Galoisian method for studying integrability of dynamical systems (generally hamiltonian), theory and practice

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Outline

1. Complete Integrability
   - Galoisian method for studying integrability of dynamical systems (generally hamiltonian), theory and practice
   - Complete Integrability (general).
   - Complete Integrability (general).
   - Variational equation
   - Ziglin, Morales & Ramis
2. Practical criteria for parametrized systems
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   - Two degrees of freedom: Kovacic
   - More degrees of freedom: local criteria
   - Factorization of symplectic systems
2. conclusion
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A ***Hamiltonian system*** on a non empty domain U of $\mathbb{R}^{2n}$:

$$(S) : \begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i}(q,p) \\ \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i}(q,p) \end{cases} \qquad i = 1, \ldots, n$$

where $H : U \to \mathbb{R}$ is the ***Hamiltonian*** function.

Condensed form: $\mathrm{x}'(t) = J\, \nabla H(\mathrm{x}(t))$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

First Integral : function $I(p,q)$ that remains constant along solutions. $\longrightarrow \sum_i \frac{dq_i}{dt}\frac{\partial I}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial I}{\partial p_i} = 0$

Liouville (complete) integrability:
"sufficiently many good first integrals".

Complete Integrability (Hamiltonian) (2).

$$(S): \qquad \mathrm{x}'(t) = J\,\nabla H(\mathrm{x}(t)) = X_H(\mathrm{x}(t))$$

`Poisson Bracket` :

$$\{G_1, G_2\} = \sum_{i=1}^{n} \left(\frac{\partial G_1}{\partial p_i}\frac{\partial G_2}{\partial q_i} - \frac{\partial G_1}{\partial q_i}\frac{\partial G_2}{\partial p_i} \right) = < \nabla G_1(\mathrm{x}), J\,\nabla G_2(\mathrm{x}) >$$

`First integral` G of (S) defined by $\{H, G\} = 0$

Definition

(S) is *completely integrable* (Liouville) in a class $\mathcal{F}$ of functions if it admits n first integrals $G_1 = H, G_2, \ldots, G_n \in \mathcal{F}$ such that

1. the G_i are functionally independent, and
2. in involution: $\{G_i, G_j\} = 0$

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Complete Integrability (Hamiltonian) (3).

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remark:

$[X_{G_1}, X_{G_2}] = X_{\{G_1, G_2\}}$: involution condition induces "symmetry"

Complete Integrability (general).

$$(1)\ x' = F(x), \quad x = (x_1, \dots, x_n), F = (F_1, \dots, F_n)$$

⋆ "old hamiltonian trick" : $H := \sum_{j=1}^{n} y_j F_j(x)$ so

$$(2)\ \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i} = F_i(x), \qquad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} = \sum_{j=1}^{n} y_j \frac{\partial F_j(x)}{\partial x_i}$$

idea 1 : (1) integrable
when (2) is completely integrable.

⋆ Better! (Bogoyavlensky, Cushman& Bates, Nguyen T-Z,...)
Vector field X on manifold M. Vector fields $X = X_1, \dots, X_p$ and $f_1, \dots, f_{n-p}$ s.t $[X_i, X_j] = 0$ and $X_i(f_j) = 0$.

∘ Most of what follows can be extended to general systems (cf. Ayoul & Nguyen TZ, 2006..)

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(Rather sketchy) historical overview

- 19th century. Kovaleskaya, Poincaré:

  Variational equation *VE* along a solution
- ~ 84 : Ziglin : monodromy of variational equation
- Yosida & others, refinements and applications
- $93 - 95$: Baider-Churchill-Rod-Singer, Morales: Galois group
- 98 : Morales & Ramis:
  the differential Galois group of *VE* is virtually abelian
- $\sim 95 \rightarrow 06$ at least 50 papers of applications (e.g Morales, Simon, Tsygvintsev, Maciejewski, Przybylska, Audin, Boucher, J.A.W, etc). Generally use the Kovacic algorithm (2nd order *VE*).
- ~ 04 Morales-Ramis-Simo: higher variational equations.

Variational equation

$(S)\,\frac{dx}{dt} = F(x)$. Particular solution $\mathrm{x}_0(t)$.

Write $x = \mathrm{x}_0(t) + \epsilon y(t)$

Definition

The *(first) variational system* (VE) along a solution $\mathrm{x}_0(t)$ of a differential system is the linear differential system:

$$(VE): \qquad y'(t) = A(t)\, y(t) \quad , \quad A(t) = \mathrm{Jac}(F)(\mathrm{x}_0(t))$$

where $\mathrm{Jac}(F)$ denotes the Jacobian matrix of F at $\mathrm{x}_0(t)$.

Junior part f° of first int. f:

$$f(x) = f(\mathrm{x}_0(t)) + \epsilon^m f^\circ(\mathrm{x}_0(t), y(t)) + h.o.t$$

$f^\circ(\mathrm{x}_0(t), y(t))$ rational (non-autonomous) function of y.

f *merom.* F.Int. of (S) $\Longrightarrow$ f° *rational* F.Int. of (VE).

Remark: $\frac{d\mathrm{x}_0(t)}{dt}$ is a particular solution, reduce order of (VE).

Ziglin's lemma

$$(S)\frac{dx}{dt} = F(x)$$

Lemma (Ziglin, simple form)

If (S) admits m algebraically independent first integrals $f_1, \ldots, f_m$, then it admits m algebraically independent first integrals $g_1, \ldots, g_m$ such that their junior parts $g_1^\circ, \ldots, g_m^\circ$ are algebraically independent.

Ziglin's lemma, Galoisian formulation

Key:

rational F. Int. of (VE) $\longleftrightarrow$ rational invariants of $DGal(VE)$.

Lemma (Ziglin revisited, Galoisian formulation)

If (S) admits m algebraically independent first integrals $f_1, \ldots, f_m$, then $DGal(VE)$ along a particular solution admits m algebraically independent rational invariants.

Rational invariants of $DGal(VE)$ are obtained from rational or (certain) exponential solutions of constructions (symmetric powers) on (VE):

$\rightarrow$ can be computed algorithmically

Hamiltonian systems: the Morales-Ramis theorem

$$(S): \qquad \mathrm{x}'(t) = J\,\nabla H(\mathrm{x}(t))$$

variational equation along a solution:

$$(VE): \qquad h'(t) = A\,h(t)$$

$A = J \cdot \mathcal{H}(H, \mathrm{x}_0(t))$, with $\mathcal{H}(H, \mathrm{x}_0(t))$=Hessian of H at $\mathrm{x}_0(t)$.

Remark : $A \in sp(2n, C) \quad \Rightarrow \quad G := DGal(VE) \subset Sp(2n, C)$

Theorem (J.J. Morales, J.P. Ramis)

Consider a Hamiltonian system (S) and the variational system (VE) : $Y' = AY$ along a solution $x_0(t)$ of (S). If the system (S) is completely integrable, then G° is abelian (G virtually abelian).

Fact: G° solvable $\leftrightarrow$ Liouvillian solutions (Singer, Ulmer, etc).

Note: reduce order to $2(n-2)$: Normal Variational Eqn (NVE).

Parametrized system: wash away the wrong parameters !

Where it gets problematic / where it gets useful:
we have a *family* of hamiltonians (parametrized by, say, masses). Want to decide for which values of the parameters the Galois group is virtually abelian (i.e select the *very few* possible values)

Undecidable problems ...
... when "local data" depend on parameters

so design toolbox ("postcard algorithm") :
wash away wrong parameters
⋆ Works quite well in practice (more than 50 successful papers) for second order, harder for higher: this is where we come in .
⋆ Algorithms for parametrized linear differential equations:
PhD of Delphine Boucher (2000)

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Two degrees of freedom: reducing to the Kovacic algorithm !

Two degrees of freedom (4 variables): (NVE) has order 2.

Diff Galois group of second order equations : Kovacic (1986) , Duval& Loday (1992), Singer & Ulmer (1993), Ulmer & Weil (1995), Fakler (1998), Berkenbosch & Hoeij & Weil (2004): very efficient (also: lists for hypergeometric equations, Lamé, etc).

this is the case where are these (at least) 50 papers on the subject ! (but can still be problematic with parameters)

What about more than two degrees of freedom ??

This is where we will use the full power of constructive diff. Galois theory.

cannot use general methods (parameters..).

1. design `"easy" smaller criteria` , easier to use
2. the `symplectic nature` of the system will help us a lot : use the *structure* of the system!

Handle systems directly (without converting to equations),

Intermezzo: local solutions

$$Y' = AY,$$

Fundamental formal solution matrix at $x = 0$: $Y = \phi . x^L e^Q$,

$$Q = diag(q_i(x^{1/r})),\ L \in \mathcal{M}_n(C),\ \phi \in GL_n(((x)))$$

i.e

solution vector $Y_i = e^{q_i(x^{1/r})} x^{\alpha_i} \phi_i(x)$, $\phi_i(x) \in C[[x]]^n$

or $Y_i = e^{q_i(x^{1/r})} x^{\alpha_i} (\phi_i(x) + c_i x^{n_i} \ln(x) \phi_{i-1}), \ldots.$

We'll say that

1. α_i is an `exponent` of Y_i ("abus de langage", cf Corel 2001)
2. the $e^{q_i(x^{1/r})}$ are `exponential parts` (determining factors)
3. r is the ramification index at the singularity

Practical computation: algorithmic, technicalities remain for parametrized cases.

A non-integrability criterion (Boucher & Weil, 2000-02)

Theorem (Boucher & Weil, 2002)

Let $Y' = AY$ be a linear differential system such that it is (gauge)-equivalent to a block-triangular system

$$Z' = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ \times & \ddots & 0 & \\ \times & \times & B_{s-1} & 0 \\ \times & \times & \times & B_s \end{pmatrix} Z$$

If there exists i in $\{1, \ldots, s\}$ such that one system $\nu_i' = B_i \nu_i$ is `completely reducible` *and has formal solutions with* `logs` *, then the differential Galois group G of the system $Y' = AY$ is not virtually abelian.*

`Nb1` : Stokes matrices also give similar obstructions.

`Nb2` : "log" property is "generic" when present: good !

non-integrability criterion: sketch of proof

1. particular case of 'one block' : $s = 1$.
   - if completely reducible system then G^0 abelian $\Rightarrow$ G^0 diagonalizable
   - if log then G^0 not diagonalizable
2. general case
   Let $k \subset F \subset K$ with
   $k \subset K$, Picard-Vessiot of k associated to the system $Y' = AY$
   $k \subset F$, Picard-Vessiot of k associated to the system $\nu' = B_i \nu$.
   Then

$$Gal^0(F/k) \text{ not abelian } \Rightarrow \; Gal^0(K/k) = G^0 \text{ not abelian}$$

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Two nice examples:

What when the criterion says that G° may be abelian ?

One may

⋆ try higher variational eqns (Morales-Ramis-Simo, $\sim$ 04)

⋆ start being intelligent and compute the (possible) first integrals (or use other criteria)

let's switch to two nice examples:

The Lorenz system (Canalis, Ramis, Rouchon, JAW)

The Friedman-Robertson-Walker cosmological model (Boucher, JAW)

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Using the symplectic structure

$$(VE) : Y' = AY\,, \;\; A \in sp(2n, C) \Longrightarrow DGal(VE) \subset Sp(2n, C).$$

(M, D) the differential module $(M = k^{2n},\ D = \frac{d}{dx} - A)$.

- ⋆ Isomorphism M to dual M^*: $Y \mapsto J.Y$.
- ⋆ $End_D(M) \cong M \otimes M^* \cong M \otimes M \cong Sym^2(M) \oplus \Lambda^2(M)$
- ⋆ $(Sym^2(M), \{.,.\}) \lesssim (sp(2n, C), [.,.])$.

polynomial invariant of degree 1 or 2 induces factorization that way

`To compute those:` need information on the exponents of solutions of $Sym^2(M)$ or $\Lambda^2(M)$. Use symplectic structure.

More symplectic structure: the exponents at singularities

$Y' = AY$, Fund. local solution matrix at $x = 0$: $Y = \phi . x^L e^Q$,
$Q = diag(q_i(x^{1/r}))$, $L \in \mathcal{M}_n(C)$, $\phi \in GL_n(C((x)))$
i.e
$Y_i = e^{q_i(x^{1/r})} x^{\alpha_i} \phi_i(x)$,
$Y_i = e^{q_i(x^{1/r})} x^{\alpha_i} (\phi_i(x) + c_i x^{n_i} \ln(x) \phi_{i-1}(x))$, ...

Lemma

1. *if $q_i(x^{1/r})$ is an exponential part, then $-q_i(x^{1/r})$ is too*
2. *if α_i exponent, then there exists $m \in \frac{1}{r}\mathbb{Z}$ s.t $m - \alpha_i$ is also an exponent.*

Interest: some sums pairwise do NOT depend on parameters.
AND these sums are exponents of $Sym^2(M)$ or $\Lambda^2(M)$.
simplifies rational (exp.) solutions in these → factorization

conclusion

1. Solid toolbox, though still at the stage of postcard algorithms: need bits of intelligence (use physical constraints on parameters)
2. Stick to the structure of the system, keeps coefficients simple
3. Allows to wash away many parameters, generally leaves "few" possible integrable configurations.
4. illustrates a use of constructive diff. Galois methods

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