Drach conjectures

(A)

The Galois groupoid of

$$\frac{dy}{dx} = \frac{x-\lambda}{x(x-1)}\frac{y(y-1)}{y-\lambda}$$

is <u>special</u> if and only if λ is a solution of P_6.

(B)

The Galois groupoid of

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{y-\lambda}$$

is <u>special</u> if and only if λ is a solution of P_1 or P_2.

$\mathcal{D}$-Lie groupoid over a differential field

$(\mathcal{L}, \partial_1, \ldots, \partial_n)$ a differential field and $const = \mathbb{C}$.

- $\mathcal{O}_{J_k^*} = \mathcal{L}^{(1)} \otimes_{\mathbb{C}} \mathcal{L}^{(2)} \left[z_i^\alpha, \frac{1}{\det z_i^{\epsilon_j}}; 1 \leq i \leq n, \alpha \in \mathbb{N}^n, 1 \leq |\alpha| \leq k \right]$
  is a "Hopf algebroid" over $\mathcal{L}$
- $J_k^* = spec\ \mathcal{O}_{J_k^*}$
  is a groupoid acting on $spec\ \mathcal{L}$
- $J^* = \varprojlim J_k^*$
  is the groupoid of transformations of $spec\ \mathcal{L}/_{\mathbb{C}}$
- $\mathcal{O}_{J^*} = \varinjlim \mathcal{O}_{J_k^*}$
  is a differential ring with derivations
  $$D_i = \partial_i^{(1)} + \sum_j z_j^{\epsilon_i} \partial_j^{(2)} + \sum_{j,\alpha} z_j^{\alpha+\epsilon_i} \frac{\partial}{\partial z_j^\alpha}$$

Definition : A $\mathcal{D}$-Lie groupoid over $\mathcal{L}$ is a subgroupoid of J^* defined by a perfect differential ideal.

The $\mathcal{D}$-Lie algebra of a $\mathcal{D}$-Lie groupoid

- $T = \mathcal{L}\partial_1 + \ldots + \mathcal{L}\partial_n$ and T^* its dual over $\mathcal{L}$
- $\mathcal{O}_{A(J_k^*)} = ST^*\,[a_i^\alpha; 1 \leq i \leq n, \alpha \in \mathbb{N}^n, 1 \leq |\alpha| \leq k]$
  the ring of order k p.d.e. on vector fields $a_1\partial_1 + \ldots + a_n\partial_n$
- $A(J_k^*) = spec\ \mathcal{O}_{A(J_k^*)}$

Fondamental isomorphism

$$TJ^*_{/p_1}|_{id} \sim A(J^*)$$

$$\mathcal{D}\text{-Lie groupoid } G \quad \rightarrow \quad \text{linear subspace } A(G)$$

Definition : The Galois groupoid of a vector field X over $\mathcal{L}$ (or an equation over $\mathcal{K}$) is the smallest $\mathcal{D}$-Lie groupoid over $\mathcal{L}$, G, such that

$$fX \in A(G), \forall f.$$

- $Gal(E/\mathcal{K}) \subset Aut(E/\mathcal{K})$
- An equation E is special if $Gal(E/\mathcal{K}) \neq Aut(E/\mathcal{K})$

The Galois groupoid of an order 1 equation

$$y' = A(y) \in \mathcal{K}(y) \ ; \ \mathbb{C}(x) \subset \mathcal{K}$$

Theorem : $E/\mathcal{K}$ is special if and only if there is ω, α, β with coefficients in $\mathcal{K}(y)$ such that

- $\omega(X) = 0$
- $d\omega = \alpha \wedge \omega$
- $d\alpha = \beta \wedge \omega$ (sl_2-triplet for $E/\mathcal{K}$)
- $d\beta = \beta \wedge \alpha$

if and only if one can find $R \in \mathcal{K}(y)$ such that

$$\frac{\partial^3 A}{\partial y^3} + \frac{\partial R}{\partial x} + A\,\frac{\partial R}{\partial y} + 2\frac{\partial A}{\partial y}\,R = 0$$

(Drach's resolvant equation)

Isomonodromic deformation

Equations satisfied by the special first integral

$$\begin{cases} S_y H = R \\ \\ \frac{\partial H}{\partial x} + A\frac{\partial H}{\partial y} = 0 \end{cases}$$

its linearization $\frac{\partial H}{\partial y} = P^{-2}$

$$\begin{cases} \frac{\partial^2 P}{\partial y^2} = -\frac{1}{2}RP \\ \\ \frac{\partial P}{\partial x} + A\frac{\partial P}{\partial y} = \frac{1}{2}\frac{\partial A}{\partial y}P \end{cases}$$

and the 2×2 system

$$d\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} -\alpha/2 & \omega/\sqrt{2} \\ \beta/\sqrt{2} & \alpha/2 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Garnier counter-example

(B) equation $\frac{dy}{dx} = \frac{1}{2}\frac{1}{y-\lambda}$ over $\mathbb{C}(x, \lambda, \lambda', \ldots)$

Set $R = \sum_{n=-2}^{m} a_n (y-\lambda)^n$

$$\text{resolvant} \Rightarrow \begin{cases} a'_m &= 0 \\ a'_{m-1} &= m a_m \lambda' \\ a'_i &= (i+1)a_{i+1}\lambda' - \frac{i}{2}a_{i+2} \\ a_1 &= 4\lambda'' \\ a_0 &= -2\lambda'^2 \\ a_{-1} &= 2\lambda' \\ a_{-2} &= -\frac{3}{2} \end{cases}$$

$$\left| \begin{array}{l} m = 1 \ \rightarrow \lambda'' = c_1 \\ m = 2 \ \rightarrow \lambda'' = c_2\lambda + c_1 \\ m = 3 \ \rightarrow \lambda'' = c_3(6\lambda^2 + x) + c_2\lambda + c_1 \\ m = 4 \ \rightarrow \lambda'' = c_4(2\lambda^3 + \lambda x)c_3(6\lambda^2 + x) + c_2\lambda + c_1 \\ m = 5 \ \rightarrow \text{order 4 equation} \end{array} \right.$$

$$4\left(\frac{\lambda'''}{\lambda'}\right)' = 6a_3\lambda' - 2a_4 - \frac{1}{2}\left(\frac{a_3}{\lambda}\right)'$$

with $\begin{cases} a_5 &= c_5 \\ a_4 &= 5c_5\lambda' + c_4 \\ a_3 &= 10c_5\lambda^2 + 4c_4\lambda - \frac{3}{2}c_5 x + c_3 \end{cases}$

<ins>Garnier</ins> : This equation has some movable branching points

$\Rightarrow$ (B) conjecture is false.

<ins>Problem</ins> : Find a counter-example for (A) using a irregular singularity at ∞.

$$R = \frac{\alpha}{(y-\lambda)^2} + \frac{\beta}{(y-\lambda)} + \sum_{a\in\{0,1,x\}} \frac{\alpha_a}{(y-a)^2} + \frac{\beta_a}{(y-a)}$$

$$\alpha = -\tfrac{3}{2},\ \beta = -\tfrac{2\lambda-1}{\lambda(\lambda-1)} + \tfrac{x(x-1)\lambda'}{\lambda(\lambda-1)(\lambda-x)},\ \alpha_a' = 0$$

$$\tfrac{x(x-1)}{x-\lambda}\beta_a' = \left(\tfrac{\lambda(\lambda-1)}{(\lambda-a)^2} - 1\right)\beta_a + 2\tfrac{\lambda(\lambda-1)}{(\lambda-a)^3}\alpha_a$$

- $\frac{x(x-1)}{x-\lambda}\left(\sum \beta_a + \beta\right)' = -\left(\sum \beta_a + \beta\right)$
- $\frac{x(x-1)}{x-\lambda}\left(\sum a\beta_a + \lambda\beta\right)' = (\lambda-1)\left(\sum \beta_a + \beta\right)$

The (A') conjecture

(A') The Galois groupoid of

$$\frac{dy}{dx} = \frac{x-\lambda}{x(x-1)}\frac{y(y-1)}{y-\lambda}$$

is <u>special</u> and <u>regular</u>
(= its "R" has order 2 poles)
if and only if λ is a solution of P_6.

$$R = \frac{\alpha}{(y-\lambda)^2} + \frac{\beta}{(y-\lambda)} + \sum_{a\in A} \frac{\alpha_a}{(y-a)^2} + \frac{\beta_a}{(y-a)}$$

$$a \in \overline{\mathcal{K}}^{alg}, \alpha_a, \beta_a \in \mathcal{K}(a)$$

$$\alpha = -\tfrac{3}{2},\ \beta = -\tfrac{2\lambda-1}{\lambda(\lambda-1)} + \tfrac{x(x-1)\lambda'}{\lambda(\lambda-1)(\lambda-x)},\ \alpha_a' = 0$$

$$\tfrac{x(x-1)}{x-\lambda}\beta_a' = \left(\tfrac{\lambda(\lambda-1)}{(\lambda-a)^2} - 1\right)\beta_a + 2\tfrac{\lambda(\lambda-1)}{(\lambda-a)^3}\alpha_a$$

- $\tfrac{x(x-1)}{x-\lambda}a' = \tfrac{a(a-1)}{a-\lambda}$
- $\sum \beta_a + \beta = 0$
- $\sum a\beta_a + \lambda\beta = const$