Integrability of homogeneous systems; Results and problems

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Homogeneous Hamiltonian equations

Integrability of Hamiltonian systems given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \qquad (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n},$$

$$V(\boldsymbol{q}) \in \mathbb{C}[\boldsymbol{q}], \qquad \deg V = k > 2,$$

$V(\boldsymbol{q})$ — homogeneous; i.e., integrability of

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i = \frac{\partial H}{\partial p_i} = p_i, \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i}, \quad i = 1, \ldots, n$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}q_i = -F_i(\boldsymbol{q}), \qquad F_i(\boldsymbol{q}) = \frac{\partial V}{\partial q_i}, \quad i = 1, \ldots, n,$$

Homogeneous Newton's equations

Integrability of the system of autonomous Newton equations

$$\ddot{\boldsymbol{q}} = -\boldsymbol{F}(\boldsymbol{q}), \qquad \boldsymbol{q} = (q_1, \ldots, q_n)^T \in \mathbb{C}^n,$$

where $\boldsymbol{F}_i \in \mathbb{C}[\boldsymbol{q}]$ are homogeneous, and

$$\deg F_i = l := k - 1, \text{ for } i = 1, \ldots, n.$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = p_i, \qquad \frac{\mathrm{d}}{\mathrm{d}t} p_i = -F_i, \qquad i = 1, \ldots, n$$

Integrability in the Liouville sense

$H_i = H_i(\boldsymbol{q}, \boldsymbol{p})$, for $i = 1, \ldots, n$ are first integrals,

$$\{H_i, H_j\} := \sum_{k=1}^{n} \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} = 0$$

Definition

We say that the Hamiltonian system with n degrees of freedom is integrable in the Liouville sense if it admits n functionally independent first integrals that commute.

Integrability in the Jacobi sense

Definition

We say that the system of n first order differential equations is integrable in the Jacobi sense if it admits an invariant n-form and possesses $n-2$ functionally independent first integrals.

$$\boldsymbol{v}(\boldsymbol{q},\boldsymbol{p}) := \sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial q_i} - F_i(\boldsymbol{q})\frac{\partial}{\partial p_i}\right), \qquad \boldsymbol{p} := \dot{\boldsymbol{q}},$$

$$\operatorname{div}(\boldsymbol{v}) := \sum_{i=1}^{n}\frac{\partial v_i}{\partial x_i} = 0,$$

invariant $2n$-form

$$\mu = \mathrm{d}q_1\wedge\cdots\wedge\mathrm{d}q_n\wedge\mathrm{d}p_1\wedge\cdots\wedge\mathrm{d}p_n$$

Fact

For integrability in the Jacobi sense of Newton's equations $2n-2$ first integrals are necessary.

Classes of equivalence

- Equivalent potentials:

$$V(\boldsymbol{q}) \sim V_A(\boldsymbol{q}) = V(A\boldsymbol{q}), \qquad A \in \mathrm{PO}(n, \mathbb{C}),$$

$$\mathrm{PO}(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C}) \,|\, AA^T = \alpha E, \quad \alpha \in \mathbb{C}\}.$$

- We say that forces $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ are equivalent if there exists a non-singular matrix $A \in \mathrm{GL}(n, \mathbb{C})$ such that $\widetilde{\boldsymbol{F}}(\boldsymbol{q}) = A^{-1}\boldsymbol{F}(A\boldsymbol{q})$.

- Integrability of one system implies the integrability of all systems belonging to the same equivalence class.

Problem

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How to find integrable Hamiltonian and Newton systems?

Known methods

- Kovalevskaya analysis $\Longrightarrow$ Painlevé analysis
- Ziglin theory
- Morales-Ramis theory

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Variational equations

Main Idea

A non-linear system leaves fingerprints of its properties in variational equations.

For a system

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, \ldots x_n)^T \in \mathbb{C}^n.$$

with a known non-equilibrium particular solution $\varphi(t)$ the substitution $\boldsymbol{x} = \varphi(t) + \xi$ is applied.

Variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi = A(t)\xi, \qquad A(t) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\varphi(t)).$$

First integrals of the system and its VEs

First implication

If the system possesses k functionally independent meromorphic first integrals $F_1, \ldots, F_k$, then VEs have k functionally independent rational first integrals.

Second implication

If system has k functionally independent first integrals which are meromorphic in a connected neighbourhood of a non-equilibrium solution $\varphi(t)$, then the differential Galois group $\mathcal{G}$ of the variational equations along $\varphi(t)$ has k functionally independent rational invariants.

Hamiltonian systems – Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ, and that the variational equations along Γ are Fuchsian. Then the identity component of the differential Galois group of the variational equations along Γ is Abelian.

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Particular solutions

Definition

Darboux point $\boldsymbol{d} \in \mathbb{C}^n$ is a solution of

$$\boldsymbol{F}(\boldsymbol{d}) = \boldsymbol{d},$$

and all $\widetilde{\boldsymbol{d}} = \gamma \boldsymbol{d}$ where $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ define the same Darboux point.

Particular solution

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \qquad \text{provided} \qquad \ddot{\varphi} = -\varphi^{k-1}.$$

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right), \quad \varepsilon \in \mathbb{C}^*.$$

Variational equations

$$\ddot{\boldsymbol{x}} = -\varphi(t)^{k-2}\boldsymbol{F}'(\boldsymbol{d})\boldsymbol{x},$$

where $\boldsymbol{F}'(\boldsymbol{d})$ is the Jacobi matrix of $\boldsymbol{F}$ calculated at $\boldsymbol{d}$. In the Hamiltonian case $\boldsymbol{F}'(\boldsymbol{d}) = V''(\boldsymbol{d})$

If $F'(\boldsymbol{d})$ is diagonalisable

$$\ddot{\eta}_i = \lambda_i \varphi(t)^{k-2}\eta_i, \qquad i = 1, \ldots, n,$$

where λ_i for $i = 1, \ldots, n$ are eigenvalues of $F'(\boldsymbol{d})$.

By homogeneity of $\boldsymbol{F}$, $\lambda_n = k - 1$.

Differential Galois group

$$\mathcal{G} = \mathcal{G}(\lambda_1) \times \cdots \times \mathcal{G}(\lambda_n) \subset \mathrm{Sp}(2n, \mathbb{C}), \qquad \mathcal{G}(\lambda_i) \subset \mathrm{Sp}(2, \mathbb{C}).$$

Transformation to hypergeometric equations

$$z := \frac{1}{k}\varphi(t)^k.$$

$$\left.\begin{array}{l} z(1-z)\eta'' + [c-(a+b+1)z]\eta' - ab\eta = 0, \\ a+b = \dfrac{k-2}{2k}, \qquad ab = -\dfrac{\lambda_i}{2k}, \qquad c = 1-\dfrac{1}{k}. \end{array}\right\}$$

$$G = G(\lambda_1)\times\cdots\times G(\lambda_n) \subset \mathrm{GL}(2n,\mathbb{C}), \qquad G(\lambda_i) \subset \mathrm{GL}(2,\mathbb{C}).$$

Isomorphism of identity components of $\mathcal{G}(\lambda_i)$ and $G(\lambda_i)$.

Other Morales-Ramis Theorem

If the Hamiltonian system with homogeneous potential is is meromorphically integrable then each (k, λ_i) belong to the following list:

$$\left(k, p + \frac{k}{2}p(p-1)\right), \qquad \left(k, \frac{1}{2}\left[\frac{k-1}{k} + p(p+1)k\right]\right),$$
$$\left(3, -\frac{1}{24} + \frac{3}{32}(1+4p)^2\right), \left(3, -\frac{1}{24} + \frac{3}{50}(1+5p)^2\right),$$
$$\left(3, -\frac{1}{24} + \frac{3}{50}(2+5p)^2\right), \left(4, -\frac{1}{8} + \frac{2}{9}(1+3p)^2\right),$$
$$\left(5, -\frac{9}{40} + \frac{5}{18}(1+3p)^2\right), \left(5, -\frac{9}{40} + \frac{1}{10}(2+5p)^2\right),$$

where $p \in \mathbb{Z}$.

What for Newton's equations?

Theorem

Assume that the Newton system with polynomial homogeneous polynomial right-hand sides of degree $l = k - 1 > 1$ is integrable in the Jacobi sense. Then if $\mathbf{d}$ is a Darboux point such that $\mathbf{F}'(\mathbf{d})$ is semi-simple and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, then (k, λ_i) for $i = 1, \ldots, n$ belong to the following list

case	k	λ	
1.	k	$p+\frac{k}{2}p(p-1)$	
2.	k	$\frac{1}{2}\left(\frac{k-1}{k}+p(p+1)k\right)$	
3.	3	$-\frac{1}{24}+\frac{1}{6}(1+3p)^2,$	$-\frac{1}{24}+\frac{3}{32}(1+4p)^2$
		$-\frac{1}{24}+\frac{3}{50}(1+5p)^2,$	$-\frac{1}{24}+\frac{3}{50}(2+5p)^2$
4.	4	$-\frac{1}{8}+\frac{2}{9}(1+3p)^2$	
5.	5	$-\frac{9}{40}+\frac{5}{18}(1+3p)^2,$	$-\frac{9}{40}+\frac{1}{10}(2+5p)^2,$

where p is an integer. Moreover, among (k, λ_i) at most two belong to item 1 of the above list.

Relations between λ_i

$\Lambda_i = \lambda_i - 1$, for $i = 1, \ldots, n-1$,
$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{d}) = (\Lambda_1, \ldots, \Lambda_{n-1}) \in \mathbb{C}^{n-1}$. τ_i, for $i = 0, \ldots, n-1$ – elementary symmetric polynomials in $(n-1)$ variables.

Theorem

Assume that $\boldsymbol{F}$ *has exactly* $D(n,k) = [(k-1)^n - 1]/(k-2)$ *Darboux points* $\boldsymbol{d} \in \mathcal{D}_{\boldsymbol{F}}$. *Then* $\boldsymbol{\Lambda}(\boldsymbol{d})$ *satisfy the following relations:*

$$\sum_{\boldsymbol{d}\in\mathcal{D}_{\boldsymbol{F}}} \frac{\tau_1(\boldsymbol{\Lambda}(\boldsymbol{d}))^r}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^n(n+k-2)^r, \quad 0 \leq r \leq n-1,$$

or, alternatively

$$\sum_{\boldsymbol{d}\in\mathcal{D}_{\boldsymbol{F}}} \frac{\tau_r(\boldsymbol{\Lambda}(\boldsymbol{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{n-r-1} \sum_{i=0}^{r} \binom{n-r-1}{r-i} (k-1)^i.$$

Λ_i are Kovalevskaya exponents

The auxiliary system related to Newton system with force field $\boldsymbol{F}$ or Hamiltonian system with $\boldsymbol{F} = V'$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = \boldsymbol{F}(\boldsymbol{q}), \qquad \boldsymbol{q} \in \mathbb{C}^n.$$

F_i are homogeneous polynomials of degree l.
The Kovalevskaya matrix $\boldsymbol{K}(\boldsymbol{d})$ at a Darboux point $\boldsymbol{d} \in \mathcal{D}_{\boldsymbol{F}}$

$$\boldsymbol{K}(\boldsymbol{d}) := \boldsymbol{F}'(\boldsymbol{d}) - \boldsymbol{E},$$

Eigenvalues $\Lambda_i = \Lambda_i(\boldsymbol{d})$, $i = 1, \ldots, n$, of the Kovalevskaya matrix $\boldsymbol{K}(\boldsymbol{d})$ are called the Kovalevskaya exponents.

By homogeneity one of them, let us say Λ_n, is trivial and it is $\Lambda_n = l - 1$.

$\boldsymbol{\Lambda}(\boldsymbol{d}) = (\Lambda_1(\boldsymbol{d}), \ldots, \Lambda_{n-1}(\boldsymbol{d}))$ – nontrivial Kovalevskaya exponents.

Guillot theorem (2004)

Theorem

Assume that the above system with homogeneous polynomial right hand sides of degree l has the maximal number of Darboux points and let S be a symmetric homogeneous polynomial in $n-1$ variables of degree less than n. Then, the number

$$R := \sum_{\boldsymbol{d} \in \mathcal{D}_{\boldsymbol{F}}} \frac{S(\boldsymbol{\Lambda}(\boldsymbol{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))},$$

depends only on the choice of S, dimension n and homogeneity degree l.

n-dimensional generalisation of the Jouanolou system:

$$\dot{x}_i = x_{i+1}^l, \qquad 1 \le i \le n, \quad x_{n+1} \equiv x_i.$$

Finiteness of choices of Λ_i

$$\sum_{\boldsymbol{d}\in\mathcal{D}_V}\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\boldsymbol{d})}=-\frac{(k-1)^n-n(k-2)-1}{(k-2)^2}.$$

Theorem

Let us assume that homogeneous system possesses the maximal number of Darboux points. Then for homogeneous Hamiltonian as well as Newton equations there exist only finite number of sets $(\Lambda_1,\ldots,\Lambda_n)$ at each Darboux point belonging to the appropriate sets of rational numbers (depending on k) satisfying the above relation.

Examples of admissible Λ_i for $n = 2$ and $k = 3, 4$

$\{\Lambda_1, \Lambda_2, \Lambda_3\}$
$\{-1, -1, 1\}$
$\{-2/3, 4, 4\}$
$\{-7/8, 14, 14\}$
$\{-2/3, 7/3, 14\}$.

$\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$
$\{-1, -1, 2, 2\}$
$\{-5/8, 5, 5, 5\}$
$\{-5/8, 2, 20, 20\}$
$\{-5/8, 27/8, 27/8, 135\}$
$\{-5/8, 2, 14, 35\}$.

Two types of integrability obstructions

1. all $\Lambda_i = \lambda_i - 1$ must belong to appropriate sets of rational numbers (tables in Morales-Ramis theorem for Hamiltonian systems and our theorem for Newton systems). This is a local analysis.
2. between Λ_i calculated at all Darboux points some universal relations exist. This is a global analysis.

Algorithm for integrability analysis

1. Fix k and postulate the general form of polynomial potential V or force field $\boldsymbol{F}$ with undetermined coefficients.
2. Determine sets for that belong all Λ_i.
3. Using finiteness relation find all sets $(\Lambda_1, \ldots, \Lambda_n)$ at each Darboux point.
4. using Λ_i reconstruct V or $\boldsymbol{F}$, i.e. find all undetermined coefficients.
5. If after reconstruction some polynomial coefficients become free parameters apply stronger integrability obstruction due to variational equations of higher orders or repeat analysis with different particular solution.
6. analyse nongeneric cases (with smaller number of Darboux points, multiple Darboux points and without Darboux points).

Case $n = 2$. Hamilton systems

Theorem

Assume that there exist $1 \leq l \leq k$ simple Darboux points and in V factor $(q_2 \pm \mathrm{i} q_1)$ can appear at most with multiplicity 1, then Λ_i satisfy

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1.$$

Theorem

Assume that there exist $1 \leq l \leq k$ simple Darboux points and that $r_\pm \neq l/2$ are multiplicities of $(q_2 \pm \mathrm{i} q_1)$ in V, then

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1 - \theta_{r_+,2}\frac{r_+}{k-2r_+} - \theta_{r_-,2}\frac{r_-}{k-2r_-},$$

$$\theta_{x,y} := \begin{cases} 0 & \text{for} \quad x < y, \\ 1 & \text{for} \quad x \geq y. \end{cases}$$

Case $n = 2$. Newton systems

Theorem

Assume that forces $\boldsymbol{F} = (F_1, F_2)$ *satisfy the following conditions*

- **A.1** $\boldsymbol{F}$ *admits* $1 \leq l \leq k$ *simple Darboux points;*
- **A.2** *if* $F_1(\boldsymbol{q})$ *and* $G(\boldsymbol{q}) := q_2F_1(\boldsymbol{q}) - q_1F_2(\boldsymbol{q})$ *have a common linear factor* $P = \alpha q_1 + \beta q_2$, $|\alpha| + |\beta| \neq 0$, *then the multiplicity of* P *in* G *is not greater than the multiplicity of* P *in* F_1.

Let us Λ_i *is the non-trivial Kovalevskaya exponent of* $\boldsymbol{F}$ *at* i*-th Darboux point for* $i = 1, \ldots, l$. *Then*

$$\sum_{i=1}^{l} \frac{1}{\Lambda_i} = -1.$$

Non-maximal number of Darboux points – Hamiltonian case

The number of Darboux points is not maximal if and only if V has a multiple factor.
A generic cases: V has a factor different that $(q_2 \pm \mathrm{i} q_1)$ with multiplicity 2. Such potential are equivalent to

$$W(q_1, q_2) = q_1^2 \widetilde{V}(q_1, q_2), \qquad \deg \widetilde{V} = k - 2.$$

Theorem

Hamiltonian system with potential W does not admit an additional rational first integral.

Non-maximal number of Darboux points – Newton case

If a force $\boldsymbol{F} = (F_1, F_2)$ does not have the maximal number of Darboux points, then polynomials F_1 and F_2 have a common factor $P \in \mathbb{C}[\boldsymbol{q}] \setminus \mathbb{C}$, i.e., $F_i = P\widetilde{F}_i$, where $\widetilde{F}_i \in \mathbb{C}[\boldsymbol{q}]$ for $i = 1, 2$.

Theorem

Assume that $\widetilde{F}_1, \widetilde{F}_2 \in \mathbb{C}[\boldsymbol{q}]$ are homogeneous polynomials of degree $k-2$. If $\deg_{q_1} \tilde{F}_2 = k-2$, then system

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = -q_2\widetilde{F}_1(q_1, q_2), \quad \dot{q}_2 = p_2, \quad \dot{p}_2 = -q_2\widetilde{F}_2(q_1, q_2),$$

does not admit two functionally independent polynomial first integrals.

Infinitely many Darboux points

In the Hamiltonian case – radial potentials i.e. $V = V(r)$, $r^2 = q_1^2 + q_2^2$.
In the Newton case a system possesses infinitely many Darboux points iff

$$F_1 = \sum_{i=0}^{k-2} f_i q_1^{k-1-i} q_2^i \quad \text{and} \quad F_2 = \sum_{i=0}^{k-2} f_i q_1^{k-2-i} q_2^{i+1}.$$

$$I_1 = q_1 p_2 - p_1 q_2.$$

For $k = 3$ and $k = 4$ the Newton equations with the above form of forces are integrable

$$I_2 = 3(f_0 p_1 + f_1 p_2)^2 + 2(f_0 q_1 + f_1 q_2)^3,$$
$$I_2 = 2(f_0 p_1^2 + f_1 p_1 p_2 + f_2 p_2^2) + (f_0 q_1^2 + f_1 q_1 q_2 + f_2 q_2^2)^2,$$

Hamiltonian systems without Darboux points

Lemma

If potential V of degree $k>2$ does not have Darboux points, then all its linear factors are multiple. Moreover, $V=V_{k,l}$

$$V_{k,l}=\alpha(q_2-\mathrm{i}q_1)^l(q_2+\mathrm{i}q_1)^{k-l},\qquad l=0,\dots,k,\quad \alpha\in\mathbb{C}^\star$$

for some l, except for the case when $k=2l$ and V has a factor $(q_2\pm\mathrm{i}q_1)$ with multiplicity l.

$$V=(q_2-\mathrm{i}q_1)^l(q_2+\mathrm{i}q_1)^p q_2^{l-p},\qquad p>0,\quad l>1,\quad l>p+1,$$

$$V=(q_2-\mathrm{i}q_1)^l(q_2+\mathrm{i}q_1)^{l-r-s}q_2^r\left(q_2-\mathrm{i}\frac{r+s}{r-s}q_1\right)^s,$$

$$l>1,\quad r>0,\quad s>0,\quad l-r-s\in\mathbb{N}_0\setminus\{1\},$$

Hamiltonian systems without Darboux points

$$
\begin{aligned}
V &= (q_2 - \mathrm{i} q_1)^l (q_2 + \mathrm{i} q_1)^{l-p-r-s} q_2^r \\
&\times \left(q_2 - \frac{\mathrm{i}(r-s)(p+r+s) - 2\sqrt{prs(p+r+s)}}{(r-s)^2 + p(r+s)} q_1 \right)^s \\
&\times \left(q_2 - \frac{\mathrm{i}(r-p)(p+r+s) + 2\sqrt{prs(p+r+s)}}{(p-r)^2 + s(p+r)} q_1 \right)^p ,
\end{aligned}
$$

$$
l > 2, \quad s \in \mathbb{N} \setminus \{1\}, \quad p \in \mathbb{N} \setminus \{1\}, \quad r > 0,
$$

$$
l - p - r - s = 0 \quad \text{or} \quad l - p - r - s > 1,
$$

$$
V = q_1 q_2 (q_2 - \mathrm{i} q_1)^l (q_2 + \mathrm{i} q_1)^{l-2}, \qquad l \in \mathbb{N} \setminus \{1, 3\},
$$

$$
V = q_1 (q_2 - \mathrm{i} q_1)^l (q_2 + \mathrm{i} q_1)^{l-r-1} \left(q_2 + \mathrm{i} \frac{r-1}{r+1} q_1 \right)^r , \quad r = 0, 2, \ldots, l -
$$

Newton systems without Darboux points

Theorem

Assume that a force $\mathbf{F} = (F_1, F_2)$, $\deg F_i = k - 1$, *does not have any Darboux point. Then*

$$F_1 = \prod_{i=1}^{k-1}(q_2 - z_i q_1), \qquad F_2 = q_2 F_1 - \prod_{i=1}^{p}(q_2 - z_i q_1)^{n_i},$$

where $1 \leq p \leq k - 1$, $z_i \in \mathbb{C}$, *and* $z_1, \ldots, z_p$ *are pairwise different;* $n_i \in \mathbb{N}$ *and* $n_1 + \cdots + n_p = k$. *Moreover, if there exist* $1 \leq j \leq p$ *such that* $n_j = 1$, *then the Newton equations with force* $\mathbf{F}$ *do not admit two functionally independent polynomial first integrals.*

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Summary

Summary

- It is possible to express the necessary integrability conditions in the language of eigenvalues Λ_i of Jacobi matrix of the right-hand sides.
- There exist universal relations between these eigenvalues.
- Using these relations one can prove that only finite number of admissible Λ_i satisfying the necessary integrability conditions is possible.
- One can find forms of the potential for Hamiltonian systems and force fields for Newton systems.

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