On the integrability and linearisation properties of certain nonlinear oscillators and systems of the form

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0$$

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Plan of the talk

- We present a new method to solve nonlinear differential equations using integrating factors, null forms, integrals of motion and linearization.
- Ex. Generalized nonlinear oscillator
- We investigate and present results for the case $q = 1$
- We then study the case $q = 2$
- Finally we consolidate the results for $q =$ arbitrary
- Conclusions

2. PS procedure

- Prelle and Singer (1983)
- Duarte et al (2001)
- Generalization (2005)

Let us consider

$$\ddot{x} = \frac{P(t,x,\dot{x})}{Q(t,x,\dot{x})}, \qquad P, Q \in \mathbf{C}[t,x,\dot{x}].$$

M. Prelle and M. Singer, Trans. Am. Math. Soc. **279**, 215, (1983)

L.G.S Durarte *et al.*, J.Phys. A. **34**, 3015, (2001)

V.K.Chandrasekar, M.Senthilvelan and M.Lakshmanan, Proc. R. Soc. London Series A **461**, 2451, (2005)

PS Procedure

- Let $I = I(t, x, \dot{x})$ be the first integral

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x}$$

- Equation of Motion

$$\frac{P}{Q} dt - d\dot{x} = 0$$

- Adding the null form

$$S(t, x, \dot{x})\dot{x} dt - S(t, x, \dot{x}) dx = 0$$

PS procedure

we obtain

$$\left(\frac{P}{Q} + \dot{x}S\right)dt - Sdx - d\dot{x} = 0$$

- Multiplying by the integrating factor $R(t, x, \dot{x})$

$$R(\phi + \dot{x}S)dt - RSdx - Rd\dot{x} = 0 \equiv dI$$

- Comparing the total differentials

$$I_t = R(\phi + \dot{x}S), \quad I_x = -RS, \quad I_{\dot{x}} = -R$$

- Compatibility conditions: $I_{tx} = I_{xt}, \; I_{t\dot{x}} = I_{\dot{x}t}, \; I_{x\dot{x}} = I_{\dot{x}x}$

PS procedure

$$D[S] = -\phi_x + \phi_{\dot{x}} S + S^2, \quad (D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}})$$

$$D[R] = -R(S + \phi_{\dot{x}})$$

$$R_x = S R_{\dot{x}} + R S_{\dot{x}}$$

- Determining equations are overdetermined
- 2 particular solutions fulfill our aim
- The method has several advantageous

Application

Consider the Generalized nonlinear equation (Lienard type)

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0, \qquad q \in R,$$

Case 1: $q = 1,$

$$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0,$$

Case 1: $q = 2$

$$\ddot{x} + (k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x = 0.$$

Case 3: $q = arbitrary$

Integrals of motion

- Integrating

$$I = r_1 - r_2 - \int \left[R + \frac{d}{d\dot{x}} (r_1 - r_2) \right] d\dot{x}$$

where

$$r_1 = \int R(\phi + \dot{x}S)dt, \quad r_2 = \int (RS + \frac{d}{dx} r_1) dx$$

- Two independent integrals guarantees integrability

Application: $q = 1$

Gen. Nonlinear Oscillator Eqn. $(q = 1)$

$$\ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0$$

Review

$k_i = 0,\ i = 1, \ldots 4$	➡	simple harmonic oscillator
$k_1,\ k_2 = 0$	➡	anharmonic oscillator
$k_1,\ k_4 = 0$	➡	force-free Duffing oscillator
$k_2,\ k_4,\ \lambda_1 = 0$	➡	MEE
$k_1,\ k_3 = 0$	➡	Helmhotz oscillator

Application: $q = 1$

- Contains both linearizable and integrable equations
- Question: Any new integrable equations beside the above?
- Answer: Yes

Analysis:

$$\text{Case 1} \qquad I_t = 0$$
$$\text{Case 2} \qquad I_t \neq 0$$

Case 1 $I_t = 0$

Null forms $\quad (I_t = 0 = R(\phi + \dot{x}S))$

$$S = \frac{-\phi}{\dot{x}} = \frac{[(k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x]}{\dot{x}}$$

Integrating Factors $\quad (D[R] = -R(S + \phi_{\dot{x}}))$

$$R_t + \dot{x}R_x - [(k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x]\, R_{\dot{x}}$$
$$= \left\{ (k_1 x + k_2) + \frac{[(k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x]}{\dot{x}} \right\} R$$

Ansatz

$$R = \frac{\dot{x}}{(A(x) + B(x)\dot{x})^r}$$

Integrating factors

(i) k_1, k_2: arbitrary; k_3, k_4 λ_1 interms of k_1, k_2

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{r}\left(\frac{k_1}{2}x^2 + k_2 x\right) + \dot{x}\right]^r}, \quad r \neq 0$$

(ii) k_1,λ_1: arbitrary; $k_2 = 0$, $k_4 = 0$, k_3 interms of k_1

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{2r}k_1 x^2 + \frac{r\lambda_1}{k_1} + \dot{x}\right]^r}, \quad r \neq 0$$

(iii) $k_1 = k_2 = 0$

$$R = \dot{x}, \quad r = 0$$

Compatible solutions

$$\text{(i)}\quad S = \frac{\left[(k_1 x + k_2)\dot{x} + \frac{(r-1)}{r^2}\left(\frac{k_1^3}{2}x^2 + \frac{3k_1k_2}{2}x^2 + k_2^2 x\right)\right]}{\dot{x}}$$

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{r}\left(\frac{k_1}{2}x^2 + k_2 x\right) + \dot{x}\right]^r}, \quad k_1, k_2 = \text{arbitrary}, \; r \neq 0$$

$$\text{(ii)}\quad S = \frac{\left[k_1 x\dot{x} + \frac{(r-1)}{2r^2}k_1^2 x^3 + \lambda_1 x\right]}{\dot{x}}, \qquad k_1 = \text{arbitrary}$$

$$R = \frac{\dot{x}}{\left[\frac{(r-1)}{2r}k_1 x^2 + \frac{r\lambda_1}{k_1} + \dot{x}\right]^r}, \qquad k_2 = 0, \quad r \neq 0$$

$$\text{(iii)}\quad S = \frac{(k_3 x^3 + k_4 x^2 + \lambda_1 x)}{\dot{x}}, \qquad R = \dot{x}, \qquad k_1,\; k_2 = 0$$

Integrable Equations

$$\text{(i)}\ \ddot{x} + (k_1 x + k_2)\dot{x} + \frac{(r-1)}{2r^2}\left(k_1^2 x^3 + 3k_1 k_2 x^2 + 2k_2^2 x\right) = 0$$

$$r \neq 0$$

$$\text{(ii)}\ \ddot{x} + k_1 x\dot{x} + \frac{(r-1)k_1^2}{2r^2}x^3 + \lambda_1 x = 0, \quad r \neq 0$$

$$\text{(iii)}\ \ddot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0, \quad r = 0$$

Transformation $x = \left(y - \frac{k_2}{k_1}\right)$ puts (i) into the form (ii)

$$\ddot{y} + k_1 y\dot{y} + \frac{(r-1)k_1^2}{2r^2}y^3 - \frac{(r-1)k_2^2}{2r^2}y = 0, \ \ r \neq 0$$

First Integrals

$$\text{(iia) } I_1 = \left(\dot{x} + \frac{(r-1)}{2r}k_1x^2 + \frac{r\lambda_1}{k_1}\right)^{-r} \left\{\dot{x}\left[\dot{x} + \frac{k_1}{2}x^2 + \frac{r^2\lambda_1}{(r-1)k_1}\right] + \frac{(r-1)}{r^2}\left[\frac{k_1}{2}x^2 + \frac{r^2\lambda_1}{(r-1)k_1}\right]^2\right\}, r \neq 0, 1, 2$$

$$\text{(iib)} I_1 = \frac{4k_1\dot{x}}{k_1^2x^2 + 4k_1\dot{x} + 8\lambda_1} - \log(k_1^2x^2 + 4k_1\dot{x} + 8\lambda_1), r = 2$$

$$\text{(iic)} I_1 = \dot{x} + \frac{k_1}{2}x^2 - \frac{\lambda_1}{k_1}\log(k_1\dot{x} + \lambda_1), \quad r = 1$$

$$\text{(iii)} I_1 = \frac{\dot{x}^2}{2} + \frac{k_3}{4}x^4 + \frac{k_4}{3}x^3 + \frac{\lambda_1}{2}x^2, \quad r = 0$$

Hamiltonian Description

Assuming the Hamiltonian

$$I(x,\dot{x}) = H(x,p) = p\dot{x} - L(x,\dot{x})$$

we have

$$\frac{\partial I}{\partial \dot{x}} = \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x}$$

so that

$$p = \int \frac{I_{\dot{x}}}{\dot{x}} d\dot{x}$$

- p is known then L and H can be fixed from the above

Lagrangian

$$\text{(iia)}\ L = \frac{1}{(2-r)(r-1)}\left(\dot{x} + \frac{k_1(r-1)}{2r}x^2 + \frac{r\lambda_1}{k_1}\right)^{2-r},$$

$$r \neq 0, 1, 2$$

$$\text{(iib)}\ L = \log(4k_1\dot{x} + 8\lambda_1 + k_1^2 x^2), \quad r = 2$$

$$\text{(iic)}\ L = \frac{\lambda_1}{k_1}\log(k_1\dot{x} + \lambda_1) + \dot{x}(\log(k_1\dot{x} + \lambda_1) - 1) - \frac{k_1}{2}x^2,$$

$$r = 1$$

$$\text{(iii)}\ L = \frac{\dot{x}^2}{2} - \frac{k_3}{4}x^4 - \frac{k_4}{3}x^3 - \frac{\lambda_1}{2}x^2, \quad r = 0$$

Hamiltonian

$$\text{(iia)}\ H = \left[\frac{((r-1)p)^{\frac{r-2}{r-1}}}{(r-2)} - p\left(\frac{(r-1)}{2r}k_1x^2 + \frac{r\lambda_1}{k_1}\right)\right],$$

$$r \neq 0, 1, 2$$

$$\text{(iib)}\ H = \frac{2\lambda_1}{k_1}p + \frac{k_1}{4}x^2p + \log\left(\frac{4k_1}{p}\right), \quad r = 2$$

$$\text{(iic)}\ H = \frac{1}{k_1}\left(e^p - \lambda_1 p + \frac{k_1^2}{2}x^2 - \lambda_1\right), \quad r = 1$$

$$\text{(iii)}\ H = \frac{p^2}{2} + \frac{k_3}{4}x^4 + \frac{k_4}{3}x^3 + \frac{\lambda_1}{2}x^2, \quad r = 0$$

Canonical momenta

$$\text{(iia,b)}\ p = \frac{1}{r-1}\left(\dot{x} + \frac{k_1(r-1)}{2r}x^2 + \frac{r\lambda_1}{k_1}\right)^{1-r}, \quad r \neq 0, 1$$

$$\text{(iic)}\ p = \log(k_1\dot{x} + \lambda_1), \quad r = 1$$

$$\text{(iii)}\ p = \dot{x}, \quad r = 0$$

Canonical transformation

$$x = \frac{2rP}{k_1 U}, \qquad p = -\frac{k_1 U^2}{4r}, \qquad r \neq 0, 1$$

$$x = \frac{P}{k_1}, \qquad p = -k_1 U, \qquad r = 1$$

Standard Hamiltonians

$$H = \begin{cases} \frac{1}{2}P^2 + \frac{(r-1)\lambda_1}{4}U^2 + \frac{(1-r)}{(r-2)}\left(\frac{(r-1)k_1 U^2}{4r}\right)^{\frac{(r-2)}{r-1}}, & r \neq 0, 1, 2 \\ \frac{1}{2}P^2 + \frac{\lambda_1}{4}U^2 + \log(\frac{32}{U^2}), & r = 2 \\ \frac{1}{2}P^2 + \lambda_1 k_1 U + e^{-k_1 U}, & r = 1 \\ \frac{1}{2}P^2 + \frac{\lambda_1}{2}U^2 + \frac{k_4}{3}U^3 + \frac{k_3}{4}U^4, & r = 0 \end{cases}$$

Equations of motion

$$\ddot{U} - 2\left(\frac{(r-1)k_1}{4r}\right)^{\frac{(2-r)}{(1-r)}} U^{\frac{(3-r)}{(1-r)}} + \frac{(r-1)\lambda_1}{2} U = 0, \quad r \neq 0, 1$$

$$\ddot{U} - k_1 e^{-U} + k_1 \lambda_1 = 0, \qquad r = 1$$

$$\ddot{U} + k_3 U^3 + \lambda_1 U = 0, \qquad r = 0$$

- Now become standard type anharmonic oscillator equations
- Liouville integrable

Case 2 $I_t \neq 0$ (Time independent)

S equation

$$S_t + \dot{x}S_x - ((k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x)S_{\dot{x}}$$
$$= (k_1 \dot{x} + 3k_3 x^2 + 2k_4 x + \lambda_1) - (k_1 x + k_2)S + S^2$$

R equation

$$R_t + \dot{x}R_x - ((k_1 x + k_2)\dot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x)R_{\dot{x}}$$
$$= -\left[-(k_1 x + k_2) + S\right] R$$

Extra determining equation

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0$$

Case $I_t \neq 0$ Time dependent integrals

(i) $k_3 = \frac{k_1^2}{9}$, $k_4 = \frac{k_1 k_2}{3}$ (k_1 k_2, λ_1: arbitrary)

(a) $$I = e^{\mp\omega t}\left(\frac{3\dot{x} - \frac{3(-k_2 \mp \omega)}{2}x + k_1 x^2}{3\dot{x} - \frac{3(-k_2 \pm \omega)}{2}x + k_1 x^2}\right), \quad \omega = (k_2^2 - 4\lambda_1)^{\frac{1}{2}}$$

(b) $$I = -t + \frac{x}{(\frac{k_2}{2}x + \frac{k_1}{3}x^2 + \dot{x})}, \quad k_2^2 = 4\lambda_1$$

(ii) $k_3 = 0$, $k_4 = \frac{k_1}{4}(k_2 \pm \omega)$, ($k_1$, k_2, λ_1: arbitrary)

$$I = \left(\dot{x} + \frac{k_2 \mp \omega}{2}x + \frac{k_1}{2}x^2\right)e^{(\frac{k_2 \pm \omega}{2})t}, \quad \omega = (k_2^2 - 4\lambda_1)^{\frac{1}{2}}$$

Case $I_t \neq 0$ Contd.

(iii) $k_1,\ k_3 = 0,\ \lambda_1 = \frac{6k_2^2}{25}$ ($k_2,\ k_4$: arbitrary)

$$I = e^{\frac{6}{5}k_2 t}\left(\frac{\dot{x}^2}{2} + \frac{2k_2}{5}x\dot{x} + \frac{2k_2^2}{25}x^2 + \frac{k_4}{3}x^3\right)$$

(iva) $k_3 = \frac{(r-1)k_1^2}{2r^2},\ k_4 = \frac{k_1 k_2}{3},\ \lambda_1 = \frac{2k_2^2}{9},\ r \neq 0$

(a) $$I = \left[\frac{k_3}{2}x^4 + (\dot{x} + \frac{k_2}{3}x)(\dot{x} + \frac{k_2}{3}x + \frac{k_1}{2}x^2)\right]$$
$$\times\left(\dot{x} + \frac{k_2}{3}x + rk_3x^2\right)^{-r} e^{\frac{2(2-r)}{3}k_2 t},\quad r \neq 2$$

Case $I_t \neq 0$ Contd.

(ivb) $k_3 = \frac{(r-1)k_1^2}{2r^2},\ k_4 = \frac{k_1 k_2}{3},\ \lambda_1 = \frac{2k_2^2}{9},\ r \neq 0$

$$\text{(b)}\quad I = \frac{2}{3}k_2 t + \log(4k_2 x + 3k_1 x^2 + 12\dot{x}) - \frac{4(k_2 x + 3\dot{x})}{(4k_2 x + 3k_1 x^2 + 12\dot{x})}, \qquad r = 2$$

(ivb) $k_1 = 0,\ k_4 = 0,\ \lambda_1 = \frac{2k_2^2}{9}, r = 0\ k_2,\ k_3$: arbitrary

$$I = e^{\frac{4}{3}k_2 t}\left[\frac{\dot{x}^2}{2} + \frac{k_2}{3}x\dot{x} + \frac{k_2^2}{18}x^2 + \frac{k_3}{4}x^4\right]$$

Transforming Time dependent integral into time independent integral

$$I = F(t, x, \dot{x}) \qquad (1)$$

$$I = F_1(t, x, \dot{x}) + F_2(t, x) \qquad \text{(Step 1)}$$

$$I = F_1\left(\frac{1}{G_2(t, x, \dot{x})}\frac{d}{dt}G_1(t, x)\right) + F_2\left(G_1(t, x)\right) \qquad \text{(Step 2)}$$

$$= F_1\left(\frac{dt}{dz}\frac{dw}{dt}\right) + F_2(w)$$

$$w = G_1(t, x), \quad z = \int_o^t G_2(t', x, \dot{x})dt'$$

$$I = F_1\left(\frac{dw}{dz}\right) + F_2(w) \qquad \text{(Step 3)}$$

Transforming Time dependent integral into time independent integral

$$F_1\left(\frac{dw}{dz}\right) = I - F_2(w)$$

$$\frac{dw}{dz} = f(w)$$

- $F_2(w) = 0$ we get linear equation. Then w and z are linearizing transformation.
- Time independent integrals
- Rewriting interms original variables we get the solution for the original equation

Application

Case (ia): $k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3},\ k_1,\ k_2$ and λ_1 : arbitrary

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x = 0$$

$$I = e^{\mp\omega t}\left(\frac{3\dot{x} - \frac{3(-k_2 \mp \omega)}{2}x + k_1 x^2}{3\dot{x} - \frac{3(-k_2 \pm \omega)}{2}x + k_1 x^2}\right), \quad \omega = (k_2^2 - 4\lambda_1)^{\frac{1}{2}}$$

$$I = -\frac{k_1 e^{\frac{k_2 \mp \omega}{2}t} x^2}{(3\dot{x} - \frac{(-k_2 \pm \omega)}{2}3x + k_1 x^2)}\left[\frac{d}{dt}\left(\left(\frac{-3}{k_1 x} + \frac{-k_2 \pm \omega}{2\lambda_1}\right)e^{\frac{-k_2 \mp \omega}{2}t}\right)\right]$$

$$I = \frac{dt}{dz}\frac{dw}{dt}, \qquad \left(\omega = \sqrt{k_2^2 - 4\lambda_1}\right)$$

Application

$$w = \left(\frac{-3}{k_1 x} + \frac{-k_2 \pm \omega}{2\lambda_1}\right) e^{\frac{-k_2 \mp \alpha}{2} t}, \; z = \left(\frac{-3}{k_1 x} + \frac{-k_2 \mp \omega}{2\lambda_1}\right) e^{\frac{-k_2 \pm \omega}{2} t}$$

$$I = \frac{dw}{dz} \Longrightarrow 0 = \frac{d^2 w}{dz^2} \Longrightarrow w = I_1 z + I_2$$

- In terms of original variables

$$x(t) = \left(\frac{6\lambda_1(1 - I_1 e^{\omega t})}{k_1 \omega (1 + I_1 e^{\omega t}) - (k_2 \pm \omega) I_2 e^{\frac{k_2 \pm \omega}{2} t} - k_1 k_2 (1 - I_1 e^{\omega t})}\right)$$

$$\omega = \sqrt{k_2^2 - 4\lambda_1}$$

Sub-cases of Case 1

Case $k_2^2 < 4\lambda_1$

$$x(t) = \frac{A\cos(\omega_0 t + \delta)}{\left[e^{\frac{k_2}{2}t} + \frac{2k_1 A}{3(k_2^2+4\omega_0^2)}\left(2\omega_0 \sin(\omega_0 t + \delta) - k_2 \cos(\omega_0 t + \delta)\right)\right]}$$

$$\omega_0 = \frac{\sqrt{4\lambda_1 - k_2^2}}{2}$$

A further restriction $k_2 = 0$ yields

$$x(t) = \frac{A\sin\left(\omega_0 t + \delta\right)}{1 - \left(\frac{k}{3\omega_0}\right)A\cos\left(\omega_0 t + \delta\right)}, \quad 0 \leq A < \frac{3\omega_0}{k}, \quad \omega_0 = \sqrt{\lambda_1}$$

- Amplitude independent frequency of oscillations

Sub-cases of Case 1

Case $2k_2^2 > 4\lambda_1$: Solution looks dissipative/front-like

$$x(t) = \left(\frac{3k_2(I_1 e^{k_2 t} - 1)}{k_1 + k_2(3I_2 + k_1 I_1 t)e^{k_2 t}} \right) \quad (\lambda_1 = 0)$$

Case (ib): $k_3 = \frac{k_1^2}{9},\ k_4 = \frac{k_1 k_2}{3},\ k_2^2 = 4\lambda_1$

$$x(t) = \left(\frac{3(I_1 + t)}{3I_2 e^{\frac{k_2}{2}t} - \frac{2k_1}{k_2^2}(2 + I_1 k_2 + k_2 t)} \right)$$

- A further restriction $k_2, \lambda_1 = 0$ yields MEE

Case (ii)

$k_3 = 0,\ k_4 = \frac{k_1}{4}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1}),\ k_1,\ k_2\ , \lambda_1 :$ arbitrary

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1}{4}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^2 + \lambda_1 x = 0$$

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \frac{k_1}{2}x^2\right) e^{\frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2}t}$$

Nothing but Riccati equation

$$\dot{x} = I e^{(\frac{-k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2})t} - \left(\frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}\right)x - \frac{k_1}{2}x^2$$

Sub-case of case (ii) $\lambda_1 = \frac{2k_2^2}{9}$

The restriction $\lambda_1 = \frac{2k_1^2}{9}$ fixes the equation of motion

$$\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1 k_2}{3} x^2 + \frac{2k_2^2}{9} x = 0$$

Integral of motion

$$I = \left(\dot{x} + \frac{k_2}{3} x + \frac{k_1}{2} x^2 \right) e^{\frac{2k_2}{3} t}$$

We identify

$$w = x e^{\frac{k_2}{3} t}, \quad z = -\frac{3}{k_2} e^{-\frac{k_2}{3} t}$$

Sub-case continued

Time dependent integral now becomes

$$\hat{I} = w' + \frac{k_1}{2} w^2$$

Integrating, we obtain

$$w(z) = \sqrt{\frac{2I}{k_1}} \tanh\left[\sqrt{\frac{k_1 I}{2}}(z - z_0)\right]$$

In terms of old variables

$$x(t) = \sqrt{\frac{2I}{k_1}} e^{-(\frac{k_2}{3})t} \tanh\left[\frac{3}{k_2}(\sqrt{\frac{k_1 I}{2}})(e^{-\frac{k_2}{3}t_0} - e^{-\frac{k_2}{3}t})\right]$$

(iii) $k_1,\ k_3 = 0,\ \lambda_1 = \frac{6k_2^2}{25},\ k_2\ , k_4\ :$ **arbitrary**

Equation of motion

$$\ddot{x} + k_2\dot{x} + k_4 x^2 + \frac{6k_2^2}{25}x = 0$$

Time dependent first integral

$$I = e^{\frac{6}{5}k_2 t}\left(\frac{\dot{x}^2}{2} + \frac{2k_2}{5}x\dot{x} + \frac{2k_2^2}{25}x^2 + \frac{k_4}{3}x^3\right)$$

$$I = \frac{1}{2}\left(\dot{x} + \frac{2k_2}{5}x\right)^2 e^{\frac{6}{5}k_2 t} + \frac{k_4}{3}x^3 e^{\frac{6}{5}k_2 t}$$

$$I = e^{\frac{2k_2 t}{5}}\left[\frac{d}{dt}\left(\frac{1}{\sqrt{2}}xe^{\frac{2k_2 t}{5}}\right)\right]^2 + \frac{k_4}{3}\left(xe^{\frac{2}{5}k_2 t}\right)^3$$

(iii) $k_1, \; k_3 = 0, \; \lambda_1 = \frac{6k_2^2}{25}, \; k_2 \; , k_4 \; :$ **arbitrary**

Transformation

$$w = \frac{1}{\sqrt{2}} x e^{\frac{2k_2 t}{5}}, \qquad z = -\frac{5}{k_2} e^{-\frac{k_2 t}{5}}$$

Transformed Integral

$$\hat{I} = w'^2 + \frac{\hat{k_4}}{3} w^3, \qquad \hat{k_4} = 2\sqrt{2} k_4$$

so that

$$w'^2 = 4w^3 - g_3, \qquad z = 2\sqrt{\frac{3}{\hat{k_4}}}\hat{z}, \;\; g_3 = -\frac{12 I_1}{\hat{k_4}}$$

Solution: in terms of Weierstrass function

Case (iv)

$k_3 = \frac{(r-1)}{2r^2}k_1^2,\ k_4 = \frac{k_1k_2}{3},\ \lambda_1 = \frac{2k_2^2}{9},\ k_1,\ k_2,\ r$: arbitrary

Equation of motion

$$\ddot{x} + (k_1x + k_2)\dot{x} + \frac{(r-1)k_1^2}{2r^2}x^3 + \frac{k_1k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0, \quad r \neq 0$$

Time dependent first integral

$$I = \begin{cases} \left(\frac{(r-1)}{4r^2}k_1^2x^4 + (\dot{x} + \frac{k_2}{3}x)(\dot{x} + \frac{k_2}{3}x + \frac{k_1}{2}x^2)\right) & r \neq 0, 2 \\ \qquad \times \left(\dot{x} + \frac{k_2}{3}x + \frac{(r-1)}{2r}k_1x^2\right)^{-r} e^{\frac{2(2-r)}{3}k_2t}, & \\ \frac{2}{3}k_2t + \log(4k_2x + 3k_1x^2 + 12\dot{x}) - \frac{4(k_2x+3\dot{x})}{(4k_2x+3k_1x^2+12\dot{x})}, & r = 2 \end{cases}$$

Case (iv)

We identify

$$w = xe^{\frac{k_2}{3}t}, \quad z = -\frac{3}{k_2}e^{-\frac{k_2}{3}t}$$

Time dependent first integral becomes

$$I = \begin{cases} \left(w' + \frac{(r-1)}{2r}k_1 w^2\right)^{-r}\left[\frac{(r-1)}{4r^2}k_1^2 w^4 + w'(w' + \frac{k_1}{2}w^2)\right], & r \neq 0,2 \\ \\ \frac{4w'}{k_1 w^2 + 4w'} - \log(k_1 w^2 + 4w'), & r = 2 \end{cases}$$

Hamiltonian

Equation of motion now becomes

$$w'' + k_1 w w' + \frac{(r-1)k_1^2}{2r^2} w^3 = 0, \quad r \neq 0 \ \text{ and } ' = \frac{d}{dz}$$

Underlying Hamiltonian

$$H = \begin{cases} \dfrac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - p\left(\frac{(r-1)}{2r} k_1 w^2\right), & r \neq 0, 1, 2, \\ \frac{k_1}{4} w^2 p + \log\left(\frac{4k_1}{p}\right), & r = 2 \\ e^p + \frac{k_1}{2} w^2, & r = 1 \\ \frac{p^2}{2} + \frac{k_3}{4} w^4, & r = 0 \end{cases}$$

Summary of results for the case $q = 1$

We identified 6 integrable systems. The following 4 are already known in the literature

(1) $\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1^2}{9}x^3 + \frac{k_1 k_2}{3}x^2 + \lambda_1 x = 0$

(2) $\ddot{x} + (k_1 x + k_2)\dot{x} + \frac{k_1 k_2}{3}x^2 + \frac{2k_2^2}{9}x = 0$

(3) $\ddot{x} + k_2\dot{x} + k_4 x^2 + \frac{6k_2^2}{25}x = 0$

(4) $\ddot{x} + k_3 x^3 + k_4 x^2 + \lambda_1 x = 0$

New integrable equations

The following 2 equations are new

$$(4) \quad \ddot{x} + k_1 x\dot{x} + k_3 x^3 + \lambda_1 x = 0$$

$$(5) \quad \ddot{x} + (k_1 x + k_2)\dot{x} + k_3 x^3 + \frac{k_1 k_2}{3} x^2 + \frac{2k_2^2}{9} x = 0$$

where $r^2 k_3 = \frac{(r-1)k_1^2}{2}$, k_1, k_2, λ_1, r are arbitrary parameters.

- Eq.(4) is the first equation in MEE hierarchy
- (5) transformed into integrable equation. Explicit solutions are known only for special values of r.

Case 2 $q = 2$

Gen. Nonlinear Oscillator Eqn. ($q = 2$)

$$\ddot{x} + (k_1 x^2 + k_2)\dot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x = 0$$

Review:

$k_3 = 0$	➡	force-free DVP oscillator
$k_2,\ k_4,\ \lambda_1 = 0$	➡	modified Emden hierarchy
$k_3 = \frac{k_1^2}{16},\ k_4 = \frac{k_1 k_2}{4}$, $\lambda_1 = \omega_0^2 + \frac{k_2}{4}$	➡	A 5th power nonlinear oscillator

Q: Any new integrable equation besides the above?

Procedure $q = 2$

Analysis:

Case 1	Time independent integral
Case 2	Time dependent integral

- We use the same ansatz (used for the case $q = 1$) to determine the functions S and R.

Time independent integrals ($q = 2$)

Integrable Equations

$$\text{(ia)}\quad \ddot{x} + (k_1 x^2 + k_2)\dot{x} + \frac{(r-1)}{3r^2} k_1^2 x^5 + \frac{4(r-1)k_1 k_2}{3r^2} x^3 + \frac{(r-1)k_2^2}{r^2} x = 0, \; r \neq 0$$

$$\text{(ib)}\quad \ddot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x = 0, \; r = 0$$

Associated Hamiltonians

$$\text{(ia)}\quad H = \left[\frac{((r-1)p)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r} p\left(\frac{k_1}{3} x^3 + k_2 x\right) \right], \quad r \neq 0, 1, 2$$

Hamiltonian ($q = 2$)

$$\text{(ib)}\quad H = \frac{k_2}{2}xp + \frac{k_1}{6}x^3p + \log(\frac{6}{p}) \quad r = 2,$$

$$\text{(ic)}\quad H = e^p + \frac{k_1}{3}x^3 + k_2x \quad r = 1$$

$$\text{(ii)}\quad H = \frac{p^2}{2} + \frac{k_3}{6}x^6 + \frac{k_4}{4}x^4 + \frac{\lambda_1}{2}x^2, \quad r = 0$$

- Canonical transformations are yet to be explored to transform the above H into standard H

Time dependent integrals ($q = 2$)

Case (ia)

$$\ddot{x} + (k_1 x^2 + k_2)\dot{x} + \frac{k_1^2}{16}x^5 + \frac{k_1 k_2}{4}x^3 + \lambda_1 x = 0$$

First Integral

$$I = e^{\mp\omega t}\left(\frac{4\dot{x} + 2(k_2 \pm \omega)x + k_1 x^3}{4\dot{x} + 2(k_2 \mp \omega)x + k_1 x^3}\right)$$

Bernoulli Equation. Solution takes the form

$$x(t) = \left(\frac{8k_2\lambda_1(e^{\omega t} - I_1)^2}{I_1^2 k_1 k_2(-k_2 + \omega) - e^{2\omega t}k_1 k_2(k_2 + \omega) + 8I_2 k_2 \lambda_1 e^{(k_2+\omega)t} + 8I_1 k_1 \lambda_1 e^{\omega t}}\right)^{\frac{1}{2}},$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$

Time dependent integral ($q = 2$)

Sub-cases:

Case: $k_2^2 < 4\lambda_1$

$$x(t) = \frac{Acos(\omega_0 t + \delta)}{\left(e^{k_2 t} - \frac{k_1 A}{4k_2} + \frac{k_1 A}{4(k_2^2+4\omega_0^2)}\left(2\omega_0 sin2(\omega_0 t + \delta) - k_2 cos2(\omega_0 t + \delta)\right)\right)^{\frac{1}{2}}}$$

Case: $k_2^2 > 4\lambda_1, \quad \lambda_1 = 0$

$$x(t) = \left(\frac{2\sqrt{k_2}(I_1 e^{k_2 t} - 1)}{(-k_1 + 2k_1 I_1 e^{k_2 t}(2 + k_2 I_1 t e^{k_2 t}) + 4k_2 I_2 e^{2k_2 t})^{\frac{1}{2}}}\right)$$

Sub-case: $k_3 = \frac{k_1^2}{16},\ k_2,\ \lambda_1 = 0$: Second equation in MEE hierarchy.

Time dependent integral ($q = 2$)

Case (ii)

$$\ddot{x} + (k_1 x^2 + k_2)\dot{x} + \frac{k_1}{6}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^3 + \lambda_1 x = 0$$

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \frac{k_1}{3}x^3\right) e^{\frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2}t}$$

Abel Equation. Solution can be obtained for the parametric choice $\lambda_1 = \frac{3}{16}k_2^2$

Case (iii): $k_1, k_2 = 0, \lambda_1 = \frac{2k_1^2}{9}$

Force free Duffing Oscillator

Time dependent integral ($q = 2$)

Case (iv)

$$\ddot{x} + (k_1 x^2 + k_2)\dot{x} + \frac{(r-1)k_1^2}{3r^2}x^5 + \frac{k_1 k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0, \quad r \neq 0.$$

Introducing the transformation

$$w = x e^{\frac{k_2}{4}t}, \quad z = -\frac{2}{k_2}e^{-\frac{k_2}{2}t}$$

We obtain

$$w'' + k_1 w^2 w' + \frac{(r-1)k_1^2}{3r^2}w^5 = 0, \quad r \neq 0, \qquad {}' = \frac{d}{dz}.$$

The transformed equation admits time independent H

Time dependent integral ($q = 2$)

$$H = \begin{cases} \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{3r} k_1 w^3 p \right], & r \neq 0,1,2, \\ \frac{k_1}{6} w^3 p + \log\left(\frac{6}{p}\right), & r = 2 \\ e^p + \frac{k_1}{3} w^3, & r = 1 \\ \frac{p^2}{2} + \frac{k_3}{6} w^6, & r = 0 \end{cases}$$

where

$$p = \begin{cases} \frac{1}{(r-1)} \left(w' + \frac{(r-1)}{3r} k_1 w^3 \right)^{(1-r)}, & r \neq 0,1 \\ \log w', & r = 1 \\ \frac{p^2}{2} + \frac{k_3}{6} w^6, & r = 0. \end{cases}$$

Summary of results ($q = 2$)

We identified 6 Integrable equations. The following 3 already known in the literature

$$(1)\ddot{x} + (k_1 x^2 + k_2)\dot{x} + \frac{k_1 k_2}{4}x^3 + \frac{3k_2^2}{16}x = 0$$

$$(2)\ddot{x} + k_2\dot{x} + k_3 x^3 + \frac{2k_2^2}{9}x = 0$$

$$(3)\ddot{x} + k_3 x^5 + k_4 x^3 + \lambda_1 x = 0$$

(1)	Force-free Duffing-van der Pol oscillator
(3)	Force-free Duffing oscillator

Summary of results ($q = 2$)

New integrable equations

$$(4)\ \ddot{x} + (k_1 x^2 + k_2)\dot{x} + k_3 x^5 + \frac{4(r-1)k_1 k_2}{3r^2} x^3 + \frac{(r-1)k_2^2}{r^2} x = 0, \; r \neq 0$$

$$(5)\ \ddot{x} + (k_1 x^2 + k_2)\dot{x} + \frac{k_1^2}{16} x^5 + \frac{k_1 k_2}{4} x^3 + \lambda_1 x = 0$$

$$(6)\ \ddot{x} + (k_1 x^2 + k_2)\dot{x} + k_3 x^5 + \frac{k_1 k_2}{4} x^3 + \frac{3k_2^2}{16} x = 0$$

$r^2 k_3 = \frac{(r-1)k_1^2}{3}$ and k_1, k_2, λ_1 and r arbitrary parameters.

- (4) Liouville integrable
- (5) General Solution is derived
- (6) Time dependent I transformed into time independent

Case $q = arbitrary$

Gen. Nonlinear Oscillator Eqn.

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{(2q+1)} + k_4 x^{q+1} + \lambda_1 x = 0$$

Review:

$k_2,\ k_4,\ \lambda_1 = 0$ ➠ MEE hierarchy

$k_3 = \frac{k_1^2}{(q+2)^2},\ k_4 = \frac{k_1 k_2}{(q+2)}$ ➠ Smith

$\lambda_1 = \omega_0^2 + \frac{k_2^2}{4}$

- Any new integrable equation besides the above?
- We use same methodology for the present case

Case 2 Time independent integrals ($q = arbitrary$)

New integrable Hamiltonian system

$$(ia)\ \ddot{x} + ((q+1)\hat{k}_1 x^q + k_2)\dot{x} + \frac{(r-1)}{r^2}[(q+1)\hat{k}_1^2 x^{2q+1} + (q+2)\hat{k}_1 k_2 x^{q+1} + k_2^2 x] = 0 \ r \neq 0$$

$$(ib)\quad \ddot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0 \ r = 0.$$

Associated Hamiltonian

(ia) $H = \left[\frac{((r-1)p)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r} p(\hat{k}_1 x^{q+1} + k_2 x)\right], \quad r \neq 0, 1, 2,$

Case 2 $q = arbitrary \; I_t = 0$

(ib) $H = \frac{k_2}{2} xp + \frac{\hat{k}_1}{2} x^{q+1} p + \log(\frac{2(q+1)}{p}) \;\; r = 2$

(ic) $H = e^p + \hat{k}_1 x^{q+1} + k_2 x, \;\; r = 1,$

(ii) $H = \frac{p^2}{2} + \frac{k_3}{2(q+1)} x^{2(q+1)} + \frac{k_4}{(q+1)} x^{q+1} + \frac{\lambda_1}{2} x^2, \;\; r = 0$

- Canonical transformations are yet to be explored to transform the above H into natural H
- We use same methodology for the present case

Case 2 Time dependent integrals ($q = arbitrary$)

Integrable Equation 1

$$\ddot{x} + ((q+2)\hat{k}_1 x^q + k_2)\dot{x} + \hat{k}_1^2 x^{2q+1} + \hat{k}_1 k_2 x^{q+1} + \lambda_1 x = 0,$$

where $k_1 = (q+2)\hat{k}_1$.

Solution

$$x(t) = \left(e^{\omega t} - I_1\right)\left[e^{\frac{q}{2}(k_2+\omega)t}\left(I_2 + \hat{k}_1 q \int \left(\frac{e^{\omega t} - I_1}{e^{\frac{1}{2}(k_2+\omega)t}}\right)^q dt\right)\right]^{\frac{-1}{q}},$$

where $\omega = \sqrt{k_2^2 - 4\lambda_1}$.

- The solution for the sub-case $k_2^2 < 4\lambda_1$ has been considered by Smith.
- In the case $k_2, \ \lambda_1 = 0$ we get the MEE hierarchy and their corresponding solutions

$I_t \neq 0$ **(**$q = arbitrary$**)**

Integrable Equation 2

$$\ddot{x} + ((q+1)\hat{k}_1 x^q + k_2)\dot{x} + \frac{\hat{k}_1}{2}(k_2 \pm \sqrt{k_2^2 - 4\lambda_1})x^{q+1} + \lambda_1 x = 0,$$

First integral

$$I = \left(\dot{x} + \frac{k_2 \mp \sqrt{k_2^2 - 4\lambda_1}}{2}x + \hat{k}_1 x^{q+1}\right)e^{\left(\frac{k_2 \pm \sqrt{k_2^2 - 4\lambda_1}}{2}\right)t},$$

where $k_1 = (q+1)\hat{k}_1$.

- We are able to integrate only for the choice $\lambda_1 = (q+1)\hat{k}_2^2$

$I_t \neq 0$ **(**$q = arbitrary$**)**

Integrable Equation 3

$$\ddot{x} + (q+4)\hat{k}_2\dot{x} + k_4 x^{(q+1)} + 2(q+2)\hat{k}_2^2 x = 0$$

where $k_2 = (q+4)\hat{k}_2$. First integral

$$I = \frac{1}{2}\left(\dot{x} + 2\hat{k}_2 x\right)^2 e^{2(q+2)\hat{k}_2 t} + \frac{k_4 x^{(q+2)}}{(q+2)} e^{2(q+2)\hat{k}_2 t}$$

where $k_1 = (q+1)\hat{k}_1$.

- Choosing $w = \frac{x}{\sqrt{2}} e^{2\hat{k}_2 t}, \; z = -\frac{1}{q\hat{k}_2} e^{-q\hat{k}_2 t}$ we can transform time dependent integral into time independent integral
- Solution also derived

$I_t \neq 0$ **(**$q = arbitrary$**)**

Integrable Equation 4

$$\ddot{x} + ((q+1)\hat{k}_1 x^q + (q+2)\hat{k}_2)\dot{x} + (q+1)(\frac{(r-1)}{r^2}\hat{k}_1^2 x^{2q}$$

$$+\hat{k}_1\hat{k}_2 x^q + \hat{k}_2^2)x = 0,\ k_1 = (q+1)\hat{k}_1,\ k_2 = (q+2)\hat{k}_2$$

Transformation

$$w = xe^{\hat{k}_2 t}, \quad z = -\frac{1}{q\hat{k}_2}e^{-q\hat{k}_2 t}.$$

Transformed Equation

$$w'' + (q+1)\hat{k}_1 w^q w' + (q+1)\frac{(r-1)}{r^2}\hat{k}_1^2 w^{2q+1} = 0, \quad r \neq 0, \quad ' = \frac{d}{dz}.$$

$q = arbitrary \; I_t \neq 0$

Hamiltonian

$$H = \begin{cases} \left[\frac{\left((r-1)p\right)^{\frac{r-2}{r-1}}}{(r-2)} - \frac{(r-1)}{r} \hat{k}_1 w^{q+1} p \right], & r \neq 0, 1, 2, \\ \frac{1}{2} \hat{k}_1 w^{q+1} p + \log\left(\frac{2(q+1)}{p}\right), & r = 2, \\ e^p + \hat{k}_1 w^{q+1}, & r = 1, \\ \frac{p^2}{2} + \frac{k_3}{2(q+1)} w^{2(q+1)}, & r = 0, \end{cases}$$

$$p = \begin{cases} \frac{1}{(r-1)} \left(w' + \frac{(r-1)}{r} \hat{k}_1 w^{q+1} \right)^{(1-r)}, & r \neq 0, 1 \\ \log(w'), & r = 1 \\ w', & r = 0, \end{cases}$$

Integrable equations in $q = arbitrary$

(1) $\ddot{x} + (k_1 x^q + (q+2)k_2)\dot{x} + k_1 k_2 x^{q+1} + (q+1)k_2^2 x = 0$

(2) $\ddot{x} + ((q+2)k_1 x^q + k_2)\dot{x} + k_1^2 x^{2q+1} + k_1 k_2 x^{q+1} + \lambda_1 x = 0$

(3) $\ddot{x} + (q+4)k_2 \dot{x} + k_4 x^{q+1} + 2(q+2)k_2^2 x = 0$

(4) $$\ddot{x} + ((q+1)k_1 x^q + k_2)\dot{x} + \frac{(r-1)}{r^2}((q+1)k_1^2 x^{2q} + (q+2)k_1 k_2 x^q + k_2^2)x = 0, \; r \neq 0$$

(5) $$\ddot{x} + ((q+1)k_1 x^q + (q+2)k_2)\dot{x} + (q+1)(k_3 x^{2q} + k_1 k_2 x^q + k_2^2)x = 0$$

where $r^2 k_3 = (r-1)k_1^2$ and $k_1, \; k_2, \; k_4, \; \lambda_1$ and r are arbitrary parameters.

Conclusions

- We have studied integrability and linearization properties
- We have shown that the generalized nonlinear oscillator equation admits both Hamiltonian and dissipative structures
- Explicit form of the Hamiltonian's are constructed
- Several New integrable equations are explored
- These nonlinear oscillators posses very interesting properties (frequency independent amplitude of oscillations)

Conclusions

- More systems may be found out by improving the ansatz.
- PS procedure is very useful in identifying time independent integrals in dissipative systems.

  1 damped harmonic oscillator

  2 modified Emden equation