Solvability of algebraic pseudogroups and differential equations
Guy Casale

The purpose of these notes is to give a another Galoisian proof of

Let $f : C^\infty \to C$ be a Liouvillian function then its closed monodromy group is almost solvable in the sense that it admits a normal tower with finite or commutative quotients.

Actually Khovanskii’s proof has a Galosian nature quoting Khovanskii it is proved using “one dimensional topological Galois theory”. We want to give a more algebraic proof using the so called Godbillon-Vey sequences for codimension $1$ foliations. This is a “technology free” way to speak about the Galois pseudogroup of a foliation. After some preparation, the problem is to prove that subgroup of solvable pseudogroups are solvable. Using usual definition of solvability for pseudogroup this is wrong. The first part of this notes is devoted to definitions of solvability and infinitesimal solvability for pseudogroups, examples, counterexamples and comparison of definitions. The second one deals with Liouvillian first integrals and Liouvillian solutions of ordinary differential equations. As an application, we give the proof of Khovanskii’s theorem for the monodromy itself which was the original motivation.

1 Solvability of algebraic pseudogroups

Roughly speaking an algebraic pseudogroup is a pseudogroup of transformations of an smooth algebraic variety $X$ over $C$ defined by algebraic partial differential equations. But the reader can easily find three different definitions of pseudogroup of transformation in the literature. The difference is about how you can glue transformations when domains are not conneres.

Example 1 Let’s have a look at the pseudogroup of transformations of the affine line generated by translations without restriction on the gluing property. Because restriction and gluing of elements of a pseudogroup must belong to the pseudogroup, one gets to much elements in this pseudogroup. For any permutation $\sigma$ of $Z$ one gets a transformation $f : U \to U$ from a neighborhood of $Z$ to itself realising $\sigma$. This pseudogroup contains a group that can not be said to be commutative.

Here is a more natural example. This kind of situation can not be prevent by any “honest” definition and justifies the need of almost commutativity instead of commutativity.

Example 2 Let $\pi : Y \to X$ be a covering of a Riemann surface, $\omega$ a meromorphic 1-form on $X$ and $\pi$ its pull back on $Y$. Locally, the pseudogroup of transformation of $Y$ preserving $\pi$ look like the previous example. Even if one does not use the gluing property, it contains all the group of deck transformations of the covering. In general, it is not a commutative group.

Usually this problem is eliminated by changing definition of commutativity in an infinitesimal way. In this part we want to distinguish the first (artificial) example from the second (natural) one. To avoid problems, we will work in the C-analytic category and take the following definitions.

Definition 1 The determinations $\alpha$ at $\beta$ in $X$ of a germ of transformation $\varphi$ at $a \in X$ are the transformations obtained by analytic continuations of $\varphi$ along paths from $a$ to $\beta$. This set is denoted by $\mathbb{D}(\varphi, \beta)$. It can be empty.

Two germs of transformations $\varphi$ at $a$ and $\psi$ at $b$ are said to be compatible if $\psi \in \mathbb{D}(\varphi, b)$. A transformation $\varphi$ is said to be compatible (with itself) if every couple of germs $\varphi_a$, $\psi_b$ obtained from $\varphi$ are compatible.

Definition 2 A pseudogroup of transformation of $X$ is a set $\mathcal{PG}$ of local transformation $\varphi : U = d(\varphi) \to V = r(\varphi)$ such that
- the restriction of a transformation $\varphi$ of $\mathcal{PG}$ to an open subset of its domain $d(\varphi)$ is in $\mathcal{PG}$;
- if $\varphi, \psi \in \mathcal{PG}$ and $r(\varphi) = d(\psi)$ then $\psi \circ \varphi$ is in $\mathcal{PG}$;
- if $\varphi \in \mathcal{PG}$ then $\varphi$ is invertible and $\varphi^{-1}$ is in $\mathcal{PG}$;
- every $\varphi$ in $\mathcal{PG}$ is compatible;
- every $\varphi$ with a determination in $\mathcal{PG}$ is in $\mathcal{PG}$.

Definition 3 A subgroup $G$ of a transformation pseudogroup $\mathcal{PG}$ is a group of composable transformations or composable germs of transformations.

Remark 4 All subgroups of the first example are commutative but the second example has noncommutative subgroups.

An algebraic pseudogroup is a pseudogroup defined by partial differential equations like the second example. For a rigorous definition the reader may have a look to [B. Malgrange, monog. 38 vol 2 de l’enseignement mathematique 465-501 (2001)].

Definition 5 An algebraic pseudogroup $\mathcal{PG}$ on $X$ is a set of local differentiormorphisms $\varphi$ defined by some algebraic partial differential equations $E_i \in \mathbb{C}(x, \varphi)[\varphi^a] = \mathbb{C}(x_1, \ldots, x_n, \varphi_1, \ldots, \varphi_m)[\varphi^a] \mid 1 \leq i \leq n, \alpha \in N^m]$: $\varphi : U \to V \mid E_i(x, \varphi(x), \varphi^a(x)) = 0 \mid i = 1, \ldots, m$ such that
1. $E_i(\varphi \circ \psi) = \sum E_i(\varphi) \cdot \frac{dE_i(\psi)}{d\varphi}$ where $c$’s and $d$’s are in $C(x, \varphi, \psi)[\varphi^a, \psi^a, \frac{dE_i}{d\varphi}, \frac{dE_i}{d\psi}]$.
2. $E_i(id_x) = 0$ for all identities maps $id_x : U \to U$.
3. $E_i(\varphi^{-1}) = \sum E_i(\varphi)$, where $e$’s are in $C(x, \varphi)[\varphi^a, \frac{dE_i}{d\varphi}]$

Such an pseudogroup has a Lie algebra $\mathcal{L}$.

Definition 6 Let $\mathcal{L}$ be an algebraic pseudogroup on $X$ with equations $E_i, i = 1 \ldots m$. Its Lie algebra is the sheaf $\mathcal{LP}$ of local analytic vector fields $\xi(\mathcal{L}) = \sum a_i(x) \frac{d}{dx}$ on $U$ such that $\mathcal{LE}(\xi) := \sum \frac{dE_i}{d\xi} (id_x) a_i = 0$ for all $i = 1 \ldots m$.

Proposition 7 The sheaf $\mathcal{LP}$ is a sheaf of Lie algebra for the usual Lie bracket of vector fields.

Definition 8 An algebraic pseudogroup $\mathcal{PG}$ is said to be inf-solvable (infinitessimally solvable) if its Lie algebra $\mathcal{L}$ is a sheaf of solvable Lie algebra.

Definition 9 A solvable coparallelism of $X$ is given by $n$ 1-forms $\omega_1, \ldots, \omega_n$ such that $d\omega_1 = 0$ mod $\omega_1, \ldots, \omega_{i-1}$. It is called exactly solvable if $\omega_i$ is relatively exact (there is a rational function $h_i$ such that $\omega_i = dh_i$ mod $\omega_1, \ldots, \omega_{i-1}$) or exponentially relatively exact (there is a rational function $h_i$ such that $\omega_i = dh_i/h_i$ mod $\omega_1, \ldots, \omega_{i-1}$).

Theorem 10 Let $\omega_1$ be a solvable coparallelism on $X$.

- The algebraic pseudogroup $\mathcal{PG}(\omega_1)$ defined by $\omega_1^a := \sum \frac{d}{d\omega_1} (id_x) a_i$ for $i = 1 \ldots n$ is infinitessimally solvable.

- If $\omega_1$ is exactly solvable then subgroups of $\mathcal{PG}(\omega_1)$ are almost solvable.

Remark 11 All inf-solvable pseudogroups are not given directly by this kind of equation but after suitable prolongation. See [B. Curtan (1965)].
Proof. — The proof of the first part is a local analytic version of the proof of the second part. Let \( G \subset PG(\omega_u) \) be a subgroup of the pseudogroup defined by a exactly solvable coparallelism on \( X \).

Let \( k_1 \) be a primitive of \( w_1 \). For each \( \varphi \in PG(\omega_u) \), one gets a constant \( c_\varphi \) by the equality \( h_1 \circ \varphi = \varphi + c_\varphi \). The map \( \varphi \mapsto c_\varphi \) is a morphism of pseudogroup onto the "pseudogroup" \( \operatorname{Inv}(dx) \) on \( C \). The restriction of this morphism on \( G \) gives a morphism of group \( m_1 \) from \( G \) to \((C,+)\) whose kernel is denoted by \( G_1 \).

Now consider \( X \) as a variety over the affine line with coordinate ring \( C[\{h_1\}] \). The relative differential is denoted by \( d_1 \). Because \( w_2 \) is exact modulo \( d_1 \), one gets a rational function \( k_1 \) on \( X \) such that \( d_1/k_1 = w_2 \). Take a \( \varphi \in PG(\omega_u)\operatorname{Inv}(h_1) \). Because differentials of \( h_2 \) and \( h_2 \varphi \) are equal modulo \( d_1 \), one gets \( h_2 \varphi = h_2 + k_1 \). This gives \( m_2 \) of pseudogroup from \( PG(\omega_u)\operatorname{Inv}(h_1) \) to the "additive" pseudogroup of analytic functions on the affine line. This morphism maps \( G_1 \) on a commutative group. By induction one gets a sequence

\[ G_n \subset \cdots \subset G_1 \subset G \]

with \( G_n/G_{n-1} \) commutative and \( G_n \) finite. \( \square \)

2) Khovanskiï’s theorem

Definition 12 Let \( C(x_1, \ldots, x_n) \) be the partial differential field of rational functions of \( n \) arguments with derivations \( \partial_1, \ldots, \partial_n \) and \( K \) be a differential extension of \( C(x_1, \ldots, x_n) \). It is said to be Liouvillean if one can find a tower of differential extensions

\[ C(x_1, \ldots, x_n) = K_0 \subset K_1 \subset \cdots \subset K_n = K \]

such that \( K_i \subset K_{i+1} \) is one of the following

- algebraic,
- additive \( K_i \subset K_i(G) \) with \( \partial_i G \in K_i \),
- multiplicative \( K_i \subset K_i(G) \) with \( \partial_i G \in K_i \).

Liouvillean functions are elements of Liouvillean extensions.

Let \( X \) be a model for a field \( L \), i.e. \( C(X) = L \). If this field is differential with derivations \( \partial_1, \ldots, \partial_n \) then \( X \) is endowed with a dimension \( n \) foliation generated by these derivations.

Lemma 13 The foliation of a Liouvillean extension of \( C(x_1, \ldots, x_n) \) of transcendence degree \( q \) is defined by a solvable family of \( q \) 1-forms. Restriction of this family to a transverse \( x_i \) is \( \epsilon_i \) gives a exactly solvable coparallelism.

Proof. — The construction of these forms is direct from the definition of Liouvillean extension. Let \( \ell_1, \ldots, \ell_q \) be a transcendence basis given by the definition then

\[ \partial_{\ell_j} = \frac{\partial}{\partial \epsilon_j} + \sum r_j(x_1, \ldots, x_n, \ell_1, \ldots, \ell_{j-1}) \partial_{\ell_j} \]

where \( \partial_{\ell_j} \) stands for \( \frac{\partial}{\partial \epsilon_j} \) in the additive case and \( \ell_j \frac{\partial}{\partial x_j} \) in the multiplicative case. The forms are \( \theta_i = d\ell_j - \sum r_j dx_j \) in additive cases or \( \theta_i = d\ell_j - \sum r_j dx_j \) in multiplicative cases. Their restriction to the transverse are exact forms or exponentially exact forms. They form a special type of exactly solvable coparallelism but as we will see in the 2') of the proof of theorem 14, only exact solvability in relevant.

\[ \square \]

Theorem 14 The monodromy \( \operatorname{Mon}(f) \) of a Liouvillean function \( f \) is almost solvable.

Proof. — Let \( L \) be a Liouvillean differential field extension of \( K = C(x_1, \ldots, x_n) \) containing \( f \).

1) Geometry of \( L \) — Let \( X \) be an affine variety over \( C \) such that \( C(X) = L \). The vector fields \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \) define a (singular) foliation \( F \) on \( X \). By construction, this foliation can be defined by a very special family of 1-form \( \theta_1, \ldots, \theta_n \) where \( k \) is the number of really differential (= non-algebraic) extension involved in the "Liouvilleanity" of \( L \). These forms satisfy the solvability property:

\[ \theta_i = d\ell_j - \sum r_j dx_j \]

From the inclusion of \( K \) in \( L \), one gets a rational map \( \pi : X \rightarrow C^n \) transversal to \( F \). The forms \( \theta_i \) give rational solvable coparallelisms on fibers of \( \pi \).