Reducibility and Galois groupoid[1]

Extended abstract

Guy Casale

IRMAR, Université de Rennes 1
Campus de Beaulieu 35042 Rennes cedex, France
guy.casale@univ-rennes1.fr

In this talk two problems of reducibility/irreduciblitity of ordinary differential equations will be presented from a 'galoisian' point of view. The problem is to determine when an ordinary differential equation can be solved by means of classical functions as defined by H. Umemura in [7].

Definition 0.1 *(Painlevé, Umemura [7]) The field of classical functions over* $\mathbb{C}(x)$ *is a differential field which is the union of all the differential fields obtained by a tower of strongly normal extensions and algebraic extensions. Strongly normal extensions are :*

- *extensions by the entries of a fundamental solution of a linear ODE,*
- *extensions by an abelian function with classical functions as arguments.*

A common belief is that an answer to this kind of question should be given by a general nonlinear differential Galois theory. In [8, 4], general definitions of what should be a nonlinear Galois group (or groupoid) are given. Because of its geometric flavour we will focus on the Malgrange's Galois

[1] The first part of this work was done when the author was supported by Marie-Curie EIF Fellowship (MEIF-CT-2005-025116). The second part was done during the PEM program of the Newton Institute, Cambridge, UK. The author is partially supported by the ANR project no. JC05 41465 and by GIFT NEST-Adventure Project no. 5006.

groupoid and use it to solve the two following problems.

Irreducibility of P_1: prove that no solution of the first Painlevé equation $y'' = 6y^2 + x$ is a classical function.

Reducibility of PP_6: explain why the Picard-Painlevé sixth equation

$$y'' = \frac{3y^2 - 2y(x-1) + x}{2y(y-1)(y-x)}(y')^2 + \left(\frac{1}{x-y} + \frac{1}{1-x} - \frac{1}{x}\right)y' + \frac{y(y-1)}{2x(x-1)(y-x)}$$

can be solved by a formula though most of its solutions are non classical. The formula to solve PP_6 looks pretty classical :

$$y = \wp(a\omega_1(x) + b\omega_2(x); \omega_1(x), \omega_2(x))$$

with a and b two constants and $\omega_{1,2}$ two periods of $z^2 = y(y-1)(y-x)$.

1 The Galois groupoid of a vector field in $\mathbb{C}^3$

Let X be a vector field in $\mathbb{C}^3$. In general it is not complete and its flows are only defined on open sets small enough. All the dynamic of this vector field is contained in the pseudogroup of transformations of $\mathbb{C}^3$ generated by these local flows. By keeping only the germs of diffeomorphisms from this pseudogroup one gets a groupoid, $TanX$, acting on $\mathbb{C}^3$.

The Galois groupoid of X is the Zariski closure of $TanX$ for a (nearly) obvious embedding of $TanX$ in a infinite dimensional algebraic variety. This variety is the space J^* of formal diffeomorphisms of $\mathbb{C}^3$ with its groupoid structure and its projections on the spaces J^*_q of order q jets of diffeomorphisms. The ring O_{J^*} of this variety is the commutative differential ring of nonlinear partial differential equations on germs of diffeomorphisms. The embedding is the Taylor expansion of elements of $TanX$.

Definition 1.1 *(Malgrange [4]) The Galois groupoid of X is defined by the ideal of O_{J^*} of all the PDEs satisfied by the flows of X.*

Using Lie-Cartan local classification of pseudogroups acting on $\mathbb{C}^2$ [1], one has the following proposition.

Proposition 1.2 *([2]) If X is divergence free and γ is the closed 2-form vanishing on X then one of the following situations occurs:*

- $Gal(X)$ is imprimitive*: there exists an algebraic 1-form θ s.t. $\theta \wedge d\theta = 0$ and $\theta(X) = 0$,*
- $Gal(X)$ is transversally affine*: there exists two algebraic 1-forms θ_1, θ_2 vanishing on X and a traceless matrix of 1-forms (θ_i^j), $i, j =$1 or 2, s.t. $d\theta_i = \theta_i^j \wedge \theta_j$ and $d\theta_i^j = \theta_i^k \wedge \theta_k^j$,*
- *the only transversal equations of $Gal(X)$ are those of the invariance of γ.*

2 Irreducibility of P_1

The discussion about the irreducibility to classical functions of the solutions of the first Painlevé equation depends on the transcendance degree of the differential field generated by these solutions over $\mathbb{C}(x)$.

This is a classical result of Painlevé that such a solution cannot be algebraic, and by the Kolchin-Kovacic lemma its transcendance degree must be two. Such a solution gives an inclusion of the field $\mathbb{C}(x, y, y')$ in a field $\mathbb{C}(x, h_i, \ldots, k_p, \ldots)$. Let's take $\mathbb{C}^3$ and $\mathbb{C}^N$ as model for these fields and let π be the dominante projection induced by the inclusion. The differential structure of the first field is given by the vector field

$$X_1 = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + (6y^2 + x) \frac{\partial}{\partial y'}$$

on $\mathbb{C}^3$ and because of its special construction the vector field on $\mathbb{C}^N$ has the following shape

$$X_c = \frac{\partial}{\partial x} + \sum a_i^j(x) h_j \frac{\partial}{\partial h_i} + \sum b_p^q(x, h) k_q \frac{\partial}{\partial k_p} + \ldots$$

The projection of X_c by π gives X_1. The main tool to prove that this projection cannot exist is the following theorem.

Theorem 2.1 *The only transversal equations of $Gal(X_1)$ are those of the invariance of γ.*

Computations on the structural equation of the Galois groupoid of X_c show that

- a quotient of $Gal(X_c)$ is included in $Gal(X_1)$.
- such a quotient must be strictly smaller than $Gal(X_1)$.

On an other side, this quotient must contain $TanX_1$, this yields a contradiction.

3 Reducibility of PP_6

This equation is also divergence free in canonical coordinates but in this case one has the following theorem. Let X_{PP} be the vector field of this equation on $\mathbb{C}^3$.

Theorem 3.1 *$Gal(X_{PP})$ is transversally affine.*

To prove this we construct two first integrals in a Picard-Vessiot extension of the differential field $\left(C(x,y,y'); \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y'}\right)$ following P. Painlevé [5]. If $y(x)$ is a solution of PP_6 the integral $\int_0^{y(x)} \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-x)}}$ is a period of $z^2 = y(y-1)(y-x)$. By pulling-back linear first integral of the linear order two equation of the periods (Picard-Fuchs) one gets:
for each solution of $v'' + \left(\frac{1}{x^2} + \frac{1}{(x-1)^2} - \frac{1}{x(x-1)}\right) v = 0$, the function

$$y'v\sqrt{\frac{x(x-1)}{y(y-1)(y-x)}} + \int \sqrt{\frac{x(x-1)}{y(y-1)(y-x)}} \left\{ \left[\frac{v}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) - v' \right] dy + v\frac{y(1-y)}{2x(y-x)(x-1)} dx \right\}$$

is a first integral of X_{PP}. The theorem follows easily.

The Galois groupoid shows that this equation is special even if its non algebraic solutions are non classical. In fact the first integrals are classical functions of three variables.

References

[1] Cartan, É. - *Les sous-groupes des groupes continus de transformations*, Ann. Sci. École Normale Sup. **25** (1908) 57–194

[2] Casale, G - *Le groupoïde de Galois de P_1 et son irréductibilité*, to appear in Commentarii Mathematici Helvetici

[3] Casale, G - *The Galois groupoid of the Picard-Painlevé VI equation*, in preparation

[4] Malgrange, B. - *Le groupoïde de Galois d'un feuilletage*, Monographie **38** vol **2** de L'enseignement mathématique (2001)

[5] Painlevé, P. - *Leçons de Stockholm* (1895), Oeuvres complètes Tome **1**, éditions du CNRS (1972)

[6] Picard, E. - *Mémoire sur la théorie des fonctions algébriques de deux variables*, Journal de Liouville **5** (1889) 135–319

[7] Umemura, H. - *Painlevé equations and classical functions* [translation of Sūgaku **47** (1995)] Sugaku Expositions. **11** (1998) 77–100

[8] Umemura, H. - *Galois theory of algebraic and differential equation*, Nagoya Math. J. vol **144** (1996) 1–58
- *Differential Galois theory of infinite dimension*, Nagoya Math. J. vol **144** (1996) 59–135