SIMPLE MEROMORPHIC FUNCTIONS ARE ALGEBRAIC

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Résumé. Nous montrons que certains germes de fonctions méromorphes en deux variables sont des germes de fonctions algébriques.

Abstract. We exhibit a class of meromorphic functions in two variables conjugated to algebraic functions using the geometry of the foliation by level curves.

We are interested in the following problem :
Let $\mathcal{F}$ be a germ of holomorphic foliation on $(\mathbb{C}^2, 0)$. *Does there exist an algebraic surface S and a point $p \in S$ such that $\mathcal{F}$ is the germ at p of an algebraic foliation on S ?*
Such germs of foliation will be said to be **algebraizable**. In [6], Y. Genzmer and L. Teyssier proved the existence of a non algebraizable germ of saddle-node foliation. After them the problem splits in two parts:

Problem. *Give an example of non algebraizable germ of singularity.*

Problem. *Identify algebraizable singularities.*

We give an answer to the second problem in a very particular case following first pages of [4] (see also [9]).

Since Mather and Yau [13], it is known that a germ of holomorphic function with isolated singularity is finitely determined thus algebraizable. Such result was extended by Cerveau and Mattei [3] to germs of meromorphic functions. A consequence of the result of this paper is to get an example of a germ of algebraic meromorphic function which is not finitely determined.

This paper is concerned with holomorphic foliations of $(\mathbb{C}^2, 0)$ with a dicritical singularity at 0. Such germs of foliations have infinitely

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many analytic invariant curves going through 0. Basic examples of dicritical foliations are foliations given by level sets of a meromorphic function on $(\mathbb{C}^2, 0)$ having indeterminacy point at 0. Among all these singularities with a meromorphic first integral, we are interested in the simplest ones *i.e.* those smooth after **one** blow-up and such that **a unique** leaf is tangent to the exceptional divisor with tangency order **one**. These singularities will be called **simple** singularities.

Example (Basic example: the cusp)**.** *The dicritical cusp is given by the rational function* $\frac{y^2-x^3}{x^2}$*:*

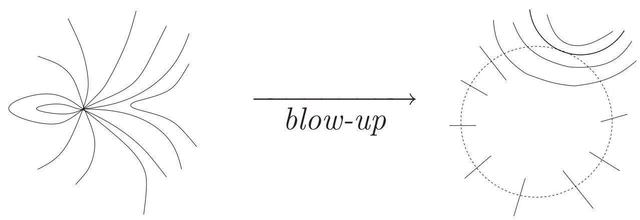

FIGURE 1.

After one blow-up one gets the function $t^2 - x$ *defining the foliation on the chart* $t = y/x$*.*

Example (Susuki's example [16, 3])**.** *This is the germ of singularity given by*

$$(y^3 + y^2 - xy)dx - (2xy^2 + xy - x^2)dy$$

It is topologically equivalent to the previous example. The topological closure of a leaf contains 0 *and is analytic. But its 'less transcendental' first integral is*

$$\frac{x}{y}e^{\frac{y(y+1)}{x}}.$$

It does not admit meromorphic first integral in any neighborhood of 0*.*

Theorem 1. *If* $\mathcal{F}$ *is a simple dicritical foliation of* $(\mathbb{C}^2, 0)$ *with a meromorphic first integral then there exist an algebraic surface* S*, a rational function* H *on* S *and a point* $p \in S$ *such that* $\mathcal{F}$ *is biholomorphic to the foliation given level curves of* H *in a neighborhood of* p*.*

Corollary 2. *If* F *is a germ of meromorphic function on* $(\mathbb{C}^2, 0)$ *with a simple type indeterminacy point at* 0 *then it is the germ of a rational function on a surface.*

Proof of the Corollary. – After a change of coordinates given by Theorem 1, the level curves of F coincide with those of a rational function H on an algebraic surface at some point 0 such that $F = f(H)$ for some function f defined on some ramified covering of $\mathbb{P}^1$. On the exceptional divisor obtained by blowing-up 0, F and H are rational functions so f is algebraic and F is a germ of rational function on a surface. □

Analytic invariant of simple singularities and basic constructions are given in §1. In §2 a proof of Theorem 1 is given. Last section contains some comments.

1. The analytic invariant of simple singularities

Following M. Klughertz [10], the description of a simple singularity up to analytic change of coordinates is done by a germ of involution at a point of $\mathbb{P}^1$ up to the action of $PGL_2(\mathbb{C})$. Let $\mathcal{F}$ be a germ of simple dicritical foliation of $(\mathbb{C}^2, 0)$. The blow-up of $(\mathbb{C}^2, 0)$ is denoted by M, the exceptional divisor by E and the strict transform of $\mathcal{F}$ by $\mathcal{F}_{-1}$. Because $\mathcal{F}_{-1}$ is smooth and has a unique leaf tangent to E at order one at some point p, leaves passing through a point in the neighborhood of p on E must cut E in another point. This gives us a germ of involution ι on the projective line. This is the holonomy of $\mathcal{F}$ on E.

The germ of involution of two analytically conjugated foliations are the same up to conjugation by an element of $PGL_2(\mathbb{C})$. Conversally if two germs of foliations with simple singularities have the same involution then they are analytically conjugated. To complete M. Klughertz' theorems, some finite determinacy properties and normal forms are obtained in [1, 15].

Let (M, E) be a pair of analytic surface and smooth rational curve $E \subset M$ with self-intersection $E \cdot E = k$. A neighborhood of E in M will still be denoted by M and called a k-neighborhood of $\mathbb{P}^1$.

Lemma 3. *Assume $k \leq 0$. Any germ of involution on $\mathbb{P}^1$ can be realized as the holonomy of a smooth holomorphic foliation of a k-neighborhood of $\mathbb{P}^1$. Two such realizations of the same involution are biholomorphic. For a involution ι, any such realization is called a k-realization of ι and denoted by $\mathcal{F}_k(\iota)$.*

Proof. – The proof is an illustration of the gluing trick of [12]. Let ι be such an involution on a disc $D_t(r)$ with coordinate t and radius r and φ be defined by $\varphi(t) = t - \iota(t)$. The involution ι is the holonomy of the foliation of $D_t(r) \times D_x(r')$ given by level curves of $\varphi(t)^2 - x$. Let r' be small enough then $U = \{t \in D_t(r) \mid |\varphi(t)^2| \leq r'\} \subset\subset D_t(r)$. Leaves

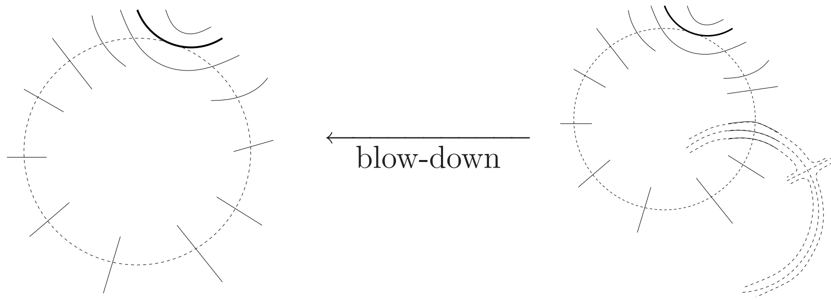

FIGURE 2. From $\mathcal{F}_k$ to $\mathcal{F}_{k-1}$

of the foliation by level curves of $\varphi(t)^2 - x$ on $(D_t - U) \times D_x$ intersect $D_t - U$ in a single point. For this reason the function $\sqrt{\varphi(t)^2 - x}$ is well defined on $(D_t - U) \times D_x$.

Let M be the manifold obtained by gluing $D_t \times D_x$ with $(\mathbb{P}^1 - U)_u \times D_y$ by

$$u = \varphi^{-1}\left(\sqrt{\varphi(t)^2 - x}\right) \text{ and } y = xt^{-k}.$$

The foliation given by $d(\varphi(t)^2 - x) = 0$ on the first chart and $du = 0$ on the second is well defined, transverse to E the divisor given by $x = 0$ and $y = 0$ but in a single point $t = 0$. The holonomy is ι by construction. When $k \leq 0$, Grauert's theorems [5, 8] give unicity of the germ of neighborhood built. □

Remark 4. *The contraction of the exceptional divisor in the* (-1)-*realization of* ι *gives a dicritical foliation of* $(\mathbb{C}^2, 0)$ *with invariant* ι.

Remark 5. *By blowing-up a point of the projective line which is not the fixed point of the involution* ι, $\mathcal{F}_k(\iota)$ *is transformed in* $\mathcal{F}_{k-1}(\iota)$.

Remark 6. *If one glues a node* $d(xy) = 0$ *to* $\mathcal{F}_{k-1}(\iota)$ *in such a way that a separatrix of the node is glued with a leaf of* $\mathcal{F}_{k-1}(\iota)$ *in a rational curve of self-intersection* (-1) *then the contraction of this leaf gives* $\mathcal{F}_k(\iota)$ *.*

Remark 7. *The involution* ι *has a rational first integral, i.e.* $R \in \mathbb{C}(\mathbb{P}^1)$ *such that* $R \circ \iota = R$, *if and only if* $\mathcal{F}_k(\iota)$ *has a meromorphic first integral.*

2. PROOF OF THEOREM 1

The strategy is to normalized a 0-realization of the involution of a dicritical singularity with a meromorphic first integral, and to prove that the meromorphic first integral of the dicritical foliation is transformed in an algebraic first integral of the 0-realization. By remarks 4–7 this is enough to prove the Theorem. Let $\mathcal{F}_{-1}(\iota)$ be a blow-up of a simple dicritical singularity with a meromorphic first integral and $\mathcal{F}_0$ be a 0-realization of the involution. It is defined on $D \times \mathbb{P}^1$ where D is a small disc around 0 in $\mathbb{C}$. Let y be a coordinate on $\mathbb{P}^1$ and x be a coordinate on D. Such a foliation is given by a differential equation

$$\frac{dy}{dx} = \frac{P_3(y)}{P_1(y)}$$

where $P_i \in \mathbb{C}\{x\}[y]$ is of degree i in y. By changing y, we can rectify three trajectories passing through p_1, p_2 and p_3 at $x = 0$ on straight lines $y = \infty$, $y = 1$ and $y = -1$ and the equation becomes

$$\frac{dy}{dx} = h(x)\frac{y^2 - 1}{y - \lambda(x)}.$$

Because the foliation is not singular, $h(0) \neq 0$. Furthermore p_1, p_2 and p_3 can be choosen such that $\lambda(0) \neq 0$. Then the trajectory passing through $0 \in \mathbb{P}^1$ at $x = 0$ is the graph of a biholomorphism and can be used as a new coordinate on the disc. The foliation $\mathcal{F}_0$ in these new coordinates is given by the equation

$$\frac{x^2 - 1}{x - \lambda(x)}\frac{dy}{dx} = \frac{y^2 - 1}{y - \lambda(x)}$$

for a germ of analytic function λ (different from the previous one) whose graph is the locus of verticality of leaves. Its phase plane is given in figure 3.

Let $H(x, y) \in \mathbb{C}(\{x\})(y)$ be a meromorphic first integral of $\mathcal{F}_0$ and d be its degree in y. Let $\mathcal{R}(d)$ be the variety of degree d rational functions on $\mathbb{P}^1$. This is a dimension $2d + 1$ algebraic variety. By restriction to vertical fibers, H gives a germ of curve $\gamma : D \to \mathcal{R}(d)$ defined by $\gamma(x) = H|_{\text{fiber above } x}$.

To prove that H is an algebraic function one has to prove that γ is included in an algebraic curve in $\mathbb{C} \times \mathcal{R}(d)$. To do so one will prove that

- γ is an integral curve of a rational vector fields $\vec{\mathcal{V}}$ on an algebraic ramified covering of $\mathbb{C} \times \mathcal{R}(d)$,
- $\vec{\mathcal{V}}$ has $2d + 1$ algebraically independent rational first integrals. Then integral curves of $\vec{\mathcal{V}}$ are algebraic.

Equations of the deformation of a rational function in the way described by the drawing of $\mathcal{F}_0$ can be written down explicitely. Let us

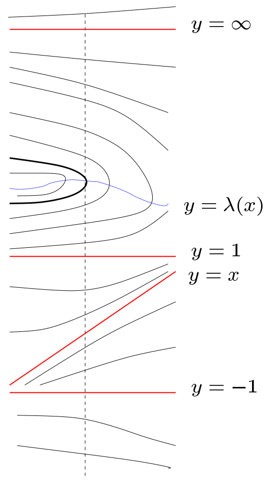

FIGURE 3. The foliation $\mathcal{F}_0$

parameterize $\mathcal{R}(d)$ by the zeros and the poles of fractions :

$$R(y)=\ell\frac{\Pi(y-a_i)}{\Pi(y-b_i)}.$$

Let $\widetilde{\mathcal{R}}$ be the ramified covering of $\mathcal{R}(d)$ where all the critical points are well-defined *i.e.* $\widetilde{\mathcal{R}}$ is the Galoisian covering of $\mathcal{R}(d)$ built from the covering of $\mathcal{R}(d)$ defined by the equation $\sum\frac{1}{p-a_i}-\sum\frac{1}{p-b_i}=0$ in $\mathcal{R}(d)\times\mathbb{C}$ with p the coordinate on $\mathbb{C}$.

The vector field

$$\vec{\mathcal{V}}=\frac{x^2-1}{x-\underline{\lambda}}\frac{\partial}{\partial x}+\sum\frac{a_i^2-1}{a_i-\underline{\lambda}}\frac{\partial}{\partial a_i}+\sum\frac{b_i^2-1}{b_i-\underline{\lambda}}\frac{\partial}{\partial b_i}$$

where $\underline{\lambda}$ is the coodinate function in $\mathbb{C}(\widetilde{\mathcal{R}})$ of a critical point, describes particular deformation of rational functions along a parameter x. If $a_i(x), b_i(x)$ and $\ell(x)$ parameterize an analytic integral curve of this vector field then the meromorphic function

$$H(x,y)=\ell(x)\frac{\Pi(y-a_i(x))}{\Pi(y-b_i(x))}$$

is a first integral of

$$\frac{x^2-1}{x-\lambda(x)}\frac{dy}{dx}=\frac{y^2-1}{y-\lambda(x)}$$

where $\lambda(x)$ is the restriction of $\underline{\lambda}$ to the integral curve.

Thus 0-realization of involutions with meromorphic first integrals of degree d in y are exactly described by trajectories of $\vec{\mathcal{V}}$ on $\mathbb{C}\times\widetilde{\mathcal{R}}$. One will denote by H_γ the meromorphic function in $\mathbb{C}\{x\}(y)$ given by an integral curve γ of $\vec{\mathcal{V}}$. Remark that trajectories realizing H_γ and $g(H_\gamma)$ for $g \in PSL_2(\mathbb{C})$ are different but describe the same 0-realization of the same involution.

The $2n+1$ independent algebraic first integrals of $\vec{\mathcal{V}}$ are buit in the following way.

Let us consider the critical values of rational functions as $2n-2$ rational functions on $\widetilde{R}$:

$$c_1, \dots, c_{2d-2}.$$

The restrictions of these functions on an integral curve γ give the critical values of H_γ with respect to y as functions of x. One can assume that $c_{2d-2}(\gamma) = \lambda$. The $2n-3$ remaining functions give the values of H_γ on leaves where H_γ ramifies so they are constant on integral curves of $\vec{\mathcal{V}}$.

The values of H_γ on the leaves $y = \infty$, $y = 1$, $y = -1$ and $y = x$ are constant hence the evaluations function $e_\infty(x, R) = R(\infty)$, $e_1(x, R) = R(1)$, $e_{-1}(x, R) = R(-1)$ and $e_x(x, R) = R(x)$ are 4 rational functions on $D \times \widetilde{\mathcal{R}}$ which are constant on integral curves of $\vec{\mathcal{V}}$.

In coordinates these functions are :

- $c_k(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(p_k - a_i)}{\Pi_i(p_k - b_i)}$
  for all $p_k \neq \underline{\lambda}$ such that $\sum \frac{1}{p_k - a_i} - \sum \frac{1}{p_k - b_i} = 0$,
- $e_\infty(...a_i...b_i...\ell) = \ell$,
- $e_1(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(1 - a_i)}{\Pi_i(1 - b_i)}$,
- $e_{-1}(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(-1 - a_i)}{\Pi_i(-1 - b_i)}$,
- $e_x(...a_i...b_i...\ell) = \ell \dfrac{\Pi_i(x - a_i)}{\Pi_i(x - b_i)}$,

These $2n+1$ funtions are algebraically independent because there is only a finite number of rational maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ with fixed critical values $c_1, \dots, c_{2d-2}$ and fixed values at -1, 1, ∞. These maps are given by the choice of the monodromies around critical values in the group of permutation of $\{1, 2, \dots, d\}$. Thus the functions $c_1, \dots, c_{2d-2}, e_1, e_{-1}, e_\infty$ are independent on $\widetilde{\mathcal{R}}$ and so are $c_1, \dots, c_{2d-3}, e_1, e_{-1}, e_\infty$ on $D \times \widetilde{\mathcal{R}}$. These functions are x-independent so are independent of e_x. This proves the functional independence of $2n+1$ rational first integrals of $\vec{\mathcal{V}}$ a vector field on a dimension $2n+2$ algebraic variety. The Lemma proved at the end of the section gives the algebraicity of integral curves of $\vec{\mathcal{V}}$.

Let S be an algebraic curve such that γ is defined on. The function H extends to a rational function on $S \times \mathbb{P}^1$. By blowing-up a point on $0 \times \mathbb{P}^1$ and blowing-down the strict transform of $0 \times \mathbb{P}^1$ one gets the dicritical singularity with rational first integral and ι as invariant.

This proves that a simple dicritical foliation given by a meromorphic function can be extended (in good coordinates) in a algebraic foliation with a rational first integral. □

The following lemma is a well-known fact but it is not easy to find a reference.

Lemma 8. *If a vector field $\vec{\mathcal{V}}$ on a dimension n algebraic variety has $n-1$ rational first integrals functionally independent then all trajectories are algebraic curves.*

Proof. – For rational functions, $H_1, \ldots, H_k$, functional independence ($dH_1 \wedge \ldots \wedge dH_k \neq 0$) and algebraic independance coincide.

Let $\mathcal{K}$ be the field of rational first integrals of $\vec{\mathcal{V}}$. It is a transcendence degree $n-1$ field. Let $H_1, \ldots H_{n-1}$ be a transcendence basis. Then, out of the dependency set $\{dH_1 \wedge \ldots \wedge dH_{n-1} = 0\}$ and the indeterminacy sets of H's, integral curves are algebraic.

Let S be a $\vec{\mathcal{V}}$-invariant irreducible hypersurface on which $dH_1 \wedge \ldots \wedge dH_{n-1} = 0$. One wants to find $n-2$ elements of $\mathcal{K}$ whose restrictions on S are algebraically independent.

Let $x \in S$ be a generic point and f a local holomorphic ireducible equation of S at x. One defines $d : \mathcal{K}^* \to \mathbb{Z}$ by $H = f^{d(H)} \frac{P}{Q}$ with P and Q holomorphic function not identically zero on S. There exists $F \in \mathcal{K}$ such that $d(\mathcal{K}^*) = d(F)\mathbb{Z}$.

Assume one gets $F_1, \ldots, F_{k-1}$ whose restrictions to S are functionnally independent. For $H \in \mathcal{K}$ one defined $\ell(H)$ such that $dH \wedge dF_1 \wedge \ldots \wedge dF_{k-1} = F^{\ell(H)} \omega$ with $\omega|_S$ a rational form in the neighborhood of S non zero on S. If such a ℓ does not exist, it is defined to by $-\infty$. If $k-1 < n-2$ there exists $H \in \mathcal{K}$ such that $F, F_1, \ldots, F_{k-1}$ and H are functionnally independent and these functions are holomorphic at x then $\ell(\mathcal{K}) \cap \mathbb{N}$ is not empty.

Let $F_k \in \mathcal{K}$ such that $\ell(F_k)$ is minimal in $\ell(\mathcal{K}) \cap \mathbb{N}$. If $\ell(F_k) = 0$ then the lemma is proved.

If $\ell(F_k) > 0$ then let $R(F_1|_S, \ldots, F_k|_S)$ be the minimal polynomial of $F_k|_S$ over $\mathbb{C}(F_1|_S, \ldots F_{k-1}|_S)$ and considere $\widetilde{F}_k = \frac{R(F_1, \ldots, F_k)}{F}$. This function is holomorphic at x because $d(F)$ si smaller than $d(R(F_1, \ldots, F_k))$. One has $d\widetilde{F}_k \wedge dF_1 \wedge \ldots \wedge dF_{k-1} = (\frac{1}{F} \frac{\partial R}{\partial F_k} dF_k + \frac{R}{F^2} dF) \wedge dF_1 \wedge \ldots \wedge dF_{k-1}$. Since R is the minimal polynomial of $F_k|_S$ $\frac{\partial R}{\partial F_k}$ can not vanish on S if R

is not zero. One has $0 \leq \ell(\widetilde{F}_k) < \ell(F_k)$ which is a condradiction then $R = 0$ and the lemma is proved. □

3. Comments

3.1. **Cerveau-Mattei finite determinacy theorem.** For special kind of functions there exists an algebraization result proved by D. Cerveau and J.-F. Mattei in [3]. Let $f_1, \ldots, f_p$ be germs of irreducible holomorphic functions in $(\mathbb{C}^n, 0)$, $\alpha_1, \ldots, \alpha_p$ be complex numbers. Considere the multivalued function

$$H = f_1^{\alpha_1} \ldots f_p^{\alpha_p}.$$

Let ω be the form $f_1 \ldots f_p \sum \alpha_i \frac{df_i}{f_i}$ and X be the hypersurface $f_1 \ldots f_p = 0$.

Definition 9. *The small critical locus of H is a subset $C'(H)$ of $Zero(\omega) \cup X$ of point p such that if $p \in X$ germs at p of f's are reducible or (X, p) is not a normal crossing germ of hypersurface.*

The function H is said to be finitely determined at order k if for every $g_1, \ldots g_p$ with $j_k(f_i - g_i) = 0$ there is a diffeomorphism u of $(\mathbb{C}^n, 0)$ such that

$$H \circ u = g_1^{\alpha_p} \ldots g_p^{\alpha_p}$$

Theorem 10 (Cerveau-Mattei [3] théorème 4.2. p 163)**.** *A function H as above is finitely determined if and only if $C'(H) \subset \{0\}$.*

In our situation, $n = 2$, $\alpha_i \in \mathbb{Z}$, finite determinacy implies conjugation to a rational function with same 'order k jet' at 0. There is a rational function which is not finitely determined *i.e.* such that a meromorphic function with the same order k jets of numerator and denominator is not conjugated to the previous rational functions. Such an example can be build using previous description of simple dicritical foliations.

Let ι be the germ of involution at $0 \in \mathbb{P}^1$ defined by

$$h \circ \iota = h$$

for some rational function h with a double zero at 0. For instance $h(t) = t^2((t-1)^2 + 1)$. The (-1)-realization of ι has a meromorphic first integral H extending h to the (-1)-neighborhood of $\mathbb{P}^1$ following $\mathcal{F}_{-1}(\iota)$. By blowing-down this function, one gets a germ of meromorphic function on $(\mathbb{C}^2, 0)$ with simple indeterminacy point.

By Cerveau-Mattei theorem such a function is finitely determined if and only if its small critical locus is contained in $\{0\}$. But the critical set of H is contained in $Zero(\omega)$ and coincides with it out of the zero and polar locus of H. If the small critical locus is included in $\{0\}$ then $Zero(\omega)$ is included in the zero and polar locus of H. This is not the case for the function built from h. This H is $\frac{f_0 f_{1+i} f_{1-i}}{f_\infty^4}$ where f_p is an equation of the leaves having slope p at 0. Its critical set is given by three leaves with slopes $3/4+i\sqrt{5}/4$, $3/4-i\sqrt{5}/4$ and ∞ at 0.

To prove that H is algebraizable by means of this finite determinacy theorem, one only needs to find a rational function f such that $f\circ H$ is finitely determinate. Critical values of $f\circ H$ are critical values of f and the image by f of those of H. If $f\circ H$ satisfies the hypothesis of the finite determinacy theorem, these critical values must be 0 and ∞. Such a f would be a (non ramified) covering of $\mathbb{P}^1-\{0,\infty\}$ by $\mathbb{P}^1-D$ where D is a finite set containing critical values of H. If $\#D\geq 3$ this is not possible. It is the case for our special H coming from h where $D\supset\{0, h(3/4+i\sqrt{5}/4), h(3/4-i\sqrt{5}/4),\infty\}$ and for this reason there is no f such that $f\circ H$ is finitely determined.

Nevertheless by our theorem, this function is a germ of algebraic function in suitable coordinates.

3.2. **The λ is not unique.** If one fixes the involution ι and three points on $\mathbb{P}^1$ then the function λ given by the normalisation of a 0-realization $\mathcal{F}_0$ is unique. So λ is an invariant of the marked involution $(\iota, p_{-1}, p_1, p_\infty)$ on $\mathbb{P}^1$. The involution ι itself is an invariant.

If four differents leaves are normalized on $y=\infty$, $y=-1$, $y=1$ and $y=x$ one gets a new function λ^*. Let us explain this change of λ. Let $u_1(x), u_{-1}(x), u_\infty(x)$ be three solutions of $\mathcal{F}_0$ and u_0 be the solution such that the crossratio of $(u_\infty(0), u_{-1}(0), u_0(0), u_1(0))$ equals the cross-ration of $(\infty,-1,0,1)$. The change of variables

$$\begin{cases} x^* &= \dfrac{u_1-u_\infty}{u_0-u_\infty}+\dfrac{u_0-u_1}{u_{-1}-u_\infty} \\ \\ y^* &= \dfrac{u_1-u_\infty}{y-u_\infty}+\dfrac{y-u_1}{u_{-1}-u_\infty} \end{cases}$$

rectifies the u's on the four lines $y^*=\infty$, $y^*=1$, $y^*=-1$ and $y^*=x^*$ and gives another normalisation of the 0-realization of ι. The involution of this realisation is ι^* defined by

$$\iota^*\left(\frac{u_1(0)-u_\infty(0)}{y-u_\infty(0)}+\frac{y-u_1(0)}{u_{-1}(0)-u_\infty(0)}\right)=\frac{u_1(0)-u_\infty(0)}{\iota(y)-u_\infty(0)}+\frac{\iota(y)-u_1(0)}{u_{-1}(0)-u_\infty(0)}$$

If λ^* is the function given by the verticality locus of this new diffferential equation then one gets

$$\lambda^*\left(\frac{u_1-u_\infty}{u_0-u_\infty}+\frac{u_0-u_1}{u_{-1}-u_\infty}\right)=\frac{u_1-u_\infty}{\lambda-u_\infty}+\frac{\lambda-u_1}{u_{-1}-u_\infty}$$

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