~~For those who were at the small meeting in Rennes in September,~~

Why does it prove our theoreme ?

This is an exercise for ~~those~~ the few who were at Rennes meeting in september —

we made the following construction ~~for the 1st~~ before the 1st proof:

$$\begin{array}{ccc} R_1V|_{\mathcal{C}} & \xrightarrow[R_1 i]{} & R_1V \circlearrowleft R_1\mathcal{F} \\ \downarrow & & \downarrow \\ \phi \circlearrowright \mathcal{C} & \xhookrightarrow[i]{} & V \circlearrowleft \mathcal{F} \end{array}$$

the fundamental variation equation.

we get

$$Mal(R_1\mathcal{F}|_{\mathcal{C}}) \subset Mal(\mathcal{F}, R_1\mathcal{F})|_{\mathcal{C}}$$

and if you believe is suitable generalisation of Anne Granier theoreme $Mal(R_1\mathcal{F}/\mathcal{C})|_p$ "means on the fiber at p"

$$\| $$

$$Gal(R_1\mathcal{F}/\mathcal{C})$$

~~And the inclusion of $Mal(R_1\mathcal{F})$~~ equality And the big one is commutative

Introduction

Let V be an algebraic variety $/\mathbb{C}$

- ω be a ~~ho~~ closed non degenerate 2-form (algebraic). (I mean without poles non rational.)
- $f: V \dashrightarrow V$ a birational map. (but dominant appears to be enough in some cases).
  s.t. $f^*\omega = \omega$

Basicly we want to undersant the dynamic of f.

This is a big, a huge, problem there is a lot of difficult and beautiful result about it from complex geometry and complex analysis

→ Usually analysts are looking orbits of f.

$$P_{-1} = f^{-1}(P_0) \to P_0 \to f(P_0) = P_1 \to f(P_1) = P_2 \dots$$

or if f is not invertible, you can also look at the total orbit and so on

Such an orbite is a discrete solution of the discrete equation.

$$S(n+1) = f(S(n)) \quad \text{with } S: \mathbb{N} \text{ or } \mathbb{Z} \to V$$

→ Some of them and some algebraist are interested in continuous solution of the equation describe by f:

$$\phi S = f(S) \quad \text{with} \quad S: \mathcal{C} \to V \quad \mathcal{C} \text{ ~~a curve~~ Riemann surface}$$

and $\phi: \mathcal{C} \to \mathcal{C}$ a non periodic automorphism

- $\mathcal{C} = \mathbb{P}^1, +1$
- $\mathcal{C} = \mathbb{P}^1, qx$ for algebraic $\mathcal{C}$ where are "Algebraic solution of f".
- $\mathcal{C} = \mathbb{E}, e_x$

Algebraic solution of such an ~~homogeneous~~ autonomous system are very rare – In general solution are very transcendental

<u>But</u> In the symplectic context one has notion of integrability meaning that the solutions are not so transcendental.

<u>One of this notion is</u>

Mischenko–Fomenko
=
non commutative
=
Isotropic

| Integrabity

The purpose of this talk is to give two proofs of the "M.R" theorem in this context.

$\boxed{\text{THM}}$ IF f is isotropically integrable
THEN The Galois group of the discrete variational equation along an algebraic solution is almost commutative.

Some remark on this theorem:

- I will give definition in some minutes ~~but~~
- It's joint work with Julien Roques.
- This is a discret analogue of

  Przybylska – Maciejewski version of Morales Ramis theorem

  wich ~~state~~ gives the condition for hamiltonian vector fields.
- 2 PROOFS, 1^a the other is a mix of Malg + MRS about higher varia

II. Définitions

Integrability of symplectic map

. A map is called integrable when there is enough invariants

. In the case a symplectic form is preserved a rational first ~~invariant~~ integral gives rise to a rational vector field which is invariant.

- ~~H~~ If H is a function on V then there is a unique vector field X_H such that $dH = \omega(X_H, \cdot)$
  This is the symplectic gradient of H

- f is called ~~isotopic~~ integrable if there are $H_1, \dots, H_{n+\ell}$ rational functions on V s.t. $\ell \geq 0$

  - $f^* \omega = \omega$ (symplectic)
  - $f^* H_i = H_i$
  - $\bigcap_{i=1}^{n+\ell} dH_i$ is generated by $X_{H_1}, \dots, X_{H_{n-\ell}}$ over $\mathbb{C}(V)$.

Some remarks about this definition.

- It is due to Mischenko-Fomenko, in cases of C^∞ vector fields.
- The level set $H_i = C_i$ are isotropic sub-manifold.
- when $l=0$ one gets Liouville integrability
- MischFom conjectured that it is equivalent to usual Liouville integrability.

It is proved in somes cases by Sadetov or Bolsinov

Maria & Andrej results give a kind of evidence of this conjecture but not a proof.

Linearization of f.

$f: V^{2n} \dashrightarrow V^{2n}$ birational map.

& $\mathcal{C}: \mathbb{P}^1 \dashrightarrow V$ a ϕ-curve.

$\mathcal{C} \circ \phi = f \circ V$ with ϕ non periodic.

we will considere the linearization in 2 forms–

$Tf: TV \dashrightarrow TV$ is the full tangent map.

$Rf: RV \dashrightarrow RV$ The bundle of frames of TV

Because $\mathcal{C}$ is f-invariant, one gets

$Tf|_{\mathcal{C}}: \mathcal{C}^*TV \dashrightarrow \mathcal{C}^*TV$ or $Rf|_{\mathcal{C}}$

By a rational gauge transformation

$Tf|_{\mathcal{C}}: \mathbb{P}^1 \times \mathbb{C}^{2n} \dashrightarrow \mathbb{P}^1 \times \mathbb{C}^{2n}$ or $Rf|_{\mathcal{C}}$ $\mathbb{P}^1 \times GL_{2n}(\mathbb{C})$

$\left(z, \begin{matrix} y_1 \\ \vdots \\ y_{2n} \end{matrix}\right) \mapsto \left(\phi z, A(z) \begin{matrix} y_1 \\ \vdots \\ y_{2n} \end{matrix}\right)$ on $(z, Y) \mapsto (z, AY)$.

variational equation fundamental var equ.

ϕ is qz or $z+1$.

Such a difference equation has a Galois group.
→ Our reference is Sv. d P. Lecture Note M. 1666.
And Morales Ramis Simo Theorem, Ziglin theorem and Churchill-Singer and their collaborator taught us ~~that if f is integrable the~~ How this Galois group ~~reflects~~ gives information about first integrals.

1st Proof "Morales Ramis type"

2 changes – Liouville to Mischenko-Fomenko.
it is not a big difficulty because we have the work of Andrey & Maria.
– The Picard-Vessiot Ring is not a domain. ~~Here you~~

Step 1 A rational first integral H of f gives rise to a rational first integral H^0 of $Tf|_{\mathcal{C}}$

In local coordinates $x_1 - x_{2n}$ on V

$t_1 - t_{2n}$ on fibers on TV

If $H = \sum_{|\alpha| \geq k} h_\alpha (x-\underline{x})^\alpha$ $k \geq 0$ and $\alpha \in \mathbb{N}^{2n}$.
in the neighborhoung of $\underline{x} \in V$.

then $H^0 = \sum_{|\alpha| = k} h_\alpha t^\alpha$ on $T_{\underline{x}} V$

and do it for all $\underline{x}$ in $\mathcal{C}$.

Step 2 • the Galois group acts on fibers of $TV|_{\mathcal{C}}$ by

$\phi Y = Y C(\phi)$ where Y is a PV fondamental matrix of $Tf|_{\mathcal{C}}$.
$C(\phi)$ is a constant matrix $GL_{2n}(\mathbb{C})$

• Fonctions $\tilde{H}^0 : \mathbb{C}^{2n} \dashrightarrow \mathbb{C}$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{2n} \end{pmatrix} \dashrightarrow H^0\left(z, Y\begin{pmatrix} c_1 \\ \vdots \\ c_{2n} \end{pmatrix}\right)$$

are invariant with respect to the action of Galois

⚠ be carefull because PV ring is not a domain

So $\tilde{H}^0$ may be not define! (8)

Lemma (known) There is a $s \in \mathbb{N}$ such that $(T\mathcal{G}|_{\mathcal{C}})^{\circ s}$ has a PV ring without zero divisors ~~and~~ the Lie algebra of the Galois group is unchange.

Step 3 Independence of the H_i^0

In this first proof we will use Ziglin lemma.

~~first~~ if $H_1 - H_k$ are functionally independent on an Z dense open subset of V

then ~~you cho~~ in any point $p \in V$, you can choose ~~H_1, H_2~~

$$F_1 = H_1$$
$$F_2 = P_2(H_1, H_2)$$
$$F_k = P_k(H_1 - H_k) \quad \text{polynomial}$$

s.t. $F_1^0 - F_k^0$ are functionally independent on T_pV hence alg ind.

First apply Ziglin to $H_1 - H_{n-l}$.

Then to $H_{n-l+1} - H_{n+l}$.

Step 4 Let v and w be 2 linear vector fields on $\mathbb{C}^{2n} \simeq T_pV$ in the algebra of Galois

The $[v, w] = 0$

(Step 4 1/2) v preserves ω_p

v preserves H_i^0) $\Rightarrow$ v preserve $X_{H_i^0}$

~~so $v = \sum$~~

v preserve $H_1^0 — H_{n+l}^0 \Rightarrow v = \sum_{i=1}^{n-l} c_i X_{H_i^0}$

Last Mich-Form cong.

c_i are in $\mathbb{C}(T_pV)$

but v preserve X_{H_i} so c_i are constant with respect to X_{H_i} $i = 1, —, n-l$.

2 such vector fields must commute –.

$\boxed{4}$

This ends the proof BUT I want to give another proof as an introduction to Malgrange pseudogroup

& Morales Ramis Simo

higher variational equation.

In the context of discret systems –.

It is expected (and partially proved) that this will give more results than "commutativity" about Gal.

~~Malgrange Morales Ramis Simo~~

Malgrange pseudo-group approach:

Let V^{2n} be an algebraic variety.

and $R_q V = J_q^*((\mathbb{C}^{2n},0) \to V)$

be the bundles of order q frame on V

= q-jets of diffeo from a neighborhood of 0 to V

$R_0 V = V$

$R_1 V$ is bundles of basis of TV (frames of TV)

$$R_{q+1} V \downarrow R_q V$$

- IF G_q is $J_q^*((\mathbb{C}^{2n},0) \to (\mathbb{C}^{2n},0))$ the linear algebraic group of q-jet of diffeo fixing 0

Then G_q acts by composition $R_q V$ and R_q is a G_q-principal bundle

- IF J_q^* is $J_q^*(V \to V)$

Then J_q^* acts by composition on $R_q V$ and in some sense it act principally –

⚠ this is just a groupoid ~ pseudo group

~~Prolongation of dynamical system~~

"Differential" structure of $R_q V$

Let $x_1 - x_{2n}$ be local coordinates on $(\mathbb{C}^{2n}, 0)$

and $r_1 - r_{2n}$ local $(\mathbb{C}^{2n}, 0) \to V$

~~then~~ can be writen as power series and r_i^α are coordinates on $R_q V$.

with $\alpha \in \mathbb{N}^{2n}$ $|\alpha| \leq q$

A function $H \in \mathbb{C}(R_q V)$ can de derivate w.r.t. x_j

and gives a funcion $D_j H$ in $\mathbb{C}(R_{q+1} V)$

by $D_j(r_i^\alpha) = r_i^{\alpha + \epsilon(j)}$ $\quad \epsilon(j) = (0 - 0, 1, 0 - 0)$

[recall that $\mathbb{C}(R_q V) = \mathbb{C}(r_i^\alpha \mid 1 \leq i \leq 2n, \alpha \in \mathbb{N}^{2n} |\alpha| \leq q)$

Prolongation

- f acts on $R_q V$ by its order q prolongation

$$R_q f : R_q V \dashrightarrow R_q V$$
$$j_q(r) \longrightarrow j_q(f \circ r)$$

- X a local analytic vector field act "locally" on transcd. top. acts also on $R_q V$ by "same formulae"

Fortunately there is a nice formula for this action

$$X = \sum a_i \frac{\partial}{\partial x_i} \to R_q X = \sum D^\alpha a_i \frac{\partial}{\partial r_i^\alpha}$$

Example $\quad R_1 X = \sum a_i \frac{\partial}{\partial r_i} + \sum \frac{\partial a_i}{\partial x_j} \cdot r_j^{\epsilon(k)} \frac{\partial}{\partial r_i^{\epsilon(k)}}$

Rough definitions.

- A differential Invariant of $\mathcal{F}$ is a rational first integral of $R_q\mathcal{F}$: $H \in \mathbb{C}(R_q V)$

$$H \circ R_q(\mathcal{F}) = H.$$

- ~~Malg~~ Malgrange pseudo-group of $\mathcal{F}$: $Mal(\mathcal{F})$ is the "subvariety of $J^*_\infty(V \to V)$" given by the equation $H \circ j_q(s) = H$ for all $q \in \mathbb{N}$ and all order q diff inv of $\mathcal{F}$.

This definition is equivalent to original Malg definition when everything is algebraic (Gabriel SGA, Pommaret DGT)

⚠ This is not a subgroupoid because in general there are some "singularities" like the indeterminacy locus of H's.

- The Lie algebra of this object is the "set" of formal vector fields X on V s.t

$$(R_q X) \cdot H = 0 \text{ for all } q \text{ all dif invariants.}$$

This is more than a set but I do not want to give more details than needed.

Remark

- In any cases
$$[R_q X_1, R_q X_2] = R_q [X_1, X_2]$$
- If X_1 and X_2 are convergent then $X_1 X_2$ ~~is~~ in $\mathrm{malg}(\mathcal{S})$
$\Rightarrow [X_1, X_2]$ in $\mathrm{malg}(\mathcal{S})$
- If one has formal vector field it's ~~more~~ subtle because one can be at a point in the indeterminacy locus of H.

~~If you believe that any solution of linear p.d.e is summable then there is no problem.~~

~~This pro~~

~~This is the only problem to solve to understand~~

This is our main problem to solve in proof n° 2.

Thm

- if H is an invariant $D^q H$ is an order $|\alpha|+q$ invariant.
- if H vanishes at order d at $p \in V$ then $D^q(H)$ vanishes at order $d-|\alpha|$ and if $|\alpha| > d$ it define a function on $R_{|\alpha|-d}(V)\big|_p$

THM - IF f is M.F. integrable
THEN malg(f) is commutative.

- The proof when you consider convergent vector fields is ~~exact~~ ~~ex~~ exactly the one of the linear case.

X must preserve H_i order 0 invariant
X ———— ω —— 1 ——
X ———— X_{H_i} —— 1 ——

write ~~those~~ a a point where H_i are defined and independant

- When X is a formal vector field is more complicated because you cannot change the point to another where every thing is OK.

IF X is define at $p \in V$ then maybe H_i are not
Morales-Ramis-Simó overcome this difficulty by using Artin theorem to approximate this formal vector field by ~~suit~~ a non holonomic section ~~the~~ jet bundles -

Let me briefly explain this –

$mal(\mathfrak{g})$ is defined by systeme of linear pde on TV

i.e. for every q you have $L_q \subset J_q(TV)$ with compatibilities –

a linear subspace ~~A~~ not a bundle.

it can ave torsion –

if you start with a formal vector field $\hat{X} = \sum \hat{a}_i \frac{\partial}{\partial x_i}$ solutions of the differential systeme.

Then $\left(\hat{a}_1, \ldots \hat{a}_n, \frac{\partial \hat{a}_1}{\partial x_1}, \ldots \frac{\partial^\alpha \hat{a}_i}{\partial x^\alpha}\right) \quad |\alpha| \leq q$.

is a formal section of $L_q/_V$ at p.

by ARTIN Theorem ~~states~~ : ~~you~~ there is a convergent section $\mathcal{Y}_{\alpha}(b_1 - b_n, b_1^{\alpha(1)} - b_1^q -) = S : (V,p) \to$ close to the formal one in the m-adic topology at p.

In general. ∂b_i it is not holonomic

$$\frac{\partial b_i^\alpha}{\partial x_j} \neq b_i^{\alpha+\varepsilon(j)}.$$

Now if X_1 and X_2 are 2 such ~~form~~ vector field,
formal vector field
in $mal(f)$
at the same point

Then let S_1 be an artin approxim of X_1
S_2 ——— X_2

on can compute $[S_1, S_2]$.

at $\tilde{p}$ ~~the~~ $S_1(\tilde{p})$ is a q jet of vector field
$S_2(\tilde{p})$ ———

$[S_1, S_2](\tilde{p})$ is a $q-1$ jet of vector field.

~~If you take q big enought.~~

by an not so easy theorem of Cartan, Kuhler, Goldschmidt ...
... Malgrange.

$S_1(\tilde{p})$ and $S_2(\tilde{p})$ are jet of ~~soluti~~ convergent solution of $mal(f)$

~~so~~ if by the easy part of the proof
$[S_1(\tilde{p}), S_2(\tilde{p})] = 0$ for almost all $\tilde{p}$