AN INTRODUCTION TO MALGRANGE PSEUDOGROUP

by

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Abstract. — The pseudogroup defined by B. Malgrange as a generalization of the differential Galois group for nonlinear differential equation is presented. It is proved that a equation integrable by quadratures has a solvable pseudgroup. From this a new proof of a theorem of M. Singer is given.

***Résumé* (Une introduction au pseudo-groupe de Malgrange).** — Nous donnons une introduction au pseudo-groupe défini par B. Malgrange généralisant aux équations différentielles non linéaires le groupe de Galois différentiel. Nous prouvons que le pseudo-groupe d'une équation intégrable par quadratures est résoluble et donnons une nouvelle preuve d'un théorème de M. Singer.

Contents

1. Introduction

Between 1887 and 1904, E. Picard [17] and E. Vessiot [23] applied ideas from Galois theory to differential equations. They succeded in getting a complete theory in case of linear differential equations nowadays known as Picard-Vessiot Theory. Almost at the same times J. Drach [6] and E. Vessiot [24] tried to extend this theory to a Differential Galois Theory involving also non linear equations. Two reciprocal pseudogroups are defined in [24, 25] by Vessiot, the specific one and the rationality one. One can find this definition in the introduction (paragraph 3.) of [24]:

> The specific group is the smallest rational group containing the equation as infinitesimal subgroup.

No one follows this direction until two independent articles of H. Umemura [22] and B. Malgrange [12]. H. Umemura infinitesimal Galois group of a differential equation is the rationality pseudogroup of Vessiot defined rigorously by a Lie-Ritt functor. B. Malgrange without knowledge of this late Vessiot's article gives almost the same definition: Galois pseudogroup of a vector field is the smallest algebraic pseudogroup containing this vector field as infinitesimal transformation so it is the specific pseudogroup of Vessiot. Note that Lie-Ritt functors are very close to the way used by E. Cartan to study Lie pseudogroups by means of what is now called Cartan connections and are also very closed to 'virtual groups' defined by B. Malgrange in [14].

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In the case of linear differential equations, these two groups appear as the Galois group and the intrinsec group of Katz [2] which are respectively the rationality group and the specific group of Vessiot. In special non linear cases these two groups appear to have been introduced by A. Pillay in [18].

The point of view presented here is the point of view of B. Malgrange with some (very) small modifications. The end of the introduction contains a first look at the linear case and the main tool used : theorem 1.3. We give the Galois/Umemura group and the Malgrange pseudogroup in the linear case and compare them. Justifications are given in the last section. It appears that this two objects are isomorphic but theirs actions are reciprocal as claimed by Vessiot.

In the second section, algebraic singular groupoids and associated objects are defined. Same examples are given after each definition to understand their relationship.

The third section ends with the definition of Malgrange pseudogroup and the projection theorem which is the only remainder of the Galois correspondence at this stage.

In the last section we prove the non linear analog of a classical result due to Kolchin in Picard-Vessiot Theory: the Galoisian object attached to an equation integrable by quadratures is solvable. Here the Galoisian object is rather the Lie algebra of Malgrange pseudogroup than the pseudogroup itself mainly because no good enough definition of solvability for pseudogroup is known.

Before first examples let us recall the set theoretical definition of a pseudogroup. A more algebraic definition will be given in the pseudogroup section.

Definition 1.1. — *A pseudogroup of transformations of an analytic manifold V is a set $\mathcal{G}$ of analytic maps between open sets of V, $\varphi : s(\varphi) \to t(\varphi)$ such that*

- *the restriction of a transformation φ of $\mathcal{G}$ to a open subset of its domain $s(\varphi)$ is in $\mathcal{G}$;*
- *if $\psi \in \mathcal{G}$ and $\varphi \in \mathcal{G}$ and $t(\varphi) = s(\psi)$ then $\psi \circ \varphi$ is in $\mathcal{G}$;*
- *if $\varphi \in \mathcal{G}$ then φ is invertible and $\varphi^{\circ -1}$, the inverse for the composition law, is in $\mathcal{G}$;*
- *if $\varphi : s(\varphi) \to t(\varphi)$ is invertible and $U \subset s(\varphi)$ is an open subset such that $\varphi|_U$ is in $\mathcal{G}$ then φ is in $\mathcal{G}$.*

1.1. Constant linear equations. — Let G an algebraic group of dimension d and X a right invariant vector field on G. The description of the Malgrange pseudogroup of X can be done as follow. Let $Y_1, \ldots, Y_d$ be a basis over $\mathbb{C}$ of left invariant vector fields on G and $H_1, \ldots, H_n$ be generators of the field of rational first integrals of X. It is an easy lemma to prove that left invariant vector fields are infinitesimal generators of right translations, see [11]. Because of this, one gets $[X, Y_i] = 0$ for all $1 \le i \le d$ and by definition $X \cdot H_j = 0$ for all $1 \le j \le n$. One gets (see definition 3.9)

$$Mal = \Big\{\varphi \text{ map between open sets } s(\varphi) \text{ and } t(\varphi) \text{ of } G \\ \Big| \varphi^* H_j = H_j, \varphi^* Y_i = Y_i; \forall 1 \le j \le n, 1 \le i \le d\Big\} \quad (1)$$

This pseudogroup is easy to define by means of subgroups of G. Let $K \subset G$ the smallest algebraic group such that X belongs to the right Lie algebra of K.

Lemma 1.2. — *Mal is the pseudogroup of analytic localisations of elements of K acting by left translation on G.*

Proof. — For $g \in G$ let rg (resp. lg) be the right (resp. left) translation by g on G. Because left invariant vector fields are infinitesimal generators of right translations one gets from invariance of Y's $\varphi \circ rg = rg \circ \varphi$. Applying to the neutral element e one gets $\varphi(g) = \varphi(e)g$ thus φ is left translation on its domains. Let K be the subgroup of all $k = \varphi(e)$ for all $\varphi \in Mal$. It is an algebraic subgroup defined by $H_i(k) = H_i(e)$ for all i.

If a smaller algebraic subgroup contains X in its Lie algebra then by Chevalley theorem it has more rational invariants. But all rational invariants of such a group are rational first integrals of X. From the beginning we get all such integrals so K is the smallest algebraic group whose Lie algebra contains X. □

Chevalley theorem can be replaced by the following more general theorem.

Theorem 1.3 **(Gomez-Mont [8]).** — *Let $\mathcal{F}$ be an holomorphic foliation of a projective variety V whose leaves are quasiprojective then there is a variety W and a rational map $H : V \dashrightarrow W$ such that the closure of general fibers of H are closure of leaves of $\mathcal{F}$.*

This theorem can replace Chevalley theorem by setting $G = V$, K is the smallest algebraic subgroup whose Lie algebra contains X and $\mathcal{F}$ is the foliation by orbits of left action of K.

1.2. Linear equations. — Let us consider a linear differential equations on the trivial vector bundle $\mathbb{C}_x \times \mathbb{C}^n_y$

$$(E) \qquad \frac{d}{dx}(y_1, \ldots, y_n) = (y_1, \ldots, y_n)A(x)$$

with $A \in \mathfrak{gl}_n(\mathbb{C}(x))$. For simplicity such system will be write using the vector field

$$\frac{\partial}{\partial x} + yA(x)\frac{\partial}{\partial y}$$

where $y = (y_1, \ldots, y_n)$ and $\frac{\partial}{\partial y} = \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}$.

Going from the equation to its fundamental form. —

Definition 1.4. — *A general solution is a vector of n holomorphic functions $y(x, c_1, \ldots, c_n)$ of $1 + n$ arguments such that $\frac{\partial}{\partial x}y = yA(x)$ and $\det \frac{\partial y}{\partial c} \not\equiv 0$.*

Because of linearity, the Jacobian satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial c}\right) = \left(\frac{\partial y}{\partial c}\right) A(x)$$

Lemma 1.5. — *Up to some change of integrating constants, one can assume that the dependency in c is linear, i.e. $y = c\frac{\partial y}{\partial c}$.*

The matrix $\frac{\partial y}{\partial c}$ is a fundamental solution of equation (E). Such a general solution is called a linear general solution

GL_n action from the right or the left. —

Lemma 1.6. — *If $c(d)$ is a invertible transformation such that for any linear general solution $y(x, c)$, $y(x, c(d))$ is another linear general solution then $c(d)$ is a linear transformation i.e. $c(d) = dC$ for a matrix $C \in GL_n(\mathbb{C})$.*

Two linear general solutions with same domain are related by a linear change of c. This lemma is classical: two fundamental solution are related by multiplication on the left by a constant coefficient invertible matrix.

Lemma 1.7. — *If $(x, \overline{y}(x, y))$ is a invertible transformation such that for any linear general solution $y(x, c)$, $\overline{y}(x, y(x, c))$ is a linear general solution then $\overline{y}(x, y)$ is a linear gauge transformation i.e. $\overline{y}(x, y) = yY(x)$ for some open set U of $\mathbb{C}$ and $Y \in GL_n(\mathcal{O}(U))$ satisfying $\frac{\partial}{\partial x}Y = [Y, A(x)]$.*

Two linear general solutions with same range are related by a linear gauge transformation. Let y be linear general solution with range over $U \subset \mathbb{C}$. Then for any $C \in GL_n(\mathbb{C})$ there is a $Y \in GL_n(\mathcal{O}(U))$ solution of $\frac{\partial}{\partial x}Y = [Y, A(x)]$ such that $y(x, dC) = y(x, d)Y$. This correspondence is one to one.

The universal solution extension. — Let $\mathcal{S}$ be the set of all germs of linear general solutions. It is an algebraic variety, it can be identify with a Zariski open set of $\mathbb{C}\times GL_n(\mathbb{C})$ by $(x_0, y(x,c)) \to (x_0, \frac{\partial y}{\partial c}(x_0))$. The equation $\frac{d}{dx}y(x,c) = y(x,c)A(x)$ on $\mathcal{S}$ gives a rational vector field on $\mathbb{C}\times GL_n(\mathbb{C})$:

$$X = \frac{\partial}{\partial x} + \sum y_i^j A_k^i(x)\frac{\partial}{\partial y_k^j}$$

where x is a coordinate on $\mathbb{C}$ and y_i^j are usual coordinates on GL_n.

Picard-Vessiot extension. — Let $H_1, \dots H_k$ be rational first integrals of X such that $\mathbb{C}(H_1, \dots, H_k) = \{H \in \mathbb{C}(\mathcal{S}) \mid X\cdot H = 0\}$ and $h_1, \dots, h_k$ be complex numbers such that the projection of $PV(h) = \{H_i = h_i,\ \forall i\} \subset \mathcal{S}$ on $\mathbb{C}_x$ is dominant. A linear change of the integrating constants gives a right translation on $GL_n(\mathbb{C})$. These translations act transitively on PV's. The field $\mathbb{C}(PV(h))$ is a Picard-Vessiot extension for the equation (E).

Malgrange pseudogroup and Galois group. — Let U be an open subset of $\mathbb{C}$. One defines

$$Mal(U) = \Big\{\text{gauge transf. } \varphi : (x,y) \mapsto (x, yY(x)) \text{ with } Y \in GL(\mathcal{O}(U)) \\ \Big| \frac{\partial}{\partial x}Y = [Y, A] \text{ and } \forall i H_i(\varphi(x,y)) = H_i(x,y)\Big\}$$

and

$$Gal(h) = \{C \in GL_n(\mathbb{C}) \mid y(x,d) \in PV(h) \text{ iff } y(x,dC) \in PV(h)\}.$$

Up to the choice of a general linear solution in $PV(h)$ on an open subset $U \subset \mathbb{C}$, the correspondence $C \leftrightarrow Y$ is an isomorphism of groups between $Gal(h)$ and $Mal(U)$, the subgroup of gauge transformations defined on U.

Action on Picard-Vessiot extension. — Malgrange pseudogroup over U and Galois group are isomorphic as group but do not act in the same way on $PV(h)$. By choosing a linear general solution $y(x,d)$ in $PV(h)$ defined at $x_0 \in \mathbb{C}$, one can identify the fiber $PV(h)_{x_0}$ with $Gal(h)$ and the action of $Gal(h)$ is by left translation but the action of Mal_{x_0} correspond to the action of $Gal(h)$ by right translation. Compare with [2] and [18, Lemma 3.9].

2. Definitions

Definitions and results presented in this section can be found in [9, 12, 15, 16, 19, 20]. All objects presented can be defined on smooth connected algebraic variety but for simplicity this variety will be an affine space. The general definitions can be obtained by restriction to affine subvarieties and gluing these local constructions.

2.1. Frame bundles. — Let V be the affine space over $\mathbb{C}$ of dimension d with coordinates $r_1, \dots, r_d$ and $(\mathbb{C}^d, 0)$ be the germ of analytic space at 0 with coordinates $x_1, \dots, x_d$. An order q frame on V is a q-jet

$$j_q r = \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\le q}} r_i^\alpha \frac{x^\alpha}{\alpha!}$$

of an invertible formal map $r : (\mathbb{C}^d, 0) \to V$ given by power series

$$r(x) = \sum_{\alpha\in\mathbb{N}^d} r_i^\alpha \frac{x^\alpha}{\alpha!}.$$

The space of q-frames is denoted by R_qV. It is a affine variety with coordinate ring

$$\mathbb{C}[R_qV] = \mathbb{C}\Big[r_i^\alpha, \frac{1}{\det(r_i^{\epsilon(j)})}\Big| 1 \le i \le d, \alpha \in \mathbb{N}^d, |\alpha| \le q\Big]$$

where $\epsilon(j)$ is the multiindex $(0,\cdots,\underset{j^{th}}{1},\cdots,0)$. By identification of r_i^0 with r_i, the q-frames space is a principal bundle over V with structural group

$$\Gamma_q^d = \left\{ j_q g \mid g : (\mathbb{C}^d,0) \to (\mathbb{C}^d,0) \text{ invertible formal map} \right\}$$

acting by 'source composition' : $j_q r \cdot j_q g = j_q(r\circ g)$, the map

$$\begin{array}{ccc} R_qV\times\Gamma_q^d & \to & R_qV\underset{V}{\times}R_qV \\ (j_qr,j_qg) & \mapsto & (j_qr\cdot j_qg, j_qr) \end{array}$$

being an isomorphism. Thanks to Faa di Bruno formulas [26], these groups and actions are algebraic.

Example 2.1. — *If V is the affine line $\mathbb{A}^1$ then $R_qV = \mathbb{A}^1\times GL_1\times\mathbb{A}^{q-1}$ with ring $\mathbb{C}[R_qV] = \mathbb{C}[r, r^1, \frac{1}{r^1}, r^2\ldots r^q]$. The group Γ_q^1 is the algebraic variety $GL_1\times\mathbb{A}^{q-1}$ with a twisted law.*

For $q=3$, the product law is

$$(h^1,h^2,h^3)\cdot(g^1,g^2,g^3) = \left(h^1g^1, h^2(g^1)^2+h^1g^2, h^3(g^1)^3+3h^2g^2g^1+h^1g^3\right).$$

The action on R_3V is

$$(r^0,r^1,r^2,r^3)\cdot(g^1,g^2,g^3) = (r^0, r^1g^1, r^2(g^1)^2+r^1g^2, r^3(g^1)^3+3r^2g^1+r^1g^3).$$

The following lemma, explaining the structure of this group, will be used in the last section. We give a proof following E. Cartan [3].

Lemma 2.2. — *A proper algebraic subgroup of Γ_q^1 has one of its equation of order less or equal three.*

- *If the minimal order is three, the equation is $2\frac{g^3}{g^1} - 3\left(\frac{g^2}{g^1}\right)^2 + c(g^1)^2 - c = 0$ for a constant c.*
- *If it is two, the equation is $\frac{g^2}{g^1} + cg^1 - c = 0$ for a constant c.*
- *If it is one, the equation is $(g^1)^k - 1 = 0$.*

Proof. — By Chevalley (or theorem 1.3) an algebraic subgroup of a linear group is characterized by its rational invariants for the action by right translation. Let q be the smallest order of such an invariant. By minimality of q the field of invariant of order q has transcendence degree 1.

The action of Γ_q^1 on itself by right translation is linear. For $g\in\Gamma_q^1$, let D_g be the matrix of the right translation *i.e.* $(h^1,h^2,h^3,\ldots)\cdot g = (h^1,h^2,h^3,\ldots)D_g$. One gets

$$D_g = \begin{bmatrix} g^1 & g^2 & g^3 & \ldots \\ 0 & (g^1)^2 & 3g^2g^1 & \ldots \\ 0 & 0 & (g^1)^3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

A basis of right invariant 1-form on Γ_q^1 are given by entries of the first column of the matrix version of Maurer-Cartan form : dgg^{-1} :

$$\begin{bmatrix} dg^1 & dg^2 & dg^3 & \ldots \\ 0 & 2g^1dg^1 & 3g^2dg^1+3g^1dg^2 & \ldots \\ 0 & 0 & 3(g^1)^2dg^1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \frac{1}{(g^1)^{q(q+1)/2}} \begin{bmatrix} (g^1)^5 & -g^2(g^1)^3 & 3(g^2)^2g^1 - g^3(g^1)^2 & \ldots \\ 0 & (g^1)^4 & -3g^2(g^1)^2 & \ldots \\ 0 & & (g^1)^3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & \ldots \\ 0 & * & * & \ldots \\ 0 & & * & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

These forms ω_i are dual to the basis of the Lie algebra of Γ^1_q given by monomial vector fields $x^i\frac{\partial}{\partial x}$ $1 \leq i \leq q$ thus

$$d\omega_i = \sum_{\substack{m>n>0\\ m+n-1=i}} (m-n)\ \omega_m \wedge \omega_n.$$

The first three forms will be used in the sequel, they are :

$$\begin{aligned}\omega_1 &= \tfrac{dg^1}{g^1},\\ \omega_2 &= -g^2\tfrac{dg^1}{(g^1)^3} + \tfrac{dg^2}{(g^1)^2},\\ \omega_3 &= 3(g^2)^2\tfrac{dg^1}{(g^1)^5} - g^3\tfrac{dg^1}{(g^1)^4} - 3g^2\tfrac{dg^2}{(g^1)^4} + \tfrac{dg^3}{(g^1)^3}.\end{aligned}$$

If H is an order q invariant for the right action of an algebraic subgroup on Γ^1_q then dH is an invariant 1-form thus one can write $dH = \sum H_i\omega_i$ and H_i are also right invariant ; because the field of order q invariant has transcendence degree 1 : $dH \wedge dH_i = 0$. Differentiating $\frac{dH}{H_q} = \sum K_i\omega_i$ one gets

$$0 = \sum_i \left(dK_i \wedge \omega_i + K_i \sum_{m+n-1=i} (m-n)\omega_m \wedge \omega_n \right).$$

Because $K_q = 1$ the term $\omega_q \wedge \omega_p$ only appears in $dK_p \wedge \omega_p$ if $p > 1$, this implies $dK_p = 0$. The first part of the sum above is a multiple of ω_1 but not the second unless $q \leq 3$. This proves the first part of the lemma.

Let us assume that the minimal order of invariant is 3 then one can normalize de differential of such an invariant H :

$$\frac{dH}{H_3} = K_1\omega_1 + K_2\omega_2 + \omega_3$$

where K_1, K_2 are order less than 3 invariants, thus functions of H. One differentiates again

$$0 = dK_1 \wedge \omega_1 + dK_2 \wedge \omega_2 + K_2\omega_2 \wedge \omega_1 + 2\omega_3 \wedge \omega_1.$$

The monomial $\omega_3 \wedge \omega_2$ appears only in the second term of the sum unless K_2 is a constant.

Because K_1 in a function of H, there is some K_{11} such that $dH = K_{11}dK_1$. By comparison of $\frac{dH\wedge\omega_1}{H_3}$ with $\frac{dK_1\wedge\omega_1}{K_{11}H_3}$ one gets, $K_2 = 0$ and $K_{11}H_3 = -2$. This means that K_1 satisfies

$$dK_1 = -2K_1\omega_1 - 2\omega_3.$$

Such equation can be solved explicitely by using integrating factors and one gets

$$K_1 = \left(-2\frac{g^3}{g^1} + 3\left(\frac{g^2}{g^1}\right)^2 \right)\frac{1}{(g^1)^2} + c\frac{1}{(g^1)^2}$$

for some contant c. The level $K_1 = c$ gives the a subgroup of Γ^1_3. The cases of order 2 and 1 invariants can be done in the same way. □

Example 2.3. — *If V is the affine plane $\mathbb{A}^2$ then $R_qV = \mathbb{A}^2 \times GL_2 \times \mathbb{A}^{n_q}$ where*

$$n_q = 2\ \#\{(\alpha_1,\alpha_2) \in \mathbb{N}^2\ 2 \leq \alpha_1 + \alpha_2 \leq q\} = (q+4)(q-1)$$

with ring

$$\mathbb{C}[R_qV] = \mathbb{C}[r_1^{00}, r_2^{00}, r_1^{10}, r_1^{01}, r_2^{10}, r_2^{01}\frac{1}{(r_1^{10}r_2^{01} - r_1^{01}r_2^{10})}, r_1^{20}, r_1^{11}, \ldots, r_2^{1(q-1)}, r_2^{0q}].$$

The group Γ^2_q is an extension of GL_2. As variety it is $GL_2 \times \mathbb{A}^{n_q}$ with coordinate ring

$$\mathbb{C}[g_1^{10}, g_1^{01}, \ldots, g_2^{0q}, \frac{1}{(g_1^{10}g_2^{01} - g_1^{01}g_2^{10})}].$$

The law given by composition formulas is more difficult to write explicitely but a classification of subgroups following lemma 2.2 is done in [3].

2.2. Groupoids. — An order q jet of local automorphism of V at $a \in V$ is

$$j_q\varphi = \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le q}} \varphi_i^\alpha \frac{(r-a)^\alpha}{\alpha!}$$

of an invertible formal maps $\varphi : (V,a) \to (V,\varphi(a))$ given by power series

$$\varphi(r) = \sum_{\alpha\in\mathbb{N}^d} \varphi_i^\alpha \frac{(r-a)^\alpha}{\alpha!}.$$

The algebraic variety

$$Aut_qV = \big\{ j_q\varphi \mid \varphi : (V,a) \to (V,b) \text{ invertible formal map} \big\}$$

with coordinate ring

$$\mathbb{C}[Aut_qV] = \mathbb{C}\Big[a_i, b_j, \varphi_j^\alpha, \frac{1}{\det(\varphi_j^{\epsilon(k)})} \ \Big|\ 1 \le i,j \le d, \alpha \in \mathbb{N}^d, 1 \le |\alpha| \le q\Big]$$

is an algebraic groupoid. The groupoid structure is given by the following morphisms

– source and target $(s,t) : Aut_qV \to V \times V$ coming from the inclusion $\mathbb{C}[a_1, ..., a_d, b_1, ..., b_d] \subset \mathbb{C}[Aut_qV]$,

– composition $c : Aut_qV \underset{sV^t}{\times} Aut_qV \to Aut_qV$; $c(j_q\varphi_1, j_q\varphi_2) = j_q(\varphi_1 \circ \varphi_2)$, defined on the fiber product of source and target projections.

– inverse $in : Aut_qV \to Aut_qV ; in(j_q\varphi) = j_q\varphi^{-1}$,

– identity $id : V \to Aut_qV ; id(r) = j_q id_r$,

satisfying natural commutative diagrams for groupoids such as:

– $c\big(\varphi, c(\overline{\varphi}, \overline{\overline{\varphi}})\big) = c\big(c(\varphi, \overline{\varphi}), \overline{\overline{\varphi}}\big)$,

– $c\big(j_q\varphi, id(t(\varphi))\big) = c\big(id(s(\varphi)), j_q\varphi\big) = \varphi$,

– $c\big(in(j_q\varphi), j_q\varphi\big) = id(t(\varphi))$ and $c\big(j_q\varphi, in(j_q\varphi)\big) = id(s(\varphi))$,

and their codiagrams at coordinate ring level insuring its algebraicity. This groupoid acts on R_qV by 'target composition' : $j_q\varphi \cdot j_qr = j_q(\varphi \circ r)$ and gives an isomorphism

$$\begin{array}{ccc} \times Aut_qV \underset{sV_q}{R} V & \to & R_qV \times R_qV \\ (j_q\varphi, j_qr) & \mapsto & (j_q(\varphi \circ r), j_qr). \end{array}$$

Example 2.4. — *For $V = \mathbb{A}^1$, Aut_qV is $\mathbb{A}^1 \times \mathbb{A}^1 \times GL_1 \times \mathbb{A}^{q-1}$ with ring $\mathbb{C}[Aut_qV] = \mathbb{C}[a, b, \varphi^1, \frac{1}{\varphi^1}, \varphi^2 \ldots \varphi^q]$. The composition law for $q = 3$ is*

$$(b, c, \psi^1, \psi^2, \psi^3) \cdot (a, b, \varphi^1, \varphi^2, \varphi^3) = \big(a, c, \psi^1\varphi^1, \psi^2(\varphi^1)^2 + \psi^1\varphi^2, \psi^3(\varphi^1)^3 + 3\psi^2\varphi^2\varphi^1 + \psi^1\varphi^3\big).$$

The action on R_3V is

$$(r_0, b, \varphi^1, \varphi^2, \varphi^3) \cdot (r^0, r^1, r^2, r^3) = (b, \varphi^1r^1, \varphi^2(r^1)^2 + \varphi^1r^2, \varphi^3(r^1)^3 + 3\varphi^2r^2r^1 + \varphi^1r^3).$$

Example 2.5. — *For $V = \mathbb{A}^2$ Aut_qV is $\mathbb{A}^2 \times \mathbb{A}^2 \times GL_2 \times \mathbb{A}^{n_q}$ with ring*

$$\mathbb{C}[Aut_qV] = \mathbb{C}[a_1, a_2, b_1, b_2, \varphi_1^{10}, \varphi_1^{01}, \ldots, \frac{1}{det}, \ldots \varphi_2^{0q}].$$

Explicite formulas for composition have same flavor but are not meaningfull enough to be written explicitely here.

2.3. Subgroupoids. — A regular subgroupoid $\mathcal{G}_q$ of Aut_qV is a closed algebraic subvariety such that the induced morphisms give an algebraic groupoid structure on $\mathcal{G}_q$, [**19**, definition 2.2.].

Such groupoid can be defined using the map

$$\begin{array}{cccc}\lambda: & R_qV\times R_qV & \to & Aut_qV \\ & (r,s) & \mapsto & r\circ s^{-1}\end{array}.$$

This is the quotient map by the diagonal action of Γ_q^d on $R_qV \times R_qV$.

This map gives a one to one correspondence between algebraic regular subgroupoid $\mathcal{G}_q$ of Aut_qV and algebraic Γ_q-invariant equivalence relation $\lambda^{-1}\mathcal{G}_q \subset R_qV \times R_qV$.

Regularity is too strong for applications to Galois theory. With this definition, level sets of a rational function are not orbits of a regular groupoid. Using the λ map this means that level sets are not equivalence classes of a algebraic equivalence relation.

A (singular algebraic) subgroupoid $\mathcal{G}_q$ with singularities on a closed subvariety S of V is a closed subvariety of Aut_qV whose localisation with source and target out of S gives a subgroupoid of $Aut_q(V-S)$ [**12**, definition 4.1.1.]. Without precisions a groupoid is a singular algebraic subgroupoid of Aut_qV.

Example 2.6. — *The subvariety $Af_2(0)$ of $Aut_2\mathbb{A}^1$ given by $\varphi^2=0$ is a regular subgroupoid. More generally the equation*

$$E(j_2\varphi)=\frac{\varphi^2}{\varphi^1}+h(b)\varphi^1-h(a)=0$$

with $h\in\mathbb{C}[\mathbb{A}^1]$ satisfies

$$E(j_2\psi\cdot j_2\varphi)=E(j_2\psi)\varphi^1+E(j_2\varphi)$$

and is the equation of a regular subgroupoid $Af_2(h)$ of $Aut_2\mathbb{A}^1$. Same equation with $h\in\mathbb{C}(\mathbb{A}^1)$ defines a groupoid on $\mathbb{A}^1$ with singularities over the polar locus of h.

Example 2.7. — *The subvarieties $Pr_3(h)$ of $Aut_3\mathbb{A}^1$ given by*

$$E(j_3\varphi)=2\frac{\varphi^3}{\varphi^1}-3\left(\frac{\varphi^2}{\varphi^1}\right)^2+h(b)(\varphi^1)^2-h(a)=0$$

with $h\in\mathbb{C}(\mathbb{A}^1)$ are groupoid with singularities at poles of h because of equality

$$E(j_3\psi\cdot j_3\varphi)=E(j_3\psi)(\varphi^1)^2+E(j_3\varphi).$$

Example 2.8. — *The subvariety $Inv_0(h)$ of $Aut_0\mathbb{A}^d=\mathbb{A}^d\times\mathbb{A}^d$ given by $h(a)=h(b)$ with $h\in\mathbb{C}(\mathbb{A}^d)$ is a groupoid. If $h=\frac{p}{q}$ with $p,q\in\mathbb{C}[\mathbb{A}^d]$ the equation of the subvariety is*

$$E(a,b)=p(a)q(b)-q(a)p(b).$$

From equalities

$$q(c)E(a,b)=q(b)E(a,c)-q(a)E(b,c)\ \text{and}\ p(c)E(a,b)=p(b)E(a,c)-p(a)E(b,c)$$

one gets stability by composition out of the indeterminacy locus of h. This subvariety is the singular locus of the groupoid

Example 2.9. — *The subvarieties $Inv_1(\omega)$ of $Aut_1\mathbb{A}^d=\mathbb{A}^d\times\mathbb{A}^d\times GL_d$ given by the equations*

$$\sum_i w_i(b)\varphi_i^{\epsilon(j)}-w_j(a)\quad\forall\ 1\leq j\leq d$$

with $\omega=\sum w_i dr_i$ a rational 1-form, is a groupoid with singularities included in the zero and polar locus of the vector $(w_1,\ldots,w_d)$.

Example 2.10. — *The subvarieties $Inv_1(\mathcal{F}_\omega)$ of $Aut_1\mathbb{A}^d$ given by the equation*

$$\Big(\sum_i w_i(b)\varphi_i^{\epsilon(j)}\Big)w_k(a)-\Big(\sum_i w_i(b)\varphi_i^{\epsilon(k)}\Big)w_j(a)\quad\forall\ 1\leq k<j\leq d$$

and $\omega = \sum w_i dr_i$ *a non zero rational 1-form, is a groupoid with singularities on intersection of indeterminacy locus of* $\frac{w_1}{w_i}, \dots, \frac{w_d}{w_i}$ *for a non zero* w_i.

Example 2.11. — *The subvarieties* $Inv_1(\eta)$ *of* $Aut_1\mathbb{A}^d$ *given by the equation*

$$\sum_{i<j} n_{i,j}(b)\big(\varphi_i^{\epsilon(k)}\varphi_j^{\epsilon(\ell)} - \varphi_i^{\epsilon(\ell)}\varphi_j^{\epsilon(k)}\big) - n_{k,\ell}(a) \quad \forall\, 1 \le k < \ell \le d$$

with $\eta = \sum n_{i,j} dr_i \wedge dr_j$ *a rational 2-form, is a groupoid.*

2.4. Invariants of a groupoid. — The easiest way to define a groupoid is to start with a rational map $K : R_qV \dashrightarrow W_q$ with values in some algebraic variety. The preimage of the diagonal by $K^{\times 2} : R_qV \times R_qV \dashrightarrow W_q \times W_q$ is a subvariety EK which is an equivalence relation on the complement of the indeterminacy locus of K. Equivalence classes are level sets of K. But this relation may not be Γ_q^d-invariant. Let $\widetilde{EK}$ be the biggest Γ_q^d-invariant algebraic closed subvariety of EK giving an equivalence relation on a Zariski open set. By [7] or by theorem 1.3, one gets a $H : R_qV \dashrightarrow \overline{W}_q$ such that $\widetilde{EK} = EH$. By Γ_q^d-invariance, regularity of H at a frame j_qr does only depend on the target point on V. Because of this EH gives rise to a singular groupoid on V using the λ map. The converse of this construction, starting from a singular groupoid and finding invariants, can be found in [7, theorem 8.1.] or by more direct arguments in [19, proposition 2.3.6.]. This is a kind of Chevalley theorem for subgroupoid of Aut_qV.

Theorem 2.12. — *Let* $\mathcal{G}_q$ *be a singular subgroupoid of* Aut_qV. *There are* $H_1, \dots, H_n$ *in* $\mathbb{C}(R_qV)$ *such that, out of a codimension one subvariety* $Z \subset V$,

$$\mathcal{G}_q = \{j_q\varphi \mid \forall\ j_qr \in R_qV\ \forall\ i,\ \ H_i(j_qr \cdot j_q\varphi) = H_i(j_qr)\}.$$

Groupoids are essentially characterized by their field of rational invariants $F_q \subset \mathbb{C}(R_qV)$.

Example 2.6. *The field of invariants of* $Af_2(h)$ *is the subfield of* $\mathbb{C}(R_2\mathbb{A}^1)$ *generated by* $\frac{r^2}{(r^1)^2} + h(r^0)$.

Example 2.7. *The field of invariants of* $Pr_3(h)$ *is the subfield of* $\mathbb{C}(R_3\mathbb{A}^1)$ *generated by* $\left(2\frac{r^3}{r^1} - 3\left(\frac{r^2}{r^1}\right)^2\right)\frac{1}{(r^1)^2} + h(r^0)$.

Example 2.8. *The field of invariants of* $Inv_0(h)$ *is the subfield of* $\mathbb{C}(R_0\mathbb{A}^d)$ *generated by* $h(r)$.

Example 2.9. *The field of invariants of* $Inv_1(\omega)$ *is the subfield of* $\mathbb{C}(R_1\mathbb{A}^d)$ *generated by* $\sum_i w_i(r)r_i^{\epsilon(j)}$ *for all* $1 \le j \le d$.

Example 2.10. *The field of invariants* $Inv_1(\mathcal{F}_\omega)$ *is the subfield of* $\mathbb{C}(R_1\mathbb{A}^d)$ *generated by :*

$$\frac{\sum_i w_i(r)r_i^{\epsilon(j)}}{\sum_i w_i(r)r_i^{\epsilon(k)}} \text{ for all } 1 \le j < k \le d.$$

Example 2.11. *The field of invariants* $Inv_1(\eta)$ *is the subfield of* $\mathbb{C}(R_1\mathbb{A}^d)$ *generated by :*

$$\sum n_{i,j}\big(r_i^{\epsilon(k)}r_j^{\epsilon(\ell)} - r_i^{\epsilon(\ell)}r_j^{\epsilon(k)}\big) \text{ for all } 1 \le j < k \le d.$$

3. Pseudogroups

3.1. Prolongations of maps of V. — Frame spaces are natural spaces is the sense that a biholomorphism $\varphi : U_1 \to U_2$ between two open sets of V induces a biholomorphism between open sets of R_qV :

$$\begin{array}{rccc} R_q\varphi & : & R_qU_1 & \to & R_qU_2 \\ & & j_qr & \mapsto & j_q(\varphi \circ r) \end{array}$$

called the order q prolongation of φ. These prolongations are defined by polynomial formulas and can be extend to formal biholomorphism $\widehat{\varphi} : \widehat{V,a} \to \widehat{V,b}$. The prolongation is $R_q\widehat{\varphi} : (\widehat{R_qV, R_qV_a}) \to (\widehat{R_qV, R_qV_b})$ a formal biholomorphism from a formal neighborhood of frames at $a \in V$ to formal neighborhood of frames at b. The jet of order k of $R_q\varphi$ at some frame j_qr at $r(0)$ only depends on the jet of order $q+k$ of φ at $r(0)$.

Example 3.1. — *Let V be the affine line. If $\varphi : U_1 \to U_2$ is a biholomorphism between open sets of V its third prolongation is $R_3\varphi : U_1 \times GL_1(\mathbb{C}) \times \mathbb{C}^2 \to U_2 \times GL_1(\mathbb{C}) \times \mathbb{C}^2$ and $R_3\varphi(r^0, r^1, r^2, r^3)$ is*

$$\Big(\varphi(r^0),\ \varphi'(r^0)r^1,\ \varphi''(r^0)(r^1)^2 + \varphi'(r^0)r^2,\ \varphi'''(r^0)(r^1)^3 + 3\varphi''(r^0)r^1r^2 + \varphi'(r)r^3\Big)$$

Let X be a holomorphic vector field on an open set U of V. Prolongations of its flows define a local 1-parameter action on R_qU as it is stated in the lemma below. The infinitesimal generator of this action is R_qX the prolongation of X.

Lemma 3.2. — *For couple of composable biholomorphisms (φ, ψ) one gets $R_q\varphi \circ R_q\psi = R_q(\varphi\circ\psi)$. For a couple of vector fields (X, Y) one gets $[R_qX, R_qY] = R_q[X, Y]$.*

Properties of prolongations can be understand by using Cartan derivations. These derivations are given by the action of the derivation $\frac{\partial}{\partial x_i}$ on $\mathbb{C}[R_qV]$, the ring of PDE in d functions, $r_1, \ldots, r_d$ of d variables $x_1, \ldots, x_d$ in the neighborhood of 0 :

$$\begin{array}{cccc} D_i & : & \mathbb{C}[R_qV] & \to & \mathbb{C}[R_{q+1}V] \\ & & r_j^\alpha & \mapsto & r_j^{\alpha+\epsilon(i)} \end{array}.$$

The proof of the following lemma is left to the reader following [19, pp258–270] or [16].

Lemma 3.3. —

– *Let $\varphi : U_1 \to U_2$ be a local biholomorphism on V and $(R_q\varphi)^* : \mathbb{C}[R_qV] \underset{\mathbb{C}[V]}{\otimes} \mathcal{O}(U_2) \to \mathbb{C}[R_qV] \underset{\mathbb{C}[V]}{\otimes} \mathcal{O}(U_1)$ the induced isomorphism of rings then*

$$D_i \circ (R_q\varphi)^* = (R_{q+1}\varphi)^* \circ D_i.$$

– *Let X be a local holomorphic vector field $U \subset V$ then*

$$D_i \circ R_qX = R_{q+1}X \circ D_i.$$

– *The order q prolongation of a vector field $X = \sum_j a_j \frac{\partial}{\partial r_j}$ is*

$$R_qX = \sum_{\substack{0\le j\le d\\ \alpha\in\mathbb{N}^d}} D^\alpha a_j \frac{\partial}{\partial r_j^\alpha}.$$

Example 3.4. — *Let V be the affine space of dimension d over $\mathbb{C}$ with coordinate ring $\mathbb{C}[r_1, \ldots, r_d]$ the order 1 frame bundle is $R_1V = V \times GL_d(\mathbb{C})$ with coordinate ring $\mathbb{C}\left[r_1, \ldots, r_d, r_1^1, \ldots, r_d^d, \frac{1}{det(r_i^j)}\right]$. If $X = \sum a_j(r)\frac{\partial}{\partial r_j}$ then*

$$R_1X = \sum a_j(r)\frac{\partial}{\partial r_j} + \sum \frac{\partial a_j}{\partial r_i}(r) r_i^k \frac{\partial}{\partial r_j^k}.$$

When $r(t)$ is a trajectory of X then the restriction of R_1X above this trajectory is

$$\frac{\partial}{\partial t} + \sum \frac{\partial a_j}{\partial r_i}(r(t)) r_i^k \frac{\partial}{\partial r_j^k}$$

i.e. the first variational equation of X along $r(t)$ in fundamental form.

3.2. Pseudogroups. — Because of projections π_q^{q+1} one can defined the formal frame bundle $RV = \underleftarrow{\lim} R_qV$ with structural group $\Gamma = \underleftarrow{\lim}\Gamma_q$. The field of rational functions of any order $\mathbb{C}(RV) = \underrightarrow{\lim}\ \mathbb{C}(R_qV)$ with Cartan derivations is a differential field. It is the D-differential field generated by $\mathbb{C}(V)$. These projections can also be used to define $AutV = \underleftarrow{\lim} Aut_qV$.

Definition 3.5. — *The groupoid $AutV$ is called the pseudogroup of points transformations of V.*

It is the proalgebraic variety of formal invertible maps between formal neighborhood of points of V. This space is a groupoid on V rather that a pseudogroup but because its elements are actual (even if formal) transformations of V, we will use the word pseudogroup. To refer to its proalgebraic structure, the more precise terminology of algebraic Lie pseudogroup will be used.

Let $\mathcal{G}_q$ be a groupoid with invariant field F_q. Let D^1F_q be the field generated by F_q and the D_iF_q in $\mathbb{C}(R_{q+1}V)$. It defines a subgroupoid $D^1\mathcal{G}_q$ of $Aut_{q+1}V$. A biholomorphism φ is said to preserve F_q if for any $H \in F_q$, $H \circ R_q\varphi = H$. If φ preserves F_q it satisfies a system of pde's of order q. Moreover it preserves the field D^1F_q and thus satisfies a system of pdes of order $q+1$. The latter is obtained from the former by derivations of equations with respect to the independent variables.

The differential field F generated by all the F_q defines a subvariety $\mathcal{G}$ of $AutV$ whose projection on Aut_qV, $(\mathcal{G})_q$, can be smaller than $\mathcal{G}_q$. By [12, theorem 4.4.1.] these $(\mathcal{G})_q$ are subgroupoid with singularities on $S \subset V$ independent of q.

Definition 3.6. — *An algebraic Lie subpseudogroup $\mathcal{G}$ of $AutV$ is a subgroupoid defined by a differential subfield of $\mathbb{C}(RV)$.*

For each q one gets a subfield of $\mathbb{C}(R_qV)$ thus a groupoid $\mathcal{G}_q$. Because the field is differential one gets the inclusion $\mathcal{G}_{q+1} \subset D^1\mathcal{G}_q$.

Example 2.6. *The pseudogroup given by the equation of this example is $Af(h)$ given by invertible formal maps φ on V such that*

$$\frac{\varphi''(r)}{\varphi'(r)} + h \circ \varphi(r)\varphi'(r) = h(r)$$

Example 2.7. *The pseudogroup given by the equation of this example is $Pr(h)$ given by invertible formal maps φ on V such that*

$$2\frac{\varphi'''(r)}{\varphi'(r)} - 3\Big(\frac{\varphi''(r)}{\varphi'(r)}\Big)^2 + h \circ \varphi(r)\big(\varphi'(r)\big)^2 = h(r).$$

Example 2.8. *The pseudogroup given by the equation of this example is given by invertible formal maps φ on V such that*

$$h \circ \varphi(r) = h(r).$$

Example 2.9. *Let $\mathcal{G}_1$ be the groupoid of this example and ω be the differential 1-form $\sum w_i(r)dr_i$. The pseudogroup $\mathcal{G}$ generated by $\mathcal{G}_1$ is given by invertible formal maps φ on V such that*

$$\varphi^*\omega = \omega.$$

This is an example where $(\mathcal{G})_1$ is different from $\mathcal{G}_1$. From differentiation and linear combinaison of equations of the order 1 system above, one gets a new order one equations :

$$\varphi^*d\omega = d\omega.$$

In general the second system is not an algebraic consequence of the first.

Assume $d = 2$ and $\omega = dr_1$ then the second system is empty and solutions are

$$\varphi(r_1, r_2) = (r_1 + c, \varphi_2(r_1, r_2))$$

where c is a constant and φ_2 *is any holomorphic function in two variables.*

Assume $d = 2$ *and* $\omega = r_2 dr_1$ *then solutions are*

$$\varphi(r_1, r_2) = (\varphi_1(r_1), \frac{r_2}{\varphi_1'(r_1)})$$

where φ_1 *is any holomorphic functions in one variable.*

These two pseudogroups are different but the difference can not be seen directly at the $\mathcal{G}_1$ *level.*

Example 2.10. *Let* ω *be the differential* 1*-form* $\sum w_i(r)dr_i$*. The pseudogroup is given by the* φ *such that*

$$\varphi^*\omega \wedge \omega = 0.$$

From differentiation and linear combinaison of equations of the order 1 system above, one get a new order one equations :

$$(\varphi^* d\omega) \wedge \omega - \varphi^*\omega \wedge d\omega = 0.$$

Assume $d = 3$ *and* $\omega = dr_1$ *then the second system is empty and solutions are* $\varphi(r_1, r_2, r_3) = (r_1 + c, \varphi_2(r_1, r_2, r_3), \varphi_3(r_1, r_2, r_3))$ *where c is a constant and* φ_2, φ_3 *are holomorphic functions in three variables.*

Assume $d = 3$ *and* $\omega = dr_1 - r_2 dr_3$ *then after differentiation one gets that a* φ *preserving* ω *also preserves* $\frac{dr_2}{r_2}$ *and* $r_2 dr_3$*. Such* φ *are* $\varphi(r_1, r_2, r_3) = (r_1 + c_1, \frac{r_2}{c_2}, c_2 r_3)$ *where* c_1, c_2 *are two constants.*

These two pseudogroups are very different.

Example 2.11. *Let* ω *and* η *be the differential* 1*-form* $\sum w_i(r)dr_i$ *and* 2*-form* $\sum n_{i,j}(r)dr_i \wedge dr_j$ *The pseudogroup is given by the* φ *such that*

$$\varphi^*\omega = \omega \quad \textit{and} \quad \varphi^*\eta = \eta.$$

Assume $d = 2$ *and* $\omega = dr_1$ *and* $\eta = dr_1 \wedge dr_2$ *then solutions are* $\varphi(r_1, r_2) = (r_1 + c, r_2 + \varphi_2(r_1))$ *where c is a constant and* φ_2 *is an holomorphic functions.*

Let F be a differential subfield of $\mathbb{C}(RV)$. Let us define

$$Iso(F) = \{\text{formal biholomorphism } \varphi : \widehat{V, a} \to \widehat{V, b} \mid \forall q, \forall H \in F_q, H \circ R_q\varphi = H\}$$

whose 'Lie algebra' is

$$\mathfrak{iso}(F) = \{\text{formal vector field } Y \text{ on } \widehat{V, a} \mid \forall q, \forall H \in F_q, R_q Y \cdot H = 0\}$$

and Iso_q, $\mathfrak{iso}_q$ the closures of theirs projections on order q jets spaces.

Remark 3.7. — *The equation* $H \circ R_q\varphi = H$ *has to be understood as* $(P \circ R_q\varphi)Q = P(Q \circ \varphi)$ *for* P *and* Q *in* $\mathbb{C}[R_qV]$ *such that* $\frac{P}{Q} = H$*.*

Theorem 3.8. — *The subspace* $Iso(F)$ *of* $AutV$ *is stable by composition and inversion. The linear space* $\mathfrak{iso}(F)$ *of formal vector fields is stable by Lie bracket.*

This last space is not a bundle of Lie algebras but Lie algebras parametrized by V in the sense of [10]. The stability claimed above is a set theoretical stability. The proalgebraic variety $Iso(F)$ is singular subgroupoid and the singularities are unavoidable and prevent stability by composition of $Iso_q(F)$. But the theorem says that the set of formal solutions is a set theoretical groupoid. Let us show the difficulty on an example.

Example 2.6. *An differential equation of definition of the pseudogroup* $Af(h)$ *is*

$$E(\varphi) = \frac{\varphi''}{\varphi'} q(r)q(\varphi) + p(\varphi(r))q(r)\varphi'(r) - p(r)q(\varphi)$$

where $h = p/q$, $p, q \in \mathbb{C}[\mathbb{A}^1]$*. Because* $q(\psi)E(\varphi \circ \psi) = q(r)E(\varphi) \circ \psi\psi' + q(\varphi \circ \psi)E(\psi)$ *Stability by composition is valid only when source at target are out of* $\{q = 0\}$*.*

When $d = 1$, pseudogroup are defined by ordinary differential equations. One can use the fact that formal solutions of ODE are asymptotic developements of holomorphic solutions to prove stability under composition from stability above a Zariski open set. When $d > 1$ we have to use weaker tools like Artin approximation theorems [1]. This is done in [5].

3.3. Malgrange pseudogroup. —

Definition 3.9. — *Let X be a rational vector field on V its field of order q differential invariants is*

$$Inv_q(X) = \{H \in \mathbb{C}(R_qV) \mid R_qX \cdot H = 0\}.$$

Let $Inv(X)$ be the differential field of all differential invariant of any order then Malgrange pseudogroup of a rational vector field X is

$$Mal(X) = Iso(Inv(X))$$

whose Lie algebra is

$$\mathfrak{mal}(X) = \mathfrak{iso}(Inv(X)).$$

Isotopy group at a point $a \in V$ is the group of elements of $Mal(X)$ whose source and target are a. It is denoted by $Mal(X)_{a,a}$. Generically its Lie algebra is given by elements of $\mathfrak{mal}(X)$ vanishing at a, these linear spaces are denoted by $\mathfrak{mal}(X)^0_a \subset \mathfrak{mal}(X)_a \subset \mathfrak{mal}(X)$.

Order one invariant are more usual to work with. They correspond to X-invariant rational tensor field on V.

Example 3.10. — *A vector field Y on V defines d functions on R_1V, $H_i(j_1r)$ $1 \leq i \leq d$ by its coordinates in a $AutV$-invariant basis:*

$$Y = \sum_i H_i(j^1r)\big(\sum_j r_j^{\epsilon(i)}\frac{\partial}{\partial r_j}\big).$$

The equations of X-invariance $\mathcal{L}_XY = 0$ or $\varphi^*Y = Y$ are equivalent to $R_1X \cdot H_i = 0$ or $H_i \circ R_1\varphi = R_1\varphi$ for all $1 \leq i \leq d$. The vector field X itself is X-invariant: $L_XX = [X,X] = 0$ thus any vector field gets invariant and $MalX$ is never $AutV$.

Example 3.11. — *If ω is a X-invariant 1-form on V. It defines d functions on R_1V : $H_i(j_1r) = \omega(\sum_j r_j^{\epsilon(i)}\frac{\partial}{\partial r_j})$ and $\mathcal{L}_X\omega = 0$ is equivalent to $R_1X \cdot H_i = 0$ for all $1 \leq i \leq d$. The pseudogroup $\mathcal{G}$ of formal maps φ preverving this form: $\varphi^*\omega = \omega$ is described by order one equations of example 2.9.*

Example 3.12. — *Let $d = 2$ and $X = \frac{\partial}{\partial r_1} + r_2\frac{\partial}{\partial r_2}$. Then dr_1 and $\frac{dr_2}{r_2}$ are X-invariant forms. Formal map $\varphi \in MalX$ preserves these two forms thus $\varphi(r_1, r_2) = (r_1+c_1, c_2r_2)$ for two constants. There is a third X-invariant tensor field X itself so $\varphi \in Mal(X)$ must satisfied $\varphi^*X = X$. On this example it is a consequence of previous equation. Because order 1 invariant are 4 independent functions on $R_1\mathbb{A}^2$, if a new invariant exist it must give a order 0 invariant i.e. a rational first integral of X, this does not exist.*

Lemma 3.13. — *Let X be a rational vector field on V and $\mathcal{F}_X$ be the set theoretical pseudogroup generated its flows on opens sets of V. The Malgrange pseudogroup is the Zariski closure of $\mathcal{F}_X$ in $AutV$.*

Proof. — Because Malgrange pseudogroup is closed and closed pseudogroup are characterized by their differential invariants field, one have to prove that the closure of $\mathcal{F}_X$ is a pseudogroup.

One gets two prolongations of X on each factor of the product $RV \times RV$. Because these prolongations commute with Γ, one gets two commuting vector field R^sX and R^tX on $AutV = (RV \times RV)/\Gamma$. The set of Taylor expansions for elements of the set theoretical pseudogroup generated by X is the union of integral curves of R^tX (or R^sX) going through identities.

Let's work on order q jets spaces. By Gomez-Mont theorem 1.3, there is a rational map $(H,s) : Aut_qV \dashrightarrow N_q \times V$ whose generic fiber is the closure of a generic integral curve of R^tX.

Let $\mathcal{L}(\varphi)$ be the integral curve through $\varphi \in Aut_qV$. Then

$$\begin{array}{rccc} R^s\varphi^{\circ -1}: & \{\psi \mid s(\psi) = s(\varphi)\} & \to & \{\psi \mid t(\psi) = s(\varphi)\} \\ & \psi & \mapsto & \psi \circ \varphi^{\circ -1} \end{array}$$

is an isomorphism sending φ to the identity at $a = t(\varphi)$. For generic values of $a \in V$ the algebraic closure of $L(id(a))$ is given by the fiber of (H,s). For source and target in U a Zariski open dense subset of V, the Zariski closure of integral curves through identities is given by equations $H = H \circ id$.

These are equations of a groupoid whose invariants are given by the map $H : R_qV \to N_q$ obtained when a generic value of the source is fixed. □

There is no complete theory developed for this object involving a Galois correspondence. A first step in this direction is the following projection theorem [4].

Theorem 3.14. — *Let (V, X) and (W, Y) be two algebraic varieties with rational vector field. If $\pi : W \dashrightarrow V$ is a dominant map such that Y si π-projectable on X. Then $Mal(Y)$ is π-projectable and*

$$\pi_* Mal(Y) = Mal(X).$$

The pseudogroup of π-projectable biholomorphisms is

$$Aut(\pi) = \{\varphi \in Aut(W) \mid \exists\ \pi_*\varphi \text{ in } Aut(V) \text{ with } \pi \circ \varphi = \pi_*\varphi \circ \pi\}.$$

A pseudogroup is said to be π-projectable if it is a subpseudogroup of $Aut(\pi)$. The vector field Y is π-projectable on X for any $f \in \mathbb{C}(V)$ $Y(f \circ \pi) = X(f)$

The pseudogroup of π-projectable morphims is given by differential invariants of order 1 and implies $Mal(Y) \subset Aut(\pi)$. This is a more general version of example 2.10.

This theorem can be readen at the algebra level: let $\mathfrak{mal}_a Y$ and $\mathfrak{mal}_{\pi(a)} X$ be the formal vector field of the Lie algebras at a generic point a then one gets a sujective morphism of Lie algebras $\pi_*\mathfrak{mal}Y_a \to \mathfrak{mal}X_{\pi(a)}$.

The difficulty of the proof comes from the fact composable projections $\pi_*\varphi$ and $\pi_*\psi$ can be projections of non composable biholomorphisms thus stability by composition of projections is not direct and is wrong in general.

Proof. — The projection π gives a projection π_* from $Aut(\pi)$ to $AutV$. Let $\pi_* Mal(Y)$ be the Zariski closure of the image of $Mal(Y)$. This is an algebraic subvariety of $AutV$ containing $\mathcal{F}_X$. By lemma 3.13 it contains $Mal(X)$.

To prove the other inclusion, let us define $R(\pi)$ to be the frames that conjugate π to the projection on first coordinates. This projection gives a map $\pi_* : R(\pi) \to RV$. Using this maps invariants of X can be lifted on $R(\pi)$ and are invariant by the restriction of RY on $R(\pi)$. Because Γ commute with RY, the foliation defined by level set of these functions can be prolonged to the whole RW. Then by 1.3 one can find first integral for this foliation *i.e.* differential invariant of Y projectable on differential invariant of X. One has $Mal(Y) \subset \pi_*^{-1}(Mal(X))$ thus the converse inclusion. This proves the theorem. □

4. Differential equations

4.1. Linear constant case. — Let X be a right invariant vector field on $GL_n(\mathbb{C})$. The matrix $r = (r_i^j)$ is the matrix of coordinates on the group and $\frac{\partial}{\partial r} = \left(\frac{\partial}{\partial r_j^i}\right)$ the corresponding matrix of derivations. There is a $n \times n$ constant matrix A such that $X = Ar\frac{\partial}{\partial r}$. The flows of such a vector field are given by left translation and leave the Maurer-Cartan matrix form $r^{-1}dr$ invariant. We get the differential invariants explicitely by writing r as functions of $x_1, \ldots x_{n^2}$ and going to the order 1 frame bundle. Let $\partial_k r$ be the n^2 matrices $((r_i^j)^{\epsilon(k)})$ then entries of all matrices $y^{-1}\partial_k y$ give n^4 independent functions on $R_1GL_n(\mathbb{C})$. These are invariants for R_1X. Because $\dim R_1GL_n(\mathbb{C}) - \dim GL_n(\mathbb{C}) = n^4$. These functions generate $\mathbb{C}(RGL_n(\mathbb{C}))$ over $\mathbb{C}(GL_n(\mathbb{C}))$ as differential field for Cartan derivations. Any other independent differential invariant of X is an ordre 0 one *i.e.* a rational first integral of X.

One gets the description given in the introduction and Malgrange pseudogroup is the algebraic hull of X : the smallest algebraic group that contains X in is right Lie algebra.

4.2. Linear case. — Let $\mathscr{C}$ be an algebraic curve over $\mathbb{C}$, $E \xrightarrow{\pi} \mathscr{C}$ a principal G-bundle, *i.e.* $E \underset{\mathscr{C}}{\times} E \sim E \times G$ over E for the first projection and G is an algebraic linear group. For a π-projectable, G-invariant rational vector field X on E with $\pi_* X \not\equiv 0$, PX denotes a closed minimal X-invariant subvariety of E dominating $\mathscr{C}$ and $Gal(X)$ its stabilizer in G.

– Two such PX are isomorphic under action of G and called Picard-Vessiot varieties of X. The field extension $\mathbb{C}(\mathscr{C}) \subset \mathbb{C}(PX)$ is usually called the Picard-Vessiot extension for X.

– The group $Gal(X)$ is well defined up to conjugation in G. It is the Galois group of X.

– Common level sets of all rational first integrals of X in $\mathbb{C}(E)$ dominating $\mathscr{C}$ are Picard-Vessiot varieties.

Malgrange pseudogroup of such a X is simple to describe. Let $Z_1, \cdots, Z_N$ be infinitesimal generators of the action of G then $X, Z_1, \ldots, Z_N$ is a X-invariant rational parallelism of E *i.e.* a basis of the $\mathbb{C}(E)$ vector space of rational vector field on E such that $[X, Z_i] = 0$. Let $\mathbb{C}(E)^X$ be the field of rational first integrals of X. One has

$$Mal(X) = \{\varphi \mid \varphi^* X = X, \forall i\ \varphi^* Z_i = Z_i, \forall F \in \mathbb{C}(E)^X\ F \circ \varphi = F\}.$$

The inclusion '$\subset$' is clear from the definition. To prove the other inclusion one remarks that X and Z's give rise to a basis of invariant of order 1, $\mathbb{C}(R_1 E)$ is generated over $\mathbb{C}(E)$ by these invariants. This implies that $\mathbb{C}(R_q E)$ is generated over $\mathbb{C}(E)$ by derivatives of these invariants. Each new differential invariant for X reduces modulo this field of invariants to order 0 invariant *i.e.* to a rational first integral of X.

Let $Mal(X)_a$ be the restriction of this pseudogroup to the fiber E_a at generic $a \in \mathscr{C}$. The fiber E_a is isomorphic to G by choosing a point $e \in G$. If $e \in Gal(X)$ this isomorphism send PX_a on $Gal(X)$.

This isomorphism conjugates the action of G on E_a to the left translation on G. Because the action of $Mal(X)$ commutes to left translation on G each $\varphi \in Mal(X)_a$ is the restriction on some open set of right translation by a $g_\varphi \in G$. But $Mal(X)_a$ must preserve $Gal(X)$ so $g_\varphi \in Gal(X)$. We have proved the following theorem.

Theorem 4.1. — *Under this isomorphism $Mal(X)_a$ equals $Gal(X)$ as pseudogroup generated by a subgroup of G.*

4.3. Equations integrable by quadratures. —

Definition 4.2. — *Let $\mathbb{C}(t)$ be the differential field of rational functions with derivations $\partial = \frac{\partial}{\partial t}$ and $(K, \overline{\partial})$ be a differential extension of $(\mathbb{C}(t), \partial)$. It is said to be Liouvillian if one can find a tower of differential extensions*

$$(\mathbb{C}(t), \partial) = (K_0, \partial_0) \subset (K_1, \partial_1) \ldots \subset (K_p, \partial_p) = (K, \overline{\partial})$$

such that $K_{i-1} \subset K_i$ is one of the following

– *algebraic,*
– *additive $K_i = K_{i-1}(z_i)$ with $\partial_i z_i \in K_{i-1}$,*
– *mutiplicative $K_i = K_{i-1}(z_i)$ with $\frac{\partial_i z_i}{z_i} \in K_{i-1}$.*

Liouvillian functions are elements of Liouvillian extensions.

Assume transcendence degree of K over $\mathbb{C}$ is $p+1$. Let W be a model for a field L *i.e.* $\mathbb{C}(W) = L$. Because this field is differential, W is endowed with a rational vector field Y.

Lemma 4.3. — *There exist p rational differential 1-forms $\theta_0, \ldots, \theta_p$ on W satisfying $d\theta_0 = 0$, $d\theta_1 = 0$ and $d\theta_i = 0 \mod \theta_j, 1 \leq j \leq i-1,\ 2 \leq i \leq p$.*

Proof. — The construction of these forms is direct from the definition of Liouvillian extension. Let $t = z_0, \ldots, z_p$ be a transcendence basis given by the definition then

$$\partial_i = \partial_{i-1} + \sum r_{i-1} \partial_{z_i}$$

where ∂_{z_i} stands for $\frac{\partial}{\partial z_i}$ in the additive case and $z_i\frac{\partial}{\partial z_i}$ in the multiplicative case and $r_{i-1} \in K_{i-1}$. The forms are $\theta_0 = dt = dz_0$, $\theta_i = dz_i - r_{i-1}dz_0$ in additive cases or $\theta_i = \frac{dz_i}{z_i} - \sum r_{i-1}dz_0$ in multiplicative cases. □

Proposition 4.4. — *The Lie algebra of the Malgrange pseudogroup of Y at a generic point of W is solvable.*

Proof. — Let a be a generic point on W. Let $s_0, \dots, s_p$ be analytic coordinates in a neighborhood of a such that $ds_i = \theta_i \mod (\theta_1, \dots, \theta_{i-1})$ then $Y \in \mathfrak{mal}X$ can be written

$$Y = c_0\frac{\partial}{\partial s_0} + c_1\frac{\partial}{\partial s_1} + c_2(s_1)\frac{\partial}{\partial s_2} + \dots + s_d(s_1, \cdots, s_{p-1})\frac{\partial}{\partial s_p}.$$

The pth derived algebra of this type of Lie algebra of formal vector field is zero. □

Remark 4.5. — *The proof of theorem 3.8 implies that such property is true at any point of W*

Definition 4.6. — *Let $y^{(n+1)} = E(t, y, y', \dots y^{(n)}) \in \mathbb{C}(t, y', \dots, y^{(n)})$ be an order $n+1$ differential equation. The equation E is said to be integrable by quadratures if there is a Liouvillian solution f with transc.deg.$\mathbb{C}(t, f, f', \dots, f^{(n)})/\mathbb{C}(t) = n+1$.*

Remark 4.7. — *It is important to allow new constants in order to get $y'' = 0$ integrable by quadratures.*

One defines the Malgrange pseudogroup of E as the one of

$$\frac{\partial}{\partial t} + y_1\frac{\partial}{\partial y} + \dots + y_n\frac{\partial}{\partial y_{n-1}} + E(t, y, y_1, \dots, y_n)\frac{\partial}{\partial y_n}.$$

A consequence of the proposition 4.4 and theorem 3.14 is

Theorem 4.8. — *If a rational ordinary differential equation is integrable by quadratures then the Lie algebra of its Malgrange pseudogroup is solvable.*

This theorem enable us to give a new proof of a theorem of M. Singer [21].

Theorem 4.9. — *If $y' = E(x, y) \in \mathbb{C}(x, y)$ is integrable by quadratures then there exist a closed rational 1-form α on $\mathbb{A}^2$ such that*

$$H = \int \exp(\int \alpha)(dy - Edx)$$

is a first integral of $X = \frac{\partial}{\partial x} + E(x, y)\frac{\partial}{\partial y}$

By writing conditions on $\exp(\int \alpha)(dy - Edx)$ to be closed on gets that $\alpha = -\frac{\partial E}{\partial y}dx + R(dy - Edx)$ for some $R \in \mathbb{C}(x, y)$ such that

$$\frac{\partial^2 E}{\partial y^2} + R\frac{\partial E}{\partial y} + \frac{\partial R}{\partial x} + E\frac{\partial R}{\partial y} = 0.$$

This equation was already known by J. Drach [6] and E. Vessiot [24] as 'resolvant equation': existence of rational solution is equivalent to existence of order two invariant for X.

Proof. — Let us go back to notations of the first part. The equation is a vector field $X = \frac{\partial}{\partial r_1} + E(r_1, r_2)\frac{\partial}{\partial r_2}$ on the affine plane $\mathbb{A}^2$. This vector field has some differential invariants which define the Malgrange pseudogroup $MalX$ and for each values of them a subgroup G of Γ^2 well defined up to conjugaison. For a generic point $p \in \mathbb{A}^2$, the isotropy group of $MalX$ at p is isomorphic to G (they are reciprocal). The Lemma 2.2 will give us the possibilities for G then one will check in each situation if $MalX$ is solvable or not.

The 1-form dr_1 is X-invariant then r_1^{10} and r_1^{01} are differential invariants of X. The vector field X is X-invariant then

$$\frac{r_2^{01} - a(r_1, r_2)r_1^{01}}{r_1^{10}r_2^{01} - r_1^{01}r_2^{10}} \quad \text{and} \quad \frac{r_2^{10} - a(r_1, r_2)r_1^{10}}{r_1^{10}r_2^{01} - r_1^{01}r_2^{10}}$$

are differential invariants of X. Let V be the subvariety of $R\mathbb{A}^2$ defined by the values $1,0,1,0$ for these four invariants and 0 for theirs derivatives. Let V_{min} a minimal RX-invariant subvariety of V. The stabilizer of V in Γ^2 is a subgroup containing G the stabilizer of V_{min}. It is defined by the equations $g_1^{10}=1$, $g_1^{nm}=0$ for all $n>1$ or $m\geq 0$, $g_2^{nm}=0$ for all $n>0$ and $m\geq 0$. Solutions of this subgroup are formal maps $(x_1,x_2)\mapsto(x_1,g_2(x_2))$ and G can be identified with a subgroup of Γ^1.

Using Lemma 2.2 one gets six possibilities, G is Γ^1, a subgroup of Γ^1_3 of one of the three types given by the lemma, it is just the identity or X has a rational first integral.

In the first case, G is not solvable so is $MalX$. In the second case the order 3 equation of G is

$$2\frac{g_2^{03}}{g_2^{01}}-3\left(\frac{g_2^{02}}{g_2^{01}}\right)^2+c(g_2^{01})^2-c=0.$$

Up to conjugaison by $h(x_1,x_2)=(x_1,\sum\frac{1}{n!}(\sqrt{-c}x_2)^n)$, one can assume $c=0$.

An invariant of its action on $R_3\mathbb{A}^2$ is

$$\left(2\frac{r_2^{03}}{r_2^{01}}-3\left(\frac{r_2^{02}}{r_2^{01}}\right)^2\right)\left(\frac{1}{r_2^{01}}\right)^2$$

and the equation of V_{min} is

$$\left(2\frac{r_2^{03}}{r_2^{01}}-3\left(\frac{r_2^{02}}{r_2^{01}}\right)^2\right)\left(\frac{1}{r_2^{01}}\right)^2=R(r_1,r_2)$$

for some $R\in\mathbb{C}(r_1,r_2)$. The sub-pseudogroup of $Aut(\mathbb{A}^2)$ preserving all these invariants is $MalX$. Let r_1,r_2 be an analytic frame whose formal Taylor expansion at $0\in\mathbb{C}^2$ is in V_{min} then in these new coordinates $\varphi\in MalX$ is a solution of

$$\varphi_1^{10}=1,\varphi_1^{nm}=0\text{ for }n\geq 2,\varphi_2^{nm}=0\text{ for }n\geq 1\text{ and }2\frac{\varphi_2^{03}}{\varphi_2^{01}}-3\left(\frac{\varphi_2^{02}}{\varphi_2^{01}}\right)^2=0.$$

These equations describe the group of homographies with Lie algebra SL_2, it is not a solvable pseudogroup.

The others cases are subcases of the third case *i.e.* the order 2 part of G is given by :

$$\frac{g_2^{02}}{g_2^{01}}+cg_2^{01}-c=0$$

for a constant c. One can assume $c=0$ by using conjugaison by $h(x_1,x_2)=(x_1,\sum\frac{1}{n!}(cx_2)^n)$. The equation of V_{min} is

$$\frac{r_2^{02}}{r_2^{01}}\frac{1}{r_2^{01}}=R(r_1,r_2)$$

for some $R\in\mathbb{C}(r_1,r_2)$. Invariance of V_{min} by R_1X is exactly the resolvant equation. Now in the coordinates given by a solution of V_{min}, the pseudogroup of invariance of these invariants is given by φ such that

$$\varphi(x_1,x_2)=(x_1+c_1,a_2x_2+c_2)$$

for some constants a_2,c_2,c_1. This pseudogroup is solvable and $MalX$ is included in it.

The Liouvillian first integral is given by $H(r_1,r_2)=x_2(r_1,r_2)$ the second coordinate of the inverse of a solution of V_{min}. □

Example 4.10. — *Let considere the equation $y'=a(x)y+b(x)$ with rational a and b. In coordinates of this article it is the vector field*

$$X=\frac{\partial}{\partial r_1}+\Big(a(r_1)r_2+b(r_1)\Big)\frac{\partial}{\partial r_2}.$$

It first prolongation is

$$\frac{\partial}{\partial r_1}+(a(r_1)r_2+b(r_1))\frac{\partial}{\partial r_2}+\Big(a'(r_1)r_2r_1^{10}+b'(r_1)r_1^{10}a(r_1)r_2^{10}\Big)\frac{\partial}{\partial r_2^{10}}+\Big(a'(r_1)r_2r_1^{01}+b'(r_1)r_1^{01}a(r_1)r_2^{01}\Big)\frac{\partial}{\partial r_2^{01}}.$$

In order to get invariants and the Malgrange pseudogroup, one needs to find the rational first integrals of the second prolongation. It is feasable but not easy. Let us check the solvability of the Malgrange pesudogroup by means of Singer's theorem.

The equation is given by the form $\omega = dr_2 - \big(a(r_1)r_2 + b(r_1)\big)dr_1$*, the coefficient of* dr_1 *has a vanishing second derivative thus the resolvante equations has a rational solution* $R = 0$*. By the theorem above,* $MalX$ *is solvable.*

On this linear equation one can check solvability directly. The α *form is* $-a'(r_1)dr_1$*. This gives a couple satifying relations* $(*)$ *:* $d\omega = \omega\wedge\alpha$*.* $d\alpha = 0$*. The form* dr_1 *is* X*-invariant so* $\varphi \in Mal(X)$ *satisfies* $\varphi(r_1, r_2) = (r_1 + c, \varphi_2(r_1, r_2))$*. The vector field* X *is* X*-invariant thus* φ_2 *must satisfy*

$$\frac{\partial\varphi_2}{\partial r_1} + (a(r_1)r_2 + b(r_1))\frac{\partial\varphi_2}{\partial r_2} = a(r_1 + c)\varphi_2(r_1, r_2) + b(r_1 + c).$$

The order two invariants are encoded in the two forms ω *and* α *in the following way. From the equation above* $\varphi \in Mal(X)$ *satisfies*

$$\varphi^*\omega = \frac{\partial\varphi_2}{\partial r_2}\omega$$

and

$$\varphi^*\alpha = -a'(r_1 + c)dr_1.$$

The couple $(\varphi^*\omega, \varphi^*\alpha)$ *gives an other couple of form satisfying relations* $(*)$*:*

$$\omega,\ \varphi^*\alpha + d\left(\frac{\partial\varphi_2}{\partial r_2}\right)\Big/\left(\frac{\partial\varphi_2}{\partial r_2}\right).$$

Writing that these two couples are the same, one get the equation :

$$\frac{\partial^2\varphi_2}{\partial r_2^2} = 0.$$

This implies that Mal *is included in the pseudogroup of diffeomorphism with the following form*

$$\varphi(r_1, r_2) = (r_1 + c_1, r_1(r_1)r_2 + a_0(r_1)).$$

Infinitesimal generators of this pseudogroups at 0 *are* $\frac{\partial}{\partial r_1}$*,* $(r_1)^n\frac{\partial}{\partial r_2}$*,* $(r_1)^n r_2\frac{\partial}{\partial r_2}$ *for all* $n \in \mathbb{N}$*. This Lie algebra is solvable.*

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