial}{\partial y} - \frac{\partial K}{\partial y} \frac{\partial}{\partial z}$ with

  $K = \frac{1}{x(x-1)} [y(y-1)(y-x)z^2 - zy(y-1) + \frac{1}{4}(y-x)]$,
  and $z = \left(\frac{x(x-1)}{y(y-1)(y-x)} \right) y' + \frac{1}{2(y-x)}$,

  is D.-V. reducible of second type (transversally affine)

D.-V. irreducibility of P_1

Ingredients

- The change of variable $u = y'^2 - 4y^3$.

$$K[y, y'] = K[y, u, \sqrt{4y^3 + u}]$$

$$X_1 = \frac{\partial}{\partial x} + \sqrt{4y^3 + u}\frac{\partial}{\partial y} + x\frac{\partial}{\partial y'}$$

- The weight p : $p(y) = 2$ and $p(u) = 6$.

<u>The first case is impossible.</u>

- ω can be supposed polynomial and $i_{X_1} d\omega = 0$.

- $\omega = \sum_m^M \omega_h$, compute ω_M, ...

  The computation of ω_{M-5} leads to a contradiction.

<u>The second case is impossible.</u>

- Θ can be supposed to be $\begin{pmatrix} dy - y'dx \\ dy' - (6y^2 + x)dx \end{pmatrix}$

and

$$\Omega = A\theta_1 + B\theta_2 + \begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix} dx$$

where A and B are matrices of polynomials s.t.

$$\begin{cases} X_1 A + 12yB - \begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix}, A\right] \\ X_1 B + A = \left[\begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix}, B\right] \\ \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y'} = [A, B] \end{cases}$$

D.-V. reducibility of PP_6

By

$$\begin{cases} x &= x \\ w &= \displaystyle\int_0^y \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-x)}} \\ w' &= \frac{y'}{\sqrt{y(y-1)(y-x)}} + \displaystyle\int_0^y \frac{d\xi}{2(\xi-x)\sqrt{\xi(\xi-1)(\xi-x)}}, \end{cases}$$

PP_6 is tranformed in $4x(x-1)w'' - 4(2x-1)w' - w = 0$. The pull-back of a linear first integral gives

$$\begin{aligned} H_{\alpha,\beta} = & \frac{y'\beta}{\sqrt{y(y-1)(y-x)}} \\ & + \int \ (\alpha + \frac{\beta}{2(y-x)})\frac{dy}{\sqrt{y(y-1)(y-x)}} \\ & \qquad - \frac{\beta y(y-1)}{2x(1-x)(y-x)}\frac{dx}{\sqrt{y(y-1)(y-x)}}. \end{aligned}$$

with

$$\begin{cases} \alpha' &= \beta\frac{-1}{4x(x-1)} \\ \beta' &= \beta\frac{1-2x}{x(x-1)} - \alpha. \end{cases}$$

Nishioka-Umemura irreducibility of P_1 by means of classical functions

Let K be a field of classical functions, (V, X_c) a model for $(K, \frac{d}{dx})$ and

$$(V, X_c) \xrightarrow{\pi} (\mathbb{C}^3, X_1)$$

a classical solution of P_1.

<u>Lemmas</u>

- π is dominant,
- $Gal(X_c) \xrightarrow{\pi_*} Gal(X_1)$,
- π_* is dominant.

Let $LG(X)$ be the Lie algebra of the Galois groupoid of X at the generic point and

$$LG(X_c)/X_c \xrightarrow{\pi_{**}} LG(X_1)/X_1$$

the induced projection.

- $LG(X_c)/X_c$ has a invariant intransitive Lie sub-algebra L such that $LG(X_c)/L$ is finite dimensional.
- $LG(X_1)/X_1$ is transitive, simple and infinite dimensional.

$\Rightarrow \pi$ cannot exist.