Nonintegrability by discrete quadratures

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Résumé

Nous nous intéressons à la notion d'intégrabilité par quadratures discrètes pour les systèmes aux différences algébriques. Nous montrons que les groupes de Galois des équations variationnelles discrètes le long des solutions algébriques d'un système aux différences intégrable sont virtuellement résolubles. Cette condition nécessaire à l'intégrabilité, dans la veine de la théorie de Morales et Ramis, est ensuite utilisée pour montrer que deux types d'équations de Painlevé discrètes ne sont pas intégrables par des quadratures discrètes.

Abstract

We give a necessary condition for integrability of a difference system by means of discrete quadratures : the discrete variational equations along algebraic solutions must have almost solvable Galois groups. This necessary condition à la Morales and Ramis is used in order to prove that q-analogues of Painlevé I and Painlevé III equations are non integrable by discrete quadratures.

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Introduction

In this paper, we develop a technique à la Morales-Ramis in order to study the integrability by discrete quadratures of general algebraic difference equations and we give two applications to q-Painlevé equations. More precisely, we prove:

Necessary condition for integrability. *If an algebraic difference equation is integrable by discrete quadratures then the Galois groups of its discrete variational equations (of all order) along algebraic particular solutions are virtually solvable.*

and we use this result in order to prove that q-analogues of Painlevé I and Painlevé III equations are non integrable by q-quadratures. For the original Morales-Ramis theory, we refer to the work of Morales and Ramis [24, 25] and of Morales, Ramis and Simo [26]; for its ramifications and its innumerable applications, see [19, 27] for instance.

Our work heavily relies on the Galois pseudogroup introduced by A. Granier [14] following ideas of B. Malgrange [21]. The basic strategy is to prove, using Artin approximation theorem, that the pseudogroup of an algebraic q-difference equation controls the Galois group of its linearization along any particular solution. The above necessary condition for integrability is then deduced from the fact that the pseudogroup a difference equation integrable by discrete quadratures is infinitesimally solvable.

This paper is organized as follows. In sections 1, 2 and 3 we recall the definition of Malgrange pseudogroup and we prove its infinitesimal solvability in the integrable case. In section 4 we recall useful results regarding the Galois theory of linear q-difference equations. In section 5 we prove the above necessary condition for integrability. In section 6 and in section 7 we prove that q-analogues of Painlevé I and Painlevé III equations are non integrable by quadratures.

1 Frame bundles and prolongations

Let M be a smooth irreducible affine algebraic variety over $\mathbb{C}$ of dimension d with coordinate ring $\mathbb{C}[M]$. Classical constructions from differential geometry will be presented algebraically by means of their functors of points [17, 21]. The formal frame bundle of M is the complex proalgebraic variety RM of all formal invertible maps $r : (\mathbb{C}^d, 0) \to M$. More precisely, for any $\mathbb{C}$-algebra A, the A-points of RM are given by

$$RM(A) = \{\text{locally invertible } \mathbb{C}\text{-algebra morphisms } f : \mathbb{C}[M] \to A[[t]]\}$$

where $A[[t]] = A[[t_1, \ldots, t_d]]$ is the ring of formal power series in the indeterminates $t_1, ..., t_d$.

A $\mathbb{C}$-algebra morphism $f : \mathbb{C}[M] \to A[[t]]$ is said to be locally invertible if its scalar extension $f_A : A[M] \to A[[t]]$ induces an isomorphism $df_A : J/J^2 \to (t)/(t)^2$ where $(t) = (t_1, \ldots, t_d)$ is the ideal of $A[[t]]$ generated by $t_1, ..., t_d$ and where $J = f_A^{-1}(t)$; it is equivalent to require that f_A induces an isomorphism $\widehat{f}_A : \widehat{A[M], J} \to A[[t]]$ (see [23]). The algebra $\mathbb{C}[RM]$ of this variety is built up from the differential algebra $\mathbb{C}\{M\}$ generated by $\mathbb{C}[M]$ with d commuting derivations $\frac{\partial}{\partial t_1}, \dots \frac{\partial}{\partial t_d}$ by adding $\det \left[\frac{\partial}{\partial t_i} x_j\right]^{-1}$ for any transcendence basis $x_1, \dots, x_d$ of $\mathbb{C}[M]$ over $\mathbb{C}$. The projection on M is given on the A-points by $f \to f/(t)$.

The ring $\mathbb{C}[RM]$ is filtered by the subrings $\mathbb{C}[RM]_k$ of differential polynomials of order less than or equal to k. The ring $\mathbb{C}[RM]_k$ is the coordinate ring of the order k frame bundle R_kM of M. For any $\mathbb{C}$-algebra A, the A-points of R_kM are given by

$$R_kM(A) = \{\text{locally invertible } \mathbb{C}\text{-algebra morphisms } f : \mathbb{C}[M] \to A[[t]]/(t)^{k+1}\}$$

A derivation or an endomorphism of $\mathbb{C}[RM]$ is said to be of degree less than or equal to n if it maps $\mathbb{C}[RM]_k$ in $\mathbb{C}[RM]_{k+n}$ for all $k \in \mathbb{N}$.

1.1 RM has a canonical parallelism

One gets a morphism from the $\mathbb{C}$-Lie algebra χ of formal vector field on $(\mathbb{C}^d, 0)$ to the $\mathbb{C}$-Lie algebra of vector fields on RM of degree less than or equal to 1. Let A be a $\mathbb{C}$-algebra. For any A-point $Z \in \chi\widehat{\otimes}A$ of χ, the map

$$\begin{array}{rcl} RM(A) & \to & RM(A[\epsilon]/(\epsilon^2)) \\ f & \mapsto & (\mathbb{I} + \epsilon Z) \circ f \end{array}$$

defines a vector field on RM. One can check that images of non vanishing vector fields (at $0 \in (\mathbb{C}^d, 0)$) have degree 1 but images of vanishing vector fields have degree 0. This Lie subalgebra is denoted by χ_0.

At each point f of RM, the image of χ spans T_fRM and its Lie subalgebra χ_0 spans the relative tangent $T_f(RM/M)$.

1.2 RM is a principal bundle

Let Γ be the complex proalgebraic group whose A-points are :

$$\Gamma(A) = \{\text{invertible } (t)\text{-continuous } A\text{-algebra morphisms } g : A[[t]] \to A[[t]]\}.$$

This group acts on RM and the quotient RM/Γ is M. This action has degree 0 and integrates the infinitesimal action of its Lie algebra χ_0. This group is the projective limit of the complex algebraic groups Γ_k indexed by $k \in \mathbb{N}$ whose A-points are

$$\Gamma_k(A) = \{\text{invertible } A\text{-algebra morphisms } g : A[[t]]/(t)^{k+1} \to A[[t]]/(t)^{k+1}\}.$$

1.3 RM is a 'natural' bundle

Let us consider two points p, p' of M. Let φ be an isomorphism from be the formal scheme $\widehat{M,p}$ with ring $\widehat{\mathbb{C}[M],p}$ to the formal scheme $\widehat{M,p'}$ with ring $\widehat{\mathbb{C}[M],p'}$. We have a natural lift $R\varphi$ of φ :

$$R\varphi : \widehat{RM,p} = \widehat{M,p} \underset{M}{\times} RM \to \widehat{RM,p'} = \widehat{M,p'} \underset{M}{\times} RM$$

defined, for any $f : \widehat{\mathbb{C}[M],p'} \to A[[t]]$, by $R\varphi(f) = f \circ \varphi$. This isomorphism has degree 0. The prolongation $\varphi \to R\varphi$ has degree 0 with respect to the induced graduations.

Let p be a point of M and let X be a formal vector field at p i.e. $\mathbb{I} + \epsilon X$ is a morphism from $\widehat{\mathbb{C}[M],p}$ to $(\widehat{\mathbb{C}[M],p})[\epsilon]/\epsilon^2$. We have a natural lift RX of X :

$$\begin{array}{rcl} RM(A) & \to & RM(A[\epsilon]/(\epsilon^2)) \\ f & \mapsto & f_\epsilon \circ (\mathbb{I} + \epsilon X) \end{array}$$

where f_ϵ is the prolongation of f sending ϵ to itself. These prolongations commute with the actions of Γ and χ.

1.4 $\mathbb{C}[RM]$ is a differential ring

The fact that $\mathbb{C}[RM]$ is a differential ring is clear from its definition but the differential structure is not canonical. Let $\mathfrak{h}$ be a Lie algebra such that $\chi = \mathfrak{h} + \chi_0$. Then $\mathbb{C}[RM]$ is a localization of the $\mathfrak{h}$-differential ring generated by $\mathbb{C}[M]$.

By duality on gets a differential $d : \mathfrak{h}^* \to \wedge^2\mathfrak{h}^*$. The action of $\mathfrak{h}$ gives a total derivative $D : \mathbb{C}[RM] \to \mathbb{C}[RM] \otimes \mathfrak{h}^*$. It can be prolong into $D : \mathbb{C}[RM] \otimes \wedge^n\mathfrak{h}^* \to \mathbb{C}[RM] \otimes \wedge^{n+1}\mathfrak{h}^*$ by $D(f \otimes h^*) = Df \wedge h^* + f \otimes dh^*$ and, by Jacobi identity, $DD = 0$.

2 Groupoids and Algebroids

2.1 The groupoid EM

The product $EM = RM \times RM$ has a structure of groupoid on RM given by

- two projections s and t onto the first and the second factor respectively,
- a composition $c : EM \underset{{}^tRM^s}{\times} EM \to EM$: the projection on the first and third factors,
- an identity $e : RM \to EM$: the diagonal,
- an inverse $i : EM \to EM$ exchanging the two factors.

These maps have order 0 and satisfy some commutative diagrams [20]. This product situation is the prototype of groupoid. Subgroupoids of EM are algebraic equivalence relations on RM.

This space is endowed with two commuting prolongation procedures for vector fields on M. The first one, the source prolongation of X, R^sX is defined by RX on $s^*\mathbb{C}[RM]$ and 0 on $t^*\mathbb{C}[RM]$. The target prolongation is defined in the same way (*mutatis mutandis*). These prolongations of vector fields can be integrated at the formal maps level.

2.2 The groupoid $AutM$

The group Γ acts diagonally on the product $RM \times RM = EM$. The quotient EM/Γ has two projections on $RM/\Gamma = M$ still denoted by s and t. The groupoid structure of the direct product induces a groupoid structure on the quotient still denoted by c, e and i. This groupoid is $AutM$. Points of this space can be identified with formal maps φ between formal neighborhoods of points in M : let f, g be two A-frames on M and let $\widehat{f}_A : A[\widehat{M], f^{-1}}(t) \to A[[t]]$, $\widehat{g}_A : A[\widehat{M], g^{-1}}(t) \to A[[t]]$ be the corresponding formal completions, then

$$\begin{array}{ccc} RM(A) \times RM(A) & \to & AutM(A) \\ (f, g) & \mapsto & (\widehat{f}_A)^{-1}\widehat{g}_A \end{array}$$

is the quotient map. Because source and target prolongations commute with Γ they provide prolongations on $AutM$ still denoted by R^s and R^t.

One can find RM from $AutM$ by choosing a closed point $p \in M$ and a formal $\mathbb{C}$-frame at p, $r : (\mathbb{C}^d, 0) \to (M, p)$. The part $Aut_{(p,M)}$ of $AutM$ above $\{p\} \times M$ for the (source,target) projection can be identified with RM by means of r. Two such isomorphisms are related by the action of Γ. All these maps are compatible with the filtrations of rings.

This groupoid has a differential structure coming from the differential structure of RM. Any section $r : M \to RM$ of the projection (a moving frame on M) can be used to trivialize some bundles:

- $TM = M \times \mathfrak{h}$ by $(p, v) \mapsto (p, T_p\widehat{r(p)}^{-1} v)$,
- $M \times RM = AutM$ by $(p, f) \mapsto f^{-1} \circ r(p)$,

and one gets

$$\mathbb{C}[AutM] \to \mathbb{C}[RM] \otimes \mathbb{C}[M] \overset{D \otimes 1}{\to} \mathbb{C}[RM] \otimes \mathfrak{h}^* \otimes \mathbb{C}[M] \to \mathbb{C}[RM] \otimes \Omega^1(M) \to \mathbb{C}[AutM] \underset{s^*\mathbb{C}[M]}{\otimes} \Omega^1(M)$$

giving the differential structure D of the ring $\mathbb{C}[AutM]$. Because gauge transformations act transitively on moving frames, this definition is independent of r.

2.3 Subgroupoids and pseudogroups

There are two possibilities to define subgroupoids of $Aut(M)$. The first one is to consider algebraic subvarieties of $Aut(M)$ (projective limits of subvarieties of $Aut_k(M)$) such that restrictions of source, target, composition, identity and inverse maps give a groupoid structure. The second one is to take limit of groupoids i.e. $G = \lim\limits_{\leftarrow} G_k$ with G_k a subvariety of $Aut_k(M)$ and a subgroupoid for k large enough.

These objects are too smooth for our purpose and for this reason we introduce singular subgroupoids.

Definition 2.1 *A singular (sub)groupoid of $AutM$ is an algebraic subvariety of $AutM$: $G = \lim\limits_{\leftarrow} G_k$ with G_k a subvariety of $Aut_k(M)$ and a subgroupoid of $Aut_k(M-S)$ for k large enough and S a closed subvariety of M independent of k.*

From the differential structure one gets also the notion of differential subvarieties. A D-variety (for short) is an algebraic subvariety defined by a differential ideal I (i.e. such that $DI \subset I \otimes \Omega^1(M)$).

Definition 2.2 *A subpseudogroup G of $AutM$ is a singular algebraic groupoid given by a differential ideal (they are called D-groupoids in [21]).*

2.4 The Lie algebroid eM

The Lie algebroid of $RM \times RM$ is the relative tangent bundle for the source projection above the diagonal. It is a vector bundle over RM canonically isomorphic to its tangent TRM.

The pull back of the usual Lie bracket of vector fields on the sections of TRM defines a Lie bracket on the sections of eM. It is compatible with the graduation i.e. it induces a bracket on the projectable vector fields on TR_kM. It is obvious that such objects satisfy the definition from [20]; we will only use concrete examples.

A Lie subalgebroid of eM is a linear subspace of eM in the sense of [15] stable by the bracket. The relative tangent bundle for the source projection of a singular subgroupoid of EM is a Lie algebroid.

Example 2.3 *A foliation on M defined by $F \subset TM$ gives by prolongation of sections a foliation $RF \subset TRM$ stable under Lie bracket. It is a Lie subgroupoid.*

Example 2.4 *An equivalence relation $E \subset M \times M$ gives by prolongation of sections an equivalence relation on EM i.e. a subgroupoid. Its Lie algebroid is the Lie algebroid defined by the foliation of M by equivalence classes of E.*

Remark 2.5 *If F has no first integrals, it cannot be the foliation by equivalence classes of an equivalence relation thus it is not the Lie algebroid of an algebraic groupoid (even singular).*

2.5 The Lie algebroid $autM$

This Lie algebroid and its Lie subalgebroids are defined from the previous one by taking quotients.

Because the identity $e : M \to AutM$ is a differential morphism for differential structures d on $\mathbb{C}[M]$ and D on $\mathbb{C}[AutM]$, the tangent bundle along e inherits a differential structure. Furthermore because these structures are compatible with source projection, the relative tangent bundle $autM$ has also a differential structure i.e. it is a $\mathbb{C}[M]$-module with connection. Lie algebroids of pseudogroups are Lie algebroids with a connection above the exterior derivative on $\mathbb{C}[M]$.

Let $G \subset AutM$ be a pseudogroup, $p \in M$ be a closed point not in the singular locus S of G and $G_{(p,M)} = s^{-1}(p) \cap G$. By choosing a formal frame $r : (\mathbb{C}^d, 0) \to (M, p)$ at p, one can identify $G_{(p,M)}$ with a subvariety of RM. By groupoid laws, this subvariety is a $G_{(p,p)}$-principal bundle over $M - S$ with $G_{(p,p)} = s^{-1}(p) \cap t^{-1}(p) \cap G$. This defines a Lie subalgebra $\mathfrak{g}$ of χ. By choosing another frame at p, one gets another Lie subalgebra conjugated to the previous one under the action of Γ. This $\mathfrak{g}$ will be called 'the' Lie algebra of G at p.

2.6 Some properties of pseudogroups

A pseudogroup G is usually not an algebraic groupoid but only out of a singular locus. Nevertheless it defines a set theoretical groupoid.

Theorem 2.6 *Let φ_1, φ_2 be two composable formal maps in neighborhood of two $\mathbb{C}$-points and let G be a pseudogroup. If φ_1 and φ_2 are in G so is $\varphi_1 \circ \varphi_2$.*

The proof of this theorem can be found in [8, theorem 1.5]. The main ingredient of the proof is Artin approximation theorem (already used in a special case in [26]).

A consequence of this theorem is that $G_{(p,p)}$ is a group for any point p and $G_{(p,M)}$ is a $G_{(p,p)}$ principal bundle over $t(G_{(p,M)}) \subset M$. The proof of the previous theorem has also the following consequence proved in [8, lemma 3.3].

Lemma 2.7 *If the Lie algebra of a pseudogroup at a generic point is solvable with a sequence of derived algebras of length N, so is its Lie algebra at any point.*

3 Malgrange's pseudogroup

3.1 Definition

A set theoretical pseudogroup on M is a set of analytic invertible maps $\varphi : U \to V$ between analytic open sets of M stable by composition (when defined), inversion, restriction, analytic continuation (under the invertibility condition).

Let $\Phi : M \dashrightarrow M$ be a rational dominant map. The pseudogroup $PG\Phi$ is the set theoretical pseudogroup generated by the restrictions of Φ to any open subset where it is invertible. Such a set of invertible maps describes a subset of $AutM$ by taking all Taylor expansions of every map at every point where it is defined. This subset of $AutM$ is still denoted by $PG\Phi$.

Definition 3.1 *Let M be a smooth complex algebraic variety and let $\Phi : M \dashrightarrow M$ be a rational dominant map. The Malgrange pseudogroup $Mal\Phi$ of Φ is the Zariski closure in $AutM$ of the set theoretical pseudogroup $PG\Phi$ generated by Φ.*

We have to prove that such an object is an algebraic pseudogroup.

Proof. – Let Z be the Zariski closure of $PG\Phi$ in $Aut(M)$. Because $Aut(M)$ acts on itself by target composition, one can look at the stabilizer of Z. It is proved in [5] that it is a singular groupoid included in Z. Let $R^t\Phi$ be the action of Φ on $Aut(M)$. It is a dominant map of order 0 from $Aut(M)$ to $Aut(M)$. Because $R^t\Phi(PG\Phi) = PG\Phi$, $R^t\Phi^{-1}(Z) \supset Z$ by minimality $Z = \overline{R^t\Phi(Z)}$. Thus formal invertible germs of Φ are in the stabilizer of Z, by minimality Z is equal to its stabilizer and Z is a singular groupoid.

Let $\varphi \in PG\Phi$ be a map defined on an open set U. By its pointwise Taylor extension φ is an analytic section of the source projection $\varphi : U \to AutM$. This section is differential for the differential structure d of $\mathcal{O}(U)$ and D of $\mathbb{C}[AutM]$ i.e. the morphism $\varphi^\# : \mathbb{C}[AutM] \to \mathcal{O}(U)$ is differential. Let I be the ideal of Z. Then $E \in I$ can be written $\varphi^\# E = 0$ for all $\varphi \in PG\Phi$, this implies $\varphi^\# DE = 0$ for all $\varphi \in PG\Phi$ and I is differential. □

Theorem 3.2 *Let M and N be two smooth complex algebraic varieties endowed with rational dominant maps $\Phi : M \dashrightarrow M$ and $\Psi : N \dashrightarrow N$. A rational dominant morphism $\pi : N \dashrightarrow M$ is a difference morphism if $\pi \circ \Psi = \Phi \circ \pi$. In this situation, one gets a rational dominant groupoid morphism*

$$\pi_\star Mal\Psi \dashrightarrow Mal\Phi.$$

Proof. – One defines the algebraic pseudogroup $Aut(\mathcal{F}_\pi)$ as the singular groupoid whose points are the formal invertible π-projectable maps :

$$Aut(\mathcal{F}_\pi) = \{\varphi \in Aut(M) \mid \exists \pi_\star\varphi \in Aut(N) \ s.t.\ \pi \circ \varphi = (\pi_\star\varphi) \circ \pi\}$$

The projection of such a map induces a map

$$\pi_\star : Aut(\mathcal{F}_\pi) \dashrightarrow AutM.$$

The map Ψ preserves the foliation given by level sets of π thus $Mal\Psi \subset Aut(\mathcal{F}_\pi)$. Let $\pi_\star Mal\Psi$ be the Zariski closure of the image of $Mal\Psi$. Because Ψ is π-projectable on Φ, this is also true for $PG\Psi$ on $PG\Phi$ and $Mal\Phi \subset \pi_\star Mal\Psi$. But $\pi_\star^{-1} Mal\Phi$ is an algebraic pseudogroup containing any map π-projectable on a map in $Mal\Phi$ thus $Mal\Psi \subset \pi_\star^{-1} Mal\Phi$. The theorem follows from this inclusion by applying $\pi_\star$. □

Remark 3.3 *In usual Galois theory this theorem just states that if $\mathbb{Q} \subset L \subset M$ is a tower of number fields such that $M/\mathbb{Q}$ and $L/\mathbb{Q}$ are galoisian extensions then the small Galois group is a quotient of the big one.*

3.2 Examples

q-Liouvillian functions

Definition 3.4 (Franke [12]) *Let $(\mathbb{C}(t), \sigma)$ be the difference field of rational functions with difference operator σ a Moebius transformation. A difference extension $(K, \overline{\sigma})$ of $(\mathbb{C}(t), \sigma)$ is said to be σ-Liouvillian if one can find a tower of differential extensions*

$$(\mathbb{C}(t), \sigma) = (K_0, \sigma_0) \subset (K_1, \sigma_1) \ldots \subset (K_n, \sigma_n) = (K, \overline{\sigma})$$

such that, for all $i \in \{1, ..., n\}$, the extension $K_{i-1} \subset K_i$ is either

- *algebraic,*
- *or additive in the sense that there exist $n_i \in \mathbb{N}$ and $z_i \in K_i$ such that $\sigma_i^{n_i} z_i - z_i \in K_{i-1}$ and $K_i = K_{i-1}(z_i, \ldots, \sigma_i^{n_i - 1} z_i)$,*
- *or multiplicative in the sense that there exist $n_i \in \mathbb{N}$ $z_i \in K_i$ such that $\frac{\sigma_i^{n_i} z_i}{z_i} \in K_{i-1}$ and $K_i = K_{i-1}(z_i \ldots, \sigma_i^{n_i - 1} z_i)$.*

The σ-Liouvillian functions are elements of σ-Liouvillian extensions.

Assume that K is Liouvillian and that the transcendence degree of K over $\mathbb{C}$ is $d = 1 + \sum_i n_i$. Let N be a model for a field L i.e. $\mathbb{C}(N) = L$. Because this field is a difference field, N is endowed with a rational map Ψ.

Proposition 3.5 *The Lie algebra of the Malgrange pseudogroup of Ψ at $p \in N$ is solvable.*

Proof. – Let m be the smallest common multiple of n_i's and $z_0 = t, z_1, \ldots, z_d$ be a transcendence basis in the partial order given by the definition. We have

$$\overline{\sigma}^m(z_i) = a_i(z_0, \ldots, z_{i-1}) + z_i$$

in the additive case and

$$\overline{\sigma}^m(z_i) = b_i(z_0, \ldots, z_{i-1}) z_i$$

in the multiplicative case.

By lemma 2.7, it is enough to prove the proposition at a generic point. There exist some special rational differential 1-forms $\theta_0, \ldots, \theta_d$ on W satisfying, for all $0 \leq i \leq d$, $(\Psi^m)^*\theta_i = \theta_i$ mod $(\theta_0, \ldots, \theta_{i-1})$. The construction of these forms is direct from the definition of the σ-Liouvillian extensions. The forms are $\theta_0 = dt = dz_0$ and, for all $1 \leq i \leq d$, $\theta_i = dz_i$ in the additive case or $\theta_i = \frac{dz_i}{z_i}$ in the multiplicative case.

The equations $(\Psi^m)^*\theta_i = \theta_i \mod (\theta_0, \ldots, \theta_{i-1})$ for $0 \leq i \leq d$ are a synthetic way of writing infinitely many algebraic equations satisfied by elements of $PG(\Psi^m)$. The Lie algebroid of $Mal(\Psi^m)$ at a point p must be included in the solutions of the linearized equations : the vector fields Y in the Lie algebroid must satisfied $\mathcal{L}_Y\theta_i = 0 \mod (\theta_0, \ldots, \theta_{i-1})$. Let p be a generic point on N and $t_1, ..., t_d$ be local analytic coordinates such that $dt_i = \theta_i$. The vector field Y can be written

$$Y = c_0\frac{\partial}{\partial t_0} + c_1(t_0)\frac{\partial}{\partial t_1} + c_2(t_0, t_1)\frac{\partial}{\partial t_2} + \cdots + c_d(t_0, \cdots, t_{d-1})\frac{\partial}{\partial t_d}.$$

The $(d + 1)$th derived algebra of this type of Lie algebra of formal vector fields is trivial.

The theorem is proved for Ψ^m. To prove it for Ψ, remark that equations of invariance of differential invariants for Φ^m give partial differential equation satisfied by Φ and all its iterated and inverse branches. The lilnearization of such equations along the identity is the linearisation of equation of invariance we started with multiplied by m. Then at the Lie algebra level $mal(\Psi) \subset mal(\Psi^m)$. The other inclusion is easy to prove to get equality. This proves the theorem. □

Rational systems integrable by σ-quadratures

Definition 3.6 *Let*

$$\begin{cases} \sigma y_1 = E_1(t, y_1, y_2, \ldots, y_m) \\ \vdots \\ \sigma y_m = E_m(t, y_1, y_2, \ldots, y_m) \end{cases}$$

be a rank m system of rational σ-difference equations. This system is said to be integrable by σ-quadratures if there is a σ-Liouvillian solution $(f_1, \ldots, f_m)$ such that $\mathbb{C}(t, f_1, \ldots, f_m)$ is σ-isomorphic to $\{\mathbb{C}(t, y_1, \ldots, y_m)\}_\sigma/I$ where $\{\cdot\}_\sigma$ is the σ-ring generated by $\cdot$ and I is the σ-ideal generated by the equations of the system.

One defines the Malgrange pseudogroup of E as that of

$$\Phi : \begin{array}{ccc} \mathbb{C}^{m+1} & \dashrightarrow & \mathbb{C}^{m+1} \\ (t, y_1, \ldots, y_m) & \mapsto & (\sigma(t), E_1, \ldots, E_m) \end{array}.$$

A consequence of proposition 3.5, theorem 3.2 and lemma 2.7 is

Theorem 3.7 *If a rational ordinary σ-difference system is integrable by σ-quadratures then the Lie algebra of its Malgrange pseudogroup is solvable.*

4 Galois theory for linear q-difference equations

In this section we collect some results concerning the Galois theory of linear q-difference equations which will be used in the next sections. We set $\sigma t = \sigma_q t = qt$ with $q \in \mathbb{C}^*$ such that $|q| > 1$. Two rather different approaches for q-difference Galois theory will be used.

4.1 Picard-Vessiot theory

Let G be a complex linear algebraic group. Let $E \xrightarrow{\pi} \mathbb{P}^1$ be a principal G-bundle i.e. $E \underset{\mathbb{P}^1}{\times} E \sim E \times G$ over E for the first projection. For a π-projectable G-invariant rational dominant map $V : E \dashrightarrow E$ such that $\sigma = \pi_* V$ is $t \mapsto qt$ for some $q \in \mathbb{C}^*$ with $|q| > 1$ or $t \to t + 1$, PV denotes a closed minimal V-invariant subvariety of E dominating $\mathbb{P}^1$ and $GalV$ its stabilizer in G.

- Two such PV are isomorphic under action of G and called Picard-Vessiot varieties of V. The ring extension $\mathbb{C}[\pi(PV)] \subset \mathbb{C}[PV]$ is usually called a Picard-Vessiot extension for V.
- The group $GalV$ is well defined up to conjugation in G. It is the Galois group of V.

- Common level sets of all invariants of V in $\mathbb{C}(E)$ dominating $\mathbb{P}^1$ are Picard-Vessiot varieties.

Up to some modification of the bundle over $\infty \in \mathbb{P}^1$, we can assume that the bundle is trivial : $E = \mathbb{P}^1 \times G$. If $G = GL_n(\mathbb{C})$, one gets $V(t,g) = (\sigma t, A(t)g)$ with $A(t) \in GL_n(\mathbb{C}(t))$. The equations of invariants curves $g = g(t)$ are linear σ-difference systems in fundamental form $g(\sigma t) = A(t)g(t)$. By looking at the first column, one gets the usual vectorial form of linear σ-difference systems : $Y(\sigma t) = A(t)Y(t)$. The construction given here of the Galois group follows [38]. In this linear situation V will stand either for the system in fundamental form or for the system in vectorial form.

4.2 Tannakian approach

Let $\mathcal{D}_q = \mathbb{C}(t)\langle \boldsymbol{\sigma}_q, \boldsymbol{\sigma}_q^{-1}\rangle$ be the non commutative algebra of non commutative polynomials with coefficients in $\mathbb{C}(t)$ satisfying to the relation $\boldsymbol{\sigma}_q f = (\sigma_q f)\boldsymbol{\sigma}_q$ for any $f \in \mathbb{C}(t)$. We denote by $\mathcal{F}$ the neutral Tannakian category over $\mathbb{C}$ of q-difference modules over $\mathbb{C}(t)$: it is the full subcategory of the category of left $\mathcal{D}_q$-modules whose objects are the left $\mathcal{D}_q$-modules which are finite dimensional as $\mathbb{C}(t)$-vector spaces. The objects of $\mathcal{F}$ can be interpreted as pairs (V, Φ) where V is a finite dimensional $\mathbb{C}(t)$-vector space V and where Φ is a σ_q-linear automorphism of V; from this point of view, the morphisms from an object (V, Φ) to an object (V', Φ') are the $\mathbb{C}(t)$-linear maps $F : V \to V'$ such that $F\Phi = \Phi' F'$.

We define similar categories $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(\infty)}$ by replacing the field $\mathbb{C}(t)$ by $\mathbb{C}(\{t\})$ and $\mathbb{C}(\{t^{-1}\})$ respectively.

We have natural localization functors $\mathcal{F} \rightsquigarrow \mathcal{F}^{(0)}$ and $\mathcal{F} \rightsquigarrow \mathcal{F}^{(\infty)}$.

A q-difference module over $\mathbb{C}(t)$ is regular singular both at 0 and at ∞ if its localization at 0 has a lattice over $\mathbb{C}\{t\}$ and its localization at ∞ has a lattice over $\mathbb{C}\{t^{-1}\}$ which are invariant under the action of $\boldsymbol{\sigma}_q$ and $\boldsymbol{\sigma}_q^{-1}$. We denote by $\mathcal{E}$ the full subcategory of $\mathcal{F}$ made of its regular singular objects; it is a neutral Tannakian subcategory of $\mathcal{F}$.

For details on what precedes, we refer to [38, 36].

For the general theory of Tannakian categories, we refer to [9]. Let ω be a complex valued fiber functor over $\mathcal{F}$. The Galois group of $\mathcal{F}$ is by definition the complex proalgebraic group $\pi_1^{q-diff} = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ and the Galois group of an object M of $\mathcal{F}$ is the the complex linear algebraic group $Gal(M) = \underline{\mathrm{Aut}}^{\otimes}(\omega|\langle M\rangle)$ where $\langle M\rangle$ denotes the Tannakian subcategory of $\mathcal{F}$ generated by M (it is the full subcategory of $\mathcal{F}$ whose objects are obtained form M by combining the following operations : tensor products $\otimes$, direct sums $\oplus$, duals $\overset{\vee}{\cdot}$, quotients, subobjects).

By Tannakian duality ([9]) ω induces an equivalence of tensor categories between $\mathcal{F}$ and the rational finite dimensional linear representations of π_1^{q-diff}; similarly, for any object M of $\mathcal{F}$, ω induces an equivalence of tensor categories between $\langle M\rangle$ and the rational finite dimensional linear representations of $Gal(M)$.

Let M be an object of $\mathcal{F}$ and let $\rho_M : \pi_1^{q-diff} \to GL(\omega(M))$ be the representation corresponding to M. We can identify $Gal(M)$ with the image of ρ_M $(\subset GL(\omega(M)))$.

We give now some properties to be used later in this article.

Proposition 4.1 *An object M of $\mathcal{F}$ is simple if and only if the corresponding rational representation ρ_M is irreducible if and only if $Gal(M)$ acts irreducibly on $\omega(M)$.*

Proof. – Straightforward by Tannakian duality. □

Proposition 4.2 *Let M be an object of $\mathcal{F}$. The determinant of M is trivial if and only if $Gal(M)$ belongs to $SL(\omega(M))$.*

Proof. – Let n be the rank of M. Let ρ_M be the rational linear representation corresponding to M. We have to prove that the determinant $\Lambda^n M$ of M is trivial if and only if the determinant $\Lambda^n \rho_M$ of ρ_M is trivial. This is indeed the case because, by Tannakian duality, the representation corresponding to $\Lambda^n M$ is $\Lambda^n \rho_M$ so $\Lambda^n M$ is trivial if and only if $\Lambda^n \rho_M$ is trivial. □

Proposition 4.3 *If an object M of $\mathcal{F}$ has a virtually solvable Galois group then any object of $\langle M\rangle$ has a virtually solvable Galois group; in particular, any subobject of M has a virtually solvable Galois group.*

Proof. – Let N be an object of $\langle M\rangle$ and let ρ_N be the rational linear representation of $Gal(M)$ corresponding to N. Since $Gal(M)^\circ$ is solvable its image by the rational linear representation ρ_N, which is $Gal(N)^\circ$, is also solvable. □

Let us now consider a complex valued fiber functor $\omega^{(0)}$ over $\mathcal{F}^{(0)}$ and take for ω the complex valued fiber functor obtained by composing $\omega^{(0)}$ with the exact, faithful and tensor localization functor $\mathcal{F} \rightsquigarrow \mathcal{F}^{(0)}$. The local Galois group at 0 of an object M of $\mathcal{F}$ is the complex linear algebraic group $G_{loc,0}(M) = \underline{\mathrm{Aut}}^{\otimes}(\omega^{(0)}|\langle M^{(0)}\rangle)$ (where $M^{(0)}$ denotes the localization of M at 0) which can be viewed, as above, as a subgroup of $GL(\omega^{(0)}(M^{(0)})) = GL(\omega(M))$.

The localization functor $\mathcal{F} \rightsquigarrow \mathcal{F}^{(0)}$ induces, for any object of M of $\mathcal{F}$, a closed immersion (this is a consequence of Proposition 2.21. in [9]) :

$$G_{loc,0}(M) \quad \hookrightarrow \quad Gal(M).$$

The following result is proved in [34]. Compare with Gabber connectedness criterion (Proposition 1.2.5 in [18]).

Theorem 4.4 *Let M be an object of $\mathcal{F}$. Then we have a natural surjective morphism :*

$$G_{loc,0}(M)/G_{loc,0}(M)^\circ \twoheadrightarrow Gal(M)/Gal(M)^\circ.$$

In particular, if $G_{loc,0}(M)$ is connected then $Gal(M)$ is connected.

The interest of this theorem is that $G_{loc,0}(M)$ is easy to describe if M is an object of $\mathcal{E}$. The following corollary will allow us to greatly simplify the calculations of Galois groups in sections 6 and 7.

Corollary 4.5 *Let M be an object of $\mathcal{E}$. Assume that with respect to some basis the action of $\boldsymbol{\sigma}_q$ on the localization $M^{(0)}$ of M at 0 is given by a matrix $A \in GL_n(\mathbb{C}\{t\})$ such that the eigenvalues of $A(0)$ belong to $q^{\mathbb{Z}}$ then $Gal(M)$ is connected.*

Proof. – Theorem 4.4 ensures that it is sufficient to prove that $G_{loc,0}(M)$ is connected. This is indeed the case, because, in virtue of Theorem 2.2.3.5. in [36], $G_{loc,0}(M)$ is generated, as a complex algebraic group, by a unipotent morphism. □

In concrete examples, we will work with q-difference systems. Let M be an object of $\mathcal{F}$. Choosing a $\mathbb{C}(t)$-basis of M, we can interpret M as the q-difference system $\sigma_q Y = AY$ where A is the inverse of the matrix representing the action of $\boldsymbol{\sigma}_q$ on M with respect to the given basis. We will also work with associated q-difference operators. Concretely, let Φ_A be the σ_q-linear operator on the n-dimensional $\mathbb{C}(t)$-vector space $V = \mathbb{C}(t)^n$ given by $\Phi_A(X) = A^{-1}\sigma_q X$. We will exhibit $e \in V$ such that $(e, \Phi_A(e), ..., \Phi_A^{n-1}(e))$ is a basis over $\mathbb{C}(t)$ of V and we will work with $P(\boldsymbol{\sigma}_q) \in \mathcal{D}_q$ where $P \in \mathbb{C}(t)[X]$ is the unique unitary polynomial of degree $n-1$ such that $P(\Phi_A)e = 0$; such a e is called a cyclic vector and the theoretical existence of cyclic vector is ensured by the so-called cyclic vector Lemma ([40, 37]).

5 A linearization theorem

5.1 Variational equations

Let $\Phi : M \dashrightarrow M$ be a dominant rational map and let $\mathscr{C}$ be an algebraic rational Φ-invariant curve with $\Phi|_{\mathscr{C}}$ being either $t \mapsto qt$ for some $q \in \mathbb{C}^*$ with $|q| > 1$ or $t \mapsto t+1$. Prolongations of Φ are dominant rational maps $R_k\Phi$ on frame bundles R_kM. The restriction of the frame bundles over $\mathscr{C}$ are Γ_k-principal bundles over $\mathscr{C}$ which have a projectable Γ_k-invariant map given by the restriction of $R_k\Phi$ over $\mathscr{C}$. This is the order k variational equation in fundamental form. Because $R_{k+1}\Phi$ is π_k^{k+1}-projectable on $R_k\Phi$, this is also true for Galois groups. We have surjective morphisms

$$Gal(R_{k+1}\Phi|_{\mathscr{C}}) \twoheadrightarrow Gal(R_k\Phi|_{\mathscr{C}}).$$

We set

$$Gal(R\Phi|_{\mathscr{C}}) = \lim_{\leftarrow} Gal(R_k\Phi|_{\mathscr{C}}).$$

Theorem 5.1 *Let p be a generic point on $\mathscr{C}$. We have*

$$Gal(R\Phi|_{\mathscr{C}}) \subset Mal\Phi_{(p,p)}.$$

Proof. – By choosing a formal chart at $p \in \mathscr{C}$, the Γ-principal bundle $RM|_{\mathscr{C}}$ is isomorphic to the subspace of $AutM_{(p,\mathscr{C})}$ with source $p \in \mathscr{C}$ and target in $\mathscr{C}$. Under this identification

- $AutM_{(p,p)}$ is Γ,
- $Gal(R\Phi|_{\mathscr{C}})$ is a subgroup acting by left translation.

The closed subvariety $Mal\Phi_{(p,\mathscr{C})}$ with source a and target in $\mathscr{C}$ of $AutM_{(p,\mathscr{C})}$ is

- $R\Phi$-invariant because $Mal\Phi$ and $\mathscr{C}$ are $R\Phi$-invariants,
- dominates $\mathscr{C}$ because its projection contains the orbits of p by Φ in $\mathscr{C}$.

This implies that $Gal(R\Phi|_{\mathscr{C}}) \subset Stab(Mal\Phi_{(p,\mathscr{C})})$ by source composition thus $Gal(R\Phi|_{\mathscr{C}}) \subset Mal\Phi_{(p,p)}$. □

5.2 Main theorem

The main theorem is now a consequence of theorem 5.1 and theorem 3.7.

Theorem 5.2 *If Φ is integrable by N q-quadratures then $Gal(R_k\Phi|_{\mathscr{C}})$ is solvable of length N.*

The order 1 variational extension is the only important one for the solvability condition because $Gal(R_k\Phi|_{\mathscr{C}})$ is a commutative extension of $Gal(R_{k-1}\Phi|_{\mathscr{C}})$.

6 Nonintegrability of a discrete Painlevé I equation

The following system of non linear q-difference equations (qPA_7' in Sakai's classification [35]) is a q-analogue of Painlevé I equation :

$$\begin{cases} y(qx) &= \dfrac{1-xz(x)}{xy(x)(z(x)-1)}z(x) \\[2ex] z(qx) &= \left(\dfrac{1-xz(x)}{xy(x)(z(x)-1)}\right)^2 z(x). \end{cases}$$

The corresponding dynamical system is :

$$\Phi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} qx \\ \dfrac{1-xz}{xy(z-1)}z \\ \left(\dfrac{1-xz}{xy(z-1)}\right)^2 z \end{bmatrix}.$$

Theorem 6.1 *The qPA_7' Painlevé system is non integrable by q-quadratures.*

Proof. – It is a consequence of sections 6.1 and 6.2 below and of Theorem 5.2. □

Note that a direct proof of nonintegrability of qPA_7' can be found in [28] where the stronger statement of irreducibility of nonalgebraic solutions is proved.

6.1 Invariant curve and discrete variational equation

Let q_4 be a 4th root of q in $\mathbb{C}$. A straightforward calculation shows that Φ leaves globally invariant the curve parameterized by :

$$\varphi : \quad t \quad \mapsto \quad \begin{bmatrix} t^2 \\ q_4/t \\ 1/t \end{bmatrix}$$

and that Φ acts on the variable t as σ_{q_2}, where $q_2 = q_4^2$, in the sense that :

$$\Phi \circ \varphi = \varphi \circ \sigma_{q_2}.$$

This leads to the following discrete variational equation :

$$Y_1(q_2 t) = D\Phi(\varphi(t))(Y_1(t)) = \begin{bmatrix} q & 0 & 0 \\ \dfrac{-1}{q_4 t^3(1-t)} & \dfrac{-1}{q_2} & \frac{-2t}{q_4(1-t)} \\ \dfrac{-2}{q_2 t^3(1-t)} & \dfrac{-2}{q_4^3} & \frac{-3t-1}{q_2(1-t)} \end{bmatrix} Y_1(t) \tag{1}$$

6.2 Non virtual solvability of the Galois group of the discrete variational equation

In this section we shall prove that the q_2-difference system (1) has a non virtually solvable Galois group. Proposition 4.3 shows that it is sufficient to prove that the following subsystem of (1) has a non virtually solvable Galois group :

$$Y(q_2 t) = A(t)Y(t), \quad A(t) = \begin{bmatrix} \frac{-1}{q_2} & \frac{-2t}{q_4(1-t)} \\ \frac{-2}{q_4^3} & \frac{-3t-1}{q_2(1-t)} \end{bmatrix} \in GL_2(\mathbb{C}(t)). \tag{2}$$

This q_2-difference system is actually a basic hypergeometric equation in disguise whose Galois group was computed in [33].

Let us recall that the q_2-hypergeometric operator with parameters $(a, b; c, d) \in (\mathbb{C}^*)^4$ is given by :

$$\begin{aligned} &\left(\frac{c}{q_2}\boldsymbol{\sigma}_{q_2} - 1\right)\left(\frac{d}{q_2}\boldsymbol{\sigma}_{q_2} - 1\right) - t(a\boldsymbol{\sigma}_{q_2} - 1)(b\boldsymbol{\sigma}_{q_2} - 1) \\ = &\left(\frac{cd}{q_2^2} - tab\right)\boldsymbol{\sigma}_{q_2}^2 + \left(-\frac{c+d}{q_2} + t(a+b)\right)\boldsymbol{\sigma}_{q_2} + (1-t). \end{aligned}$$

Coming back to our concrete situation, we claim that $e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2 \subset \mathbb{C}(t)^2$ is a cyclic vector for (2) and that the corresponding q_2-difference operator is q_2-hypergeometric.

We have

$$A^{-1} = \begin{bmatrix} -q_2\frac{3t+1}{(1-t)} & \frac{2q_4^3 t}{(1-t)} \\ 2q_4 & -q_2 \end{bmatrix}.$$

Hence

$$\Phi_A(e) = \begin{bmatrix} \frac{2q_4^3 t}{(1-t)} \\ -q_2 \end{bmatrix}.$$

Consequently, e is a cyclic vector and we have

$$\begin{array}{rcl}
\Phi_A^2(e) & = & A^{-1}\begin{bmatrix} \frac{2q_4^5t}{(1-q_2t)} \\ -q_2 \end{bmatrix} \\
& = & \begin{bmatrix} -q_2\frac{3t+1}{(1-t)}\frac{2q_4^5t}{(1-q_2t)}+\frac{2q_4^3t}{(1-t)}(-q_2) \\ 2q_4\frac{2q_4^5t}{(1-q_2t)}+(-q_2)(-q_2) \end{bmatrix} \\
& = & \begin{bmatrix} \frac{2q_4^3t}{(1-t)}(-\frac{q(3t+1)}{(1-q_2t)}-q_2) \\ \frac{4q_2^3t}{(1-q_2t)}+q \end{bmatrix} \\
& = & \begin{bmatrix} \frac{2q_4^3t}{(1-t)}(-\frac{q(3t+1)}{(1-q_2t)}-q_2) \\ -q_2(-\frac{q(3t+1)}{(1-q_2t)}-q_2) \end{bmatrix} + \begin{bmatrix} 0 \\ q_2(-\frac{q(3t+1)}{(1-q_2t)}-q_2)+\frac{4q_2^3t}{(1-q_2t)}+q \end{bmatrix} \\
& = & -(\dfrac{2qt+(q+q_2)}{(1-q_2t)})\Phi_A(e)-\dfrac{q_2^3(1-t)}{(1-q_2t)}e.
\end{array}$$

So we get the q_2-difference operator $\boldsymbol{\sigma}_{q_2}^2+\frac{2qt+(q+q_2)}{(1-q_2t)}\boldsymbol{\sigma}_{q_2}+\frac{q_2^3(1-t)}{(1-q_2t)}$. Using the gauge transformation $y=ty$, we see that $\boldsymbol{\sigma}_{q_2}^2+\frac{2qt+(q+q_2)}{(1-q_2t)}\boldsymbol{\sigma}_{q_2}+\frac{q_2^3(1-t)}{(1-q_2t)}$ is equivalent to $\boldsymbol{\sigma}_{q_2}^2+\frac{2t+(1+q_2^{-1})}{(q_2^{-1}-t)}\boldsymbol{\sigma}_{q_2}+\frac{(1-t)}{(q_2^{-1}-t)}$. This is the q_2-hypergeometric operator with parameters $(a,b;c,d)=(1,1;-q_2,-1)$. It is proved in [33] that its Galois group is $SL_2(\mathbb{C})$ (to be precise, in [33] the Galois group computed is that of the operator obtained by permuting (a,b) and (c,d) in the above operator but this is of course inoffensive).

7 Nonintegrability of a discrete Painlevé III equation

The following system of non linear q-difference equations is a q-analogue of Painlevé III equation :

$$\left\{\begin{array}{rcl} y(qx) & = & \dfrac{1}{y(x)z(x)}\dfrac{1+a_0xz(x)}{a_0x+z(x)} \\ \\ z(q^{-1}x) & = & \dfrac{1}{y(x)z(x)}\dfrac{qa_1x^{-1}+y(x)}{1+qa_1x^{-1}y(x)} \end{array}\right. \tag{3}$$

where $a_0,a_1\in\mathbb{C}^*$. Note that (3) is equivalent to :

$$\left\{\begin{array}{rcl} y(qx) & = & \dfrac{1}{y(x)z(x)}\dfrac{1+a_0xz(x)}{a_0x+z(x)} \\ \\ z(qx) & = & \dfrac{1}{y(qx)z(x)}\dfrac{a_1x^{-1}+y(qx)}{1+a_1x^{-1}y(qx)} \\ & = & \dfrac{y(x)(a_0x+z(x))(x+a_0x^2z(x)+a_0a_1xy(x)z(x)+a_1y(x)z(x)^2)}{(1+a_0xz(x))(xy(x)z(x)^2+a_0x^2y(x)z(x)+a_0a_1xz(x)+a_1)}. \end{array}\right.$$

The corresponding discrete dynamical system is :

$$\Phi:\begin{bmatrix} x \\ y \\ z \end{bmatrix}\mapsto\begin{bmatrix} qx \\ \dfrac{1}{yz}\dfrac{1+a_0xz}{a_0x+z} \\ \dfrac{y(a_0x+z)(x+a_0x^2z+a_0a_1xyz+a_1yz^2)}{(1+a_0xz)(xyz^2+a_0x^2yz+a_0a_1xz+a_1)} \end{bmatrix}.$$

Theorem 7.1 *If $a_0a_1\notin -q^{-\mathbb{N}}$ then the qP_{III} equation (3) is non integrable by q-quadratures.*

Proof. – It is a consequence of sections 7.1 and 7.2 below and of Theorem 5.2. □

7.1 Invariant curve and discrete variational equation

A straightforward calculation shows that Φ leaves globally invariant the curve parameterized by :

$$\varphi: \quad t \quad \mapsto \quad \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix}$$

and that it acts on the variable t as σ_q in the sense that :

$$\Phi \circ \varphi = \varphi \circ \sigma_q.$$

The discrete variational equation of Φ along φ is easily seen to be given by :

$$Y_1(qt) = D\Phi(\varphi(t))(Y_1(t)) = \begin{bmatrix} q & 0 & 0 \\ 0 & -1 & \dfrac{2a_1}{t+a_1} \\ 0 & -\dfrac{2}{a_0t+1} & -\dfrac{a_0t^2+(1+a_0a_1)t-3a_1}{(a_0t+1)(t+a_1)} \end{bmatrix} Y_1(t) \tag{4}$$

7.2 Non virtual solvability of the Galois group of the discrete variational equation

In this section we shall prove that the q-difference system (4) has a non virtually solvable Galois group. In this purpose, Proposition 4.3 ensures that it is sufficient to prove that the following (regular singular) subsystem of (4) has a non virtually solvable Galois group :

$$Y(qt) = A(t)Y(t), \quad A(t) = \begin{bmatrix} -1 & \frac{2a_1}{t+a_1} \\ -\frac{2}{a_0t+1} & -\frac{a_0t^2+(1+a_0a_1)t-3a_1}{(a_0t+1)(t+a_1)} \end{bmatrix} \in GL_2(\mathbb{C}(t)). \tag{5}$$

We claim that the Galois group of (5) is $SL_2(\mathbb{C})$. This assertion will be a consequence of the following three observations.

First observation: G is connected – Indeed, a simple calculation shows that the complex eigenvalues of $A(0)$ belong to $q^{\mathbb{Z}}$. Corollary 4.5 ensures that G is connected.

Second observation: G acts irreducibly on $\mathbb{C}^2$ – Indeed, G acts irreducibly if and only if (see Proposition 4.1) (5) is irreducible if and only if some q-difference operator associated to (5) is irreducible over $\mathbb{C}(t)$. Let us now determine an explicit q-difference operator associated to (5).

We claim that $e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2 \subset \mathbb{C}(t)^2$ is a cyclic vector for (5). Indeed, we have:

$$A^{-1} = \begin{bmatrix} -\frac{a_0t^2+(1+a_0a_1)t-3a_1}{(a_0t+1)(t+a_1)} & -\frac{2a_1}{t+a_1} \\ \frac{2}{a_0t+1} & -1 \end{bmatrix},$$

so :

$$\Phi_A(e) = A^{-1}\sigma_q(e) = \begin{bmatrix} -\frac{2a_1}{t+a_1} \\ -1 \end{bmatrix}$$

is non $\mathbb{C}(t)$-colinear with e.

Moreover, we have:

$$\begin{aligned} \Phi_A^2(e) &= A^{-1} \begin{bmatrix} -\frac{2a_1}{qt+a_1} \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2a_1(a_0(1+q)t^2+(1+2a_0a_1+q)t-2a_1)}{(t+a_1)(qt+a_1)(a_0t+1)} \\ \frac{a_0qt^2+(a_0a_1+q)t-3a_1}{(qt+a_1)(a_0t+1)} \end{bmatrix} \\ &= -\frac{a_0(1+q)t^2+(1+2a_0a_1+q)t-2a_1}{(qt+a_1)(a_0t+1)}\Phi_A(e) - \frac{t+a_1}{qt+a_1}e. \end{aligned}$$

Hence:

$$\begin{aligned} L &= \boldsymbol{\sigma}_q^2 + \frac{a_0(1+q)t^2 + (1+2a_0a_1+q)t - 2a_1}{(qt+a_1)(a_0t+1)}\boldsymbol{\sigma}_q + \frac{t+a_1}{qt+a_1} \\ &= \boldsymbol{\sigma}_q^2 + \alpha(t)\boldsymbol{\sigma}_q + \beta(t) \end{aligned}$$

is a q-difference operator associated to (5). In order to prove that L is irreducible over $\mathbb{C}(t)$, we follow closely the method presented in [1]. Assume at the contrary that L is reducible over $\mathbb{C}(t)$ i.e. that there exist r and s in $\mathbb{C}(t)^{\times}$ such that :

$$L = (\sigma_q - s)(\sigma_q - r).$$

Then we have:

$$r(qt)r(t) + \alpha(t)r(t) + \beta(t) = 0. \tag{6}$$

According to [1], r can be decomposed as follows :

$$r = \lambda \frac{u(t)}{v(t)} \frac{c(qt)}{c(t)}$$

for some $\lambda \in \mathbb{C}^*$ and some unitary polynomials $u, v, c \in \mathbb{C}[t] \setminus \{0\}$ such that:

a) $\forall n \in \mathbb{N}$, $u(t) \wedge v(q^n t) = 1$;

b) $u(t) \wedge c(t) = 1$;

c) $v(t) \wedge c(qt) = 1$;

d) $c(0) \neq 0$.

Then equation (6) becomes:

$$\lambda^2 \frac{u(qt)u(t)}{v(qt)v(t)} \frac{c(q^2t)}{c(t)} + \lambda\alpha(t)\frac{u(t)}{v(t)}\frac{c(qt)}{c(t)} + \beta(t) = 0 \tag{7}$$

and, clearing the denominators, we get:

$$\begin{aligned} &\lambda^2(qt+a_1)(a_0t+1)u(qt)u(t)c(q^2t) \\ &+\lambda(a_0(1+q)t^2 + (1+2a_0a_1+q)t - 2a_1)u(t)v(qt)c(qt) \\ &+(a_0t+1)(t+a_1)v(t)v(qt)c(t) = 0. \end{aligned} \tag{8}$$

We see that u is a unitary polynomial dividing $(a_0t+1)(t+a_1)$ and that v is a unitary polynomial dividing $(t+a_1)(a_0q^{-1}t+1)$. Moreover, we claim that we necessary have $\deg(u) = \deg(v)$. Indeed, if $\deg(u) > \deg(v)$ (the case that $\deg(u) < \deg(v)$ is similar), then we would have the following inequalities:

$$\begin{aligned} & \deg(\lambda^2(qt+a_1)(a_0t+1)u(qt)u(t)c(q^2t)) \\ = \; & 2 + 2\deg(u) + \deg(c) \\ > \; & 2 + \deg(u) + \deg(v) + \deg(c) \\ = \; & \max\{\deg(\lambda(a_0(1+q)t^2 + (1+2a_0a_1+q)t - 2a_1)u(t)v(qt)c(qt)), \\ & \qquad \deg((a_0t+1)(t+a_1)v(t)v(qt)c(t))\} \\ \geq \; & \deg(-\lambda(a_0(1+q)t^2 + (1+2a_0a_1+q)t - 2a_1)u(t)v(qt)c(qt) \\ & \qquad -(a_0t+1)(t+a_1)v(t)v(qt)c(t)) \end{aligned}$$

contradicting (8). Hence, the only possibilities for (u,v) are $(1,1)$, $(t+a_0^{-1}, t+a_1)$, $(t+a_0^{-1}, t+qa_0^{-1})$, $(t+a_1, t+a_1)$, $(t+a_1, t+qa_0^{-1})$ and $((t+a_0^{-1})(t+a_1), (t+a_1)(t+qa_0^{-1}))$. Properties a) - d) listed above allow us to reduce the possibilities for (u,v) to $(1,1)$, $(t+a_0^{-1}, t+a_1)$ and $(t+a_1, t+qa_0^{-1})$. We now consider each case separately.

- Case $(u,v)=(1,1)$. Considering equation (7) at $t=\infty$, we obtain $\lambda^2 q^{2deg(c)}+\lambda q^{deg(c)}(1+\frac{1}{q})+\frac{1}{q}=0$ i.e. $\lambda=-q^{-deg(c)}$ or $-q^{-deg(c)-1}$. On the other hand, evaluating (7) at $t=0$ we obtain $\lambda^2-2\lambda+1=0$ i.e. $\lambda=1$. So -1 belongs to the $q^{-\mathbb{N}}$: contradiction.

- Case $(u,v)=(t+a_0^{-1},t+a_1)$. As above we have $\lambda=-q^{-deg(c)}$ or $-q^{-deg(c)-1}$. On the other hand, evaluating (7) at $t=0$ we obtain $\lambda^2(\frac{1}{a_0a_1})^2-2\lambda\frac{1}{a_0a_1}+1=0$ i.e. $\lambda=a_0a_1$. Hence a_0a_1 belongs to $-q^{-\mathbb{Z}}$: contradiction.

- Case $(u,v)=(t+a_1,t+qa_0^{-1})$. As above we have $\lambda=-q^{-deg(c)}$ or $-q^{-deg(c)-1}$. On the other hand, evaluating (7) at $t=0$ we obtain $\lambda^2(q^{-1}a_0a_1)^2-2\lambda q^{-1}a_0a_1+1=0$ i.e. $\lambda=\frac{1}{q^{-1}a_0a_1}$. Hence a_0a_1 belongs to $-q^{-\mathbb{Z}}$: contradiction.

Third observation: $G\subset SL_2(\mathbb{C})$. Indeed, this is a direct application of Proposition 4.2 since the determinant A is equal to 1.

Hence, G is a connected algebraic subgroup of $SL_2(\mathbb{C})$ acting irreducibly on $\mathbb{C}^2$. The only possibility is $G=SL_2(\mathbb{C})$.

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