Timoshenko beam under Winkler foundation

Hariz Marwan[a], Casale Guy [b], Le Marrec Loïc[c], Lerbet Jean[d].

a. Université de Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France.
E-mail: marwan.hariz@univ-rennes1.fr
b. Université de Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France.
E-mail: guy.casale@univ-rennes1.fr
c. Université de Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France.
E-mail: loic.lemarrec@univ-rennes1.fr
d. Université d'Evry, CNRS, LaMME - UMR 8071, F-91037, Evry, France.
E-mail: jlerbet@gmail.com

Abstract :

We are interested in buckling for Timoshenko beam supported along its length by an elastic wall (Winkler foundation) and subjected to a longitudinal force. We use analytical methods to determine buckling load and mode shape rather than numerical methods. Haringx and Engesser models are compared. We show that the rigidity of the wall solely gouverns the phenomena and that the two models are equivalent whatever the parameters of the problem.

Keywords: Timoshenko beam, buckling, elastic foundation, mode shape.

1 Problem statement

We consider a plane Timoshenko model for an homogeneous straight beam of length L surrounded by an elastic wall exerting a transverse elastic force distribution $\mathbf{f}$. A longitudinal force $\mathbf{P}$ is imposed at their ends in order to reach a buckling behavior.

1.1 Cosserat formulation

The kinematics may be described in a Cartesian frame $(\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})$ where $\mathbf{e_z}$ is oriented along the beam axis in the stress-free configuration, and the motion of the beam lies in the $(\mathbf{e_x}, \mathbf{e_z})$-plane. However, for such a Cosserat-like structure it is justified to use a moving director frame basis $(\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3})$ for which $\mathbf{d_2} = \mathbf{e_y}$ and $\mathbf{d_3}$ is normal to the cross-section, last this basis is orthonormal $\mathbf{d_1} = \mathbf{d_3} \wedge \mathbf{d_2}$. As the orientation of the beam is not uniform, this basis depends on the curvilinear abscissa S of the beam and in contrary to Euler-Bernoulli model $\mathbf{d_3}$ is not necessarily tangent to the center line. The kinematics of the beam is governed by the displacement $\underline{\mathbf{u}}(S) = \underline{u}_1\mathbf{d_1} + \underline{u}_3\mathbf{d_3}$ of any point of the center-line and rotation $\boldsymbol{\phi}(S) = \phi\,\mathbf{d_2}$ of the section. With the same formalism, the internal force acting on the beam is of the form $\mathbf{N} = N_1\mathbf{d_1} + N_3\mathbf{d_3}$ where N_1 is the shear force and N_3 is the normal force and the moment $\mathbf{M} = M_2\mathbf{d_2}$ where M_2 is the bending moment. In terms of constitutive laws :

$$N_1 = GA(\frac{\partial \underline{u}_1}{\partial S} - \phi), \qquad N_3 = EA\frac{\partial \underline{u}_3}{\partial S}, \qquad M_2 = EI\frac{\partial \phi}{\partial S},$$

where A and I are the area and the quadratic moment of the cross-section, E and G are the Young modulus and shear modulus G (including eventually a shear correction factor) of the beam material. For elastic foundation, the force density $\mathbf{f} = -K\underline{u}_1\mathbf{d_1}$ where K is the rigidity modulus of the wall [1, 6, 7]. The external compression load is $\mathbf{P} = -P\mathbf{d_3}$, $P \geq 0$. Of course for linearized theory $\mathbf{d_1} \simeq \mathbf{e_x}$ and $\mathbf{d_3} \simeq \mathbf{e_z}$.

1.2 Equilibrium relations

Equilibrium relations for this static problem states [2] :

$$\frac{\partial \mathbf{N}}{\partial S} + \mathbf{f} = 0, \qquad \frac{\partial \mathbf{M}}{\partial S} + (\boldsymbol{\varepsilon} + \mathbf{d_3}) \wedge \mathbf{N} = 0,$$

for which $\boldsymbol{\varepsilon} = \varepsilon_1 \mathbf{d_1} + \varepsilon_3 \mathbf{d_3}$, where $\varepsilon_1 = \frac{\partial \underline{u}_1}{\partial S} - \phi$ is the shear strain and $\varepsilon_3 = \frac{\partial \underline{u}_3}{\partial S}$ is the longitudinal strain. Projecting along directors, we obtain the following system:

$$\begin{aligned} \frac{\partial N_3}{\partial S} - \kappa_2 N_1 &= 0, \\ \frac{\partial N_1}{\partial S} + \kappa_2 N_3 - K\underline{u}_1 &= 0, \\ \frac{\partial M_2}{\partial S} + (1+\varepsilon_3)N_1 - \varepsilon_1 N_3 &= 0, \end{aligned}$$

where $\kappa_2 = \frac{\partial \phi}{\partial S}$ and $\frac{\partial \mathbf{d_i}}{\partial S} = \kappa_2 \mathbf{d_2} \wedge \mathbf{d_i}$ has been used. For finite longitudinal force $\kappa_2 N_1$ may be neglected in first approximation, leading to a uniform longitudinal force imposed by the boundary conditions $N_3 = -P$. In the same spirit $\varepsilon_3 N_1$ may be neglected in the last equation (*inextensible* approximation). Hence we obtain a system of linear equation that may be expressed using infinitesimal kinematical variable:

$$\begin{aligned} GA\left(\frac{\partial^2 \underline{u}_1}{\partial S^2} - \frac{\partial \phi}{\partial S}\right) - \frac{\partial \phi}{\partial S}P - K\underline{u}_1 &= 0, \\ EI\frac{\partial^2 \phi}{\partial S^2} + \left(\frac{\partial \underline{u}_1}{\partial S} - \phi\right)(GA + P) &= 0. \end{aligned} \tag{1}$$

This is exactly the application of Haringx model for Winkler foundation [3, 4]. Note that this has been recovered via a geometrically exact Timoshenko model justified for both small or large transformations.

1.3 Non-dimensionalization procedure

We introduce non-dimensional formulation of the problem thanks to the following variables:

$$\varrho = \sqrt{\frac{I}{A}}, \quad g = \frac{G}{E}, \quad \kappa = \frac{K}{E}\frac{I}{A^2}, \quad \eta = \frac{P}{EA}. \tag{2}$$

For any material $g \simeq \frac{1}{2(1+\nu)}$ where ν is the Poisson ratio then $\frac{1}{3} \lesssim g \lesssim \frac{1}{2}$, For compression in the elastic regime $0 < \eta < \eta_{yield}$ where η_{yield} is nothing else than the limit strain for which irreversible transformation occurs, lastly $\kappa = 0$ in absence of foundation and $\kappa = \infty$ for a rigid foundation. The kinematical variable becomes in a non-dimensional form:

$$s = \frac{S}{\varrho}, \quad \ell = \frac{L}{\varrho}, \quad u(s) = \frac{\underline{u}_1(S)}{\varrho}, \quad \theta(s) = \phi(S).$$

Note that $\varrho = \frac{R}{2}$ for circular cross-section of radius R therefore ℓ is twice the slenderness ratio (then $\ell \gg 1$). Therefore (1) takes the form

$$\boxed{\begin{array}{rcl} g(u'' - \theta') - \kappa u - \eta\theta' & = & 0, \\ \theta'' + (g+\eta)(u' - \theta) & = & 0, \end{array}} \tag{3}$$

where we have used the convention $\frac{\partial f}{\partial s} \equiv f'$ for any function $f(s)$.
Another model widely used for buckling is proposed by Engesser [5] for which the non-dimensional equilibrium relations are in case of Winkler foundations:

$$\boxed{\begin{array}{rcl} g(u'' - \theta') - \kappa u - \eta u'' & = & 0, \\ \theta'' + g(u' - \theta) & = & 0. \end{array}} \tag{4}$$

2 Secular relations and eigenfunctions

For harmonic solution $u(s) = Ue^{iks}$ and $\theta(s) = \Theta e^{iks}$, the linear differential system becomes $\mathbb{K}\mathbf{V} = 0$ where $\mathbf{V} = (U, \Theta)^T$ and the (hermitian) rigidity matrix is, for (3) and (4) respectively:

$$\mathbb{K}^H = \begin{pmatrix} gk^2 + \kappa & ik(g+\eta) \\ -ik(g+\eta) & k^2 + g + \eta \end{pmatrix}, \qquad \mathbb{K}^E = \begin{pmatrix} k^2(g-\eta) + \kappa & ikg \\ -ikg & k^2 + g \end{pmatrix}. \tag{5}$$

Non-trivial solutions arise if $\det(\mathbb{K}) = 0$ what may be written as a secular equation :

$$\begin{array}{rcl} \mathcal{P}^H(\eta,\kappa) & = & g(k^4 - k^2\eta + \kappa) + k^2(\kappa - \eta^2) + \eta\kappa. \\ \mathcal{P}^E(\eta,\kappa) & = & g(k^4 - k^2\eta + \kappa) + k^2(\kappa - k^2\eta). \end{array} \tag{6}$$

By solving $\mathcal{P}(\eta) = 0$ we find a polynomial with respect to η whose real positive roots are

$$\eta^H = \frac{-gk^2 + \kappa + \sqrt{(gk^2+\kappa)(gk^2 + 4k^4 + \kappa)}}{2k^2}, \qquad \eta^E = \frac{gk^2}{g+k^2} + \frac{\kappa}{k^2}. \tag{7}$$

They are presented in figure (1) for various values of κ. Conversely, for fixed η, $\mathcal{P}$ is a second degree polynomial with respect to k^2. Hence $u(s)$ and $\theta(s)$ have the general form

$$\begin{array}{rcl} u(s) & = & ae^{ik_1 s} + be^{-ik_1 s} + ce^{ik_2 s} + de^{-ik_2 s}, \\ \theta(s) & = & \Xi(k_1)(ae^{ik_1 s} - be^{-ik_1 s}) + \Xi(k_2)(ce^{ik_2 s} - de^{-ik_2 s}), \end{array} \tag{8}$$

where a, b, c and d will be defined by boundary conditions. Here $\pm k_1$ $\pm k_2$ are the two roots of $\mathcal{P}(k^2) = 0$. Note that we used $\mathbb{K}_{11}U + \mathbb{K}_{12}\Theta = 0$ therefore $\Theta = i\Xi U$ where $i\Xi = -\frac{\mathbb{K}_{11}}{\mathbb{K}_{12}}$, in details

$$\Xi^H = \frac{gk^2 + \kappa}{k(g+\eta)}, \qquad \Xi^E = \frac{k^2(g-\eta) + \kappa}{kg}. \tag{9}$$

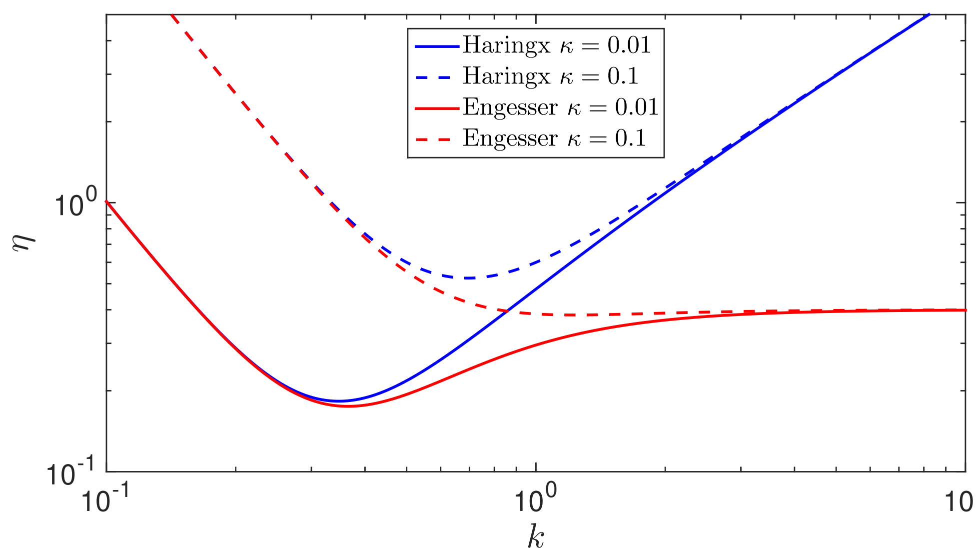

Figure 1: Secular equations (7) for various κ and $g = \frac{2}{5}$.

3 Boundary conditions and eigenmodes

Without lost of generality, we consider a pinned-pinned beam for which (non-dimensional) boundary conditions are in terms of kinematical variables $u(s) = 0$, $\theta'(s) = 0$ at $s = 0$ and $s = \ell$. This forms a set of four linear equations in terms of $X = (a, b, c, d)^T$ that may be written algebraically as $\mathbb{M}X = 0$ with

$$\mathbb{M} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ k_1\Xi(k_1) & k_1\Xi(k_1) & k_2\Xi(k_2) & k_2\Xi(k_2) \\ e^{ik_1\ell} & e^{-ik_1\ell} & e^{ik_2\ell} & e^{-ik_2\ell} \\ k_1\Xi(k_1)e^{ik_1\ell} & k_1\Xi(k_1)e^{-ik_1\ell} & k_2\Xi(k_2)e^{ik_2\ell} & k_2\Xi(k_2)e^{-ik_2\ell} \end{pmatrix}. \tag{10}$$

Again, non trivial solutions exist if $\det(\mathbb{M}) = 0$ what gives us the following relation:

$$\big(k_1\Xi(k_1) - k_2\Xi(k_2)\big)^2 \sin(k_1\ell)\sin(k_2\ell) = 0. \tag{11}$$

Direct computation shows that $k_1\Xi(k_1) \neq k_2\Xi(k_2)$. First it is observed that k_1 and k_2 play a similar role therefore we focus on $k \equiv k_1$ in the following. According to (11) non-trivial solutions exist only if $\sin(k\ell) = 0$, this implies that k must be real and more precisely

$$k = \frac{n\pi}{\ell}, \qquad n \in \mathbb{N}^*. \tag{12}$$

We should notice that in order to respect linearization hypothesis we should guarantee that the mode number is less than the slenderness ratio ($n < \ell$).

According to equation (10), the modal amplitude may be obtained up to an arbitrary constant, by solving:

$$\mathbb{A}Y = -aZ, \qquad \text{where} \qquad Y = (b, c, d)^T,$$

with $\mathbb{A}_{ij} = M_{ij}$ and $Z_i = M_{i1}$ for $1 \leq i \leq 3$ and $2 \leq j \leq 4$. Fixing $a = 1$ and $k = \frac{n\pi}{\ell}$ we find:

$$u(s) = \cos(\frac{n\pi}{\ell}s), \tag{13}$$
$$\theta(s) = -\Xi(\frac{n\pi}{\ell})\sin(\frac{n\pi}{\ell}s). \tag{14}$$

Hence general form of the eigenmode of each models is identical even if $\Xi(k, \eta, \kappa)$ may apriori differs.

4 Interpretation and conclusion

Till now n is not fixed, in practice the first buckling mode is defined as $\eta_0 = \min_n(\eta(k))$ where $k = \frac{n\pi}{\ell}$ what leads to the determination of n that counts the number of arches when buckling occurs. Since $\eta(k)$ is a convex function (see Fig.1), we can replace this approach by a continuous one by finding k_{min} such that $\left.\frac{\partial\eta}{\partial k}\right|_{k_{min}} = 0$. Then n is the rounded value of $k_{min}\frac{\ell}{\pi}$ and of course still $\eta_0 = \eta(\frac{n\pi}{\ell})$. In practice, according to (7):

$$\begin{aligned}
&\text{Haringx model} \quad && \frac{\partial\eta^H}{\partial k} = \frac{g\left(2k^6 - \kappa k^2\right) - \kappa\left(\kappa + \sqrt{(gk^2+\kappa)(gk^2+\kappa+4k^4)}\right)}{k^3\sqrt{(gk^2+\kappa)(gk^2+\kappa+4k^4)}} = 0, \\
&\text{Engesser model} \quad && \frac{\partial\eta^E}{\partial k} = \frac{2g^2k}{(g+k^2)^2} - \frac{2\kappa}{k^3} = 0
\end{aligned} \tag{15}$$

Using Taylor expansion for small k, this conditions leads to $k^4 = \kappa$ for the two models. This requirement is independent to the material parameter g and gives a simple estimation of the number of arches n:

$$n = \left\lfloor \frac{\ell}{\pi}\kappa^{\frac{1}{4}} \right\rceil. \tag{16}$$

Using Taylor expansion of (7) up to $\mathcal{O}(k^4)$, the critical load becomes for both models:

$$\eta = 2\sqrt{\kappa} = 2\,k^2. \tag{17}$$

Lastly, combining with $k^4 = \kappa$, the modal amplitudes are approached by

$$\Xi^H = \Xi^E = k(1 - \frac{k^2}{g}) \qquad \left(= \kappa^{\frac{1}{4}}(1 - \frac{\sqrt{\kappa}}{g})\right). \tag{18}$$

This shows the high influence of the wall rigidity that uniquely defines the buckling load (17), the modal amplitude (18) and the wavelength of buckling modes as $\lambda = \frac{2\pi}{k} = 2\pi\kappa^{-\frac{1}{4}}$. Without wall the critical Euler load is determined by the slenderness ratio as the first arch is alway involved, then $\frac{\ell}{\pi}\sqrt{\eta} = 1$, in our case $n = \frac{\ell}{\pi}\sqrt{\frac{\eta}{2}}$ and no buckling occurs if $\frac{\ell}{\pi}\sqrt{\frac{\eta}{2}} < 1$.

In figure 2 the behavior of k and η satisfying $\frac{\partial\eta}{\partial k} = 0$ are plotted for various κ. For small rigidity of the foundation $\kappa \ll 1$ the two models coincides with the approximation proposed. This is not the case as the rigidity of the wall increases. However, it must be noticed that the proposed models are accompanied by some physical hypotheses. First, $0 < \eta < \eta_{yield}$ where $\eta_{yield} \simeq 2\cdot10^{-3}$ for steel-like material or $\simeq 5\cdot10^{-2}$ for fiber reinforced composite. Second $k \lesssim 1$ in order to respect linearized hypothesis. These criteria are

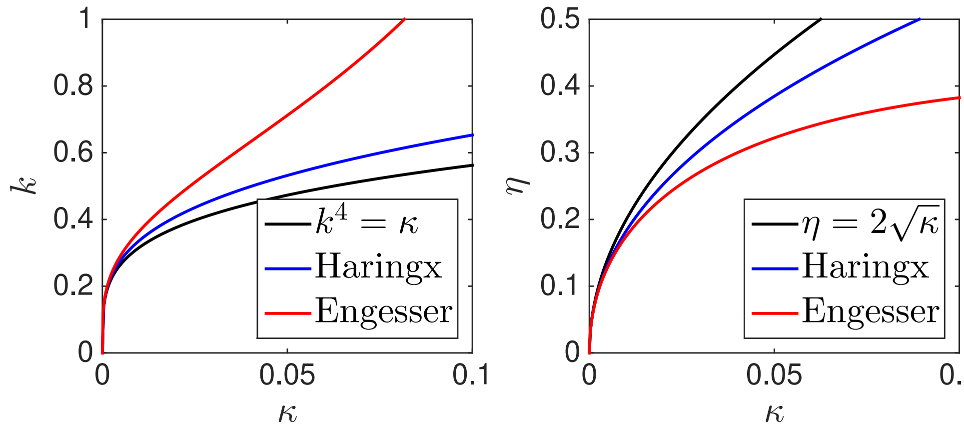

Figure 2: Solutions k and η for which $\frac{\partial \eta}{\partial k} = 0$ versus κ. Comparison between Haringx and Engesser model with $g = \frac{2}{5}$.

satisfied only for $\kappa \ll 1$ and the yield limit is always more restrictive. In other words, buckling may occurs only if the foundations are not too rigid and the maximum rigidity atteignable in order to have buckling is $\kappa_{max} = \eta_{yield}^2/4$. According to the magnitude of η_{yield} for any material, the approximation proposed in this section are all justified. In conclusion, Engesser and Haring models can't be distinguished through buckling analysis of beam supported by elastic foundations.

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