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Integrability of Hamiltonian systems with homogeneous potentials of degree zero

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ABSTRACT

We derive necessary conditions for integrability in the Liouville sense of classical Hamiltonian systems with homogeneous potentials of degree zero. We obtain these conditions through an analysis of the differential Galois group of variational equations along a particular solution generated by a non-zero solution $\boldsymbol{d} \in \mathbb{C}^n$ of nonlinear equation $\operatorname{grad} V(\boldsymbol{d}) = \boldsymbol{d}$. We prove that when the system is integrable the Hessian matrix $V''(\boldsymbol{d})$ has only integer eigenvalues and is diagonalizable.

1. Introduction

In this Letter we consider classical Hamiltonian systems with n degrees of freedom for which the Hamiltonian function is of the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \tag{1.1}$$

where $\boldsymbol{q} := (q_1, \ldots, q_n)$, $\boldsymbol{p} := (p_1, \ldots, p_n)$ are the canonical coordinates, and V is a homogeneous function of degree $k \in \mathbb{Z}$. Although systems of this form arising from physics and applied sciences are generally understood to involve only real variables, we will assume (1.1) is defined on the complex symplectic manifold $M = \mathbb{C}^{2n}$ equipped with the canonical symplectic form

$$\omega = \sum_{i=1}^{n} \mathrm{d}q_i \wedge \mathrm{d}p_i.$$

Thus, Hamilton's equations have the canonical form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = \boldsymbol{p}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = -V'(\boldsymbol{q}), \tag{1.2}$$

where V' denotes the gradient of V. Moreover, in our setting the time variable t is complex.

Assume the system of equations

$$V'(\boldsymbol{q}) = \boldsymbol{q}, \quad \text{where } V'(\boldsymbol{q}) := \operatorname{grad} V(\boldsymbol{q}), \tag{1.3}$$

has a non-zero solution $\boldsymbol{d} \in \mathbb{C}^n$. Then $\boldsymbol{d}$ is called a proper Darboux point of the potential and defines a two-dimensional plane

$$\Pi(\boldsymbol{d}) := \{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n} \mid (\boldsymbol{q}, \boldsymbol{p}) = (q\boldsymbol{d}, p\boldsymbol{d}),\ (q, p) \in \mathbb{C}^2\}, \tag{1.4}$$

which is invariant with respect to system (1.2). Eqs. (1.2) restricted to $\Pi(\boldsymbol{d})$ have the form

$$\dot{q} = p, \qquad \dot{p} = -q^{k-1}. \tag{1.5}$$

For $k \in \mathbb{Z}^\star$, the phase curves of this one degree of freedom Hamiltonian system are

$$\Gamma_{k,\varepsilon} := \left\{(q, p) \in \mathbb{C}^2 \,\middle|\, \frac{1}{2}p^2 + \frac{1}{k}q^k = \varepsilon\right\} \subset \mathbb{C}^2, \quad \varepsilon \in \mathbb{C}. \tag{1.6}$$

Thus, a solution $(q, p) = (q(t), p(t))$ of (1.5) gives rise a solution $(\boldsymbol{q}(t), \boldsymbol{p}(t)) := (q\boldsymbol{d}, p\boldsymbol{d})$ of Eqs. (1.2) with the corresponding phase curve

$$\boldsymbol{\Gamma}_{k,\varepsilon} := \{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n} \mid (\boldsymbol{q}, \boldsymbol{p}) = (q\boldsymbol{d}, p\boldsymbol{d}),\ (q, p) \in \Gamma_{k,\varepsilon}\} \subset \Pi(\boldsymbol{d}). \tag{1.7}$$

In [15] J.J. Morales-Ruiz and J.P. Ramis analyzed the integrability of Hamiltonian systems of this form. Specifically, they investigated the linearized equation of (1.2) along the phase curves $\boldsymbol{\Gamma}_{k,\varepsilon}$ for $\varepsilon \neq 0$ and proved the following theorem.

Theorem 1.1 *(Morales–Ramis). Assume the Hamiltonian system defined by (1.1), in which the potential function $V \in \mathbb{C}(\boldsymbol{q})$ is homogeneous of degree $k \in \mathbb{Z}^\star$, satisfies the following conditions:*

1. *there exists a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$; and*
2. *the system is integrable in the Liouville sense with first integrals which are meromorphic in a connected neighborhood of $\boldsymbol{\Gamma}_{k,\varepsilon}$, with $\varepsilon \in \mathbb{C}^\star$.*

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Then for each eigenvalue λ of the Hessian matrix $V''(\boldsymbol{d})$, the pair (k, λ) corresponds to one of the cases within the table

Case	k	λ	
1.	± 2	*arbitrary*	
2.	k	$p + \frac{k}{2}p(p-1)$	
3.	k	$\frac{1}{2}\left(\frac{k-1}{k} + p(p+1)k\right)$	
4.	3	$-\frac{1}{24} + \frac{1}{6}(1+3p)^2$,	$-\frac{1}{24} + \frac{3}{32}(1+4p)^2$
		$-\frac{1}{24} + \frac{3}{50}(1+5p)^2$,	$-\frac{1}{24} + \frac{3}{50}(2+5p)^2$
5.	4	$-\frac{1}{8} + \frac{2}{9}(1+3p)^2$	
6.	5	$-\frac{9}{40} + \frac{5}{18}(1+3p)^2$,	$-\frac{9}{40} + \frac{1}{10}(2+5p)^2$
7.	-3	$\frac{25}{24} - \frac{1}{6}(1+3p)^2$,	$\frac{25}{24} - \frac{3}{32}(1+4p)^2$
		$\frac{25}{24} - \frac{3}{50}(1+5p)^2$,	$\frac{25}{24} - \frac{3}{50}(2+5p)^2$
8.	-4	$\frac{9}{8} - \frac{2}{9}(1+3p)^2$	
9.	-5	$\frac{49}{40} - \frac{5}{18}(1+3p)^2$,	$\frac{49}{40} - \frac{1}{10}(2+5p)^2$

(1.8)

in which p is an integer.

The above theorem is one of the most beautiful applications of the differential Galois approach to the integrability studies—the so-called Morales–Ramis theory, see [2–5,12–14].

Systematic investigations of the integrability of homogeneous potentials were initiated by H. Yoshida [17,18]. He applied the Ziglin theory [19,20]. A substantial part of the proof of Theorem 1.1 is based on his ideas and results. Let us remark that Theorem 1.1 can be proved without differential Galois theory; such a proof was given by S.L. Ziglin in [21]. It is based on an analysis of the monodromy group of the variational equations.

The aim of this Letter is to find necessary conditions for the integrability of homogeneous potentials with degree of homogeneity $k = 0$ which are excluded by assumptions of Theorem 1.1.

Our main result is the following theorem.

Theorem 1.2. *Assume $V \in \mathbb{C}(\boldsymbol{q})$ in (1.1) is homogeneous of degree $k = 0$ and that the following conditions are satisfied:*

1. *there exists a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$; and*
2. *the system is integrable in the Liouville sense with rational first integrals.*

Then:

1. *all eigenvalues of $V''(\boldsymbol{d})$ are integers; and*
2. *the matrix $V''(\boldsymbol{d})$ is diagonalizable.*

The fact that new obstructions to integrability appear when the Hessian matrix $V''(\boldsymbol{d})$ is not diagonalizable was observed recently in [8], where the following theorem was proved.

Theorem 1.3 (Duval, Maciejewski). *Let $V(\boldsymbol{q})$ be a homogeneous potential of degree $k \in \mathbb{Z} \setminus \{-2, 0, 2\}$ having the property that there exists a non-zero solution $\boldsymbol{d} \in \mathbb{C}^n$ of equation $V'(\boldsymbol{d}) = \boldsymbol{d}$. If the Hamiltonian system generated by (1.1) is integrable in the Liouville sense with first integrals which are meromorphic in a connected neighborhood of $\boldsymbol{\Gamma}_{k,\varepsilon}$, with $\varepsilon \in \mathbb{C}^\star$, then the Hessian matrix $V''(\boldsymbol{d})$ satisfies the following conditions:*

1. *For each eigenvalue λ of $V''(\boldsymbol{d})$, the pair (k, λ) belongs to table (1.8).*
2. *The matrix $V''(\boldsymbol{d})$ does not have an elementary Jordan block of size $m \geqslant 3$.*
3. *If $V''(\boldsymbol{d})$ admits an elementary Jordan block of size $m = 2$ with corresponding eigenvalue λ, then (k, λ) satisfies the conditions in one of cases 3–9 of table (1.8).*

By an elementary Jordan block of size m with eigenvalue λ we mean a Jordan block of the form

$$B(\lambda, m) := \begin{bmatrix} \lambda & 0 & 0 & \cdots\cdots & 0 \\ 1 & \lambda & 0 & \cdots\cdots & 0 \\ 0 & 1 & \lambda & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots \; 1 & \lambda \end{bmatrix} \in \mathbb{M}(m, \mathbb{C}), \tag{1.9}$$

where $\mathbb{M}(m, \mathbb{C})$ denotes the set of $m \times m$ complex matrices.

In the next section we derive the variational equations. The proof of Theorem 1.2 is contained in Sections 3 and 4, where we investigate the differential Galois groups of subsystems of the variational equations. The last section gives an application of Theorem 1.2 to two-dimensional potentials.

2. Variational equations

Let us assume that a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ satisfies $V'(\boldsymbol{d}) = \boldsymbol{d}$, and $k = 0$. Then, $\Pi(\boldsymbol{d})$ defined by (1.4) is invariant with respect to the system (1.2). However, for $k = 0$, the phase curve corresponding to a solution $(q, p) = (q(t), p(t))$ of one degree of freedom Hamiltonian system (1.5) is not algebraic. In fact, for $k = 0$, the phase curves of (1.5) are given by

$$\Gamma_\varepsilon := \left\{ (q, p) \in \mathbb{C}^2 \;\middle|\; \frac{1}{2}p^2 + \ln q = \varepsilon \right\}, \quad \varepsilon \in \mathbb{C}. \tag{2.1}$$

A particular solution $(q, p) = (q(t), p(t))$ of (1.5) which lies on Γ_ε gives a particular solution $(\boldsymbol{q}(t), \boldsymbol{p}(t)) := (q\boldsymbol{d}, p\boldsymbol{d}) \in \Pi(\boldsymbol{d})$ of (1.2) which lies on the phase curve

$$\boldsymbol{\Gamma}_\varepsilon := \left\{ (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n} \;\middle|\; (\boldsymbol{q}, \boldsymbol{p}) = (q\boldsymbol{d}, p\boldsymbol{d}),\; (q, p) \in \Gamma_\varepsilon \right\} \subset \Pi(\boldsymbol{d}). \tag{2.2}$$

The variational equations along $\boldsymbol{\Gamma}_\varepsilon$ have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{y}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = -\frac{1}{q^2}V''(\boldsymbol{d})\boldsymbol{x},$$

or simply

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x} = -\frac{1}{q^2}V''(\boldsymbol{d})\boldsymbol{x}. \tag{2.3}$$

We can make a variable substitution $\boldsymbol{x} = \boldsymbol{C}\boldsymbol{z}$ so that the matrix $\boldsymbol{\Lambda} = \boldsymbol{C}^{-1}V''(\boldsymbol{d})\boldsymbol{C}$ appearing in the transformed equations

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{z} = -q^{-2}\boldsymbol{\Lambda}\boldsymbol{z} \tag{2.4}$$

is the Jordan form of $V''(\boldsymbol{d})$.

From now on we work with a fixed value $\varepsilon \in \mathbb{C}$. In order to apply the differential Galois theory we have to introduce an appropriate differential field of functions defined on Γ_ε. We assume that the considered first integrals are rational functions, i.e., elements of the field $\mathbb{C}(\boldsymbol{q}, \boldsymbol{p})$. This is why we choose as our base field the restriction of field $\mathbb{C}(\boldsymbol{q}, \boldsymbol{p})$ to $\boldsymbol{\Gamma}_\varepsilon$. The restriction of $\mathbb{C}(\boldsymbol{q}, \boldsymbol{p})$ to

$\Pi(\boldsymbol{d})$ gives a field $\mathbb{C}(q,p)$, which together with derivation $\frac{\mathrm{d}}{\mathrm{d}t}$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}q = p, \qquad \frac{\mathrm{d}}{\mathrm{d}t}p = -\frac{1}{q}, \tag{2.5}$$

is a differential field. The restriction of $\mathbb{C}(q,p)$ to Γ_ε gives the field

$$\mathbb{C}\big(\exp(\varepsilon - p^2/2), p\big) \simeq \mathbb{C}\big(p, \exp\big(p^2/2\big)\big).$$

This field equipped with derivation $\dot{p} = -\mathrm{e}^{-\varepsilon}\exp(p^2/2)$ is our base differential field. For the remainder of the paper q denotes $\exp(\varepsilon - p^2/2)$.

Recall that according to the main theorem of the Morales–Ramis theory, if the system is integrable in the Liouville sense, then the differential Galois group of the variational equations along a particular solution is virtually Abelian, i.e. the identity component of this group is Abelian. Thus, we have to check whether the differential Galois group $\mathcal{G}$ over the field $\mathbb{C}(q,p)$ of system

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{z} = -\frac{1}{q^2}\boldsymbol{\Lambda}\boldsymbol{z}, \tag{2.6}$$

is virtually Abelian. Notice that for each eigenvalue λ of $V''(\boldsymbol{d})$, the above system contains as a subsystem equation of the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x = -\frac{\lambda}{q^2}x, \tag{2.7}$$

and, if $\boldsymbol{\Lambda}$ has a Jordan block with the corresponding eigenvalue λ, then it contains the following subsystem

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x = -\frac{\lambda}{q^2}x, \tag{2.8}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y = -\frac{\lambda}{q^2}y - \frac{1}{q^2}x. \tag{2.9}$$

To make the above observations useful we invoke the following fact: the differential Galois group of a subsystem is a quotient of the differential Galois group of the system. This implies that if the differential Galois group of a system is virtually Abelian, then the differential Galois group of its subsystem is virtually Abelian, see Section 1.4 in [8]. Thus, to find obstructions to the integrability of the considered systems it is enough to investigate the above subsystems.

It is convenient to give another form of (2.7) and (2.8). We take p as independent variable in those systems. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{p}\frac{\mathrm{d}}{\mathrm{d}p} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \ddot{p}\frac{\mathrm{d}}{\mathrm{d}p} + \dot{p}^2\frac{\mathrm{d}^2}{\mathrm{d}p^2}.$$

Thus, using (2.5) we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{p}{q^2}\frac{\mathrm{d}}{\mathrm{d}p} + \frac{1}{q^2}\frac{\mathrm{d}^2}{\mathrm{d}p^2}.$$

So, Eq. (2.7) transforms into

$$x'' + px' + \lambda x = 0, \tag{2.10}$$

where prime denotes derivation with respect to p. System (2.8)–(2.9) transforms into

$$\left.\begin{aligned} &x'' + px' + \lambda x = 0,\\ &y'' + py' + \lambda y + x = 0. \end{aligned}\right\} \tag{2.11}$$

We need to emphasize that when considering both of the above systems we have to determine their differential Galois groups over the field $\mathbb{C}(q,p)$, not over $\mathbb{C}(p)$!

3. Rank 2 subsystems

Let G denotes the differential Galois group of Eq. (2.10) over $(\mathbb{C}(p), \frac{\mathrm{d}}{\mathrm{d}p})$. We show the following.

Proposition 3.1. *If $\lambda \notin \mathbb{Z}$, then $G = \mathrm{GL}(2,\mathbb{C})$. If $\lambda \in \mathbb{Z}$, then $G = \mathbb{C}^* \ltimes \mathbb{C}$.*

Proof. Let $z = -p^2/2$. Then, after this change of independent variable, Eq. (2.10) is of the form

$$z\frac{\mathrm{d}^2x}{\mathrm{d}z^2} + (c-z)\frac{\mathrm{d}x}{\mathrm{d}z} + ax = 0 \quad \text{where } c = \frac{1}{2},\ a = -\frac{\lambda}{2}. \tag{3.1}$$

This is the confluent hypergeometric equation in the Kummer form [9]. Its differential Galois group over $\mathbb{C}(z)$ was investigated in [16], see also [6,7]. From those investigations we know that

- if $\lambda \notin \mathbb{Z}$, then the Galois group is $\mathrm{GL}(2,\mathbb{C})$,
- if $\lambda \in \mathbb{Z}$, then Galois group is $\mathbb{C}^* \ltimes \mathbb{C}$.

This result gives the Galois group of Eq. (2.10) over the field $\mathbb{C}(p^2)$, hence over $\mathbb{C}(p)$. □

For an integer λ we can characterize solutions of Eq. (2.10) in the following proposition.

Proposition 3.2. *Let $\lambda \in \mathbb{Z}$. Then*

$$x_\lambda := \begin{cases} qH_{\lambda-1}(p), & \text{for } \lambda \geqslant 1,\\ H_{-\lambda}(-\mathrm{i}p), & \text{for } \lambda \leqslant 0, \end{cases} \tag{3.2}$$

where H_n is the Hermite polynomial of degree n, is a solution of Eq. (2.10). Its other solution is given by

$$\widetilde{x} := x_\lambda \int \frac{q}{x_\lambda^2}\,\mathrm{d}p.$$

Proof. If we put $x = qy$, then Eq. (2.10) transforms into

$$y'' - py' + ny = 0 \quad \text{where } n = \lambda - 1.$$

This is the Hermite differential equation which has for integer $n = \lambda - 1$ polynomial solution $H_n(p)$. If $\lambda \leqslant 0$ we make transformation $p \mapsto -\mathrm{i}p$, which transforms Eq. (2.10) into the Hermite equation with $n = -\lambda$. The formula for the second solution is standard. □

Note that the Hermite polynomials H_n used in this Letter are denoted by He_n in [1].

Now, we investigate the differential Galois group $\widehat{G}$ of Eq. (2.10) over our base field $\mathbb{C}(q,p)$. More precisely we determine its dimension. Let us recall that according to the Kolchin theorem the dimension of the differential Galois group of an equation is equal to the transcendence degree of the Picard–Vessiot extension of the field solving the equation. Knowing the dimension we can decide whether the group is virtually Abelian.

Lemma 3.3. *If $\lambda \notin \mathbb{Z}$, then $\widehat{G}$ is not virtually Abelian. If $\lambda \in \mathbb{Z}$, then $\widehat{G}$ is virtually Abelian.*

Proof. Let K be the Picard–Vessiot extension of $\mathbb{C}(p)$ solving the linear system

$$\begin{cases} x'' + px' + \lambda x = 0,\\ u' + pu = 0, \end{cases} \tag{3.3}$$

so

$$K := \mathbb{C}\big(p, u, x_1, x_2, x_1', x_2'\big),$$

for a basis of solutions. Let K_1 be the subextension of $K/\mathbb{C}(p)$ generated by u, i.e., $K_1 = \mathbb{C}(p, u)$, and K_2 be the differential subextension of $K/\mathbb{C}(p)$ generated by x_1 and x_2, i.e.,

$$K_2 := \mathbb{C}\big(p, x_1, x_2, x_1', x_2'\big).$$

Note, that according to our notation $K_1 = \mathbb{C}(p, u) = \mathbb{C}(p, \exp(-p^2/2)) \simeq \mathbb{C}(q, p)$.

We have two towers of extensions

$$\mathbb{C}(p) \subset K_1 \subset K, \tag{3.4}$$

and

$$\mathbb{C}(p) \subset K_2 \subset K. \tag{3.5}$$

Our aim is to determine the transcendence degree $\mathrm{tr.deg}(K/K_1)$ of extension K/K_1. This number is the dimension of the differential Galois group $\widehat{G}$.

Using basic properties of the transcendence degree, see, e.g. Chapter 8 in [10], from the first tower (3.4) we obtain

$$\mathrm{tr.deg}\big(K/\mathbb{C}(p)\big) = \mathrm{tr.deg}(K/K_1) + \mathrm{tr.deg}\big(K_1/\mathbb{C}(p)\big). \tag{3.6}$$

We know that $\mathrm{tr.deg}(K_1/\mathbb{C}(p)) = 1$ because $u = \exp(-p^2/2)$ is transcendental over $\mathbb{C}(p)$. Thus, we need to determine $\mathrm{tr.deg}(K/\mathbb{C}(p))$. But from the tower (3.5) we have

$$\mathrm{tr.deg}\big(K/\mathbb{C}(p)\big) = \mathrm{tr.deg}(K/K_2) + \mathrm{tr.deg}\big(K_2/\mathbb{C}(p)\big),$$

and this gives us

$$\mathrm{tr.deg}(K/K_1) = \mathrm{tr.deg}(K/K_2) + \mathrm{tr.deg}\big(K_2/\mathbb{C}(p)\big) - 1. \tag{3.7}$$

Two cases have to be distinguished:

- If λ is not integer, then by Proposition 3.1, the transcendence degree of $K_2/\mathbb{C}(p)$ is 4. So the transcendence degree of K/K_1 must be greater or equal to 3. The Galois group of this extension is a dimension 3 subgroup of $\mathrm{GL}(2, \mathbb{C})$, so it is not virtually Abelian.
- If λ is integer, then by Proposition 3.2, $q \in K_2$. Thus, the transcendence degree of K/K_2 is 0. As, by Proposition 3.1, $\mathrm{tr.deg}(K_2/\mathbb{C}(p)) = 2$, the transcendence degree of K/K_1 is 1. The Galois group of the variational equation is virtually Abelian. □

Corollary 3.4. *If an eigenvalue of $V''(\boldsymbol{d})$ at a Darboux point is not an integer, then the Hamiltonian system (1.2) is not integrable.*

4. Rank 4 subsystems

In this section we investigate the differential Galois group over $\mathbb{C}(q, p)$ of system (2.8)–(2.9) under assumption that $\lambda \in \mathbb{Z}$. At first we show the following.

Proposition 4.1. *If $\lambda \in \mathbb{Z}$, then Eq. (2.8) has a solution in $\mathbb{C}(p, q)$, and its differential Galois group over $\mathbb{C}(p, q)$ is the additive group G_a.*

Proof. If we take p as an independent variable in Eq. (2.8), then it becomes Eq. (2.10). Now, by Proposition 3.2 its solution $x_\lambda \in \mathbb{C}(q, p)$. We know from the proof of Lemma 3.3 that the dimension of the Galois group is 1, and this proves the second statement. □

Proposition 4.2. *If $\lambda \in \mathbb{Z}$, and the differential Galois group over $\mathbb{C}(q, p)$ of system (2.8)–(2.9) is virtually Abelian, then there exists $a \in \mathbb{C}$ such that the integral*

$$R := \int \left(a\frac{1}{x_\lambda^2} - \frac{1}{q^2} x_\lambda^2 \right) \mathrm{d}t = a \int \frac{q}{x_\lambda^2}\, \mathrm{d}p - \int \frac{x_\lambda^2}{q}\, \mathrm{d}p, \tag{4.1}$$

belongs to $\mathbb{C}(q, p)$.

Proof. By Proposition 4.1, the differential Galois group of Eq. (2.8) is the additive group G_a. So, we can apply point 2 of Theorem 2.3 from [8]. The condition (α) in this theorem is, in our case, $R \in \mathbb{C}(q, p)$, for a certain $a \in \mathbb{C}$. □

Lemma 4.3. *For any $a \in \mathbb{C}$ the integral $R(q, p)$ does not belong to the field $\mathbb{C}(p, q)$.*

Proof. We prove our lemma by a contradiction. Let us assume that there exists $a \in \mathbb{C}$ such that $R(q, p) \in \mathbb{C}(p, q)$.

At first we consider the case $\lambda \geqslant 1$. In this case the solution of Eq. (2.8) in $\mathbb{C}(p, q)$ is $x_\lambda = qH_{\lambda-1}(p)$ (see Proposition 3.2), and thus

$$R = \int \left(a\frac{1}{q}\frac{1}{H_{\lambda-1}(p)^2} - qH_{\lambda-1}(p)^2 \right) \mathrm{d}p. \tag{4.2}$$

As a rational function of the transcendent $q = \exp(\varepsilon - p^2/2)$ with coefficients in $\mathbb{C}(p)$, R can be written in the form

$$R = \sum_{l=0}^{m} \alpha_l q^l + \frac{N(p, q)}{D(p, q)},$$

where $\alpha_k \in \mathbb{C}(p)$, N and D are elements of $\mathbb{C}(p)[q]$, and $\deg_q(N) < \deg_q(D)$. Moreover, this decomposition is unique. Differentiating both sides of (4.1) we obtain

$$a\frac{1}{q}\frac{1}{H_{\lambda-1}^2} - qH_{\lambda-1}^2 = \sum_{l=0}^{m} (\alpha_l' - lp\alpha_l) q^l + \left(\frac{N(p, q)}{D(p, q)} \right)'. \tag{4.3}$$

As

$$0 > \mathrm{ord}_q \frac{N}{D} := \deg_q N - \deg_q D \geqslant \mathrm{ord}_q \left(\frac{N}{D} \right)',$$

we must have $m = 1$, because the decomposition in both sides of (4.3) is unique. In particular, we have

$$\alpha_1' - p\alpha_1 = -H_{\lambda-1}^2. \tag{4.4}$$

Because this equation is regular, $\alpha_1 \in \mathbb{C}[p]$, and it can be written as a linear combination of Hermite polynomials

$$\alpha_1 = \sum_{n=0}^{N} \gamma_n H_n, \quad \text{where } \gamma_n \in \mathbb{C}.$$

Hermite polynomials satisfy the following relation

$$H_{n+1}(p) = pH_n(p) - H_n'(p),$$

so we have

$$\alpha_1' - p\alpha_1 = \sum_{n=0}^{N} \gamma_n \big(H_n' - pH_n\big) = -\sum_{n=0}^{N} \gamma_n H_{n+1}.$$

Thus, we can rewrite (4.4) in the form

$$H_{\lambda-1}^2 = \sum_{n=0}^{N} \gamma_n H_{n+1},$$

and so

$$\int_{-\infty}^{\infty} \mathrm{e}^{-p^2/2} H_{\lambda-1}^2(p)\, \mathrm{d}p = \sum_{n=0}^{N} \gamma_n \int_{-\infty}^{\infty} \mathrm{e}^{-p^2/2} H_{n+1}(p)\, \mathrm{d}p.$$

But this gives a contradiction

$$\sqrt{2\pi}\,(\lambda - 1)! = 0,$$

because

$$\int_{-\infty}^{\infty} e^{-p^2/2} H_{n+1}(p)\,dp = \int_{-\infty}^{\infty} e^{-p^2/2} H_{n+1}(p) H_0(p)\,dp = 0,$$

for $n \geqslant 0$. For $\lambda \geqslant 1$ our lemma is proved.

In the case $\lambda \leqslant 0$ the solution of Eq. (2.8) in $\mathbb{C}(p,q)$ is $x_\lambda = H_\lambda(-ip)$, so integral (4.1) reads

$$R = \int \left(\frac{aq}{H_{-\lambda}(-ip)^2} - \frac{1}{q} H_{-\lambda}(-ip)^2 \right) dp. \tag{4.5}$$

If we set

$$v = ip, \qquad u = e^{\varepsilon - v^2/2} = e^{2\varepsilon}/q,$$
$$\widetilde{\lambda} = 1 - \lambda, \quad \text{and} \quad \widetilde{a} = ae^{4\varepsilon}, \tag{4.6}$$

then we transform the considered integral into the following one

$$R = ie^{-2\varepsilon} \int \left(\frac{\widetilde{a}}{uH_{\widetilde{\lambda}-1}(v)^2} - uH_{\widetilde{\lambda}-1}(v)^2 \right) dv. \tag{4.7}$$

But this integral, after renaming variables, is proportional to that one already considered for $\lambda \geqslant 1$, see (4.2). So, it is not rational and this finishes the proof. □

As corollaries we have:

Corollary 4.4. *If $\lambda \in \mathbb{Z}$, then the differential Galois group over $\mathbb{C}(q,p)$ of system (2.8)–(2.9) is not virtually Abelian.*

Corollary 4.5. *If $V''(\boldsymbol{d})$ at a Darboux point has an elementary Jordan block with integer eigenvalue, then Hamiltonian system (1.2) is not integrable.*

Now, our main result given by Theorem 1.2 follows directly from Corollaries 3.4 and 4.5.

5. Examples

We will consider the case $n = 2$ in detail. Our aim is to characterize those homogeneous potentials $V \in \mathbb{C}(q_1, q_2)$ of degree $k = 0$ which satisfy the necessary conditions of Theorem 1.2.

Darboux points of V are non-zero solutions of equations

$$\frac{\partial V}{\partial q_1} = q_1, \qquad \frac{\partial V}{\partial q_2} = q_2. \tag{5.1}$$

As it was explained in [11] it is convenient to consider Darboux points as points in the projective line $\mathbb{CP}^1$. Let $z = q_2/q_1$, $q_1 \neq 0$, be the affine coordinate on $\mathbb{CP}^1$. Then, we can rewrite system (5.1) in the form

$$v'(z)z = -q_1^2, \qquad v'(z) = zq_1^2, \tag{5.2}$$

where $v(z) := V(1,z)$. From the above formulae it follows that $z_\star$ is a Darboux point of V, if and only if $z_\star \in \{-i, i\}$, and $v'(z_\star) \neq 0$. Thus, the location of Darboux points does not depend on the form of potential!

If $z_\star$ is the affine coordinate of a Darboux point $\boldsymbol{d}$ of V, then the Hessian matrix $V''(\boldsymbol{d})$ expressed in this coordinate has the form

$$V''(\boldsymbol{d}) = \begin{bmatrix} -v''(z_\star)x_\star^{-2} - 2 & -[v'(z_\star) + z_\star v''(z_\star)]x_\star^{-2} \\ -[v'(z_\star) + z_\star v''(z_\star)]x_\star^{-2} & v''(z_\star)x_\star^{-2} \end{bmatrix} \tag{5.3}$$

where

$$x_\star^2 = -v'(z_\star)z_\star = v'(z_\star)/z_\star.$$

Vector $\boldsymbol{d}$ is an eigenvector of $V''(\boldsymbol{d})$ with corresponding eigenvalue $\lambda = -1$. As $\operatorname{Tr} V''(\boldsymbol{d}) = -2$, $\lambda = -1$ is the only eigenvalue of $V''(\boldsymbol{d})$. Thus the first condition of Theorem 1.2 is satisfied. If $V''(\boldsymbol{d})$ is diagonalizable, then it is diagonal. Hence the second condition of Theorem 1.2 is satisfied iff

$$v'(z_\star) + z_\star v''(z_\star) = 0. \tag{5.4}$$

Let us apply the above criterion for potential

$$V = \frac{q_2}{q_1^3}(q_2 - aq_1)(q_2 - bq_1) \quad \text{where } a \neq b,$$

assuming that it has two Darboux points with affine coordinates $\pm i$. An easy calculation shows that condition (5.4) is satisfied for $z_\star = \pm i$ iff

$$V = \frac{q_2}{q_1^3}\left(9q_1^2 + q_2^2\right).$$

We did an explicit search for first integrals which are polynomials in the momenta of degree at most four. None were found, but of course this does not prove non-integrability for the associated Hamiltonian system. Readers must keep in mind that Theorem 1.2 gives necessary conditions for integrability, but makes no claim regarding sufficiency.

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