This talk is about symplectic mappings and their integrability in the Liouville sense

One ~~gets~~ has a ~~symplectic variety~~ ~~com~~
complexe algebraic smooth variety ~~dim 2n~~.
~~with a closed non degenerate 2-form ω~~
(V, ω)

and $f : V \dashrightarrow V$ birrational.

we want to find rational first integrals $H : V \dashrightarrow \mathbb{C}$
with $H \circ f = H$.

Following Morales-Ramis theorems, we want to find obstruction to the existence of first integrals by using the galois groupe of the linearization of f along some particular solution.

* particular solutions will be an invariant projective line $\mathbb{P}^1 \hookrightarrow V$ in V such that $f|_{\mathbb{P}^1}$ is not periodic
* Linearization of f along $\mathbb{P}^1$ will be a difference systeme.

because of this it is a application of previous talk.
and it is a joint work with Julien Roques.

Discret dynamical system. (algebraic).

. Let V be an algebraic complex variety (smooth).
as we will used only rational functions, rational transformation..., smoothness will be assumed.

and $f: V \dashrightarrow V$ a ~~bir~~ rational map.
we will assume f is birational but dominant might be enough.

Basicly We want to understand the dynamic of f. This is a huge problemes.
There is a ~~extensif~~ lot of works and results using complex geometry and complex analysis on this problema ~~and~~ beautiful.

~~The point of view usually~~

to understand the dynamics is to understant the orbits of point $p \in V$

$\to p_2 \to p_1 \to p_0 \to p_1 = f(p) \quad p_2 = f(p_1) \cdots$) equation ~~$p_n = f(p_{n-1})$~~
$\tilde{p}_1 \quad \tilde{p}_0 \quad \tilde{p}_1$ and if the map is not birational. This is the really discret point of view.

~~Discret~~ Equation
$p_{n+1} = f(p_{n})$ i.e. discret Solutions are maps $\mathbb{N} \xrightarrow{s} V$
such that $s(n+1) = f \circ s$.

- for more than 1 than f you get more than 1 index and your equation is a lattice equation.

Our point of view in this talk is to look at "continuous" solution.

we will only look at the first two types and with $|q| \neq 1$

a) $s(z+1) = f(s)$ $\quad s: \mathbb{P}^1 \to V$ for the additive time mathematician.
a) $s(qz) = f(s)$ $\quad$ for the multiplicative —

a) $s(ez) = f(s)$ $\quad s: E \to V$ for the elliptic mathematician.
- - -

These are homogeneous non linear difference equations.
as it was presented in numbers of talk befor this one.

a solution is an invariant ~~Riemann curve~~ C is V
~~such that $\delta|_C$ is not periodic~~.

For this ~~talk~~ the curve will be algebraic & rational and $\delta|_C$ not periodic.
i.e. $\mathbb{P}^1$

the ~~hyper~~ algebraic hypothesis

Isotropic Integrability of a symplectic map.

(non commutative, Mischenko-Fomenko).

A map is called integrable when there is enough "invariants"

or..

→ symplectic V and symplectic gradient.

IF There ~~are a close non degenerate 2 form on~~ V^{2n} ~~and~~ ω

$H_1, \dots, H_{n+l}$ $n+l$ rational functions on V ~~(...)~~

$(l \geq 0)$ such that $f^*\omega = \omega$ (symplectic)

~~and~~ $f^* H_i (= H_i \circ f) = H_i$

and H_i first integrals.

~~the f is said to be integrable in the thi~~ $\cap\, dH_i$ is generated by $X_{H_1} - X_{H_{n-l}}$ over $\mathbb{C}(V)$.

- the last condition says that the level set $H_i = c_i$ are isotropic sub manifold.
- when $l = 0$ it is the usual Liouville integrability.

Some remarks on this definition.

- Misch Fom conjectured that it is equivalent to usual Liouville.
  → there is proofs in some cases by Sadetov or Bolsinov.
- ~~then In the continuous cases. For a vector field~~

Now we want a discrete ~~version~~ analogue of Przybylska - Maciejewski "version of Morales-Ramis theorem.

To state this statement and for peoples ~~who didn't~~ absents in previous talk I will ~~give~~ recall some definitions.

Some Definition $\quad \phi\, V^{2n} \dashrightarrow V^{2n}$ and $T\phi : TV \dashrightarrow V$
on tangent bundle.

let $i: \mathbb{CP}^1 \dashrightarrow V$ be a ϕ adapted curve.

(1). then: if i is ϕ invariant $\Rightarrow TV|_C$ is $T\phi$ invariant.

and $T\phi$ ~~gives a~~ can be restricted on i^*TV a rank $2n$.
vector bundle on $\mathbb{P}^1$ and it is fibrewise linear.
so using a rational "gauge transform" this bundle is trivial.

and $T\phi|_i$ is $\left(z, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) \to (\phi z, A(x) Y)$.

where ϕ is $z \to z+1$ and $A(x) \in GL_n(\mathbb{C}(x))$.
or $z \to qz$.

This is the discrete variational ~~invariant~~ equation.

(2) and H ~~such~~ a rational function such that $H \circ \phi = H$.
and $H(p)$ defined.
if $p \in V$ the initial part of H at p: H_p^0 is the first non zero.
homogeneous polynomial in its taylor serie as a function on T_pV
if $H = \frac{F}{G}$ then we define H_p^0 by $\frac{F_p^0}{G_p^0}$.

(3) Definition of the Galois group of the linear differen system.

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Thm.

IF $g: V \to V$ is integrable then its variational Equation has a almost commutative Galois Group.

$\|$

means that its Lie algebra is commutative.

The first proof follow M-R theorem with some with 2 changes

1) - change Liouville to non comm. but here most of the work was done by Andrei & Flavia.

2°) - the Picard Vessiot Ring is not a domain, and we have to be careful.

Step 1. if H is a 1st I of g the H^{o}_{i} is a 1st I of $Tg|_i$

Step 2. IF $H^{o}_{i}(z, Y)$ is a rational 1st Integral of $Tg|_i$

then $\tilde{H}: \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \to H^{o}_{i}\left(z, F\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}\right)$ with F a fundamental matrix of $Tg|_i$ and $\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$ ~~are constants~~

is Galois invariant for the usual action on $\mathbb{C}^{2n}$.

⚠ because PV extension is not a domain this function in not always define.

Lemme (S-vdP). There is a $g \in \mathbb{N}$ such that the PV ring of $Tg|_{i}^{g}$ is a domain.

(6)

Step 3. Independence of the H_c^o

~~As it It is overcom~~ this difficulty is overcome by Ziglin lemma.

In 2 ~~steps~~ 1) Independence of the $(H_1)_c^o$ — $(H_{n-l})_c^o$

initial part of the n-l first integrales.

by Ziglin lemma there symplectic gradien span the same distribution.

2) Independence of the ~~n~~ (2l) remaining initial parts.

independence of the $(H_i)_c^o$ ~~simply~~ independance of the $(\tilde{H}_i)_c^o$

Step 4. Let $\mathcal{O}_1$ be a linea vector field on $\mathbb{C}^{2n}$ coordinate $\begin{pmatrix} z_1 \\ \vdots \\ z_{2n} \end{pmatrix}$ in the lie algebra of Galois

after T_pV for som $p \in \mathcal{C}$

then. $\mathcal{O}$ preserve the symplectic for ω

$\mathcal{O}$ preserve $(H_1)_c^o \sim (H_1)_p^o$

$\vdots$

$(H_{n-l})_c^o \Rightarrow \mathcal{O}_1 = \sum_{i=1}^{n-l} c_i X_{H_i}$

c_i are constantes with respect to X_{H_i} $i = 1, \dots, n-l$.

$\Rightarrow$ two such vector fields commute.

This is the complete proof but I want to give you an other proof. not completly complet by using Malgrange groupoid.

This approach is ~~no really~~ not really completly different by different enough to give some other results like the evolution of the ~~size~~ dimension of de Galois groupe of higher order variational equation.

And Relation ~~to~~ with Gabber theorem.

Malgrange groupoid Approach.

$f : V^{2n} \dashrightarrow V^{2n}$

Let $R_q V$ be the bundle of ~~order~~ order q frame on V.

$= J_q^*((\mathbb{C}^{2n},0) \to V)$ ~~J_1^*~~

$R_0 V = V$ just the target

$R_1 V =$ open set of ~~$\{\ldots\}$~~

$TV \times_V TV \times_V \cdots \times_V TV$ of basis of T_pV for all $p \in V$.

R_q: order 2 frames are ~~frame~~ a point $p \in V$ a frame of T_pV

and are way to move this frame order 2

but the important formula is $\boxed{J_q^*[(\mathbb{C}^{2n},0) \to V]}$

- order q diffeomorphisme at 0 $\underset{G_q}{\underset{\shortparallel}{J_q^*[(\mathbb{C}^{2n},0) \to (\mathbb{C}^{2n},0)]}}$ acts by composition.

and R_q is a G_q-principal bundle.

- order q diffeomorphismes of V $J_q^*(V \to V)$ acts by composition on the R_q

Def prolongation.

~~Asa~~ Remark that f itself act on R_q

Let call $R_q f : R_q V \dashrightarrow R_q V$ this action.

$j_q(x) \to j_q(f \circ x)$.

~~Def vector field.~~

Def A first integral of $R_q f$ is called a differential invariant of f.

Rem Tensor invariants are particular order 1 diff invariant.

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- The malgrange groupoid of f is the subvariety of $J^*_\infty(V \to V)$ $= \varprojlim J^*_q(V \to V)$ given by the equation.

$$H \circ f_q(s) = H \quad \text{for all } q \in \mathbb{N} \text{ and all order } q\text{-invariant of } f.$$

This definition is equivalent to original definition from B Malgrange when. ~~V~~ every things is algebraic.

- Its (~~sheaf of~~) lie algebra is the sheaf of analytic vector fields. X on transcendental open set of V such that

$$(R_q X).H = 0 \quad \text{for all } q \text{ and order } q\text{-invariant of } f.$$

<u>Thm</u> ~~(???)~~ IF f is Mísch-Form Integrable then its Malgrange groupoid is infinitesimally commutative.
$\rightsquigarrow$ more ~~or~~ less already done by J-P RAMIS.

<u>How does it imply commutativity for the Galois group of the linearized?</u>

The ~~???~~ total linearization of f is $R_1 f : R_1 V \to R_1 V$.
If $c : \mathbb{P}^1 \to V$ is an f-invariant curve then one can restrict $R_1 f$ above $c(\mathbb{P}^1)$.
Let B_q be the pull back $c^* R_q V$ then. • B_1 is the frame bundle on $c^* TV$
• $R_1 f|_{B_1}$ is the variation equation in the fundamental forme. it can be written $(z, (Y_i^j)) \mapsto (\phi z, A(z) Y)$

~~Now fix a point $p \in V$ then the fiber of $R_1 V$ is $J_1((\mathbb{C}^n,0) \to (V,p))$.~~

~~with action of $J_1((\mathbb{C}^n,0) \to (\mathbb{C}^n,0))$ Gl_n at source.~~

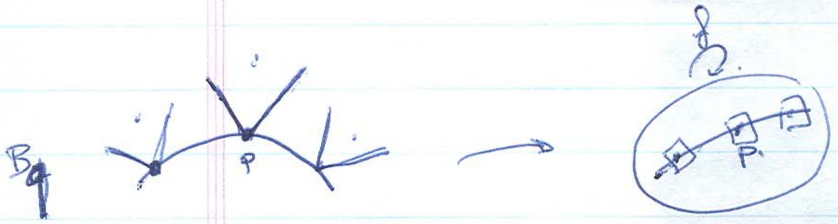
~~and action of $J_1((V,p) \to (V,p))$ at target~~

~~(this two groupes are Gl_n) ~~but I will used~~~~

~~Among all the equations $H \cdot j_1(s) = H$ and its~~

Pb Understand the restrictions above $\mathcal{C}$ of all the differential invariants of $\mathcal{F}$ - because some of them ~~gives~~ are order 1 above $\mathcal{C}$ -

~~And in particular these~~



Let X be a ~~vertical~~ vector ~~space~~ field on B_q in the Lie algebra of the Galois groupoid of the variational equation. at $p \in \mathcal{C}$.

then • X preserves all the restriction of invariant of $\mathcal{F}$ above $\mathcal{C}$ { when the restriction is defined -

~~• X is $j_q(\nu)$ for some formal vector field ν on V at p~~

~~the prolongations of X preserve all the $\tilde{q} > q$ invariants~~

Let $J_q(TV)$ denote the ~~spa~~ order q jet of vector field on V.

order q jet of ~~vectors~~ fields in the Lie algebra of Malgrange groupoid are solutions of a linear subspace $L_q \subset J_q(TV)$.

$$L_q \to V$$

because invariants are ~~not~~ rational and not always independant.

the dimensions of fibers $L_q \to V$ ~~is not~~ are not constant.

~~are gets~~ there is a fiberwise bracket on L_q $[j_q(X), j_q(Y)] = j_{q-1}[X, Y]$.

~~we pro~~ Even if the bracket is zero above.