The Lorenz system:
Variational equation, Bessel dynamics..
(calculation by M. Canalis, J.P Ramis, P. Rouchon and J.A Weil, dec. 2001)

```
> with(plots):with(linalg):with(DEtools):_Envdiffopdomain:=[Dx,x]:
> F:=[-alpha*x+alpha*y,beta*x-y-x*z,gamma*z+x*y];
  Ft:=subs(x=x(t),y=y(t),z=z(t),F):
```

$$F := [-\alpha x+\alpha y, \beta x-y-x z, \gamma z+x y] \quad (1)$$

First we try to solve "brutally" with maple's *dsolve* function: we obtain one special trajectory (a straight line).

```
 infolevel[dsolve]:=1:
print(` The Lorenz system : `);
sys:={diff(x(t),t)=Ft[1],diff(y(t),t)=Ft[2],diff(z(t),t)=Ft[3]};
ds:=[dsolve(sys , {x(t),y(t),z(t)})]:
```

The Lorenz system :

$$sys := \left\{ \frac{d}{dt} x(t) = -\alpha x(t) + \alpha y(t), \frac{d}{dt} y(t) = \beta x(t) - y(t) - x(t) z(t), \frac{d}{dt} z(t) = \gamma z(t) + x(t) y(t) \right\}$$

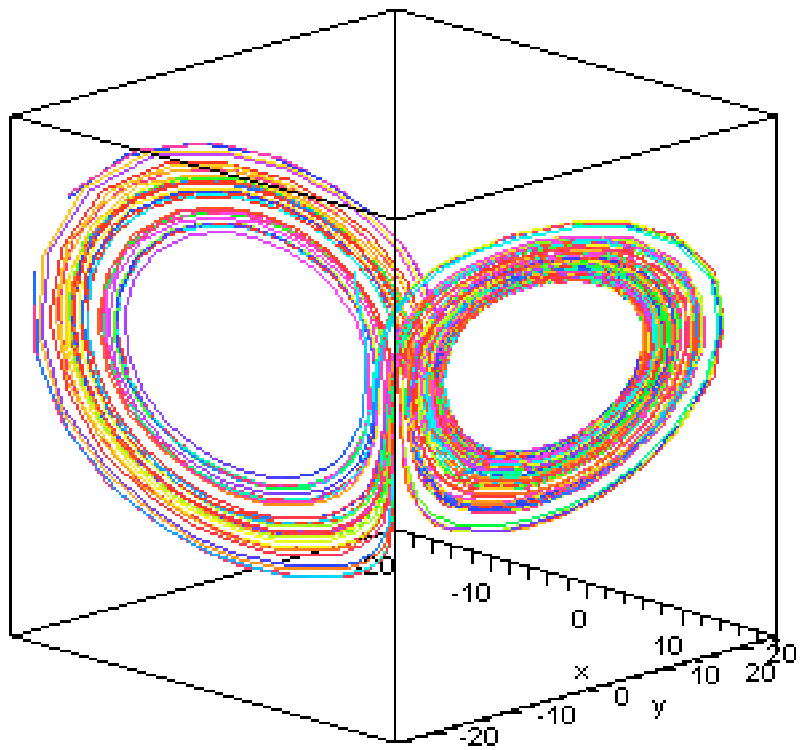
```
Warning,  computation interrupted
> ## a particular solution is even found by maple:
  ds[1];
```

$$[\{x(t)=0\}, \{y(t)=0\}, \{z(t)=\_C1\, e^{\gamma t}\}] \quad (2)$$

```
>
```

```
> display([d1,d2]);
```

Lorenz Chaotic Attractor



▶ Graphical behavior

```
>
```

The three singular points of the system are the following: two of them are "central" to the strange attractors

```
solve(map(rhs,sys),{x(t),y(t),z(t)});
```

$$\{y(t)=0, x(t)=0, z(t)=0\}, \{y(t)=\mathrm{RootOf}\left(-\gamma+\gamma\beta+\_Z^2, label=\_L1\right), z(t)=-1+\beta, x(t)=\mathrm{RootOf}\left(-\gamma+\gamma\beta+\_Z^2, label=\_L1\right)\} \quad (3)$$

We associate a cannonical Hamiltonian system with the Lorez system by introducing three "dummy" variables.

```
> p:=[x,y,z]:
  q:=[q1,q2,q3]:
  H:=add(F[i]*q[i],i=1..3);
```

$$H := (-\alpha x+\alpha y)\, q1+(\beta x-y-x z)\, q2+(\gamma z+x y)\, q3 \quad (4)$$

The "generic" linearized system is given by the following

```
Hsys:=matrix(6,1,[seq(diff(H,q[i]),i=1..3),seq(-diff(H,p[i]),i=1..3)
]);
```

$$Hsys := \begin{bmatrix} -\alpha x+\alpha y \\ \beta x-y-x z \\ \gamma z+x y \\ \alpha q1-(\beta-z)\, q2-y\, q3 \\ -\alpha q1+q2-x\, q3 \\ x\, q2-\gamma q3 \end{bmatrix} \tag{5}$$

Now we specify the trajectory and compute the matrix A of the linearized system along this trajectory:

```
trajectoire := x=0,y=0,z=exp(-gamma*t):
LIN:=matrix(3,3,[seq([seq(coeff(Hsys[i+3,1],q[j],1),j=1..3)],i=1..3)
]);
A:=subs(trajectoire , evalm(LIN) );
```

$$LIN := \begin{bmatrix} \alpha & -\beta+z & -y \\ -\alpha & 1 & -x \\ 0 & x & -\gamma \end{bmatrix}$$

$$A := \begin{bmatrix} \alpha & -\beta+e^{-\gamma t} & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & -\gamma \end{bmatrix} \tag{6}$$

> ## This system is decoupled (third indeterminate is independent: it corresponds to the "solution for free" obtained by linearizing.
In this case, computing a Normal Variational Equation amounts to studying the system for the first two indeterminates.
We now study this system. A simple way is to convert it to an equation:
"Integrability" of the original system would imply the existence of a first integral for this linearized system.
We will see that this is not the case:

```
V1:=matrix(3,1,[1,0,0]): V2:=evalm(A&*V1) : V3:=evalm(A&*V2):
P:=augment(V1,V2,V3):
nullspace(P)[1]:
L:=%[1]*y(t)+%[2]*diff(y(t),t)+%[3]*diff(y(t),t$2);
```

$$L := \left(-\alpha\beta+\alpha e^{-\gamma t}+\alpha\right) y(t)+(-\alpha-1)\left(\frac{d}{dt}y(t)\right)+\frac{d^2}{dt^2}y(t) \tag{7}$$

We solve this system and see that no rational first integral exists. In fact, we have two singularities and very specific special functions arising.
From the dynamic of those, we may expect to understand more of the local dynamic of the Lorenz system:
This "special function" feature can be obtained from Bronstein/Lafaille 2002 (combined with Bronstein/Fredet 1999).

```
locsol:=simplify( dsolve(L,y(t)) , symbolic);
   <- No Liouvillian solutions exists
```

$$locsol := y(t) = e^{\frac{1}{2}(\alpha+1)t}\left(\_C1\,\mathrm{BesselJ}\left(-\frac{\sqrt{\alpha^2-2\alpha+1+4\alpha\beta}}{\gamma}, \frac{2\sqrt{\alpha}\,e^{-\frac{1}{2}\gamma t}}{\gamma}\right) + \_C2\,\mathrm{BesselY}\left(-\frac{\sqrt{\alpha^2-2\alpha+1+4\alpha\beta}}{\gamma}, \frac{2\sqrt{\alpha}\,e^{-\frac{1}{2}\gamma t}}{\gamma}\right)\right) \quad (8)$$

\>

pullback routine

\>

\> # We transform the system to make it a system with rational functions coordinates. For this we perform an exponential pullback by letting x^2=exp(-gamma*t).

```
simplify(pullback(de2diffop(L,y(t)),-1/gamma*ln(x)),symbolic):
L2:=collect(%/lcoeff(%,Dx),Dx, X->collect(X,x,factor));
```

$$L2 := Dx^2 + \frac{(\alpha+1+\gamma)\,Dx}{\gamma x} + \frac{\alpha}{\gamma^2 x} - \frac{\alpha(-1+\beta)}{\gamma^2 x^2} \quad (9)$$

```
> LL2:=pullback(L2,gamma^2*x^2/4/alpha):
LL2:=collect(LL2,Dx,X->collect(X,x,factor));
```

$$LL2 := Dx^2 + \frac{(2\alpha+2+\gamma)\,Dx}{\gamma x} + 1 - \frac{4\alpha(-1+\beta)}{\gamma^2 x^2} \quad (10)$$

```
> ds2:=dsolve(diffop2de(LL2,y(x)),y(x));
<- No Liouvillian solutions exists
```

$$ds2 := y(x) = \_C1\,x^{\frac{-\alpha-1}{\gamma}}\,\mathrm{BesselJ}\left(\frac{\sqrt{1+\alpha^2+(-2+4\beta)\alpha}}{\gamma}, x\right) + \_C2\,x^{\frac{-\alpha-1}{\gamma}}\,\mathrm{BesselY}\left(\frac{\sqrt{1+\alpha^2+(-2+4\beta)\alpha}}{\gamma}, x\right) \quad (11)$$

We may push further the sudy of the dynamics for this equation and decompose how the Bessel solutions are obtained.

the original Lorenz system

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