Local irreducibility of the first Painlevé equation

Guy Casale

Abstract

In this article the local irreducibility of the first Painlevé equation ($P_1 : y'' = 6y^2 + x$) is investigated. The notion of irreducibility used concern the reducibility of the general solution following Painlevé-Nishioka-Umemura [22, 21, 28]. By local, we mean irreducibility over any ordinary differential extension K of $\mathbb{C}(x)$ of finite type. Such a field may contain any finite set of solutions of the equation.

The main tool used is the Galois groupoid of P_1 over K along lines given in [19, 6]. In order to adapt the previous calculations to this more general framework, the degeneration of P_1 on an elliptic equation is replaced by the use of weight on the dependent variables following H. Umemura [28]. The result can be interpreted as follows. The knowledge of any finite set of solutions of P_1 does not give any differential-algebraic informations about the dependency of the general solution on the integrating constants.

Contents

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Address: IRMAR, Université de Rennes 1 Campus de Beaulieu 35042 Rennes cedex France

E-mail address: guy.casale@univ-rennes1.fr

1 The Galois groupoid of a foliation

Let L be a differential field with commuting derivations $\partial_1, \ldots, \partial_n$. We will assume L is a finite transcendence degree extension of $\mathbb{C}$ but we will use this assumption as less as we can. The field of constants of L is supposed to be algebraically closed and its characteritic is zero. In this article we assume this field is $\mathbb{C}$. Let $Der(L)$ be the L-vector space generated by the ∂'s and $Der^*(L)$ be its dual over L with dual base $d_1, \ldots, d_n$.

There is two (dual) ways to define a foliation on L. The first one is to give the equations of an involutive distribution. These equations are given by a L-subspace N of $Der^*(L)$ stable by the exterior differential $d : Der^*(L) \to \Lambda^2 Der^*(L)$ *i.e.* $dN \subset Der^*(L) \wedge N$. The second one is to give the solutions of these equations over L *i.e.* a subspace of $Der(L)$ stable under the Lie bracket. In this article, a foliation must be thought as a particular $\mathcal{D}$-Lie algebra. As general $\mathcal{D}$-Lie algebras can have no solution over L, we will emphase the former point of view.

1.1 $\mathcal{D}$-variety over L

Let $\mathbb{A}^m$ be the affine space of dimension m over L with coordinates $z_1, \ldots, z_m$. The space of order q jets of sections of $\mathbb{A}^m$ over L : $J_q(\mathbb{A}^m{}_{/L}) = \mathsf{spec}\, L[J_q]$ is the variety defined by the L-algebra

$$L[J_q] = L\left[\ z_i^\alpha \mid\ 1 \leq i \leq m,\ \alpha \in \mathbb{N}^n,\ 1 \leq |\alpha| \leq q\right].$$

These varieties form a projective system $\pi_q^p : J_p \to J_q$ for $p \geq q$. The space of jets of sections $J(\mathbb{A}^m{}_{/L}) = \varinjlim J_q(\mathbb{A}^m{}_{/L})$ is a scheme of 'countable' type over L. The derivations of L act on $L[J]$ by the following formulae

$$D_i : L[J_q] \to L[J_{q+1}], \quad 1 \leq i \leq n$$

$$D_i = \partial_i + \sum_{j,\alpha} z_j^{\alpha+\epsilon_i} \frac{\partial}{\partial z_j^\alpha}.$$

where ϵ_i is the multi-index $(0\ldots 0, \underset{i^{th}}{1}, 0 \ldots 0))$.

Definition 1.1 *An affine $\mathcal{D}$-variety over L is a sub-variety $\mathcal{Z}$ of $J(\mathbb{A}^m{}_{/L})$ defined by a differential ideal (i.e. a ideal stable under the action of the D_i's).*

Example 1 *Let Z be an affine variety over L in $\mathbb{A}^n$. Its ideal I generates a differential ideal $\mathcal{I}$ of $L[J(\mathbb{A}^m{}_{/L})]$. This $\mathcal{D}$-variety is the space of jet of sections of Z over L denoted by $J(Z_{/L})$.*

The varieties $\mathcal{Z}_q = (\pi_q^\infty)_* \mathcal{Z}$ are defined by the order q equations defining $\mathcal{Z}$.

Definition 1.2 *Let $\mathcal{Z}$ be a $\mathcal{D}$-variety over L and $L \subset E$ be a fields extension.*
- *A E-point of $\mathcal{Z}$ is a morphism $L[\mathcal{Z}] \to E$ over the inclusion.*
- *A differential point is given by a differential morphism in a differential field.*

Remark 1 *A E-point of $\mathcal{Z}$ is a section of $\mathcal{Z}$ over E but a $\mathcal{D}$-E-point is the jet of a section of $\mathcal{Z}_0$ over E satisfying the differential equations encoded in the $\mathcal{Z}_q$, $q > 0$.*

This construction is a special case of prolongation sequences defined in differential algebraic framework by J. Johnson [14]. Johnson's construction allows us to replace $\mathbb{A}^m_{/L}$ by any L-algebra like $L \underset{\mathbb{C}}{\otimes} L$.

1.2 $\mathcal{D}$-linear space over L

Let V be a finite dimensional L-linear space. In this article V will stand both for the 'abstract' linear space and for the 'concrete' variety defined by the L-algebra $SymV^*$ of symetric powers of its dual.

Definition 1.3 *A $\mathcal{D}$-linear space over L is a $\mathcal{D}$-variety $\mathcal{V}$ over L with a linear structure :*

– there is a L-linear $\mathcal{D}$-invariant subspace $Lin\mathcal{V} \subset L[\mathcal{V}]$ such that $L[\mathcal{V}_q] = Sym\, Lin\mathcal{V}_q$ its symetric powers ring,

– the actions of the ∂'s on $Lin\mathcal{V}$ satisfy Liebniz rules.

For a L-vector space V, the set $\underline{V} = \{V(E)|\ E$ is a extension of $L\}$ is the set of 'local sections' of V on L.

1.3 $\mathcal{D}$-Lie groupoid over L

Let's build the space of order q jets of 'point transformations of $\mathsf{spec}\, L$': $J_q^*(L)$ (J_q^* for short). Let $L^{(1)}$ et $L^{(2)}$ be two copies of L and $L[J_q^*]$ be the L-algebra

$$L^{(1)} \underset{\mathbb{C}}{\otimes} L^{(2)} \left[z_i^\alpha, \frac{1}{det(z_i^{\epsilon_j})} \ |1 \leq i \leq n,\ \alpha \in \mathbb{N}^n,\ 1 \leq |\alpha| \leq q \right].$$

The spaces $J_q^* = \mathsf{spec}\, L[J_q^*]$ are groupoids over L. This groupoid structure is given by the following maps. The projections s (for *source*) on $\mathsf{spec}\, L^{(1)}$ and t (for *target*) on $\mathsf{spec}\, L^{(2)}$ are self-explained. The composition $c : (J_q^*, s) \times_L (J_q^*, t) \to J_q^*$ is defined on the order 0 jets by the inclusion of the product $L \otimes_{\mathbb{C}} L$ in $L \otimes_{\mathbb{C}} L \otimes_{\mathbb{C}} L$ in the first and third place. On higher order jets it is defined by using the formulae for composition of n formal power series of n variables. The inversion is defined by the inversion formulae for formal power series. These maps satisfy some commutative diagrams [18] (obvious in the framework of jet spaces). Moreover, one gets derivations

$$D_i : L[J_q^*] \to L[J_{q+1}^*], \quad 1 \leq i \leq n$$

defined by

$$D_i = \partial_i^{(1)} + \sum_j z_j^{\epsilon_i} \partial_j^{(2)} + \sum_{j,\alpha} z_j^{\alpha+\epsilon_i} \frac{\partial}{\partial z_j^\alpha}$$

We set $J^* = \underset{\leftarrow}{\lim}\ J_q^*$ with ring $L[J^*] = \underset{\rightarrow}{\lim}\ L[J_q^*]$. This space is a scheme of 'countable' type with a structure of groupoid over L and a structure of $\mathcal{D}$-variety over $L^{(1)}$.

Definition 1.4 *A $\mathcal{D}$-Lie groupoid over L is a subgroupoid of J^* defined by a perfect differential ideal.*

Remark 2 *The hypothesis 'perfect' is not relevant. By a theorem of B.Malgrange, every non-reduced $\mathcal{D}$-Lie groupoid is in fact reduced. This is proved in the analytic framework in [19].*

Example 2 *The $\mathcal{D}$-Lie groupoid defined by the ideal (0), i.e. J^* itself, is called the groupoid of point transformations over L on $\mathbb{C}$. Sometimes, it will be denoted by $Aut(L/\mathbb{C})$.*

1.4 Prolongation and differential invariants

Definition 1.5 *The space of order q frame on $\mathsf{spec}\, L$ is $J_q^*(\widehat{\mathbb{C}^n, 0} \to \mathsf{spec}\, L)$ or R_q for short. It is defined by the L-algebra*

$$L[R_q] = L \left[r_i^\alpha, \frac{1}{det(r_i^{\epsilon_j})}\ ;\ 1 \leq i \leq n,\ \alpha \in \mathbb{N}^n,\ 1 \leq |\alpha| \leq k \right].$$

This space is a principal homogeneous space over L with structural group the linear algebraic group $\Gamma_q(\mathbb{C}^n) = J_q^*(\widehat{\mathbb{C}^n, 0} \to \widehat{\mathbb{C}^n, 0})$ acting by source composition.

Definition 1.6 *Let E be a differential extension of L and $v = \sum v_i \partial_i \in E \underset{L}{\otimes} Der(L)$. One defines the prolongation of v on R_q by*

$$R_q v = v + \sum_{\substack{i,\alpha \\ |\alpha| \le q}} D^\alpha v_i \frac{\partial}{\partial r_i^\alpha}.$$

This prolongation is $\Gamma_q(\mathbb{C}^n)$-invariant and compatible with the Lie bracket. This allows us to prolong any Lie algebra of vector fields.

Lemma 1.7 *Let $N \subset Der^*(L)$ be the equations of a foliation over L. The equations of the prolongation of the foliation is the $L[R_q]$ ideal $R_q N \subset T^* R_q = Der^*(L) \underset{L}{\oplus} T^*(R_q/L)$ generated by*

$$\omega^\alpha = \sum_{\beta_1 + \beta_2 = \alpha} \binom{\alpha}{\beta_1} D^{\beta_1} \omega_i \; dr_i^{\beta_2}$$

with $\omega = \sum \omega_i d_i \in N$, $dr_i^0 = d_i$

Definition 1.8 *Let $N \subset Der^*(L)$ be the equations of a foliation on L. Rational differential invariants of N (or of $\mathcal{F}_N$: the foliation described by N) are the invariants (i.e. rational first integrals) of $R_q N$ in $L(R_q)$.*

1.5 $\mathcal{D}$-Lie algebras over L

Some notations – Let $SDer^*(L) = L[a_1, \ldots, a_n]$ be the symetric powers ring of the vector space of differentials of L. The space $T_L = \mathsf{spec}\, SDer^*(L)$ is the tangent space of L. For an extension E of L, $T_L(E)$ is the space of E-point of T_L. The space of order q jets of sections of T_L over L, $J_q(T_{L/L})$ ($J_q T$ for short), is defined by the following ring:

$$L[J_q T] = L\left[a_i^\alpha \; ; 1 \le i \le n, \; \alpha \in \mathbb{N}^n, \; 0 \le |\alpha| \le q\right].$$

The space JT and its ring is defined by taking limits. Its ring $L[JT]$ is a $\mathcal{D}$-algebra, the derivation $Der(L)$ of L act on it by

$$D_i = \partial_i + \sum_{j,\alpha} a_j^{\alpha+\epsilon_i} \frac{\partial}{\partial a_j^\alpha}.$$

It is a $\mathcal{D}$-vector space: the linear stucture is given by the L-vector space of linear partial differential equations $LinJT \subset L[JT]$, *i.e.* the differential L-vector space generated by $Der^*(L)$.

The Lie bracket on the vector fields over L with coefficients in E defines a Lie bracket on the space $JT_L(\mathcal{D}\text{-}E)$ of differential E-points.

Temporary definition 1.9 *A $\mathcal{D}$-Lie algebra over L is a sub-$\mathcal{D}$-vector space $\mathcal{L}$ of JT such that the differential points of $\mathcal{L}$ are stable under Lie bracket.*

A less 'differential' definition will be given following B.Malgrange [19]. This definition will use a prolongation of the Lie bracket on $\underline{JT}$ called the Spencer bracket and the stability condition will be on the 'ordinary' points of the jet space.

1.5.1 Brackets on $\underline{JT}$

There is two brackets defined on $\underline{J_q T}$. The first bracket is defined on $\underline{J_q T}$ and takes values in $\underline{J_q T}$. It is called Spencer bracket. It allows us to named $J_q T$ a Lie algebroid $\overline{[18]}$. The second one is $\overline{\text{defined}}$ on fibers of $J_q T$ and takes values in $J_{q-1} T$. It is called the fiberwise bracket. By duality each bracket defines a differential on the system of dual vector spaces.

Spencer bracket – The construction of this bracket (denoted by $[\,.\,,.\,]$) follows the diagonal method [16, 19]. Let $R_q^{(1)}$ et $R_q^{(2)}$ be two copies of R_q

$$\lambda : R_q^{(1)} \times R_q^{(2)} \to J_q^*$$

defined for couples (r, s) of q frames by $r \circ s^{-1}$ and the morphism of ring induced. This map is the quotient by the diagonal action of $\Gamma_q(\mathbb{C}^n)$ by source composition. The tangent of λ :

$$T\lambda : T(R_q^{(1)} \times R_q^{(2)})_{/R_q^{(1)}} \to TJ^*_{q\,/L^{(1)}}$$

identifies vector fields on $R_q^{(1)} \times R_q^{(2)}$ in the kernel of the first projection and invariant under the action of $\Gamma_q(\mathbb{C}^n)$ and vector fields on J_q^* in the kernel of the *source*.

Because the constructions of the 'vertical' tangent and the jet space commute, one have

$$T(J^*_{q\,/L^{(1)}})|_{id} \sim J_q T.$$

From an other side, the identification of TR_q with $T(R_q^{(1)} \times R_q^{(2)})_{/R_q^{(1)}}|_{diag}$ is equivariant under the action of $\Gamma_q(\mathbb{C}^n)$. From these identifications, one gets $\underline{\lambda} : TR_q \to J_q T$; it is the quotient by $\Gamma_q(\mathbb{C}^n)$.

Definition 1.10 *The Spencer bracket on sections of $J_q T$ is the bracket induced by the Lie bracket on R_q.*

By duality, this bracket gives a differential on $Lin\, J_q T$.

Fiberwise bracket – This bracket (denoted by $\{\,.\,,.\,\}$) is defined by the formulae giving $j_{q-1}[X, Y]$ in terms of $j_q X$ et $j_q Y$ for two vector fields on $\mathbb{C}^n$.

There are several formulae which characterized the Spencer bracket. The relation between the two brackets is the following

$$[f j_q u, g j_q v] = fg\{j_q u, j_q v\} + f\ L_u(g)\ j_q v - g\ L_v(f)\ j_q u$$

with f et g in $E \supset L$, u and v are any E-point of $T_{/L}$, $j_q u$ stands for the corresponding E-point of $J_q T$ and L_u is the Lie derivative along u.

1.5.2 $\mathcal{D}$-Lie algebra over L

Definition 1.11 *A $\mathcal{D}$-Lie algebra over L is a sub-$\mathcal{D}$-vector space $\mathcal{L}$ of JT such that the points of $\mathcal{L}$ are stable under Spencer bracket.*

As in the differential case, the $\mathcal{D}$-Lie algebra of a $\mathcal{D}$-Lie groupoid is defined by the vertical tangent along the identity [18].

Theorem 1.12 ([19]) *Let $\mathcal{Z}$ be a $\mathcal{D}$-Lie groupoid over L. After identification $J_q T \sim TJ^*_{q\,/L^{(1)}}|_{id}$, the $\mathcal{D}$-vector space $T\mathcal{Z}_{/L^{(1)}}|_{id}$ is a $\mathcal{D}$-Lie algebra over L.*

Foliations over L are special $\mathcal{D}$-Lie-algebra. Here is the definition used in this article.

Remark 3 *A foliation $\mathcal{F}$ is a $\mathcal{D}$-Lie-algebra differentially defined by $\mathcal{F}_0 \subset J_0 T$.*

1.6 The Galois groupoid

As for algebraic Lie groups and Lie algebras, the main problem for dealing with $\mathcal{D}$-Lie groupoids and $\mathcal{D}$-Lie algebras is the lack of *Lie third theorem*. In general a $\mathcal{D}$-Lie algebra over L is not the algebra of a $\mathcal{D}$-Lie groupoid over L.

Definition 1.13 *Let $\mathcal{L}$ be a $\mathcal{D}$-Lie algebra over L. The smallest $\mathcal{D}$-Lie groupoid over L whose $\mathcal{D}$-Lie algebra contains $\mathcal{L}$ is the $\mathcal{D}$-envelope of $\mathcal{L}$.*

When $\mathcal{L}$ is the $\mathcal{D}$-Lie algebra of a $\mathcal{D}$-Lie groupoid over L, it is called integrable over L.

As foliations are particular $\mathcal{D}$-Lie-algebras, one sets the following definition.

Definition 1.14 *Let $\mathcal{F}$ be a foliation over L. Its $\mathcal{D}$-envelope is called the Galois groupoid of $\mathcal{F}$ over L.*

This definition generalizes the definition of the differential Galois group of Picard-Vessiot theory. If $\mathcal{L}$ is the $\mathcal{D}$-Lie-algebra of the Galois groupoid of $\mathcal{F}$, $\mathcal{F}$ is an ideal of $\mathcal{L}$. The transversal Lie algebroid $\mathcal{L}/\mathcal{F}$ measures the lack on integrability.

The proof of the existence of such a minimal groupoid is done using Noetherianity properties. That the reason of the hypothese 'L is finite type field over $\mathbb{C}$'.

2 Maurer-Cartan form and Structural equation

2.1 Maurer-Cartan form

The groupoid J_q^* acts on itself by *target* composition. Let $L^{(1)}$, $L^{(2)}$ be the *source* and *target* of a first copy of $J_q^{*(1)}$ and $L^{(2)}$, $L^{(3)}$ be the *source* and *target* of a second copy, $J_q^{*(2)}$. The action of the second jet space on the first one is given by the following map :

$$J_q^{*(1)} \times_{L^{(2)}} J_q^{*(2)} \to J_q^{*(1)}.$$

The tangent gives $TJ^{*(1)}_{q\,/L^{(1)}} \times_{T_{L^{(2)}}} TJ_q^{*(2)} \to TJ^{*(1)}_{q\,/L^{(1)}}$. Thanks to the connection given by J_1, one gets a morphism $T_{L^{(2)}} \times_{L^{(2)}} J_1(J^{*(2)}_{q\,/L^{(2)}}) \to TJ_q^{*(2)}$. These morphisms, the inclusion of ${J_{q+1}^*}^{(2)} \to J_1(J^{*(2)}_{q\,/L^{(2)}})$ and the trivial identification $TJ^{*(1)}_{q\,/L^{(1)}} \sim TJ^{*(1)}_{q\,/L^{(1)}} \times_{T_{L^{(2)}}} T_{L^{(2)}}$ give

$$TJ^{*(1)}_{q\,/L^{(1)}} \times_{L^{(2)}} {J_{q+1}^*}^{(2)} \to TJ^{*(1)}_{q\,/L^{(1)}}.$$

By restriction of this morphism on the vertical tangent of $J_q^{*(1)}$ along the identity (on which $L^{(1)} = L^{(2)}$), one gets an isomorphism $J_qT \times_L J_{q+1}^* \to TJ^*_{q\,/L} \times_{J_q^*} J_{q+1}^*$ which induces a form Θ on the pull back of the vertical tangent of $TJ^*_{q\,/L}$ on J_{q+1}^* with values in J_qT. By associativity this form is invariant under the action of J_{q+1}^* by target composition. By construction it is compatible with (fiberwise) Lie bracket.

Definition 2.1 *The form $\Theta : TJ^*_{q\,/L} \times_{J_q^*} J_{q+1}^* \to J_qT$ is the order q (fiberwise) Maurer-Cartan form of J^*.*

The (fiberwise) Maurer-Cartan form is the limit $\Theta : \widetilde{T}J^*{}_{/L} \to JT$ where $\widetilde{T}$ stand for the shifted tangent.

Definition 2.2 *Let $\mathcal{Z}$ be a $\mathcal{D}$-Lie groupoid with $\mathcal{D}$-Lie algebra $\mathcal{L}(\mathcal{Z})$. The restriction of Θ on $\mathcal{Z}$ takes values in $\mathcal{L}(\mathcal{Z})$. It is the (fiberwise) Maurer-Cartan form of $\mathcal{Z}$.*

In the special case of a Galois groupoid $\mathcal{Z}$ of a foliation $\mathcal{F}$, one defines the transversal (fiberwise) Maurer-Cartan form in the following way. The foliation is a ideal of $\mathcal{L}(\mathcal{Z})$ one can get the quotient and

$$\Theta_{\mathcal{Z}} : \widetilde{T}\mathcal{Z}_{/L} \to \mathcal{L}(\mathcal{Z})/\mathcal{F}$$

is the transversal Maurer-Cartan form.

`To give the definition of 'non-fiberwise' Maurer-Cartan form ??`

2.2 Cartan's structural equation

Let $\widetilde{\Theta} : \widetilde{T}J^*_{q+1/L} \to J_{q-1}T$ be the order q Maurer-Cartan form followed by the projection. The fiberwise structural equation is (see [12] for a proof)

$$d\widetilde{\Theta} = \frac{1}{2}\{\widetilde{\Theta} \wedge \widetilde{\Theta}\}$$

where d is the relative differential over L. In the special case of a Galois groupoid $\mathcal{Z}$ of a foliation $\mathcal{F}$, the transversal structural equation

$$d\widetilde{\Theta}_{\mathcal{Z}} = \frac{1}{2}\{\widetilde{\Theta}_{\mathcal{Z}} \wedge \widetilde{\Theta}_{\mathcal{Z}}\}$$

is satisfied by the transversal Maurer-Cartan form.

`To give the 'non-fiberwise' structural equations ??`

2.3 Godbillon-Vey sequences

In order to define Godbillon-Vey sequences, one will assume that $L = \mathbb{C}(V)$ for some (pro)variety. Let $\mathcal{Z}$ be a groupoid containing the Galois groupoid of $\mathcal{F}$ and

$$s : \mathsf{spec}\left(L^{(1)} \otimes L^{(2)}\right) \to \mathcal{Z}$$

be a section over $\mathsf{spec}\, L^{(1)}$. Then by pull-back on gets a $L^{(1)}$-linear map

$$s^*\widetilde{\Theta}_{\mathcal{Z}} : L^{(1)} \otimes Der(L^{(2)}) \to \mathcal{L}(\mathcal{Z})/\mathcal{F}$$

By using a point of V, it can be specialized on $\mathbb{C}$ and one gets

$$GV : Der(L^{(2)}) \to \big(\mathcal{L}(\mathcal{Z})/\mathcal{F}\big)\big|_v.$$

A direct computation shows that $\big(\mathcal{L}(\mathcal{Z})/\mathcal{F}\big)\big|_v$ is isomorphic to a sub-Lie-algebra $\widehat{\mathfrak{g}} \subset \widehat{\chi}^d$ of formal vector fields in $d = codim\mathcal{F}$ variables.

By construction $dGV = GV \wedge GV$, first integrals of $\widehat{\mathfrak{g}}_0$ give first integrals of $\mathcal{Z}$ and the projection of GV onto $\widehat{\mathfrak{g}}_0 \subset \mathbb{C}^d$ gives $d-$(number of independent first integrals) 1-forms defining $\mathcal{F}$.

By choosing a basis of first integrals and a basis of $\widehat{\mathfrak{g}}_0$ over $\mathbb{C}$ one gets a sequence of 1-forms in $Der^*(L)$ called a Godbillon-Vey sequence for $\mathcal{F}$ given by $\mathcal{Z}$ and s.

2.4 For codimension 2 foliation.

By choosing for all order q a base of sections of $\mathcal{L}(\mathcal{Z})/\mathcal{F}$ over $L^{(1)}$, the transversal Maurer-Cartan form gives rise to a family of form in $\Omega^1_{\mathcal{Z}/L^{(1)}}$ satisfying coordinate version of the structural equation. Such families of forms are classified in [3] in the case of codimension 2 foliations by combinatorial arguments (see also [6]). The main consequence of this classification used in this paper is the following theorem

Theorem 2.3 *If $\mathcal{F}$ is defined by a closed 2-form $\gamma \in \Lambda^2 Der^*(L)$ then at least one of the following situation occurs*

- *the Galois groupoid of $\mathcal{F}$ is intransitive: there is a first integral in L;*
- *the Galois groupoid is imprimive in codimension one: there is a form $\omega \subset Der^*(L)$such that $\omega \wedge d\omega = 0$ and $\omega = 0$ on $\mathcal{F}$.*

- *the Galois groupoid is transversally affine: there are an algebraic extension $\widetilde{L}$ of L, two 1-forms θ_1 and θ_2 in $\widetilde{L} \otimes Der^*(L)$ vanishing on $\mathcal{F}$ and three 1-forms $\omega_{1,1}, \omega_{1,2}, \omega_{2,1}$ in $\widetilde{L} \otimes Der^*(L)$ such that*

$$d\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\omega_{1,1} & \omega_{1,2}\\ \omega_{2,1} & -\omega_{1,1}\end{pmatrix} \wedge \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \quad d\begin{pmatrix}\omega_{1,1} & \omega_{1,2}\\ \omega_{2,1} & -\omega_{1,1}\end{pmatrix} = \begin{pmatrix}\omega_{1,1} & \omega_{1,2}\\ \omega_{2,1} & -\omega_{1,1}\end{pmatrix} \wedge \begin{pmatrix}\omega_{1,1} & \omega_{1,2}\\ \omega_{2,1} & -\omega_{1,1}\end{pmatrix}$$

- *the Galois groupoid of $\mathcal{F}$ is the groupoid of invariance of γ.*

3 The Galois groupoid of P_1 over K

The theorem 2.3 will be used to compute the Galois groupoid of the first Painlevé equation: $y'' = 6y^2 + x$ over any ordinary differential extension K on $\mathbb{C}(x)$. The derivations is denoted by $\frac{\partial}{\partial x}$. The vector field over $K(y, y')$ defining the first Painlevé equation is

$$X_1 = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + (6y^2 + x)\frac{\partial}{\partial y'}.$$

Its foliation $\mathcal{F}_1$ is defined by the closed 2-form

$$\gamma = dy \wedge dy' - y'dx \wedge dy' + (6y^2 + x)dx \wedge dy.$$

Theorem 3.1 *For any differential extension K of $\mathbb{C}(x)$ of finite type, the Galois groupoid of $\mathcal{F}_1$ over $L = K(y, y')$ is the groupoid of invariance of γ.*

In order to study the properties of the vector field X_1, weight on variables are introduced following [28]. The elements of K have weight 0. The variables y and y' have respectively weights $\mathsf{w}y = 2$ and $\mathsf{w}y' = 3$. The degree induced by this weight will be denoted by δ_{w}. The vector field has a decomposition $X_1 = \frac{\partial}{\partial x} + X_0 + x\frac{\partial}{\partial y'}$ into homogeneous composants of weights 0, 1 and -3. The 'simplified' vector field

$$X_0 = y'\frac{\partial}{\partial y} + 6y^2\frac{\partial}{\partial y'}$$

comes from the Hamiltonian $u = y'^2 - 4y^3$ on $\mathbb{C}^2$. For this reason it is easier to study X_1 by mean of X_0.

3.1 Some lemmas on X_0

The missing proofs of the following lemmas can be found in [28].

Lemma 3.2 *If R is a homogeneous first integral of X_0 in $K[y, y']$ then $R = a\,u^p$ with $a \in K$.*

Lemma 3.3 *If R is homogeneous and satisfies $X_0R = a\ u^p$ for some $a \in K$ then $R = a = 0$.*

Lemma 3.4 *If R is homogeneous and $X_0R = ay$ with $a \in K$ then $R = a = 0$.*

Lemma 3.5 *The equation $X_0R = \frac{a}{y'^2} + b$, with a and b in $K(u)$ has a solution if and only if $a = \frac{3}{2}c\ u$ et $b = -\frac{1}{2}c$ with $c \in K$. In this case, the homogeneous solution is $R = c\frac{y}{y'}$.*

Proof. – Set $R = \frac{2}{3}\frac{a}{u}\frac{y}{y'} + R_0$ then $X_0R = \frac{a}{y'^2} - \frac{1}{3}\frac{a}{u} + X_0R_0$. Lemma 3.3 asserts that $X_0R_0 = b + \frac{1}{3}\frac{a}{u}$ has no non zero solution. □

Lemma 3.6 *The equation* $X_0 R = y\left(\frac{a}{y'^2}+b\right)$*, with* a *and* b *in* $K(u)$ *has a solution if and only if* $a = 3ub$*. In this case, the homogeneous solution is* $R = 2b\frac{y^2}{y'}$*.*

Proof. – Let R be such a rational function and write $R = 2b\frac{y^2}{y'} + (a-3bu)R_0$ then $X_0R_0 = \frac{y}{y'^2}$. In coordinates y, u, R_0 is equal to $A + \sqrt{4y^3+u}B$ with A ans B in $K(y)$ and the equation is

$$\frac{\partial A}{\partial y} = 0 \text{ and } (4y^3+u)\frac{\partial B}{\partial y} + 6y^2 B = \frac{y}{4y^3+u}.$$

Then A is in $K(u)$ and B can be written $B = \frac{P}{(4y^3+u)^m}$ for a $P \in K(u)[y]$ of degree n and some integer m. A direct computation shows thant m must be 1 and n can not be integer. This proves the lemma. □

3.2 The Galois groupoid is transitive

Proposition 3.7 *There is no first integral of* X_1 *in* $K[y, y']$

Proof. – Let $R = \sum_{h=m}^{M} R_h$ be the decomposition of R in homogeneous composantes with $R_M \neq 0$. The equation

$$X_1 R = \sum_{h=m-3}^{M} \frac{\partial R_{h+1}}{\partial x} + X_0 R_h + x\frac{\partial R_{h+3}}{\partial y} = 0$$

implies $\frac{\partial R_{h+1}}{\partial x} + X_0 R_h + x\frac{\partial R_{h+3}}{\partial y} = 0$, $\forall h \in \{m-3, \ldots, M\}$.

- $h = M$. The equation is $X_0 R_M = 0$. By lemma 3.2, $R_M = a\ u^k$ with $a \in K$ and $M = 6k$.
- $h = M-1$. The equation is $X_0 R_{M-1} = -a'\ u^k$. By lemma 3.3, $R_{M-1} = a' = 0$.
- $h = M-2$ and $M-3$. The equation is $X_0 R_h = 0$. By lemma 3.2, R_h=0.
- $h = M-4$. The equation is $X_0 R_{M-4} = -xa2k\ u^{k-1}y'$. By lemma 3.2, $R_{M-4} = -xa2k\ u^{k-1}y$.
- $h = M-5$. The equation is $X_0 R_{M-5} = a2k\ u^{k-1}y$. By lemma 3.4, $R_{M-5} = a = 0$.

This gives a contradiction and proves the proposition. □

Corollary 3.8 *There is no invariant divisor for* X_1

Proof. – Suppose that there exist P and L in $K[y, y']$ for K a differential extension of $\mathbb{C}(x)$ such that $X_1 P = LP$. Because $\delta_{\mathsf{w}}(X_1 P) \leq \delta_{\mathsf{w}} P + 1$, one has $\delta_{\mathsf{w}} L \leq 1$. But in $K[y, y']$ there is no weight 1 element and $L = \ell \in K$ has weight zero. Let K‘ be an extension of K by a solution of $e' + e\ell = 0$. One checks that eP is a first integral of X_1 in K‘$[y, y']$. By the previous proposition, P is zero. □

Corollary 3.9 *There is no first integral of* X_1 *in* $K(y, y')$

3.3 The Galois groupoid is imprimitive in codimension one

If there is an integrable 1-form on $\mathbb{C}^3$ vanishing on X_1 with coefficient in $K(y, y)$, $\omega = Adx + Bdy + Cdy'$, it can be supposed to be polynomial and the coefficients have no common divisor.

Lemma 3.10 *If this form* ω *exists then for a extension* K^* *of* K*, there is a 1-form* ω^* *over* $K^*[y, y']$ *satisfying* $i_{X_1}\omega^* = 0$ *and* $i_{X_1}d\omega^* = 0$*.*

Proof. – Let ω be such a 1-form. Because it is integrable, $d\omega = \alpha \wedge \omega$ for a 1-form α with coefficients in $K(y, y')$. Take the inner product with X_1 : $i_{X_1} d\omega = L\omega$ with $L = \alpha(X_1)$. Because these two 1-forms are polynomial and the coefficients of ω do not have common divisor, L must be in $K[y, y']$. Its degree $\delta_{\mathsf{w}} L$ is strictly less than two then $L = \ell \in K$. Let $K^* = K(e)$ for a solution of $e' + e\ell = 0$ and $\omega^* = e\omega$. Then $i_{X_1} d\omega^* = 0$. □

Proposition 3.11 *There is no integrable 1-form over $K(y, y')$ such that $i_{X_1}\omega = 0$ and $i_{X_1} d\omega = 0$.*

Proof. – As for the proposition 3.7, computations will be decomposed following the weight. Let ω be a 1-form given by lemma 3.10. The weight decomposition is $\omega = \sum_{h=m}^{M} \omega_h$. Let's have a look to the equations satisfied by the five terms of highest weight.

- $h = M$. The equations are $\omega_M(X_0) = 0$ and $i_{X_0} d\omega_M = 0$. From these equations and lemma 3.2,

$$\omega_M = a_M\ u^p dx + b_M\ u^{p-1} du,\ a_M \text{ and } b_M \text{ in } K.$$

- $h = M - 1$. From the first equation $\omega_{M-1}(X_0) = -a_M\ u^p$, one gets

$$\omega_{M-1} = a_{M-1} dx + b_{M-1} du - \frac{a_M\ u^p}{y'} dy.$$

The second equation is $i_{X_0} d\omega_{M-1} + i_{\frac{\partial}{\partial x}} d\omega_M = 0$. Computing the terms of this sum one gets

- $i_{X_0} d\omega_{M-1} = (X_0 a_{M-1} + a'_M\ u^p)\, dx + \left(X_0 b_{M-1} + a_M p\ u^{p-1} - \frac{a_M\ u^p}{2y'^2}\right) du$,
- $i_{\frac{\partial}{\partial x}} d\omega_M = (b'_M\ u^{p-1} - a_M p\ u^{p-1})\, du$.

This gives

- $X_0 a_{M-1} + a'_M u^p = 0$, by lemma 3.3, $a'_M = a_{M-1} = 0$;
- $X_0 b_{M-1} = \frac{a_m\ u^p}{2y'^2} - b'_M u^{p-1}$, by lemma 3.5, $6b'_M = a_M$ and $b_{M-1} = 2b'_M\ u^{p-1} \frac{y}{y'}$.

One has

$$\omega_{M-1} = a_M\ u^{p-1} \frac{y}{3y'} du - a_M\ u^p \frac{1}{y'} dy.$$

- $h = M - 2$ and $M - 3$. The equations are $\omega_h(X_0) = 0$ and $i_{X_0} d\omega_h = 0$, by lemma 3.2, $\omega_h = 0$.
- $h = M - 4$. The first equation is $\omega_{M-4}(X_0) + \omega_M(x \frac{\partial}{\partial y'}) = 0$. The fourth 1-form can be written

$$\omega_{M-4} = a_{M-4} dx + b_{M-4} du - 2b_M x\ u^{p-1} dy.$$

The second equation is $i_{X_0} d\omega_{M-4} + i_{x\frac{\partial}{\partial y'}} d\omega_M = 0$. One computes these two terms:

- $i_{X_0} d\omega_{M-4} = (X_0 a_{M-4} + 2y'\ u^{p-1}(b_M + x b'_M))\, dx + (X_0 b_{M-4} + 2xy'\ (p-1) u^{p-2} b_M)\, du$;
- $i_{x\frac{\partial}{\partial y'}} d\omega_M = (a_M 2xy'\ p u^{p-1} - b'_M 2xy'\ u^{p-1})\, dx$.

This gives

- $X_0 a_{M-4} = -2u^{p-1} y'(b_M + p a_M x)$;
- $X_0 b_{M-4} = -2u^{p-2} y'(p-1) x b_M$.

These equations are easily solved and

$$\omega_{M-4} = -2(b_M + pa_M x)yu^{p-1}dx - 2(p-1)b_M xy\ u^{p-2}du - 2b_M x\ u^{p-1}dy.$$

• $h = M-5$. The first equation is $\omega_{M-5}(X_0) + \omega_{M-4}(\frac{\partial}{\partial x}) + \omega_{M-1}(x\frac{\partial}{\partial y'}) = 0$.
From $\omega_{M-4}(\frac{\partial}{\partial x}) = -2(b_M + pa_M x)y\ u^{p-1}$ and $\omega_{M-1}(x\frac{\partial}{\partial y'}) = \frac{2}{3}a_M xy\ u^{p-1}$, one gets

$$\omega_{M-5} = a_{M-5}dx + b_{M-5}du + \left(2b_M + (2p - \frac{2}{3})a_M x\right)\frac{y}{y'}u^{p-1}dy.$$

The second equation is $i_{X_0}\omega_{M-5} + i_{\frac{\partial}{\partial x}}\omega_{M-4} + i_{x\frac{\partial}{\partial y'}}\omega_{M-1} = 0$ and gives $i_{\frac{\partial}{\partial x}}i_{X_0}\omega_{M-5} + i_{\frac{\partial}{\partial x}}i_{x\frac{\partial}{\partial y'}}\omega_{M-1} = 0$. This equation is equivalent to $X_0 a_{M-5} = (2p - \frac{1}{3})a_M y u^{p-1}$. By lemma 3.4, $a_M = a_{M-5} = 0$ and the second equation is $X_0 b_{M-5} = u^{p-2}b_M y\left(2(p-1) - \frac{u}{y'^2}\right)$. This implies $b_M = 0$ by lemma 3.6 and gives a contradiction. □

3.4 The Galois groupoid is not transversally affine

The aim of this section is to prove the following proposition

Proposition 3.12 *There does not exist* als_2*-sequence for* $\mathcal{F}_{X_1}$.

The proof will be decomposed in several lemmas.

Lemma 3.13 *If there exists a* asl_2*-sequence for the foliation of* P_1*, there exists a polynomial one with* $\theta_1 = dy - y'dx$ *and* $\theta_2 = dy' - (6y^2 + x)dx$.

Proof. – Let $\widetilde{\Theta}$ the vector of forms $(\widetilde{\theta}_1, \widetilde{\theta}_2)^T$ and $\widetilde{\Omega}$ be a asl_2-sequence for P_1 with coeffcients in an algebraic extension L of $K(y, y')$. One has $\Theta = F\widetilde{\Theta}$ for a matrix with coefficients in L. Because P_1 has no first integrals, $\det F$ must be a constant. Then Θ can be completed in a asl_2-sequence for P_1 by the matrix $\Omega = dFF^{-1} + F\widetilde{\Omega}F^{-1}$.

Suppose that the coefficients of Ω are not in $K(y, y')$. One can find two matrices Ω and $\widetilde{\Omega}$ satisfying the asl_2 equations beginning with Θ. This implies that the $\mathcal{D}$-Lie groupoid of invariance of these two transversally affine stucture admits a order one equation. From E. Cartan [] (see also []), one gets a codimension one foliation vanishing on X_1. Then section 3.3 proves that the coefficients of Ω are in $K(y, y')$.

Let f be the equation of an irreductible component of the polar locus of Ω. Let's write $\Omega = \frac{1}{f^n}\Omega_p + \Omega_0$ with Ω_0 et Ω_p polynomial and Ω_p not divisible by f. From the asl_2-equations, one gets $\Omega_p \wedge \Theta = 0$ and

$$\frac{-n}{f^{n+1}}\Omega_p \wedge df + \frac{1}{f^n}d\Omega_p + d\Omega_0 = \frac{1}{f^{2n}}\Omega_p \wedge \Omega_p + \frac{1}{f^n}\left(\Omega_0 \wedge \Omega_p + \Omega_p \wedge \Omega_0\right) + \Omega_0 \wedge \Omega_O.$$

The contraction by X gives

$$\frac{1}{f^n}i_X d\Omega_p + \frac{n}{f^{n+1}}Xf\Omega^p + i_x d\Omega_0 = \frac{1}{f^n}\left(\Omega_0(X)\Omega_p - \Omega_p\Omega_0(X)\right) + \Omega_0(X)\Omega_0 - \Omega_0\Omega_0(X)$$

thus Xf is divisible by f and f is a constante.

□

Because

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix} = \begin{pmatrix} 0 & dx \\ 12y dx & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

one has

$$\Omega = A\theta_1 + B\theta_2 + \begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix} dx,$$

where A and B are matrices of polynomials such that $A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Writing the equation $d\Omega = \Omega \wedge \Omega$ in the basis (dx, θ_1, θ_2), one gets the following system of p.d.e.'s on A and B :

$$(E) \quad \begin{cases} XA + 12yB - \begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix}, A \right] \\ XB + A = \left[\begin{pmatrix} 0 & 1 \\ 12y & 0 \end{pmatrix}, A \right] \\ \dfrac{\partial B}{\partial y} - \dfrac{\partial A}{\partial y'} = [A, B] \end{cases}$$

Lemma 3.14 $\delta_{\mathsf{w}} A = \delta_{\mathsf{w}} B + 1$.

Proof. – This comes from the previous system of equations. From the first one, one gets $max(\delta_{\mathsf{w}} A + 1, \delta_{\mathsf{w}} B + 2, 0) \geq \delta_{\mathsf{w}} S$ where S stand for the right hand side of the first equation from (E). If $\delta_{\mathsf{w}} A + 1 \neq \delta_{\mathsf{w}} B + 2$ then one gets equality. One gets also $\delta_{\mathsf{w}} S \leq \delta_{\mathsf{w}} A + 2$. This implies $\delta_{\mathsf{w}} A \geq \delta_{\mathsf{w}} B$. From the second equation of (E), one gets $\delta_{\mathsf{w}} A \leq \delta_{\mathsf{w}} B + 2$.

If $\delta_{\mathsf{w}} B = 0$ then $A = A_1 y + A_0$ with A_1 and A_0 in $M_{2\times 2}(K)$. The first equation gives

$$A_1' y + A_1 y' + A_0' + 12yB - \begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix}, A_1 \right] y^2 + \left(\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_0 \right] + \left[\begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix}, A_1 \right] \right) y + \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_0 \right].$$

This implies

$$A_1 = 0, B = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_0 \right] \text{ and } A_0' - \begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_0 \right].$$

The second equation gives

$$B' + A_0 = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B \right] + y \left[\begin{pmatrix} 0 & 0 \\ 12 & 0 \end{pmatrix}, B \right].$$

This implies

$$B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \text{ and } A_0 = \begin{pmatrix} b & 0 \\ -b' & -b \end{pmatrix}.$$

From the equality $A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, one gets $b = 0$ and $A = 0$. This is a contradiction.

In the case $\delta_{\mathsf{w}} A = \delta_{\mathsf{w}} B$, let A_m and B_m be the homogeneous part of weight $m = \delta_{\mathsf{w}} A$. Taking homogeneous part of the equations (E), one gets :

$$\begin{cases} B_m = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_m \right] \\ 0 = \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_m \right] \\ A_m \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B_m \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}.$$

From the second equation, one gets $B_m = \begin{pmatrix} 0 & 0 \\ b_m & 0 \end{pmatrix}$. The third one implies $A_m = \begin{pmatrix} -b_m & 0 \\ a_m & b_m \end{pmatrix}$. The first one gives the contradiction.

In the case $\delta_{\mathsf{w}} A = \delta_{\mathsf{w}} B + 2$, computation are done in the same way. The weight $\delta_{\mathsf{w}} A + 2$ equations gives $A_m = \begin{pmatrix} 0 & 0 \\ a_m & 0 \end{pmatrix}$. The weight $\delta_{\mathsf{w}} A + 1$ equations are $X_0 A_m = 12y \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{m-1} \right]$ and $A_{m-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. This implies $A_{m-1} = \begin{pmatrix} 0 & 0 \\ a_{m-1} & 0 \end{pmatrix}$ and $X_0 a_m = 0$. Finally the weight $\delta_{\mathsf{w}} A$ equations are :

$$\begin{cases} \frac{\partial A_m}{\partial x} + X_0 A_{m-1} + 12y B_{m-2} = 12y \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{m-2} \right] + \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_m \right] \\ A_m = 12y \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_{m-2} \right] \\ 0 = [A_m, B_{m-2}] \end{cases}$$

The third equation implies $\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_{m-2} \right] = 0$ then $A_m = 0$. This contradiction proves the lemma. ☐

Proof of propotion 3.12. – Let A_m the homogeneous part of weight $m = \delta_{\mathsf{w}} A$ of A. Homogeneous part of weight $m+2$ of equations (E) gives

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_m \right] = 0 \text{ thus } A_m = \begin{pmatrix} 0 & 0 \\ a_m & 0 \end{pmatrix}.$$

Homogeneous part of weight $m+1$ of equations (E) gives

$$\begin{cases} X_0 A_m + 12y B_{m-1} = 12y \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{m-1} \right] \\ 0 = 12y \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_{m-1} \right] \\ A_{m-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B_{m-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

For the second and third equation one gets

$$B_{m-1} = \begin{pmatrix} 0 & 0 \\ b_{m-1} & 0 \end{pmatrix} \text{ and } A_{m-1} = \begin{pmatrix} -b_{m-1} & 0 \\ a_{m-1} & b_{m-1} \end{pmatrix}.$$

Then the first equation is

$$\boxed{X_0 a_m = -36y b_{m-1}}$$

Homogeneous part of weight m of equations (E) gives

$$\begin{cases} \frac{\partial A_m}{\partial x} + X_0 A_{m-1} + 12y B_{m-2} = 12y \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{m-2} \right] + \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_m \right] \\ X_0 B_{m-1} + A_m = 12y \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_{m-2} \right] \\ [A_m, B_{m-2}] + [A_{m-1}, B_{m-1}] = 0 \\ A_{m-2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B_{m-2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}.$$

The third equation gives

$$\left[\begin{pmatrix} 0 & 0 \\ a_m & 0 \end{pmatrix}, B_{m-2}\right] = \begin{pmatrix} 0 & 0 \\ 2b_{m-1}^2 & 0 \end{pmatrix} \text{ thus } B_{m-2} = \begin{pmatrix} b_{m-1}^2/a_m & 0 \\ b_{m-2} & -b_{m-1}^2/a_m \end{pmatrix}.$$

The fourth gives

$$A_{m-2} = \begin{pmatrix} -b_{m-2} & b_{m-1}^2/a_m \\ a_{m-2} & b_{m-2} \end{pmatrix},$$

and the second gives Then the first equation is

$$\boxed{X_0 b_m + a_m = 24y\frac{b_{m-1}^2}{a_m}}.$$

Let's have a look to the equations satisfy by a_m and b_{m-1}, denoted by a and b in the sequel :

$$\boxed{\begin{aligned} &X_0 a = -36yb \\ &X_0 b = 24y\frac{b^2}{a} - a \end{aligned}}$$

These equations imply

$$X_0\left(\frac{b}{a}\right) = 60y\left(\frac{b}{a}\right)^2 - 1$$

Lemma 3.15 *With previous notation,* $y^2\frac{b}{a} \in K[y, y']$.

Proof. – Let's show that for any $n \in \mathbb{N}$, $y^{2n-1}\frac{b^{n+1}}{a^n}$ is a polynomial. This is true for $n = 1$ because $X_0 b = 24y\frac{b^2}{a} - a$ and a, b are polynomials and X_0 has polynomial coefficients. For the same reason, it si true for $n = 2$:

$$X_0\left(y^2\frac{b^2}{a}\right) = 84y^3\frac{b^3}{a^2} - 2y^2 b + 2y'y\frac{b^2}{a}.$$

Now assume that it is true for $y^{2n-3}\frac{b^n}{a^{n-1}}$ and $y^{2n-5}\frac{b^{n-1}}{a^{n-2}}$. Then because

$$X_0\left(y^{2n-2}\frac{b^n}{a^{n-1}}\right) = (2n-2)y^{2n-3}y'\frac{b^n}{a^{n-1}} + y^{2n-2}\left((60n-36)y\frac{b^{n+1}}{a^n} - n\frac{b^{n-1}}{a^{n-2}}\right)$$

it is true for $y^{2n-1}\frac{b^{n+1}}{a^n}$. Let's write $b = y^p \Pi \beta_i^{n_i}$ and $a = y^q \Pi \alpha_j^{m_j}$ the factorization in irreducible elements, one gets $(n+1)p + 2n - 1 \geq nq$ and $(n+1)n_i \geq nm_i$. Let n be large enough, this prove that $p + 2 \geq q$ and $n_i \geq m_i$. This proves the lemma. □

Now one can finish the proof on proposition 3.12. Becaus a is homogeneous of weight m, b of weight $m-1$ and y of weight 2, $y^2\frac{b}{a}$ is a homogeneous polynomial of weight 3. Such a polynomial is ky' for somme $k \in K$ thus $\frac{b}{a} = k\frac{y'}{y^2}$. One can compute

$$X_0\frac{b}{a} = -2k - 2k\frac{u}{y^3}$$

and

$$60y\left(\frac{b}{a}\right)^2 - 1 = 240k^2 - 1 + 60k^2\frac{u}{y^3}.$$

Such a k cannot exist. This proves the proposition

4 Local irreducibility of P_1

A differential equation is said locally reducible if there exists a field extens

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