FIRST INTEGRALS OF DICRITICAL SINGULARITIES

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We are interested in the following problem :
Let $\mathcal{F}$ be a germ of holomorphic foliation on $(\mathbb{C}^2, 0)$*. Does there exist a biholomorphism* φ *such that* $\varphi^*\mathcal{F}$ *is a germ of algebraic foliation on a surface.*
In [6], Y. Genzmer and L. Teyssier prove the existence of a 'non algebraizable' germ of saddle-node foliation. After this the problem splits in two parts

Problem. *Give an example of 'non algebraizable' germ of singularity.*

Problem. *Identify 'algebraizable' singularities.*

We give an answer to the second problem in a very particular case following first pages of [4] (see also [8]).

This paper is concerned with holomorphic foliations of $(\mathbb{C}^2, 0)$ with a dicritical singularity at 0. Such germs of foliations have infinitely many analytic invariant curves going through 0. Basic examples of dicritical foliations are foliations given by level sets of a meromorphic function on $(\mathbb{C}^2, 0)$. Among all these singularities, we are interested in the simplest ones *i.e.* those smooth after **one** blow-up and such that **a unique** leaf is tangent to the exceptional divisor with tangency order **one**. These singularities will be called **simple** singularities.

Example (Basic exemple : the cusp)**.** *The dicrital cusp is given by the rational function* $\frac{y^2-x^3}{x^2}$ *:*

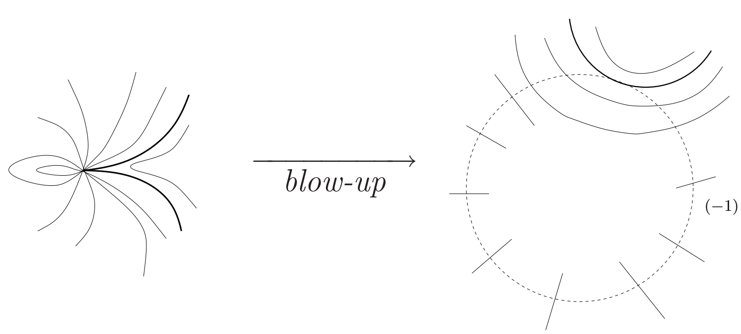

Fig. 1.

After one blow-up one gets the function $t^2 - x$ defining the foliation on the chart $t = y/x$.

Example (Susuki's example [14, 3])**.** *This is the germ of singularity given by*

$$(y^3 + y^2 - xy)dx - (2xy^2 + xy - x^2)dy$$

It is topologically equivalent to the previous example and leaves in neighborood of 0 *go through* 0 *and are analytic at* 0*. But its 'less transcendental' first integral is*

$$\frac{x}{y}e^{\frac{y(y+1)}{x}}.$$

It does not admit meromorphic first integral in any neighborood of 0*.*

Theorem 1. *If $\mathcal{F}$ is a simple dicritic foliation of* $(\mathbb{C}^2, 0)$ *with a meromorphic first integral then there exist an algebraic surface V, a rational function H on V and a point $p \in V$ such that $\mathcal{F}$ is biholomorphic to the foliation given level curves of H in a neighborood of p.*

Corollary 2. *If F is a germ of meromorphic function on* $(\mathbb{C}^2, 0)$ *with a simple type indeterminacy point in* 0 *then it is the germ of a rational function on a surface.*

Proof. – After a change of coordinates given by Theorem 1, the level curves of F coincide with those of a rational function H on an algebraic surface such that $F = f(H)$ for some function f defined on some ramified covering of $\mathbb{P}^1$. On the exceptional divisor obtained by blowing up 0, F and H are rational functions so f is algebraic and h is a germ of rational function on a surface. □

Analytic invariant of simple singularities and basic constructions are given in §1. In §2 a proof of Theorem 1 is given. Last section contains comments and partial proofs.

1. The analytic invariant of simple singularities

Following M. Klughertz [9], the description of a simple singularity up to analytic change of coordinates is done by a germ of involution at a point of $\mathbb{P}^1$ up to the action of $PGL_2(\mathbb{C})$. Let $\mathcal{F}$ be a germ of simple dicritical foliation of $(\mathbb{C}^2, 0)$. The blow-up of $(\mathbb{C}^2, 0)$ is denoted by M, the exceptional divisor by E and the strict transform of $\mathcal{F}$ by $\mathcal{F}_{-1}$. Because $\mathcal{F}_{-1}$ as a unique leaf tangent to E at order one at some point p, leaves passing through a point in the neigborhood of p on E must cut E in another point. This gives us a germ of involution ι on a projective line. This is the holonomy of $\mathcal{F}$ on E.

The germ of involution of two analytically conjugated foliations are the same up to conjugation by an element of $PGL_2(\mathbb{C})$. Conversally if two foliations have the same involution then they are analytically conjugated.

Let $\iota : (\mathbb{P}^1, p) \to (\mathbb{P}^1, p)$ be a germ of involution on $\mathbb{P}^1$. Such an involution gives rise to a dicritical singularity $\mathcal{F}(\iota)$ on $(\mathbb{C}^2, 0)$ gluing trick.

Let (M, E) be a pair of analytic surface and smooth rational curve with self-intersection $E.E = k$. A neighborood of E in M will still be denoted by M and called a k-neighborood of $\mathbb{P}^1$.

Lemma 3. *Assume $k \leq 0$. Any germ of involution on $\mathbb{P}^1$ can be realized as the holonomy of a smooth holomorphic foliation of a k-neighborood of $\mathbb{P}^1$. Two such realizations of the same involution are biholomorphic. For a involution ι, any such realization is called a k-realization of ι and denoted by $\mathcal{F}_k(\iota)$.*

Proof. – The proof is an illustration of the gluing trick of [11]. Let ι be such an involution on a disc $D_t(r)$ with coordinate t and radius r and $\varphi(t) = t - \iota(t)$. The involution ι is the holonomy of the foliation of $D_t(r) \times D_x(r')$ given by level curves of $\varphi(t)^2 - x$. Let r' be small enough then $U = \{t s.t. |\varphi(t)^2| \leq r'\} \subset\subset D_t(r)$. Leaves of the foliation by level curves of $\varphi(t)^2 - x$ on $(D_t - U) \times D_x$ intersect $D_t - U$ in a single point. For this reason the function $\sqrt{\varphi(t)^2 - x}$ is well defined on $(D_t - U) \times D_x$.

Let's M be the manifold obtained by gluing $D_t \times D_x$ with $(\mathbb{P}^1 - U)_u \times D_y$ by

$$u = \varphi^{-1}\left(\sqrt{\varphi(t)^2 - x}\right) \text{ and } y = xt^{-k}.$$

The foliation given by $d(\varphi(t)^2 - x) = 0$ on the first chart and $du = 0$ on the second is well defined, transverse to E the divisor given by $x = 0$ and $y = 0$ but in a single point $t = 0$. The holonomy is ι by construction. When $k \leq 0$, Grauert's theorems [5, 7] give unicity of the manifold built. □

Remark 4. *The contraction of the exceptional divisor in the (-1)-realization of ι gives a dicritical foliation of $(\mathbb{C}^2, 0)$ with invariant ι.*

Remark 5. *By blowing up a point of the projective line which is not the fixed point of the involution ι, $\mathcal{F}_k(\iota)$ is transformed in $\mathcal{F}_{k-1}(\iota)$. See the figure below.*

Remark 6. *If one glues a node $d(xy) = 0$ to $\mathcal{F}_{k-1}(\iota)$ in such a way that a separatrix of the node is glued with a leaf of $\mathcal{F}_{k-1}(\iota)$ in a rational*

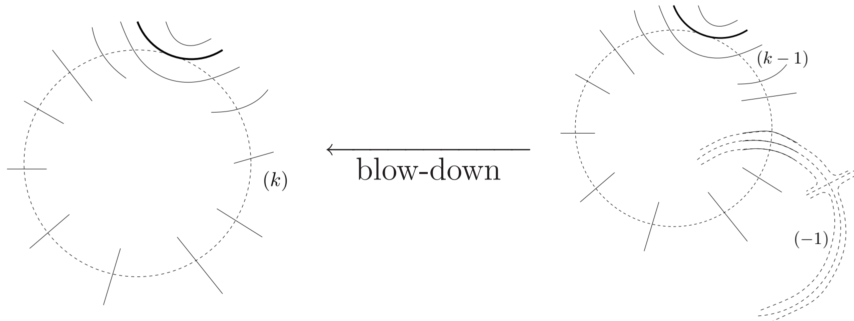

FIG. 2. From $\mathcal{F}_k$ to $\mathcal{F}_{k-1}$

curve of self-intersection (-1) *then the contraction of this leaf gives* $\mathcal{F}_k(\iota)$ *. See the figure 2 below.*

Remark 7. *The involution* ι *has a rational first integral i.e.* R *such that* $R \circ \iota = R$ *if and only if* $\mathcal{F}_k(\iota)$ *has a meromorphic first integral.*

The involution ι *leaves invariant a rational* 1*-form i.e.* η *such that* $\iota^*\eta = \eta$ *if and only if* $\mathcal{F}_k(\iota)$ *is defined by a closed meromorphic* 1*-form.*

2. The proof of Theorem 1

The strategy is to normalized a 0-realization of the involution of a dicritical singularity with a meromorphic first integral, and to prove that the meromorphic first integral of the dicritical foliation is transformed in a rational first integral of the 0-realization. By remarks 4–7 this is enought to prove the corollary. Let $\mathcal{F}_{-1}(\iota)$ be a blow-up of a simple dicritical singularity with a meromorphic first integral and $\mathcal{F}_0$ be a 0-foliation of the involution. It is defined on $D \times \mathbb{P}^1$ where D is a small disc around 0 in $\mathbb{C}$. Let y be a coordinate on $\mathbb{P}^1$ and x be a coordinate on D. Such a foliation is given by a differential equation

$$\frac{dy}{dx} = \frac{P_3(y)}{P_1(y)}$$

where $P_i \in \mathbb{C}\{x\}[y]$ of degree i. By changing y we can rectify three trajectories passing through p_1, p_2 and p_3 on $\mathbb{P}^1$ at $x = 0$ on the straight lines $y = \infty$, $y = 1$ and $y = -1$ and the equation is

$$\frac{dy}{dx} = h(x)\frac{y^2 - 1}{y - \lambda(x)}$$

because the foliation is not singular, $h(0) \neq 0$ and trajectory passing throught $0 \in \mathbb{P}^1$ at $x = 0$ is the graph of a diffeomorphism and can be

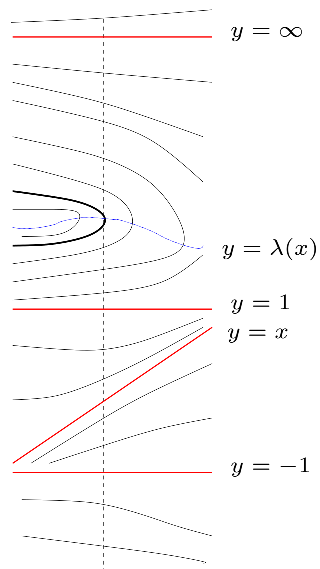

FIG. 3. The foliation $\mathcal{F}_0$

used as a new coordinate on the disc. The foliation $\mathcal{F}_0$ in these new coordinates is given by the equation

$$\frac{x^2-1}{x-\lambda(x)}\frac{dy}{dx} = \frac{y^2-1}{y-\lambda(x)}.$$

for a germ of analytic function λ whose graph is the locus of verticality of leaves. Its phase plane is figure 3.

Let $H(x,y) \in \mathbb{C}(\{x\})(y)$ be a meromorphic first integral of $\mathcal{F}_0$, d its degree in y and $\mathcal{R}(d)$ the variety of degree d rational functions on $\mathbb{P}^1$. This is a dimension $2d+1$ variety. By restriction to vertical fibers, h gives a germ of curve $\gamma : D \to \mathcal{R}(d)$ by $\gamma(x) = H|_{\text{fiber above } x}$.

Let us consider the critical values of rational functions as $2n-2$ algebraic functions on $\mathcal{R}(d)$:

$$c_1, \ldots, c_{2d-2}.$$

The restrictions of these functions on the curve give the critical values of H with respect to y as functions of x. One can assume that $c_{2d-2}(\gamma) = \lambda$. The $2n-3$ remaining functions give the values of H on leaves where H ramifies so they are constant on γ.

The values of H on leaves $y = \infty$, $y = 1$, $y = -1$ and $y = x$ are constant hence the evaluations function $e_\infty(x,R) = R(\infty)$, $e_1(x,R) = R(1)$, $e_{-1}(x,R) = R(-1)$ and $e_x(x,R) = R(x)$ are 4 rational functions on $D \times \mathcal{R}(d)$ witch are constant on the graph of γ.

If these $2n+1$ funtions are functionnally independent then γ must be algebraic. This is true because there is a finite number of rational maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ with fixed critical values $c_1, \ldots, c_{2d-2}$ and fixed values at -1, 1, ∞. These maps are given by the choices of the monodromies around critical values in the group of permutation of $\{1, 2, \ldots, d\}$. Thus the functions $c_1, \ldots, c_{2d-2}, e_1, e_{-1}, e_\infty$ are functionnally independent on $\mathcal{R}(d)$ and so are $c_1, \ldots, c_{2d-3}, e_1, e_{-1}, e_\infty$ on $D \times \mathcal{R}(d)$. These functions are x-independent so are independent of e_x. This proves the functional independence and the algebraicity of γ.

Let S be an algebraic curve such that γ is defined on S. The function H extends to a rational function on $S \times \mathbb{P}^1$. By blowing up a point on $0 \times \mathbb{P}^1$ and blowing-down the strict transform of $0 \times \mathbb{P}^1$ one gets the dicritical singularity with rational first integral and φ as invariant.

This proves that a simple dicritical foliation given a meromorphic fonction can be extended (in good coordinates) in a algebraic foliation with a rational first integral. Let's give a 'coordinate' description of the proof.

Equations of the deformation of a rational function in the way described by the drawing of $\mathcal{F}_0$ can be written down explicitely. Let parametrize $\mathcal{R}(d)$ by the zeros and the poles of fractions R :

$$R(y) = \ell \frac{\Pi(y-a_i)}{\Pi(y-b_i)}.$$

The vector field

$$\vec{\mathcal{V}} = \frac{x^2-1}{x-\underline{\lambda}} \frac{\partial}{\partial x} + \sum \frac{a_i^2-1}{a_i-\underline{\lambda}} \frac{\partial}{\partial a_i} + \sum \frac{b_i^2-1}{b_i-\underline{\lambda}} \frac{\partial}{\partial b_i}$$

where $\underline{\lambda}$ is an algebraic function of a's and b's such that $\sum \frac{1}{\underline{\lambda}-a_i} - \sum \frac{1}{\underline{\lambda}-b_i} = 0$ describes particular deformation of rational functions along a parameter x. If $a_i(x), b_i(x)$ et $\ell(x)$ are an analytic solution of this equation then the meromorphic function

$$H(x,y) = \ell(x) \frac{\Pi(y-a_i(x))}{\Pi(y-b_i(x))}$$

is a first integral of

$$\frac{x^2-1}{x-\lambda(x)} \frac{dy}{dx} = \frac{y^2-1}{y-\lambda(x)}$$

where $\lambda(x) = \underline{\lambda}(\ldots a_i(x))\ldots b_i(x)\ldots)$. So 0-realization of involutions with rational first integrals on $\mathbb{P}^1$ are exactly described by trajectories of $\vec{\mathcal{V}}$ on $\mathbb{C} \times \mathcal{R}(d)$. Be carefull that $\underline{\lambda}$ is only algebraic so to be completly right one must replace $\mathcal{R}(d)$ by a ramified covering where $\underline{\lambda}$ is well defined.

Remark that trajectories given by H and $g(H)$ for $g \in PSL_2(\mathbb{C})$ are different but describe the same 0-realization of the same involution.

The $2n+1$ independent algebraic first integrals of $\vec{\mathcal{V}}$ are :

– $c_k(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(p_k - a_i)}{\Pi_i(p_k - b_i)}$
  for all $p_k \neq \underline{\lambda}$ such that $\sum \frac{1}{p_k - a_i} - \sum \frac{1}{p_k - b_i} = 0$,

– $e_\infty(...a_i...b_i...\ell) = \ell$,

– $e_1(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(1 - a_i)}{\Pi_i(1 - b_i)}$,

– $e_{-1}(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(-1 - a_i)}{\Pi_i(-1 - b_i)}$,

– $e_x(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(x - a_i)}{\Pi_i(x - b_i)}$,

It is not easy to verify that these functions are functionally independent by a direct computation. This is done by a geometric interpretation previously. □

3. Comments

3.1. **Cerveau-Mattei finite determinacy theorem.** For special kind of functions one gets an 'algebrization' result proved by D. Cerveau and J.-F. Mattei in [3]. Let $f_1, \dots, f_p$ be germs of holomorphic functions in $(\mathbb{C}^n, 0)$, $\alpha_1, \dots, \alpha_p$ be complex numbers. We will look at the multivalued function

$$H = f_1^{\alpha_1} \dots f_p^{\alpha_p}.$$

Let ω be the form $f_1 \dots f_p \sum \alpha_i \frac{df_i}{f_i}$ and X be the hypersurface $f_1 \dots f_p = 0$.

Definition 8. *The small critical locus of H is a subset $C'(H)$ of $Zero(\omega) \cup X$ of point p such that if $p \in X$ germs at p of f's are reducible or (X, p) is not a normal crossing germ of hypersurface.*

The function H is said finitely determinate at order k if for every $g_1, \dots g_p$ with $j_k(f_i - g_i) = 0$ there is a diffeomorphism u of $(\mathbb{C}^n, 0)$ such that

$$H \circ u = g_1^{\alpha_p} \dots g_p^{\alpha_p}$$

Theorem 9 (Cerveau-Mattei [3] théorème 4.2. p 163)**.** *A function H as above is finitely determinate if and only if $C'(H) \subset \{0\}$.*

In our situation, $n = 2$, $\alpha_i \in \mathbb{Z}$, finite determinacy implies conjugation to a rational function with same 'order k jet' at 0. There is some rational function which are not finitely determinate *i.e.* such that a meromorphic with the same order k jets of numerator and denominator is not conjugated to the previous rational functions. Such an example can be build using previous description of simple dicritic foliations.

Let ι be the germ of involution at $0 \in \mathbb{P}^1$ defined by

$$h \circ \varphi = h$$

for some rational function h with a double zero at 0. For instance $h(t) = t^2((t-1)^2 + 1)$. The (-1)-realization of ι has a meromorphic first integral H extending h to the (-1)-neighborood of $\mathbb{P}^1$ following $\mathcal{F}_{-1}(\iota)$. By blowing down this function one gets a germ of meromorphic function on $(\mathbb{C}^2, 0)$ with simple indeterminacy point.

By Cerveau-Mattei theorem such a function is finitely determinated if and only if its small critical locus is contained in $\{0\}$. But the critical set of H is contained in $Zero(\omega)$, if it is not in the small critical locus it means that is is included in the zero and polar locus of H. This is not the case for the function build from h.

To prove that h is algebraizable by means of this finite determinacy theorem, one only needs to find a rational function f such that $f \circ H$ is finitely determinate. Such a f would be a (non ramified) covering of $\mathbb{P}^1 - \{0, \infty\}$ by $\mathbb{P}^1 - D$ where D is a finite set containing critical values of H. If $\#D \geq 3$ this is not possible. It is the case for our special H coming from h where $D = \{0, 1, \infty\}$ and for this reason there is no f such that $f \circ H$ is finitely determinate.

Nevertheless by our theorem, this function is a germ of algebraic function in suitable coordinates.

3.2. **The λ is not unique.** If one fixes the involution ι and three points on $\mathbb{P}^1$ then the function λ given by the normalisation of a 0-realization $\mathcal{F}_0$ is unique. So λ is an invariant of the marked involution $(\iota, p_{-1}, p_1, p_\infty)$ on $\mathbb{P}^1$. the invoution ι itself is an invariant.

If four differents leaves are normalized on $y = \infty$, $y = -1$, $y = 1$ and $y = x$ one gets a new function λ^*. Let us explain this change of λ. Let $u_1(x), u_{-1}(x), u_\infty(x)$ be three solutions of $\mathcal{F}_0$ and u_0 be the solution such that the cross ration of $(u_\infty(0), u_{-1}(0), u_0(0), u_1(0))$ equals the cross-ration of $(\infty, -1, 0, 1)$. The change of variables

$$\begin{cases} x^* = \dfrac{u_1 - u_\infty}{u_0 - u_\infty} + \dfrac{u_0 - u_1}{u_{-1} - u_\infty} \\[2ex] y^* = \dfrac{u_1 - u_\infty}{y - u_\infty} + \dfrac{y - u_1}{u_{-1} - u_\infty} \end{cases}$$

rectifies the u's on the four lines $y^* = \infty$, $y^* = 1$, $y^* = -1$ and $y^* = x^*$ and gives another normalisation of the 0-realization of ι. The involution of this realisation is ι^* defined by

$$\iota^*\left(\frac{u_1(0)-u_\infty(0)}{y-u_\infty(0)}+\frac{y-u_1(0)}{u_{-1}(0)-u_\infty(0)}\right)=\frac{u_1(0)-u_\infty(0)}{\iota(y)-u_\infty(0)}+\frac{\iota(y)-u_1(0)}{u_{-1}(0)-u_\infty(0)}$$

If λ^* is the function given by the verticality locus of this new diffferential equation then one gets

$$\lambda^*\left(\frac{u_1-u_\infty}{u_0-u_\infty}+\frac{u_0-u_1}{u_{-1}-u_\infty}\right)=\frac{u_1-u_\infty}{\lambda-u_\infty}+\frac{\lambda-u_1}{u_{-1}-u_\infty}$$

Can we get a invariant for the involution from this ?

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