# Solvability of algebraic pseudogroups and differential equations

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The purpose of these notes is to give a another Galoisian proof of

**Theorem** [**A. Khovanskii**, J. Dyn. and Con. Systems **1** 99–132 (1995)]
*Let $f : \mathbb{C}^n \to \mathbb{C}$ be a Liouvillian function then its closed monodromy group is almost solvable in the sense that it admits a normal tower with finite or commutative quotients.*

Actually Khovanskii's proof has a Galoisian nature quoting Khovanskii it is proved using "one dimensional topological Galois theory". We want to give a more algebraic proof using the so called Godbillon-Vey sequences for codimension $q$ foliations. This is a "technology free" way to speak about the Galois pseudogroup of a foliation. After some preparation, the problem is to prove that subgroup of solvable pseudogroups are solvable. Using usual definition of solvability for pseudogroup this is wrong. The first part of this notes is devoted to definitions of solvability and infinitesimal solvability for pseudogroups, examples, counterexamples and comparaison of definitions. The second one deals with Liouvillian first integrals and Liouvillian solutions of ordinary differential equations. As an application, we give the proof of Khovanskii's theorem for the monodromy itself which was the original motivation.

## 1 Solvability of algebraic pseudogroups

Roughly speaking an algebraic pseudogroup is a pseudogroup of transformations of an smooth algebraic variety $X$ over $\mathbb{C}$ defined by algebraic partial differential equations. But the reader can easily find three different definitions of pseudogroup of transformation in the litterature. The difference is about how you can glue transformations when domains are not connexe.

**Example 1** *Let's have a look at the pseudogroup of transformations of the affine line generated by translations without restriction on the gluing property. Because restriction and gluing of elements of a pseudogroup must belong to the pseudogroup, one gets to much elements in this pseudogroup. For any permutation $\sigma$ of $\mathbb{Z}$ one gets a transformation $\varphi : U \to U$ from a neighborhood of $\mathbb{Z}$ to itself realising $\sigma$. This pseudogroup contains a group that can not be said to be commutative.*

Here is a more natural example. This kind of situation can not be prevent by any "honest" definition and justifies the need of almost commutativity instead of commutativity.

**Example 2** *Let $\pi : Y \to X$ be a covering of a Riemann surface, $\omega$ a meromorphic 1-form on $X$ and $\varpi$ its pull back on $Y$. Locally, the pseudogroup of transformation of $Y$ preserving $\varpi$ look like the previous example. Even if one does not use the gluing property, it contains all the group of deck transformations of the covering. In general, it is not a commutative group.*

Usually this problem is eliminated by changing definition of commutativity in an infinitesimal way. In this part we want to distinguish the first (artificial) example from the second (natural) one. To avoid problems, we will work in the $\mathbb{C}$-analytic category and take the following definitions.

**Definition 1** *The determinations at $b \in X$ of a germ of transformation of $X$ $\varphi$ at $a \in X$ are the transformations obtained by analytic continuations of $\varphi$ along any pathes from $a$ to $b$. This set is denoted by $\mathcal{D}(\varphi, b)$, it can be empty.*

*Two germs of transformations $\varphi$ at $a$ and $\psi$ at $b$ are said to be compatible if $\psi \in \mathcal{D}(\varphi, b)$. A transformation $\varphi$ is said to be compatible (with itself) if every couples of germs $\varphi_a$, $\varphi_b$ obtained from $\varphi$ are compatible.*



**Definition 2** *A pseudogroup of transformation of $X$ is a set $PG$ of local transformation $\varphi : U = d(\varphi) \to V = r(\varphi)$ such that*
- *the restriction of a transformation $\varphi$ of $PG$ to a open subset of its domain $d(\varphi)$ is in $PG$;*
- *if $\phi \in PG$ and $\varphi \in PG$ and $r(\varphi) = d(\psi)$ then $\psi \circ \varphi$ is in $PG$;*
- *if $\phi \in PG$ then $\varphi$ is invertible and $\varphi^{\circ -1}$ is in $PG$;*
- *every $\varphi$ in $PG$ is compatible;*
- *every $\varphi$ with a determination in $PG$ is in $PG$.*

**Definition 3** *A subgroup $\mathsf{G}$ of a transformation pseudogroup $PG$ is a group of composable transformations or composable germs of transformations.*

**Remark 4** *All subgroups of the first example are commutative but the second example has noncommutative subgroups.*

An algebraic pseudogroup is a pseudogroup defined by partial differential equations like the second example. For a rigourous definition the reader may have a look to [**B.Malgrange**, monog. **38** vol **2** de L'enseignement mathématique 465–501 (2001)].

**Definition 5** *An algebraic pseudogroup $PG$ on $X$ is a set of local diffeomorphisms $\varphi$ defined by some algebraic partial differential equations $E_i \in \mathbb{C}(x,\varphi)[\varphi^\alpha] = \mathbb{C}(x_1,\dots,x_n,\varphi_1,\dots,\varphi_n)\left[\varphi_i^\alpha \mid 1 \le i \le n, \alpha \in \mathbb{N}^n\right]$:*

$$\{\varphi : U \to V \mid E_i(x, \varphi(x), \varphi^\alpha(x)) = 0 \ ; \ i = 1,\dots,m\}$$

*such that*

1. *$E_i(\varphi \circ \psi) = \sum c_i^j E_j(\varphi) + d_i^k E_k(\psi)$,*
   *where $c$'s and $d$'s are in $\mathbb{C}(x,\varphi,\psi)[\varphi^\alpha, \psi^\alpha, \frac{Dx}{D\varphi}, \frac{Dx}{D\psi}]$.*
2. *$E_i(id_U) = 0$ for all identities maps $id_U : U \to U$,*
3. *$E_i(\varphi^{-1}) = \sum e_i^j E_j(\varphi)$,*
   *where $e'$s are in $\mathbb{C}(x,\varphi)[\varphi^\alpha, \frac{Dx}{D\varphi}]$*

Such an pseudogroup has a Lie algebra :

**Definition 6** *Let $PG$ be an algebraic pseudogroup on $X$ with equations $E_i$, $i = 1\dots m$. Its Lie algebra is the sheaf $\mathcal{L}PG$ of local analytic vector fields $\vec{a} = \sum a_j(x)\frac{\partial}{\partial x_j}$ on $U$ such that*

$$\mathcal{L}E_i(\vec{a}) := \sum \frac{\partial E_i}{\partial \varphi_j^\alpha}(id_U) a_j^\alpha = 0 \text{ for all } i = 1\dots m.$$

**Proposition 7** *The sheaf $\mathcal{L}PG$ is a sheaf of Lie algebra for the usual Lie bracket of vector fields.*

**Definition 8** *An algebraic pseudogroup $PG$ is said to be inf-solvable (infinitesimally solvable) if its Lie algebra is a (sheaf of) solvable Lie algebra.*

**Definition 9** *A solvable coparallelism of $X$ is given by $n$ 1-forms $\omega_\bullet : \omega_1,\dots,\omega_n$ such that $d\omega_i = 0 \mod \omega_1,\dots,\omega_{i-1}$. It is called exactly solvable if $\omega_i$ is relatively exact (there is a rational function $h_i$ such that $\omega_i = dh_i \mod \omega_1,\dots,\omega_{i-1}$) or exponentially relatively exact (there is a rational function $h_i$ such that $\omega_i = dh_i/h_i \mod \omega_1,\dots,\omega_{i-1}$).*

**Theorem 10** *Let $\omega_\bullet$ be a solvable coparallelism on $X$.*
- *The algebraic pseudogroup $PG(\omega_\bullet)$ defined by*

$$\varphi^*\Big(\bigwedge_{j=1\dots i} \omega_j\Big) = \bigwedge_{j=1\dots i} \omega_j \quad \text{for } i = 1\dots n$$

*is infinitesimally solvable.*
- *If $\omega_\bullet$ is exactly solvable then subgroups of $PG(\omega_\bullet)$ are almost solvable.*

**Remark 11** *All inf-solvable pseudogroups are not given directly by this kind of equation but after suitable prolongation. See* [**É.Cartan** (1905)].

*Proof.* – The proof of the first part is a local analytic version of the proof of the second part. Let $\mathsf{G} \subset PG(\omega_\bullet)$ be a subgroup of the pseudogroup defined by a exactly solvable coparallelism on $X$.

Let $h_1$ be a primitive of $\omega_1$. For each $\varphi \in PG(\omega_\bullet)$, one gets a constant $c_\varphi$ by the equality $h_1 \circ \varphi = \varphi + c_\varphi$. The map $\varphi \mapsto c_\varphi$ is a morphism of pseudogroup onto the "pseudogroup" $Inv(dx)$ on $\mathbb{C}$. The restriction of this morphism on $\mathsf{G}$ gives a morphism of group $m_1$ from $\mathsf{G}$ to $(\mathbb{C}, +)$ whose kernel is denoted by $\mathsf{G}_1$.

Now considere $X$ as a variety over the affine line with coordinate ring $\mathbb{C}[h_1]$. The relative differential is denoted by $d_/$. Because $\omega_2$ is exact modulo $dh_1$, one gets a rational function $h_2$ on $X$ such that $d_/ h_2 = \omega_2$. Take a $\varphi \in PG(\omega_\bullet) \cap Inv(h_1)$. Because differentials of $h_2$ and $h_2 \circ \varphi$ are equal modulo $dh_1$, one gets $h_2 \circ \varphi = h_2 + \underline{c}_\varphi(h_1)$ for an analytic function $\underline{c}_\varphi$. This gives a morphism $m_2$ of pseudogroup from $PG(\omega_\bullet) \cap Inv(h_1)$ to the "additive" pseudogroup of analytic functions on the affine line. This morphism maps $\mathsf{G}_1$ on a commutative group. By induction one gets a sequence

$$\mathsf{G}_n \subset \ldots \subset \mathsf{G}_1 \subset \mathsf{G}$$

with $\mathsf{G}_i / \mathsf{G}_{i-1}$ commutative and $\mathsf{G}_n$ finite. □

## 2 Khovanskii's theorem

**Definition 12** *Let $\mathbb{C}(x_1, \ldots, x_n)$ be the partial differential field of rational functions of $n$ arguments with derivations $\partial_{x_1} \ldots \partial_{x_n}$ and $\mathcal{K}$ be a differential extension of $\mathbb{C}(x_1, \ldots, x_n)$. It is said to be Liouvillian if one can find a tower of differential extensions*

$$\mathbb{C}(x_1, \ldots, x_n) = \mathcal{K}_0 \subset \mathcal{K}_1 \ldots \subset \mathcal{K}_p = \mathcal{K}$$

*such that $K_i \subset K_{i+1}$ is one of the following*
- *algebraic,*
- *additive $K_{i+1} = K_i(G)$ with $\partial_{x_i} G \in K_i$,*
- *mutiplicative $K_{i+1} = K_i(G)$ with $\frac{\partial_{x_i} G}{G} \in K_i$.*

*Liouvillian functions are elements of Liouvillian extensions.*

Let $X$ be a model for a field $L$ *i.e.* $\mathbb{C}(X) = L$. If this field is differential with derivations $\partial_{x_1} \ldots \partial_{x_n}$ then $X$ is endowed with a dimension $n$ foliation generated by these derivations.

**Lemma 13** *The foliation of a Liouvillian extension of $\mathbb{C}(x_1, \ldots x_n)$ of transcendence degree $q$ is defined by a solvable family of $q$ 1-form. Restriction of this family to a transverse $x_i = c_i$ gives a exactly solvable coparallelism.*

*Proof.* – The construction of these forms is direct from the definition of Liouvillian extension. Let $\ell_1, \ldots, \ell_q$ be a transcendence basis given by the definition then

$$\partial_{x_i} = \frac{\partial}{\partial x_i} + \sum r_{i,j}(x_1 \ldots x_n, \ell_1 \ldots \ell_{j-1}) \partial_{\ell_j}$$

where $\partial_{\ell_j}$ stands for $\frac{\partial}{\partial \ell_j}$ in the additive case and $\ell_j \frac{\partial}{\partial \ell_j}$ in the multiplicative case. The forms are $\theta_i = d\ell_j - \sum r_{i,j} dx_i$ in additive cases or $\theta_i = \frac{d\ell_j}{\ell_j} - \sum r_{i,j} dx_i$ in multiplicative cases. Their restriction to the transverse are exact forms or exponentially exact forms. They form a special type of exactly solvable coparallelism but as we will see in the 2°) of the proof of theorem 14, only exact solvability in relevant. □

**Theorem 14** *The monodromy $Mon(f)$ of a Liouvillian function $f$ is almost solvable.*

*Proof.* – Let $L$ be a Liouvillian differential field extension of $K = \mathbb{C}(x_1, \ldots x_n)$ containing $f$.

1°) **Geometry of $L$** – Let $X$ be an affine variety over $\mathbb{C}$ such that $\mathbb{C}(X) = L$. The vector fields $\frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_n}$ define a (singular) foliation $\mathcal{F}$ on $X$. By constrution, this foliation can be defined by a very special family of 1-form $\theta_\bullet : \theta_1, \ldots \theta_k$ where $k$ is the number of really differential (= non-algebraic) extension involved in the "Liouvillianity" of $L$. These forms satisfy the solvability property :

"$\theta_i$ is closed modulo $\theta_1, \ldots \theta_{i-1}$".

From the inclusion of $K$ in $L$, one gets a rational map $\pi : X \dashrightarrow \mathbb{C}^n$ transversal to $\mathcal{F}$. The forms $\theta_\bullet$ give rational solvable coparallelisms on fibers of $\pi$.



2°) **Holonomy of $L/K$** – The forms $\theta_\bullet$ define a transversal geometric structure on $(X, \mathcal{F})$. The isometries pseudogroup of this structure, is $PG(\theta_\bullet)$. If $\vec{a}$ is a vector field tangent to $\mathcal{F}$ then from $d(\bigwedge_{j=1\ldots i} \omega_j) = 0$ and $\iota_{\vec{a}}(\bigwedge_{j=1\ldots i} \omega_j) = 0$ on gets $L_{\vec{a}}(\bigwedge_{j=1\ldots i} \omega_j) = 0$ so its flows belong to $PG(\theta_\bullet)$.

Let $\mathrm{Hol}(\mathcal{F})$ be the holonomy pseudogroup of $\mathcal{F}$ computed on a generic fiber $T$ of $\pi$. Because holonomy maps are built from flows of tangent vector fields, one gets $\mathrm{Hol}(\mathcal{F}) \subset PG(\theta_\bullet)\big|_T = PG(\theta_\bullet\big|_T)$.

3°) **Monodromy of $f$** – By definition $f$ is a rational function in the coordinates of a leaf $\mathcal{L}$ of $\mathcal{F}$ and its monodromy group is a quotient of the monodromy group of $\mathcal{L}$. This group is the subgroup of permutation of $\mathcal{L} \cap T$ generated by regular $\pi$-preimages of loops on $\mathbb{C}^n$ pointing at $\pi(T)$, thus $\mathrm{Mon}(\mathcal{L}) \subset \mathrm{Hol}(F) \subset PG(\theta_\bullet\big|_T)$ as pseudogroups on $T$. Remark that $\mathrm{Mon}(\mathcal{L})$ is not *a priori* a transformation group on $T$ but on $T \cap \mathcal{L}$. One can use the holonomies along pathes involved in definition of an element of $\mathrm{Mon}(\mathcal{L})$ to realize this group as a subgroup of $\mathrm{Diff}(T, T \cap \mathcal{L}) \cap PG(\theta_\bullet\big|_T)$. The theorem 10 can be applied to prove almost solvability of $Mon(\mathcal{L})$. Because $Mon(f)$ is a quotient of $Mon(\mathcal{L})$, it is almost solvable too. □

## References


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