Finiteness of integrable n-dimensional homogeneous polynomial potentials

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Outline

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Darboux points of homogeneous systems

Let us consider a system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = \boldsymbol{F}(\boldsymbol{q}), \qquad \boldsymbol{q} \in \mathbb{C}^n.$$

where F_i are homogeneous polynomials of degree k, i.e.

$$F_i(\lambda x_1, \ldots, \lambda x_n) = \lambda^k F_i(x_1, \ldots, x_n)$$

Definition

A non-zero vector $\boldsymbol{d} \in \mathbb{C}^n$ is called a Darboux point of system $\boldsymbol{F}$ if $\boldsymbol{d}$ is parallel to $\boldsymbol{F}(\boldsymbol{d})$ and $\boldsymbol{F}(\boldsymbol{d}) \neq \boldsymbol{0}$ i.e.

$$\boldsymbol{F}(\boldsymbol{d}) = \gamma \boldsymbol{d}, \qquad \gamma \in \mathbb{C}^\star.$$

Darboux point – some remarks

- Let $\boldsymbol{d}$ and $\widetilde{\boldsymbol{d}}$ represent the same Darboux point, i.e. $\boldsymbol{d} = \alpha\widetilde{\boldsymbol{d}}$ for some $\alpha \in \mathbb{C}^\star$. Thus we have

$$\boldsymbol{F}(\boldsymbol{d}) = \gamma\boldsymbol{d}, \qquad \boldsymbol{F}(\widetilde{\boldsymbol{d}}) = \widetilde{\gamma}\widetilde{\boldsymbol{d}}, \quad \text{where} \quad \widetilde{\gamma} = \frac{\gamma}{\alpha^{k-2}}.$$

Definition

Darboux point is a solution of

$$\boldsymbol{F}(\boldsymbol{d}) = \boldsymbol{d}. \tag{1}$$

- Equation (1) has in generic case $k^n - 1$ non-zero solutions.

Number of Darboux points

If $\boldsymbol{d} \neq \mathbf{0}$ is a solution of equation

$$\boldsymbol{F}(\boldsymbol{d}) = \boldsymbol{d}.$$

then by homogeneity of $\boldsymbol{F}$, also $\widetilde{\boldsymbol{d}} := \varepsilon \boldsymbol{d}$ is a solution of this equation provided ε is a $(k-1)$-th root of the unity. Thus if the above equation has m different solutions, then they define only $m/(k-1)$ different Darboux points.

Fact

n-dimensional dynamical system with homogeneous right-hand sides of degree k possesses maximally $D_{\boldsymbol{F}}(n,k) := (k^n - 1)/(k-1)$ isolated Darboux points.

Kovalevskaya exponents

The Kovalevskaya matrix $\boldsymbol{K}(\boldsymbol{d})$ at a Darboux point $\boldsymbol{d} \in \mathcal{D}_{\boldsymbol{F}}$

$$\boldsymbol{K}(\boldsymbol{d}) := \boldsymbol{F}'(\boldsymbol{d}) - \boldsymbol{E},$$

Eigenvalues $\Lambda_i = \Lambda_i(\boldsymbol{d})$, $i = 1, \ldots, n$, of the Kovalevskaya matrix $\boldsymbol{K}(\boldsymbol{d})$ are called the Kovalevskaya exponents.

By homogeneity one of them, let us say Λ_n, is trivial and it is $\Lambda_n = k - 1$.

$\boldsymbol{\Lambda}(\boldsymbol{d}) = (\Lambda_1(\boldsymbol{d}), \ldots, \Lambda_{n-1}(\boldsymbol{d}))$ – nontrivial Kovalevskaya exponents.

Guillot theorem (2004)

Theorem

Assume that the above system with homogeneous polynomial right hand sides of degree k has the maximal number of simple Darboux points and let S be a symmetric homogeneous polynomial in $n-1$ variables of degree less than n. Then, the number

$$R := \sum_{\boldsymbol{d} \in \mathcal{D}_F} \frac{S(\boldsymbol{\Lambda}(\boldsymbol{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))},$$

depends only on the choice of S, dimension n and homogeneity degree k.

Guillot, A., Un théorème de point fixe pour les endomorphismes de l'espace projectif avec des applications aux feuilletages algébriques, *Bull. Braz. Math. Soc. (N.S.)*, 35(3):345–362, 2004.

How to find explicit form of relations between Λ-s?

1. find a homogeneous system for any $k \geq 2$ and $n \geq 2$ with maximal number of Darboux points

Answer

n-dimensional generalisation of the Jouanolou system:

$$\dot{x}_i = x_{i+1}^k, \qquad 1 \leq i \leq n, \quad x_{n+1} \equiv x_1.$$

2. choose some standard homogeneous polynomials in $n-1$ variables of degree less than n

Answer

elementary symmetric polynomials in $(n-1)$ variables of degree r i.e. $\tau_0(\boldsymbol{x}) := 1$ and

$$\tau_r(\boldsymbol{x}) := \tau_r(x_1, \ldots, x_{n-1}) = \sum_{1 \leq i_1 < \cdots < i_r \leq n-1} \prod_{s=1}^{r} x_{i_s}, \ 1 \leq r \leq n-1.$$

Darboux points $d = (d_1, \ldots, d_n)$ of Jouanolou system

$$d_n = s, \quad d_{n-1} = s^k, \quad d_{n-2} = s^{k^2}, \ldots, d_2 = s^{k^{n-2}}, \quad d_1 = s^{k^{n-1}}, \tag{2}$$

where s is a primitive root of unity of degree $k^n - 1$, i.e. s is a solution of the cyclotomic equation

$$s^{k^n - 1} - 1 = 0. \tag{3}$$

Equation (2) has $k^n - 1$ complex solutions but as was mentioned only $(k^n - 1)/(k - 1)$ of them are Darboux points.

Kovalevskaya matrix for Jouanolou system

$$\boldsymbol{K}(\boldsymbol{d}) = \begin{pmatrix} -1 & kd_2^{k-1} & 0 & \cdots & \cdots & 0 \\ 0 & -1 & kd_3^{k-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & -1 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 & kd_n^{k-1} \\ kd_1^{k-1} & 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}.$$

Kovalevskaya exponents

Characteristic polynomial of Kovalevskaya matrix

$$P(\Lambda) = \det\left(\boldsymbol{K}(\boldsymbol{d}) - \Lambda \boldsymbol{E}\right) = (-1)^n(\Lambda+1)^n$$
$$+(-1)^{n-1}k^n d_1^{k-1}\cdots d_n^{k-1} = (-1)^n[(\Lambda+1)^n - k^n].$$

All Darboux points have the same Kovalevskaya exponents

$$\Lambda_i = k\varepsilon^{n-i} - 1, \qquad 1 \leq i \leq n, \tag{4}$$

where ε is a primitive n-th root of the unity.

Elementary symmetric functions of Kovalevskaya exponents

Factorisation of characteristic polynomial

$$P(\Lambda) = (-1)^n(\Lambda + 1 - k)Q(\Lambda, n, k),$$

$$Q(\Lambda, n, k) = \sum_{p=0}^{n-1}\sum_{i=p}^{n-1}\binom{i}{p}k^{n-1-i}\Lambda^p$$

$$= \sum_{q=0}^{n-1}\sum_{i=n-1-q}^{n-1}\binom{i}{n-1-q}k^{n-1-i}\Lambda^{n-1-q}.$$

Symmetric functions $\tau_r(\mathbf{\Lambda})$ are up to the sign coefficients of the above polynomial

$$\tau_r(\mathbf{\Lambda}) = (-1)^r\sum_{i=n-1-r}^{n-1}\binom{i}{n-1-r}k^{n-1-i}, \qquad r = 0, \ldots, n-1.$$

Elementary symmetric polynomials of Kovalevskaya exponents

Lemma

Let $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_{n-1})$ denotes the non-trivial Kovalevskaya exponents calculated at a Darboux point of the Jouanolou system. Then the elementary symmetric polynomials of $\mathbf{\Lambda}$ take the following values

$$\tau_r(\mathbf{\Lambda}) = (-1)^r \sum_{i=0}^{r} \binom{n-i-1}{r-i} k^i, \tag{5}$$

for $0 \leq r \leq n-1$.

Explicit forms of relations

Proposition

For $0 \leq r \leq n-1$ *we have*

$$\sum_{\boldsymbol{d}\in\mathcal{D}_V} \frac{\tau_1(\boldsymbol{\Lambda}(\boldsymbol{d}))^r}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{n-1}(1-n-k)^r, \tag{6}$$

and

$$\sum_{\boldsymbol{d}\in\mathcal{D}_V} \frac{\tau_r(\boldsymbol{\Lambda}(\boldsymbol{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{n+r+1}\sum_{i=0}^{r}\binom{n-i-1}{r-i}k^i. \tag{7}$$

Outline

Homogeneous Hamiltonian equations

Hamiltonian systems defined by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \qquad (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n},$$

$$V(\boldsymbol{q}) \in \mathbb{C}[\boldsymbol{q}], \qquad \deg V = k > 2,$$

$V(\boldsymbol{q})$ — homogeneous polynomial

$$\frac{\mathrm{d}}{\mathrm{d}\,t} q_i = \frac{\partial H}{\partial p_i} = p_i, \qquad \frac{\mathrm{d}}{\mathrm{d}\,t} p_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i}, \quad i = 1, \dots, n$$

Kovalevskaya exponents for homogeneous Hamilton systems

Idea

With a Hamiltonian system generated by homogeneous potential V we relate the following auxiliary system with homogeneous polynomial right-hand sides of degree $k-1$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q} = V'(\boldsymbol{q}). \tag{8}$$

and use the mentioned relations.

The Kovalevskaya matrix $\boldsymbol{K}(\boldsymbol{d})$ at a Darboux point $\boldsymbol{d} \in \mathcal{D}_V$

$$\boldsymbol{K}(\boldsymbol{d}) := V''(\boldsymbol{d}) - \boldsymbol{E},$$

where $V''(\boldsymbol{d})$ is the Hessian of potential V

Existence of polynomials with maximal number of Darboux points

Lemma

Let $\mathcal{G}_k \subset \mathcal{H}_k$ be a set of all homogeneous potentials of degree $k > 2$ such that

1. *if $V \in \mathcal{G}_k$, then V has maximal number of Darboux points $\boldsymbol{d}_1, \ldots \boldsymbol{d}_s$, where*

$$s = D(n, k-1) = \frac{(k-1)^n - 1}{k-2},$$

and

2. *for each Darboux point all the Kovalevskaya exponents are different from zero.*

Then set $\mathcal{G}_k$ is open and non-empty.

Proof of the lemma (idea)

- $\mathbb{C}$-linear space $\mathcal{H}_k$ of all homogeneous polynomials of degree k has dimension $d = \binom{n+k-1}{k}$.
- fix an monomial ordering $\prec$ of variables $q_1, \ldots, q_n$. Then every homogeneous polynomial V of degree k can be uniquely written in the form

$$V = \sum_{i=1}^{d} v_i \boldsymbol{q}^{\boldsymbol{\alpha}_i},$$

  where $\boldsymbol{q}^{\boldsymbol{\alpha}_1} \prec \cdots \prec \boldsymbol{q}^{\boldsymbol{\alpha}_d}$ are all monomials of degree k.
- we identify $\mathcal{H}_k$ with $\mathbb{C}^d$ identifying V with $(v_1, \ldots, v_d) \in \mathbb{C}^d$.
- fixing in $\mathbb{C}^d$ an arbitrary norm we convert $\mathcal{H}_k$ into a complete normed space.
- $\mathcal{H}_k$ is not empty

$$V_0 = \sum_{i=1}^{n} q_i^k.$$

Proof of the lemma – continuation

- To prove that that $\mathcal{G}_k$ is open we have to show that for every $V \in \mathcal{G}_k$ all potentials close enough to V also belongs to $\mathcal{G}_k$.
- a Darboux point $\boldsymbol{d}$ of $V \in \mathcal{H}_k$ is a zero of

$$\boldsymbol{G}(\boldsymbol{q}) := V'(\boldsymbol{q}) - \boldsymbol{q}. \tag{9}$$

- if $V \in \mathcal{G}_k$, then a Darboux point $\boldsymbol{d}$ of V is an isolated zero of $\boldsymbol{G}$.
  the Jacobian of $\boldsymbol{G}$ calculated at $\boldsymbol{d}$ is not singular as $\boldsymbol{G}'(\boldsymbol{d}) = \boldsymbol{K}(\boldsymbol{d})$ and by assumption $\det \boldsymbol{K}(\boldsymbol{d}) \neq 0$. Let $V \in \mathcal{G}_k$ and

$$W = \sum_{i=1}^{d} \varepsilon_i \boldsymbol{q}^{\alpha_i}.$$

  Darboux points of $V + W$ are solutions $\boldsymbol{d}(\varepsilon)$ of

$$\boldsymbol{G}(\boldsymbol{d}, \varepsilon) := V'(\boldsymbol{d}) + W'(\boldsymbol{d}) - \boldsymbol{d} = \boldsymbol{0}, \qquad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d).$$

Proof of the lemma – continuation

- By assumption that $V \in \mathcal{G}_k$ we have that for $\varepsilon = \mathbf{0}$ the above equation has s isolated solutions $\boldsymbol{d}(\mathbf{0})$.
- Hence for $\|\varepsilon\|$ small enough there exist s solutions $\boldsymbol{d}(\varepsilon)$. This exactly means that for an arbitrary $V \in \mathcal{G}_k$ there exists an open subset of $\mathcal{G}_k$ containing V, i.e., $\mathcal{G}_k$ is open.

Relations between Kovalevskaya exponents for homogeneous potentials

Theorem

Let us assume that a homogeneous polynomial potential $V \in \mathbb{C}[\boldsymbol{q}]$ of degree k has $D(n, k-1)$ Darboux points $\boldsymbol{d} \in \mathcal{D}_V$. Then non-trivial Kovalevskaya exponents $\Lambda(\boldsymbol{d})$ satisfy the following relations:

$$\sum_{\boldsymbol{d}\in\mathcal{D}_V} \frac{\tau_1(\Lambda(\boldsymbol{d}))^r}{\tau_{n-1}(\Lambda(\boldsymbol{d}))} = (-1)^{n-1}(2-n-k)^r, \tag{10}$$

or, alternatively

$$\sum_{\boldsymbol{d}\in\mathcal{D}_V} \frac{\tau_r(\Lambda(\boldsymbol{d}))}{\tau_{n-1}(\Lambda(\boldsymbol{d}))} = (-1)^{n+r+1}\sum_{i=0}^{r}\binom{n-i-1}{r-i}(k-1)^i, \tag{11}$$

How to use these relations for integrability analysis of homogeneous Hamilton equations?

Fact

Without additional assumptions on Λ-s the existence of the mentioned relations is only interesting observation.

How to find additional obstructions on Λ-s ?

Answer

To apply Morales-Ramis theory.

Outline

Variational equations

Main Idea

A non-linear system leaves fingerprints of its properties in variational equations.

For a system

$$\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, \ldots x_n)^T \in \mathbb{C}^n.$$

with a known non-equilibrium particular solution $\boldsymbol{\varphi}(t)$ the substitution $\boldsymbol{x} = \boldsymbol{\varphi}(t) + \boldsymbol{\xi}$ is applied.

Variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} = A(t)\boldsymbol{\xi}, \qquad A(t) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}(\boldsymbol{\varphi}(t)).$$

Hamiltonian systems – Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ*, and that the variational equations along* Γ *are Fuchsian. Then the identity component of the differential Galois group of the variational equations along* Γ *is Abelian.*

This theorem yields the necessary conditions of integrability in the class of meromorphic functions.

Particular Solutions

Particular solutions:

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \qquad \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d},$$

$$\ddot{\varphi} = -\varphi^{k-1}.$$

Energy level:

$$H(\varphi(t)\boldsymbol{d}, \dot{\varphi}(t)\boldsymbol{d}) = e \in \mathbb{C}^{\star}.$$

Hyperelliptic phase curve:

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right), \qquad \varepsilon = ke.$$

Variational equations

$$\ddot{\boldsymbol{x}} = -\varphi(t)V''(\boldsymbol{d})\boldsymbol{x}.$$

If $V''(\boldsymbol{d})$ is diagonalisable, then in an appropriate base

$$\ddot{y}_i = -\lambda_i \varphi(t)^{k-2} y_i, \qquad 1 \leq i \leq n, \tag{12}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $V''(\boldsymbol{d})$

By homogeneity of V, $\lambda_n = k - 1$.

Transformation to hypergeometric equations

$$z := \frac{1}{k}\varphi(t)^k.$$

$$\left.\begin{array}{l} z(1-z)\eta'' + [c-(a+b+1)z]\eta' - ab\eta = 0, \\ a+b = \dfrac{k-2}{2k}, \qquad ab = -\dfrac{\lambda_i}{2k}, \qquad c = 1-\dfrac{1}{k}. \end{array}\right\} \tag{13}$$

Other Morales-Ramis Theorem

If the Hamiltonian system with homogeneous potential is is meromorphically integrable then each (k, λ_i) belong to the following list:

$$\left(k, p+\frac{k}{2}p(p-1)\right), \qquad \left(k, \frac{1}{2}\left[\frac{k-1}{k}+p(p+1)k\right]\right),$$
$$\left(3, -\frac{1}{24}+\frac{1}{6}(1+3p)^2\right), \left(3, -\frac{1}{24}+\frac{3}{32}(1+4p)^2\right),$$
$$\left(3, -\frac{1}{24}+\frac{3}{50}(1+5p)^2\right), \left(3, -\frac{1}{24}+\frac{3}{50}(2+5p)^2\right),$$
$$\left(4, -\frac{1}{8}+\frac{2}{9}(1+3p)^2\right), \left(5, -\frac{9}{40}+\frac{5}{18}(1+3p)^2\right),$$
$$\left(5, -\frac{9}{40}+\frac{1}{10}(2+5p)^2\right), \qquad p \in \mathbb{Z}.$$

Weakeness of this theorem in applications

$$V = \frac{1}{3}aq_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{1}{3}cq_2^3.$$

$$\lambda_1 = \frac{1}{c}, \qquad \lambda_{2,3} = \frac{2c-1}{1+a^2 \mp \Delta}, \qquad \Delta = \sqrt{a^2(2+a^2-2c)}$$

$$\lambda_1, \lambda_2, \lambda_3 \in \left\{ p + \frac{3}{2}p(p-1) \right\} \cup \left\{ \frac{1}{2}\left(\frac{2}{3} + 3p(p+1) \right) \right\}$$
$$\cup \left\{ -\frac{1}{24} + \frac{1}{6}(1+3p)^2 \right\} \cup \left\{ -\frac{1}{24} + \frac{3}{32}(1+4p)^2 \right\}$$
$$\cup \left\{ -\frac{1}{24} + \frac{3}{50}(1+5p)^2 \right\} \cup \left\{ -\frac{1}{24} + \frac{3}{50}(2+5p)^2 \right\}$$

$$c = \frac{1}{\lambda_1}, \quad a = \frac{\lambda_1 + \lambda_1\lambda_i - 2}{\sqrt{2\lambda_1\lambda_i(2-\lambda_1-\lambda_i)}}, \qquad i = 2, 3.$$

Two types of integrability obstructions

1. all $\Lambda_i = \lambda_i - 1$ must belong to appropriate sets of rational numbers (items in Morales-Ramis theorem for Hamiltonian systems). This is a local analysis.
2. between Λ_i calculated at all Darboux points some universal relations exist. This is a global analysis.

Integrability obstructions

1. $\Lambda_i = \lambda_i - 1$, for $i = 1, \ldots, n-1$ must belong to appropriate sets of rational numbers
2. the following relations must be satisfied

$$\sum_{\boldsymbol{d}\in\mathcal{D}_{\boldsymbol{F}}} \frac{\tau_1(\boldsymbol{\Lambda}(\boldsymbol{d}))^r}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^n(n+k-2)^r, \quad 0 \leq r \leq n-1,$$

or, alternatively

$$\sum_{\boldsymbol{d}\in\mathcal{D}_{\boldsymbol{F}}} \frac{\tau_r(\boldsymbol{\Lambda}(\boldsymbol{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{n+r+1}\sum_{i=0}^{r}\binom{n-r-1}{r-i}(k-1)^i.$$

Finiteness of choices of Λ_i

$$\sum_{\boldsymbol{d}\in\mathcal{D}_V}\sum_{i=1}^{n-1}\frac{1}{\Lambda_i(\boldsymbol{d})} = -\frac{(k-1)^n - n(k-2) - 1}{(k-2)^2}.$$

Theorem

Let us assume that homogeneous system possesses the maximal number of simple Darboux points. Then for homogeneous Hamiltonian equations there exist only finite number of sets $(\Lambda_1, \ldots, \Lambda_{n-1})$ at each Darboux point belonging to the appropriate sets of rational numbers (depending on k) satisfying the above relation.

How to find admissible Λ-s? (case $n = 3$)

Task

to find all solutions of

$$\sum_{i=1}^{s} X_i = -c_k, \quad \text{where} \quad c_k > 0 \tag{14}$$

assuming that $X_i = 1/\Lambda_i \in A_k \subset \mathbb{Q}$. For given k set A_k is obtained from Morales-Ramis table.

1. $A_k = A_k^- \cup A_k^+$, and A_k^- is finite
2. if $a \in A_k$, then $a = p/q$ where q belongs to a finite subset of $\mathbb{N}$.
3. if $a \in A_k^-$, then $a \leq -1$,
4. if $a \in A_k^+$, then $a \leq 1$,

Some number of X_i must be negative. We have a finite number of choices of them.

Example for $n = 3$ and $k = 3$

For $n = k = 3$ we have 7 Darboux points, so $s = D(3, 3-1)(3-1) = 14$ and $c_k = 4$. By 3 and 4 at most 9 of X_i can be negative. Assume that $X_1, \ldots, X_7$ are negative. Taking $X_1, \ldots, X_7 \in A_3^-$ we obtain 2426 different sums of negative elements

$$X_1 + \cdots + X_7 < -4 \qquad (15)$$

How to find admissible Λ-s – continuation

Thus we reduce the problem to finding solutions of

$$\sum_{i=1}^{r} X_i = c, \quad \text{where} \quad c > 0, \qquad X_i \in A_k^+. \tag{16}$$

If $(X_1, \ldots, X_r)$ is a solution of the above equation then we can assume that $X_1 \geq X_2 \geq \cdots \geq X_r$. Hence $X_1 \geq c/r$ and we have only a finite number of choices for X_1. For each of this choice we have to find solution of equation (16) where instead of r is $r - 1$. Repeating this reasoning we can conclude that there is only a finite number of solutions of (16). The problem is that we do not know *a priori* the lower bound of X_r.

How to find admissible Λ-s – continuation

We can order elements of A_k^+ into decreasing sequence $\{a_i^{(k)}\}$. Then $a_i^{(k)}$ decreases with i as i^{-2}. Assume that

$$X_1 + X_2 = \varepsilon$$

then $\varepsilon > X_1 \geq \varepsilon/2$ and the number of admissible choices increases as $1/\varepsilon$. It appears that ε can be of order 10^{-15} and smaller.

If we have a solution of (14) then it gives a finite number of choices of $(\Lambda_1^{(i)}, \ldots, \Lambda_{n-1}^{(i)})$, $i = 1, \ldots, s$ where s is the number of Darboux points.

Next step: to check if $(\Lambda_1^{(i)}, \ldots, \Lambda_{n-1}^{(i)})$, $i = 1, \ldots, s$ that we found satisfy the remaining relations.

Example for $n = 3$ and $k = 3$ – continuation

In the case consider in Example we have to check all choices of 7 pairs from 14 numbers. It gives 105 pairs, thus we have to check if the remaining relations are satisfied about 10^{14} times.

Equivalent potentials

Definition

$$V(\boldsymbol{q}) \sim V_A(\boldsymbol{q}) = V(A\boldsymbol{q}), \qquad A \in \mathrm{PO}(n, \mathbb{C}),$$

$$\mathrm{PO}(n, \mathbb{C}) := \{A \in \mathrm{GL}(n, \mathbb{C}) \,|\, AA^T = \alpha E, \quad \alpha \in \mathbb{C}\}.$$

- Integrability of a Hamiltonian system with some homogeneous potential implies the integrability of all systems with homogeneous potentials belonging to the same equivalence class.

Equivalent potentials – some properties

- Let $\boldsymbol{d}$ be a Darboux point of potential V. Then an equivalent potential V_A has a Darboux point $\widetilde{\boldsymbol{d}} = A^T\boldsymbol{d}$ as we have

  $$V_A'(\widetilde{\boldsymbol{d}}) = A^T V'(AA^T\boldsymbol{d}) = A^T V'(\alpha\boldsymbol{d}) = \alpha^{k-1}A^T V'(\boldsymbol{d}) = \widetilde{\gamma}\widetilde{\boldsymbol{d}},$$

  where $\widetilde{\gamma} = \gamma\alpha^{k-1}$. Moreover, $\gamma^{-1}V''(\boldsymbol{d})$ and $\widetilde{\gamma}^{-1}V_A''(\widetilde{\boldsymbol{d}})$ have the same eigenvalues.
- Number of Darboux points and eigenvalues λ_i (as well as Λ_i) characterise classes of equivalent potentials.

Finiteness of Λ-s versus finiteness of admissible potentials. $n = 2$

$$V = \sum_{i=0}^{k} v_{k-i} q_1^{k-i} q_2^i$$

- number of unknown coefficients $d_c = k + 1$.
- by assumption that number of Darboux points is maximal $v_1 \neq 0$, by rescalling $v_1 = 1$ and $d_c = k$.
- number of Darboux points $d_D = k$.
- one relation between Λ-s calculated at all Darboux points. Thus $d_D = k - 1$ algebraic conditions on the potential coefficients.
- $d_c - d_D = 1$ and generally obstructions on coefficients give one-parameter families of reconstructed potentials. But we have at our disposal one parameter family of generalised rotations.

Finiteness for $n > 2$

- number of unknown coefficients $d_c = \binom{n+k-1}{k}$
- some condition on maximal number of Darboux points.
- number of Darboux points $d_D = ((k-1)^n - 1)/(k-2)$ and at every Darboux point $n-1$ Λ-s, i.e.

$$d_D = \frac{(k-1)^n - 1}{k-2} \times (n-1)$$

  algebraic conditions
- but between them n relations and as result we have the following number of obstructions

$$d_D = \frac{(k-1)^n - 1}{k-2} - n$$

- Still at our disposal $n(n-1)/2$ parameter family of generalised rotations.
- Number of algebraic conditions d_D grows faster with n and k than number of conditions d_c.

Examples of admissible Λ_i for $n = 2$ and $k = 3, 4$

$\{\Lambda_1, \Lambda_2, \Lambda_3\}$
$\{-1, -1, 1\}$
$\{-2/3, 4, 4\}$
$\{-7/8, 14, 14\}$
$\{-2/3, 7/3, 14\}$.

$\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$
$\{-1, -1, 2, 2\}$
$\{-5/8, 5, 5, 5\}$
$\{-5/8, 2, 20, 20\}$
$\{-5/8, 27/8, 27/8, 135\}$
$\{-5/8, 2, 14, 35\}$.

Examples of admissible pairs of $\{\Lambda_1, \Lambda_2\}$ for $n = 3$ and $k = 3$

$$\{3\times\{-1,-1\}, 3\times\{-1,1\}, \{1,1\}\}$$

$$\left\{\{-1,-1\}, \left\{-1,-\frac{2}{3}\right\}, \left\{1,-\frac{2}{3}\right\}, 2\times\{-1,4\}, 2\times\{1,4\}\right\}$$

$$\left\{\{-1,-1\}, \left\{-1,-\frac{7}{8}\right\}, \left\{1,-\frac{7}{8}\right\}, 2\times\{-1,14\}, 2\times\{1,14\}\right\}$$

$$\left\{\{-1,-1\}, \left\{-1,-\frac{2}{3}\right\}, \left\{1,-\frac{2}{3}\right\}, \left\{-1,\frac{7}{3}\right\}, \left\{1,\frac{7}{3}\right\}, \{-1,14\}, \{1,14\}\right\}$$

$$\left\{\left\{-\frac{2}{3},-\frac{2}{3}\right\}, 2\times\left\{-\frac{2}{3},\frac{7}{3}\right\}, \{-1,4\}, \{1,4\}, 2\times\{4,14\}\right\}$$

$$\left\{\left\{-\frac{2}{3},-\frac{7}{8}\right\}, \left\{-\frac{7}{8},4\right\}, \left\{-\frac{3}{8},4\right\}, 2\times\left\{\frac{7}{3},4\right\} 2\times\{4,21\}\right\}$$

$$\left\{\left\{-\frac{2}{3},-\frac{2}{3}\right\}, 3\times\left\{-1,\frac{7}{3}\right\}, 3\times\{6,14\}\right\}$$

Problem

Formulation

To do as complete as possible classification of integrable potentials in a certain general class of functions without restriction on degree with respect to the momenta.

Problem seems to be intractable. Thus we make the restriction.

Formulation

To do as complete as possible classification of homogeneous integrable potentials up to generalised rotations.

Why homogeneous potentials are such important?

- some physical systems are described by homogeneous potentials e.g.

$$V = \frac{\alpha}{3}x_1^3 + x_1 x_2^2, \quad \text{Hénon-Heiles system,}$$

$$V = m^2 q_1^2 q_2^2 + \frac{\Lambda}{2} q_1^4 + \frac{\lambda}{2} q_2^4, \quad \text{FRW cosmological model}$$

see e.g. the recent paper of Delphine Boucher and Jacques-Arthur Weil about FRW system.

- integrability of inhomogeneous systems versus integrability of homogeneous systems

$$V(\boldsymbol{q}) = V_{\min}(\boldsymbol{q}) + \ldots + V_{\max}(\boldsymbol{q}).$$

Fact

If Hamiltonian system

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q})$$

with polynomial potential $V(\boldsymbol{q})$ admits a polynomial (meromorphic) first integral, then also Hamiltonian systems

$$H_{\min} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V_{\min}(\boldsymbol{q}), \qquad H_{\max} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V_{\max}(\boldsymbol{q}),$$

possess polynomial (rational) first integrals.

Problem

Problem

How to find integrable homogeneous potentials?

Solution

Apply Morales-Ramis theory combined with relations on Λ-s.

Algorithm for integrability analysis for homogeneous potentials

1. Fix k and postulate the general form of polynomial potential V with undetermined coefficients.
2. Determine sets for which belong all Λ_i (subset from Morales-Ramis table)
3. Using finiteness relation find all admissible $(\Lambda_1, \ldots, \Lambda_{n-1})$ at each Darboux point.
4. using Λ_i reconstruct V i.e. find all undetermined coefficients.
5. If after reconstruction some polynomial coefficients become free parameters apply stronger integrability obstruction due to variational equations of higher orders.
6. analyse nongeneric cases.

Nongeneric cases

Additional analysis must be done for potentials with

1. infinitely many Darboux points (Darboux points are not isolated),
2. non-maximal number of isolated Darboux points,
3. multiple Darboux points,
4. without Darboux points.

Outline

$n = 2$ **and** $\deg V(\boldsymbol{q}) = 3$

Known integrable potentials of degree 3:

Case	Potential
1	$(q_1 + \mathrm{i}q_2)^l(q_1 - \mathrm{i}q_2)^{3-l},\ l = 0, 1$
2	$q_1^3 + cq_2^3/3$
3	$q_1^2q_2/2 + q_2^3$
4	$q_1^2q_2/2 + 8q_2^3/3$
5	$\mathrm{i}\sqrt{3}q_1^3/18 + q_1^2q_2/2 + q_2^3$

Results of our analysis

Theorem

Hamiltonian system with homogeneous potential of degree 3 is meromorphically integrable if and only if it belongs to items 1–5.

Necessary integrability conditions are satisfied for potentials belonging to items 1–5 as well as for

$$V = \frac{1}{3}aq_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{1}{3}q_2^3.$$

Using the Morales-Ramis theory with higher order variational equations one can prove the nonintegrability of the above potential.

$n = 3$ **and** $\deg V(\boldsymbol{q}) = 3$

first family of Λ-s yields

$$V = a_1 q_1^3 + a_5 q_2^3 + \frac{1}{3} q_3^3, \qquad a_1 a_5 \neq 0. \tag{17}$$

The second family of pairs of Λ-s yield

$$V = \frac{a_5}{2} q_1^2 q_2 + a_5 q_2^3 + \frac{1}{3} q_3^3, \qquad a_5 \neq 0. \tag{18}$$

$n = 3$ and $\deg V(\boldsymbol{q}) = 3$ continuation

third family of Λ-s yields

$$V = \frac{3a_5}{16} q_1^2 q_2 + a_5 q_2^3 + \frac{1}{3} q_3^3, \quad a_5 \neq 0. \tag{19}$$

fourth family of Λ-s yields

$$V = a_5(\pm \mathrm{i}\sqrt{3} q_1^3 + 9 q_1^2 q_2 + 18 q_2^3) + \frac{1}{3} q_3^3, \quad a_5 \neq 0, \tag{20}$$

For the next families of pairs of Λ-s reconstruction of potentials is very complicated. Thus we apply the higher order variational equations from the beginning around a particular solution at the infinity with the pair of integers $\{\Lambda_1, \Lambda_2\}$.

Higher order variational equations corresponding to $\{1,4\}$

application of the third and fourth order variational equations yield

$$
\begin{aligned}
V &= \frac{5\mathrm{i}}{4}q_1^2q_2 + \frac{7\mathrm{i}}{3}q_2^3 + q_1^2q_3 + \frac{5}{2}q_2^2q_3 + \frac{1}{3}q_3^3,\\
F_1 &= 16p_1(p_3 - \mathrm{i}p_2)q_1 + (3q_1^2 + 2(q_2 - 2\mathrm{i}q_3)^2)(3q_1^2\\
&+ 2q_2(5q_2 - 4\mathrm{i}q3)) + 16\mathrm{i}p_1^2(q_2 + \mathrm{i}q_3) + 32(p_2 - 2\mathrm{i}p_3)(p_3q_2 - p_2q_3),
\end{aligned}
\tag{21}
$$

Continuation

$$
\begin{aligned}
F_2 = {} & 1296p_1^4 + 675q_1^6 + 4(-126(p_2^2 + p_3^2)^2 - 90\mathrm{i}(p_2 - \mathrm{i}p_3)(7p_2 \\
& - 23\mathrm{i}p_3)q_1^2 q_2 - 1176\mathrm{i}(p_2^2 + p_3^2)q_2^3 + 315q_1^2 q_2^4 + 2744q_2^6) \\
& + 24(6(p_2 - \mathrm{i}p_3)(61p_2 - 14\mathrm{i}p_3)q_1^2 + 315\mathrm{i}q_1^4 q_2 - 210(p_2^2 + p_3^2)q_2^2 \\
& + 558\mathrm{i}q_1^2 q_2^3 - 980\mathrm{i}q_2^5)q_3 + 18(363q_1^4 + 1140q_1^2 q_2^2 - 700q_2^4)q_3^2 \\
& - 32(21(p_2^2 + p_3^2) + 90\mathrm{i}q_1^2 q2 + 98\mathrm{i}q_2^3)q_3^3 + 96(36q_1^2 - 35q_2^2)q_3^4 \\
& - 224q_3^6 + 48p_1^2((p_2 + \mathrm{i}p_3)(29p_2 - 79\mathrm{i}p_3) + 27q_1^2(5\mathrm{i}q_2 + 4q_3) \\
& + 2(2\mathrm{i}q_2 + q_3)(13q_2^2 - 61\mathrm{i}q_2 q_3 - 7q_3^2) + 600p_1 q_1(p_2(-3\mathrm{i}q_1^2 \\
& + 10\mathrm{i}q_2^2 - 20q_2 q_3 + 8\mathrm{i}q_3^2) + p_3(3q_1^2 + 38q_2^2 + 4\mathrm{i}q_2 q_3 + 16q_3^2))
\end{aligned}
$$

Continuation

$$V = \frac{2\mathrm{i}}{3}q_1^3 + \frac{5\mathrm{i}}{2}q_1 q_2^2 + q_1^2 q_3 + \frac{5}{2}q_2^2 q_3 + \frac{1}{3}q_3^3$$

$$F_1 = 3p_2^2 + 6p_3(p3 + \mathrm{i}p_1) + (q3 + \mathrm{i}q_1)(2q_1^2 + 15q_2^2 + 2\mathrm{i}q_1 q_3 + 4q_3^2),$$
$$F_2 = p_1^2 q_2 - p_1(2\mathrm{i}p_3 q_2 + p_2(q_1 - \mathrm{i}q_3)) + p_2 p_3(q_3 + \mathrm{i}q_1)$$
$$- q_2(p_3^2 - (q_3 + \mathrm{i}q_1)^3)$$

Continuation

$$
\begin{aligned}
V &= q_1^2 q_3 + \frac{5}{2} q_2^2 q_3 + \frac{1}{3} q_3^3 + a_4 q_1 q_2^2 + \frac{7\sqrt{-25-4a_4^2}}{15} q_2^3 + \frac{4a_4^2 - 25}{30 a_4} q_1^3, \\
F_1 &= 75 p_2^2 + 50 q_3^3 + 60 a_4 p_1 p_3 + 20 a_4 q_1 (q_1^2 + 3 q_3^2) - 8 a_4^2 q_3^3 \\
&+ 75 q_2^2 (5 q_3 + 2 a_4 q_1) + p_3^2 (75 - 12 a_4^2) + 70 q_2^3 \sqrt{-25 - 4a_4^2}, \\
F_2 &= -2000 p_3^2 q_2 + 36 q_2^2 (5 q_3 + 2 a_4 q_1)^2 \sqrt{-25 - 4a_4^2} \\
&+ 5 q_2^4 (-25 - 4a_4^2)^{3/2} - 24 q_2^3 (5 q_3 + 2 a_4 q_1)(25 + 4a_4^2) \\
&+ 40 p_2 (5 q_3 + 2 a_4 q_1)(4 a_4 p_1 + p_2 \sqrt{-25 - 4a_4^2}) + 16 q_2 (125 q_3^3 \\
&+ 150 a_4 q_1 q_3^2 - 20 a_4^2 p1^2 + 60 a_4^2 q_1^2 q_3 + 8 a_4^3 q_1^3 \\
&- 5 a_4 \sqrt{-25 - 4a_4^2 p_1 p_2}) - 200 p_3 (8 a_4 p_1 q_2 + p_2 (-10 q_3 \\
&- 4 a_4 q_1 + \sqrt{-25 - 4a_4^2 q_2}))
\end{aligned}
$$

Higher order variational equations corresponding to $\{4, 21\}$

Integrable potential

$$V = 364\sqrt{17}q_1^3 + 2835\mathrm{i}\sqrt{17}q_1^2 q_2 + 1560\sqrt{17}q_1 q_2^2 + 6552\mathrm{i}\sqrt{17}q_2^3 + 4335q_1^2 q_3 + 19074q_2^2 q_3 + 578q_3^3.$$

1. F_1 quadratic in the momenta $\times$ linear combination of coordinates,
2. F_2 quartic in the momenta $\times$ linear combination of coordinates.

Computational problems with analysis of the seventh family of pairs of Λ-s.